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Determinants of matrices related to the Pascal triangle

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Résumé. On étudie les déterminants de matrices associées au triangle de Pascal.

Abstract. The aim of this paper is to study determinants of matrices related to the Pascal triangle.

1. The Pascal triangle

Let P be the infinite symmetric "matrix" with entries $p_{i,j} = \binom{i+j}{i}$ for $0 \leq i, j \in \mathbf{N}$. The matrix P is hence the famous Pascal triangle yielding the binomial coefficients and can be recursively constructed by the rules $p_{0,i} = p_{i,0} = 1$ for $i \geq 0$ and $p_{i,j} = p_{i-1,j} + p_{i,j-1}$ for $1 \leq i, j$.

In this paper we are interested in (sequences of determinants of finite) matrices related to P.

The present section deals with some minors (determinants of submatrices) of the above Pascal triangle P, perhaps slightly perturbed.

Sections 2-6 are devoted to the study of matrices satisfying the Pascal recursion rule $m_{i,j} = m_{i-1,j} + m_{i,j-1}$ for $1 \leq i, j < n$ (with various choices for the first row $m_{0,j}$ and column $m_{i,0}$). Our main result is the experimental observation (Conjecture 3.3 and Remarks 3.4) that given such an infinite matrix whose first row and column satisfy linear recursions (like for instance the Fibonacci sequence $1, 1, 2, 3, 5, 8, 13, 21, \ldots$), then the determinants of a suitable sequence of submatrices seem also to satisfy a linear recursion. We give a proof if all linear recursions are of length at most 2 (Theorem 3.1).

Section 7 is seemingly unrelated since it deals with matrices which are "periodic" along strips parallel to the diagonal. If such a matrix consists only of a finite number of such strips, then an appropriate sequence of determinants satisfies a linear recursion (Theorem 7.1).

Section 8 is an application of section 7. It deals with matrices which are periodic on the diagonal and off-diagonal coefficients satisfy a different kind of Pascal-like relation.

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We come now back to the Pascal triangle P with coefficients $p_{i,j} = \binom{i+j}{i}$. Denote by $P_{s,t}(n)$ the $n \times n$ submatrix of P with coefficients $\binom{i+j+s+t}{i+s}$, $0 \leq i, j < n$ and denote by $D_{s,t}(n) = \det(P_{s,t}(n))$ its determinant.

Theorem 1.1. *We have*

$$D_{s,t}(n) = \prod_{k=0}^{s-1} \frac{\binom{n+k+t}{t}}{\binom{k+t}{t}} \quad , s, t, n \geq 0 \ .$$

In particular, the function $n \longmapsto D_{s,t}(n)$ is a polynomial of degree st in n.

This Theorem follows for instance from the formulas contained in section 5 of [GV] (a beautiful paper studying mainly determinants of finite submatrices of the matrix T with coefficients $t_{i,j} = \binom{j}{i}$). We give briefly a different proof using the so-called "condensation method" (cf. for instance the survey paper [K1]).

Proof of Theorem 1.1. The definition $\binom{a+b}{a} = \frac{(a+b)!}{a!\, b!}$ and a short computation show that Theorem 1.1 boils down to

$$\det(A_k(n)) = \prod_{i=0}^{n-1} i!\ (i+k)!$$

where $A_k(n)$ has coefficients $a_{i,j} = (i+j+k)!$ for $0 \leq i, j < n$ with $k = s+t$. The condensation identity (cf. Proposition 10 in [K1])

$$\det(M)\ \det(M_{1,n}^{1,n}) = \det(M_1^1)\ \det(M_n^n) - \det(M_1^n)\ \det(M_n^1)$$

(with $M_{i_1,\dots,i_k}^{j_1,\dots,j_k}$ denoting the submatrix of the $n \times n$ matrix M obtained by erasing lines $i_1, \dots, i_k$ and columns $j_1, \dots, j_k$) allows a recursive (on n) computation of $\det(A_k(n))$ establishing the result. $\square$

Theorem 1.1 has the following generalization. Let

$$q(x,y) = \sum_{s,t=0} c_{s,t} \binom{x}{s} \binom{y}{t} \in \mathbf{Q}[x,y]$$

be a polynomial in two variables x, y and let $Q(n)$ be the matrix with coefficients $q_{i,j} = q(i,j),\ 0 \leq i, j < n$.

Elementary operations on rows and columns show easily the following result.

Proposition 1.2. *One has for all n*

$$\det(P_{0,0}(n) + Q(n)) = \det(C_Q(n) + \mathrm{Id}_n)$$

where $C_Q(n)$ has coefficients $c_{i,j}$ for $0 \leq i, j < n$ and where Id_n denotes the identity matrix of order n.

In particular, the sequence of determinants

$$\det(P_{0,0}(0)) = 1, \det(P_{0,0}(1)) = 1 + c_{0,0}, \det(P_{0,0}(2)), \dots$$

becomes constant for $n \geq \mu$ *where* $\mu = \min(\mathrm{degree}_x(Q), \mathrm{degree}_y(Q))$ *with* $\mathrm{degree}_x(Q)$ *(respectively* $\mathrm{degree}_x(Q)$*) denoting the degree of* Q *with respect to* x *(respectively* y*).*

In general, the function

$$n \longmapsto \det(P_{s,t}(n) + Q(n))$$

seems to be polynomial of degree $\leq st$ in n for n huge enough.

Consider the symmetric matrix G of order k with coefficients $g_{i,j} = \sum_{s=0}^{n+k-1} \binom{s}{i}\binom{s}{j}$ for $0 \leq i,j < k$. Theorem 1.1 implies $\det(G) = D_{k,k}(n)$ (with $D_{k,k}(n)$ given by the formula of Theorem 1.1).

Let us also mention the following computation involving inverses of binomial coefficients. Given three integers $s,t,n \geq 0$ let $d_{s,t}(n)$ denote the determinant of the $n \times n$ matrix M with coefficients

$$m_{i,j} = \binom{i+s+j+t}{i+s}^{-1} \qquad \text{for } 0 \leq i,j < n \ .$$

Theorem 1.3. *One has*

$$d_{s,t}(n) = (-1)^{\binom{n}{2}} \frac{1}{\prod_{k=0}^{n-1} \binom{2k+s+t}{k+s}\binom{2k-1+s+t}{k}} \ .$$

Sketch of proof. For $0 \leq k \in \mathbf{N}$ introduce the symmetric matrix $A_k(n)$ of order n with coefficients $a_{i,j} = \frac{1}{(i+j+k)!}$, $0 \leq i,j < n$. A small computation shows then that Theorem 1.3 is equivalent to the identity

$$\det(A_k(n)) = (-1)^{\binom{n}{2}} \prod_{i=0}^{n-1} \frac{i!}{(n+k+i-1)!}$$

(with $k = s+t$) which can be proven recursively on n by the condensation method (cf. proof of Theorem 1.1).

Let us now consider the following variation of the Pascal triangle. Recall that a complex matrix of rank 1 and order $n \times n$ has coefficients $\alpha_i \beta_j$ (for $0 \leq i,j < n$) where $\alpha = (\alpha_0, \dots, \alpha_{n-1})$ and $\beta = (\beta_0, \dots, \beta_{n-1})$ are two complex sequences, well defined up to $\lambda\alpha$, $\frac{1}{\lambda}\beta$ for $\lambda \in \mathbf{C}^*$.

Given two infinite sequences $\alpha = (\alpha_0, \alpha_1, \dots)$ and $\beta = (\beta_0, \beta_1, \dots)$ consider the $n \times n$ matrix $A(n)$ with coefficients $a_{i,j} = a_{i-1,j} + a_{i,j-1} + \alpha_i \beta_j$ for $0 \leq i,j < n$ (where we use the convention $a_{i,-1} = a_{-1,i} = 0$ for all i).

Proposition 1.4. (i) *The coefficient* $a_{i,j}$ *(for* $0 \leq i,j < n$*) of the matrix* $A(n)$ *is given by*

$$a_{i,j} = \sum_{s=0}^{i} \sum_{t=0}^{j} \alpha_{i-s} \beta_{j-t} \binom{s+t}{s} \ .$$

(ii) *The matrix* $A(n)$ *has determinant* $(\alpha_0 \beta_0)^n$*.*

Proof. Assertion (i) is elementary and left to the reader.

Assertion (ii) obviously holds if $\alpha_0 = 0$ or $\beta_0 = 0$. We can hence suppose $\beta_0 = 1$. Theorem 1.1 and elementary operations on rows establish the result easily for arbitrary α and $\beta = (1, 0, 0, 0, \dots)$. The case of an arbitrary sequence β with $\beta_0 = 1$ is then reduced to the previous case using elementary operations on columns. □

Another variation on the theme of Pascal triangles is given by considering the $n \times n$ matrix $A(n)$ with coefficients $a_{i,0} = \rho^i,\ a_{0,i} = \sigma^i,\ 0 \le i < n$ and $a_{i,j} = a_{i-1,j} + a_{i,j-1} + x\ a_{i-1,j-1},\ 1 \le i, j < n$. Setting $x = 0,\ \rho = \sigma = 1$ we get hence the matrix defined by binomial coefficients considered above. One has then the following result, due to C. Krattenthaler ([K2] and Theorem 1 of [K3]) which we state without proof.

Theorem 1.5. *One has* $\det(A(n)) = (1+x)^{\binom{n-1}{2}}\ (x + \rho + \sigma - \rho\sigma)^{n-1}$.

Let now $B(n)$ be the skew-symmetric $n \times n$ matrix defined by $b_{i,i} = 0,\ 0 \le i < n,\ b_{i,0} = -b_{0,i} = \rho^{i-1},\ 1 \le i < n,\ b_{i,j} = b_{i-1,j} + b_{i,j-1} + x\ b_{i-1,j-1},\ 1 \le i, j < n$.

The computation of the determinant of $B(2n)$ is again due to Krattenthaler ([K2] and Theorem 2 of [K3]):

$$\det(B(2n)) = (1+x)^{2(n-1)^2}\ (\rho + x)^{2n-2} .$$

2. Generalized Pascal triangles

Let $\alpha = (\alpha_0, \alpha_1, \dots)$ and $\beta = (\beta_0, \beta_1, \dots)$ be two sequences starting with a common first term $\gamma_0 = \alpha_0 = \beta_0$. Define a matrix $P_{\alpha,\beta}(n)$ of order n with coefficients $p_{i,j}$ by setting $p_{i,0} = \alpha_i,\ p_{0,i} = \beta_i$ for $0 \le i < n$ and $p_{i,j} = p_{i-1,j} + p_{i,j-1}$ for $1 \le i, j < n$.

It is easy to see that the coefficient $p_{i,j}$ of $P_{\alpha,\beta}(n)$ is also given by the formula

$$p_{i,j} = \gamma_0 \binom{i+j}{i} + \Big(\sum_{s=1}^{i} (\alpha_s - \alpha_{s-1}) \binom{i-s+j}{j}\Big) + \Big(\sum_{t=1}^{j} (\beta_t - \beta_{t-1}) \binom{i+j-t}{i}\Big) .$$

We call the infinite "matrix" $P_{\alpha,\beta}(\infty)$ the *generalized Pascal triangle* associated to α, β.

We will mainly be interested in the sequence of determinants

$$(\det(P_{\alpha,\beta}(1)) = \gamma_0, \det(P_{\alpha,\beta}(2)) = \gamma_0(\alpha_1 + \beta_1) - \alpha_1\beta_1, \dots,\\ \det(P_{\alpha,\beta}(n)), \dots) .$$

Example 2.1. Take an arbitrary sequence $\alpha = (\alpha_0, \alpha_1, \dots)$ and let β be the constant sequence $\beta = (\alpha_0, \alpha_0, \alpha_0, \dots)$. Proposition 1.4 implies $\det(A_{(\alpha_0,\alpha_1,\dots),(\alpha_0,\alpha_0,\dots)}(n)) = \alpha_0^n$ (using perhaps the convention $0^0 = 1$). This yields an easy way of writing down matrices with determinant 1 by choosing a sequence $\alpha = (\alpha_0 = 1, \alpha_1, \dots)$. The finite sequence $\alpha = (1, -2, 5, 11)$ for instance yields the determinant 1 matrix $\begin{pmatrix} 1 & 1 & 1 & 1 \\ -2 & -1 & 0 & 1 \\ 5 & 4 & 4 & 5 \\ 11 & 15 & 19 & 24 \end{pmatrix}$.

3. Linear recursions

This section is devoted to general Pascal triangles constructed from sequences satisfying linear recursions. Conjecturally, the sequence of determinants of such matrices satisfies then again a (generally much longer) linear recursion. We prove this in the particular case where the defining sequences are of order at most 2.

Definition. A sequence $\sigma = (\sigma_0, \sigma_1, \sigma_2, \dots)$ satisfies a *linear recursion of order d* if there exist constants $D_1, D_2, \dots, D_d$ such that

$$\sigma_n = \sum_{i=1}^{d} D_i \; \sigma_{n-i} \quad \text{ for all } n \geq d \ .$$

The polynomial

$$z^d - \sum_{i=1}^{d} D_i \; z^{d-i}$$

is then called the *characteristic polynomial* of the linear recursion.

Let us first consider generalized Pascal triangles defined by linear recursion sequences of order at most 2:

Given $\gamma_0, \alpha_1, \beta_1, A_1, A_2, B_1, B_2$ we set $\alpha_0 = \beta_0 = \gamma_0$ and consider the square matrix $M(n)$ of order n with entries

$$\begin{aligned} &m_{i,0} = \alpha_i, \ 0 \leq i < n \text{ where } \alpha_k = A_1\alpha_{k-1} + A_2\alpha_{k-2}, \ k \geq 2, \\ &m_{0,i} = \beta_i, \ 0 \leq i < n \text{ where } \beta_k = B_1\beta_{k-1} + B_2\beta_{k-2}, \ k \geq 2, \\ &m_{i,j} = m_{i-1,j} + m_{i,j-1}, \ 1 \leq i, j < n \ . \end{aligned}$$

The matrix $M(3)$ for instance is hence given by $M(3) =$

$$\begin{pmatrix} \gamma_0 & \beta_1 & B_1\beta_1 + B_2\gamma_0 \\ \alpha_1 & \alpha_1 + \beta_1 & \alpha_1 + \beta_1 + B_1\beta_1 + B_2\gamma_0 \\ A_1\alpha_1 + A_2\gamma_0 & \alpha_1 + \beta_1 + A_1\alpha_1 + A_2\gamma_0 & m_{3,3} \end{pmatrix}$$

where $m_{3,3} = 2\alpha_1 + 2\beta_1 + A_1\alpha_1 + B_1\beta_1 + A_2\gamma_0 + B_2\gamma_0$.

We have hence $M(n) = P_{\alpha,\beta}(n)$ where $P_{\alpha,\beta}$ is the generalized Pascal triangle introduced in the previous section.

We set $d(n) = \det(M(n))$ for $n \geq 1$ and introduce the constants

$$\begin{aligned} D_1 = -(&A_1\beta_1 + B_1\alpha_1 - 2(\alpha_1 + \beta_1) \\ &\qquad + \gamma_0 \left(A_1 B_2 + A_2 B_1 - (A_2 + B_2) + A_2 B_2\right)) \\ D_2 = -&\left(A_2\gamma_0 + \alpha_1 + (1 - A_1 - A_2)\beta_1\right)\left(B_2\gamma_0 + \beta_1 + (1 - B_1 - B_2)\alpha_1\right). \end{aligned}$$

Theorem 3.1. *The sequence $d(n)$, $n \geq 1$ defined as above satisfies the following equalities*

$$\begin{aligned} d(1) &= \gamma_0 \ , \\ d(2) &= \gamma_0(\alpha_1+\beta_1) - \alpha_1\beta_1 \ , \\ d(n) &= D_1 \ d(n-1) + D_2 \ d(n-2) \ \textit{for all } n \geq 3 \ . \end{aligned}$$

Theorem 3.1 will be proven below.

Example 3.2 (a) The sequence $(\det(P_{\alpha,\beta}(n)))_{n=1,2,\dots}$ of determinants associated to two geometric sequences

$$\begin{aligned} \alpha &= (1, A, A^2, \dots, \alpha_k = A^k, \dots) \\ \beta &= (1, B, B^2, \dots, \beta_k = B^k, \dots) \end{aligned}$$

is given by

$$\det(P_{\alpha,\beta}(n)) = (A + B - AB)^{n-1} \ .$$

Let $\alpha = (\alpha_0, \alpha_1, \dots)$ and $\beta = (\beta_0, \beta_1, \dots)$ be two sequences satisfying $\alpha_0 = \beta_0 = \gamma_0$ and linear recursions

$$\begin{aligned} \alpha_n &= \textstyle\sum_{i=1}^a A_i \alpha_{n-i} \text{ for } n \geq a \ , \\ \beta_n &= \textstyle\sum_{i=1}^b B_i \beta_{n-i} \text{ for } n \geq b \end{aligned}$$

of order a and b.

Theorem 3.1 and computations suggest that the following might be true.

Conjecture 3.3. *If two sequences $\alpha = (\alpha_0, \alpha_1, \dots), \beta = (\beta_0, \beta_1, \dots)$ satisfy both linear recurrence relations then there exist a natural integer $d \in \mathbf{N}$ and constants $D_1, \dots, D_d$ (depending on α, β) such that*

$$\det(P_{\alpha,\beta}(n)) = \sum_{i=1}^{d} D_i \ \det(P_{\alpha,\beta}(n-i)) \quad \textit{for all } n > d \ .$$

Remarks 3.4. (i) Generically, (i.e. for α and β two generic sequences of order a and b such that $\alpha_0 = \beta_0$) the integer d of Conjecture 3.3 seems to be given by $d = \binom{a+b-2}{a-1}$.

(ii) Generically, the coefficient D_i seems to be a homogeneous form (with polynomial coefficients in $A_1, \dots, A_a, B_1, B_b$) of degree i in $\gamma_0, \alpha_1, \alpha_{a-1}, \beta_1, \dots, \beta_{b-1}$. For non-generic pairs of sequences (try $\beta = -\alpha$ with $\alpha = (0, \alpha_1, \dots)$ satisfying a linear recursion of order 3) the coefficients D_i may be rational fractions in the variables.

(iii) If $a = b > 1$ and if the recursive sequences α, β are generic, then the coefficients $D_0 = -1, D_1, \dots, D_d$ of the linear recursion in Conjecture 3.3 seem to have the symmetry

$$D_{d-i} = q^{(d-2i)/2} D_i$$

where q is a quadratic form in $\gamma_0, \alpha_i, \beta_i$ factorizing into a product of two linear forms which are symmetric under the exchange of parameters α_i with β_i and A_i with B_i (this corresponds to transposing $P_{\alpha,\beta}$).

Theorem 3.1 shows that the generic quadratic form q_2 working for $a = b = 2$ is given by

$$q_2 = (A_2\gamma_0 + \alpha_1 + (1 - A_1 - A_2)\beta_1)\,(B_2\gamma_0 + \beta_1 + (1 - B_1 - B_2)\alpha_1)\ .$$

The generic quadratic form q_3 working for $a = b = 3$ seems to be

$$\begin{aligned} q_3 \ = \ & (A_3\gamma_0 + \alpha_1 + \alpha_2 + (1 - A_1 - A_2 - A_3)\beta_1) \\ & \quad (B_3\gamma_0 + \beta_1 + \beta_2 + (1 - B_1 - B_2 - B_3)\alpha_1)\ . \end{aligned}$$

Example 3.5. Consider the 3−periodic sequence $\alpha = (\alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_k = \alpha_{k-3}, \ldots)$. The sequence $d(n) = \det(P_{\alpha,\alpha}(n))$ seems then to satisfy the recursion relation

$$\begin{aligned} d(n) = D_1\ d(n-1) + D_2\ d(n-2) - (\alpha_0 + \alpha_1 + \alpha_2)D_2\ d(n-3) \\ -(\alpha_0 + \alpha_1 + \alpha_2)^3 D_1\ d(n-4) + (\alpha_0 + \alpha_1 + \alpha_2)^5\ d(n-5) \end{aligned}$$

where

$$\begin{aligned} D_1 &= 11\alpha_1 + 5\alpha_2 \\ D_2 &= -(3\alpha_0^2 + 37\alpha_1^2 + 3\alpha_2^2 + 15\alpha_0\alpha_1 + 5\alpha_0\alpha_2 + 24\alpha_1\alpha_2)\ . \end{aligned}$$

In the general case

$$\begin{aligned} \alpha &= (\alpha_0 = \gamma_0, \alpha_1, \alpha_2, \ldots, \alpha_k = \alpha_{k-3}, \ldots) \\ \beta &= (\beta_0 = \gamma_0, \beta_1, \beta_2, \ldots, \beta_k = \beta_{k-3}, \ldots) \end{aligned}$$

of two 3−periodic sequences (starting with a common value γ_0) one seems to have

$$\begin{aligned} d(n) = D_1 d(n-1) + D_2 d(n-2) + D_3 d(n-3) + q D_2 d(n-4) \\ + q^2 D_1 d(n-5) - q^3 d(n-6) \end{aligned}$$

where

$$\begin{aligned} q &= (\gamma_0 + \alpha_1 + \alpha_2)(\gamma_0 + \beta_1 + \beta_2) \\ D_1 &= \gamma_0 + 6(\alpha_1 + \beta_1) + 3(\alpha_2 + \beta_2) \\ D_2 &= -\big(3\gamma_0^2 + 12(\alpha_1^2 + \beta_1^2) + 13\gamma_0(\alpha_1 + \beta_1) + 5\gamma_0(\alpha_2 + \beta_2) \\ &\quad + 9(\alpha_1\alpha_2 + \beta_1\beta_2) + 11(\alpha_1\beta_2 + \alpha_2\beta_1) + 24\alpha_1\beta_1 + 8\alpha_2\beta_2\big) \\ D_3 &= 6\gamma_0^3 + \gamma_0^2(18(\alpha_1 + \beta_1) + 8(\alpha_2 + \beta_2)) + \gamma_0(25(\alpha_1^2 + \beta_1^2) + 3(\alpha_2^2 + \beta_2^2) \\ &\quad + 18(\alpha_1\alpha_2 + \beta_1\beta_2) + 54\alpha_1\beta_1 + 26(\alpha_1\beta_2 + \alpha_2\beta_1) + 10\alpha_2\beta_2) \\ &\quad + 9(\alpha_1^3 + \beta_1^3) + 9(\alpha_1^2\alpha_2 + \beta_1^2\beta_2) + 28(\alpha_1^2\beta_1 + \alpha_1\beta_1^2) + 22(\alpha_1^2\beta_2 + \alpha_2\beta_1^2) \\ &\quad + 3(\alpha_1\beta_2^2 + \alpha_2^2\beta_1) + 3(\alpha_2^2\beta_2 + \alpha_2\beta_2^2) + 30(\alpha_1\alpha_2\beta_1 + \alpha_1\beta_1\beta_2) \\ &\quad + 24(\alpha_1\alpha_2\beta_2 + \alpha_2\beta_1\beta_2) \end{aligned}$$

Let us briefly explain how Conjecture 3.3 can be tested on a given pair α, β of linear recurrence sequences.

First Step. Guess d.

Second step. Compute at least $2d+1$ terms of the sequence

$$w_1 = \det(P_{\alpha,\beta}(1)), w_2 = \det(P_{\alpha,\beta}(2)), \dots \ .$$

Third step. Check that the so-called *Hankel matrix*

$$H_{d+1}(w) = \begin{pmatrix} w_1 & w_2 & w_3 & \dots & w_{d+1} \\ w_2 & w_3 & w_4 & \dots & w_{d+2} \\ \vdots & & & & \\ w_{d+1} & w_{d+2} & w_{d+3} & \dots & w_{2d+1} \end{pmatrix}$$

of order $d+1$ has zero determinant (otherwise try again with a higher value for d) and choose a vector of the form

$$L = (D_d, D_{d-1}, D_{d-2}, \dots, D_2, D_1, -1)$$

in its kernel. One has then by definition

$$\det(P_{\alpha,\beta}(n)) = \sum_{i=1}^{d} D_i \ \det(P_{\alpha,\beta}(n-i))$$

for $d+1 \leq n \leq 2d+1$.

Finally, check (perhaps) the above recursion for a few more values of $n > 2d+1$.

3.1. Proof of Theorem 3.1. The assertions concerning $d(1)$ and $d(2)$ are obvious. One checks (using for instance a symbolic computation program on a computer) that the recursion relation holds for $d(3)$, $d(4)$ and $d(5)$.

Introduce now the lower and upper triangular square matrices

$$T_A = \begin{pmatrix} 1 & 0 & 0 & 0 & \dots \\ -A_1 & 1 & 0 & 0 & \dots \\ -A_2 & -A_1 & 1 & 0 & \dots \\ 0 & -A_2 & -A_1 & 1 & \dots \\ \vdots & & \ddots & & \end{pmatrix}$$

$$T_B = \begin{pmatrix} 1 & -B_1 & -B_2 & 0 & 0 & \dots \\ 0 & 1 & -B_1 & -B_2 & 0 & \dots \\ 0 & 0 & 1 & -B_1 & -B_2 & \dots \\ & \vdots & & \ddots & & \end{pmatrix}$$

of order n and set $\tilde{M} = T_A\ M\ T_B$. The entries $\tilde{m}_{i,j},\ 0 \le i,j < n$ of $\tilde{M}$ satisfy $\tilde{m}_{i,j} = \tilde{m}_{i-1,j} + \tilde{m}_{i,j-1},\ (i,j) \ne (2,2)$ for $2 \le i,j < n$. One has

$$\tilde{M} = \begin{pmatrix} \gamma_0 & \beta_1 - B_1\gamma_0 & 0 & 0 & 0 & \dots \\ \alpha_1 - A_1\gamma_0 & \tilde{m}_{1,1} & \tilde{m}_{1,2} & \tilde{m}_{1,3} & \tilde{m}_{1,4} & \dots \\ 0 & \tilde{m}_{2,1} & \tilde{m}_{2,2} & \tilde{m}_{2,3} & \tilde{m}_{2,4} & \dots \\ \vdots & & \vdots & & & \end{pmatrix}$$

where

$$\begin{aligned}
&\tilde{m}_{1,1} = \alpha_1 + \beta_1 - A_1\beta_1 - B_1\alpha_1 + A_1B_1\gamma_0 \\
&\tilde{m}_{1,2} = \tilde{m}_{1,3} = \tilde{m}_{1,4} = \dots = (1 - B_1 - B_2)\alpha_1 + \beta_1 + B_2\gamma_0 \\
&\tilde{m}_{2,1} = \tilde{m}_{3,1} = \tilde{m}_{4,1} = \dots = (1 - A_1 - A_2)\beta_1 + \alpha_1 + A_2\gamma_0 \\
&\tilde{m}_{2,2} = (2 - B_1)\alpha_1 + (2 - A_1)\beta_1 + (A_2 + B_2 - A_1B_2 - A_2B_1 - A_2B_2)\gamma_0
\end{aligned}$$

Developing the determinant $d(n) = \det(\tilde{M})$ along the second row of $\tilde{M}$ one obtains

$$d(n) = (\gamma_0(\alpha_1 + \beta_1) - \alpha_1\beta_1)\overline{d}(n-2) + \gamma_0\ \det(P(n-1))$$

where $\overline{d}(n-2) = \det(\overline{M}(n-2))$ with coefficients $\overline{m}_{i,j} = \tilde{m}_{i+2,j+2}$ for $0 \le i,j < n-2$ (i.e. $\overline{M}(n-2)$ is the principal submatrix of $\tilde{M}$ obtained by erasing the first two rows and columns of $\tilde{M}$) and where $P(n-1)$ is the square matrix of order $(n-1)$ with entries $p_{0,0} = 0$ and $p_{i,j} = \tilde{m}_{i+1,j+1}$ for $0 \le i,j < n-1, (i,j) \ne (0,0)$.

The matrix $\overline{M}(m)$ $(m \le n-2)$ is a generalized Pascal triangle associated to the linear recursion sequences $\overline{\alpha} = (\overline{\alpha}_0, \overline{\alpha}_1, \dots)$ and $\overline{\beta} = (\overline{\beta}_0, \overline{\beta}_1, \dots)$ of order 2 defined by

$$\begin{aligned}
\overline{\alpha}_0 = \overline{\beta}_0 &= (2 - B_1)\alpha_1 + (2 - A_1)\beta_1 \\
&\qquad + (A_2 + B_2 - A_1B_2 - A_2B_1 - A_2B_2)\gamma_0, \\
\overline{\alpha}_1 &= (3 - B_1)\alpha_1 + (3 - 2A_1 - A_2)\beta_1 \\
&\qquad + (2A_2 + B_2 - A_1B_2 - A_2B_1 - A_2B_2)\gamma_0 \\
\overline{\beta}_1 &= (3 - 2B_1 - B_2)\alpha_1 + (3 - A_1)\beta_1 \\
&\qquad + (A_2 + 2B_2 - A_1B_2 - A_2B_1 - A_2B_2)\gamma_0 \\
\overline{\alpha}_n &= 2\overline{\alpha}_{n-1} - \overline{\alpha}_{n-2} \quad \text{for } n \ge 2\ , \\
\overline{\beta}_n &= 2\overline{\beta}_{n-1} - \overline{\beta}_{n-2} \quad \text{for } n \ge 2\ .
\end{aligned}$$

Induction on n and a computation (with $\overline{A}_1 = \overline{B}_1 = 2,\ \overline{A}_2 = \overline{B}_2 = -1$) shows the equality

$$\overline{d}(m) = D_1\ \overline{d}(m-1) + D_2\ \overline{d}(m-2)$$

for $3 \le m < n$ where D_1 and D_2 are as in the Theorem.

Introducing the $(n-1)\times(n-1)$ lower triangular square matrix

$$T_P = \begin{pmatrix} 1 & 0 & 0 & \dots \\ -1 & 1 & 0 & \dots \\ 0 & -1 & 1 & \dots \\ & \vdots & & \ddots \end{pmatrix}$$

we get $\tilde{P} = T_P\ P(n-1)\ T_P^t$ with coefficients $\tilde{p}_{i,j},\ 0 \le i,j < n-1$ given by

$$\begin{aligned}
&\tilde{p}_{0,0} = \tilde{p}_{i,0} = \tilde{p}_{0,i} = 0 \text{ for } 2 \le i < n-1,\\
&\tilde{p}_{0,1} = (1-B_1-B_2)\alpha_1 + \beta_1 + B_2\gamma_0\\
&\tilde{p}_{1,0} = (1-A_1-A_2)\beta_1 + \alpha_1 + A_2\gamma_0\\
&\tilde{p}_{1,i} = \tilde{p}_{0,1} \text{ for } 2 \le i < n-1,\\
&\tilde{p}_{i,1} = \tilde{p}_{1,0} \text{ for } 2 \le i < n-1,\\
&\tilde{p}_{2,2} = (2-B_1)\alpha_1 + (2-A_1)\beta_1 + (A_2+B_2-A_1B_2-A_2B_1-A_2B_2)\gamma_0\\
&\tilde{p}_{i,j} = \tilde{p}_{i-1,j} + \tilde{p}_{i,j-1}\ , 2 \le i,j < n-1,\ (i,j) \ne (2,2)\ .
\end{aligned}$$

Let $\overline{P}(n-3)$ denote the square matrix of order $(n-3)$ with coefficients $\overline{p}_{i,j} = \tilde{p}_{i+2,j+2}\ , 0 \le i,j < n-3$ (i.e. $\overline{P}(n-3)$ is obtained by erasing the first two rows and columns of $\tilde{P}(n-1)$). One checks the equality

$$\overline{P}(n-3) = \overline{M}(n-3)$$

where $\overline{M}(n-3)$ is defined as above. This implies the identity

$$\begin{aligned} d(n) = (\gamma_0(\alpha_1+\beta_1) - \alpha_1\beta_1)\overline{d}(n-2) - \gamma_0((1-B_1-B_2)\alpha_1+\beta_1+B_2\gamma_0)\\ \times((1-A_1-A_2)\beta_1+\alpha_1+A_2\gamma_0)\overline{d}(n-3)\ . \end{aligned}$$

Using the recursion relation $\overline{d}(m) = D_1\ \overline{d}(m-1) + D_2\ \overline{d}(m-2)$ (which holds by induction for $3 \le m < n$) we can hence express $d(n)$ as a linear function (with polynomial coefficients in $\gamma_0, \alpha_1, \beta_1, A_1, A_2, B_1, B_2$) of $\overline{d}(n-4)$ and $\overline{d}(n-5)$.

Comparing this with the linear expression in $\overline{d}(n-4)$ and $\overline{d}(n-5)$ obtained similarly from $D_1\ d(n-1) + D_2\ d(n-2)$ finishes the proof. □

4. Symmetric matrices

Take an arbitrary sequence $\alpha = (\alpha_0, \alpha_1, \dots)$. The generalized Pascal triangle associated to the pair of identical sequences α, α is the *generalized symmetric Pascal triangle* associated to α and yields symmetric matrices $P_{\alpha,\alpha}(n)$ by considering principal submatrices consisting of the first n rows and columns of $P_{\alpha,\alpha}$.

The main example is of course the classical Pascal triangle obtained from the constant sequence $\alpha = (1,1,1,\dots)$. Other sequences satisfying linear recursions like for instance the Fibonacci sequence

$$(0,1,1,2,3,5,8,\dots)$$

and shifts of it yield also nice examples.

Conjecture 3.3 should of course also hold for symmetric matrices obtained by considering the generalized symmetric Pascal triangle associated to a sequence satisfying a linear recurrence relation.

The generic order $d_s(a)$ (where a denotes the order of the defining linear recursion sequence) of the linear recursion satisfied by $\det(P_{\alpha,\alpha}(n))$ seems however usually to be smaller than in the generic non-symmetric case. Examples yield the following first values

$$\begin{array}{rcccccc} a = & 1 & 2 & 3 & 4 & 5 & 6 \\ d_s(a) = & 1 & 2 & 5 & 14 & 41 & 122 \end{array}$$

and suggest that perhaps $d_s(a) = (3^{a-1}+1)/2$.

The coefficients D_i seem still to be polynomial in α_i and A_i.

The symmetry relation has also an analogue (in the generic case) which is moreover somewhat simpler in the sense that it is given by a linear form ρ (in $\alpha_0, \dots, \alpha_{a-1}$) and we seem to have

$$D_{d_s-i} = \rho^{d_s-2i}\ D_i$$

(where $D_0 = -1$).

Example 4.1. If a sequence

$$\alpha = (\alpha_0, \alpha_1, \alpha_2, \dots, \alpha_k = A_1\alpha_{k-1} + A_2\alpha_{k-2} + A_3\alpha_{k-3}, \dots)$$

satisfies a linear recursion relation of order 3, then the sequence $d(n) = \det(P_{\alpha,\alpha}(n))$ (the matrix $P_{\alpha,\alpha}(n)$ has coefficients $p_{0,i} = p_{i,0} = \alpha_i$, $0 \le i < n$ and $p_{i,j} = p_{i-1,j} + p_{i,j-1}$ for $1 \le i,j < n$) of the associated determinants seems to satisfy

$$\begin{aligned} d(1) &= \alpha_0\ , \\ d(2) &= 2\alpha_0\alpha_1 - \alpha_1^2\ , \\ d(3) &= (2\alpha_1 - \alpha_2)\left(\alpha_0(2\alpha_1+\alpha_2) - 2\alpha_1^2\right)\ , \\ d(n) &= D_1\ d(n-1) + D_2\ d(n-2) + \rho D_2\ d(n-3) + \rho^3\ D_1\ d(n-4) \\ &\qquad -\rho^5\ d(n-5) \end{aligned}$$

where

$$\begin{aligned} \rho\ &= -A_3\alpha_0 + (-2 + 2A_1 + A_2 + A_3)\alpha_1 - \alpha_2\ , \\ D_1\ &= A_3(1 - 2A_1 - 2A_2 - A_3)\alpha_0 \\ &\quad +(10 - 10A_1 - A_2 + A_3 + 4A_1^2 + 2A_1A_2)\alpha_1 + (5 - 4A_1 - 2A_2)\alpha_2\ , \\ D_2\ &= c_{0,0}\alpha_0^2 + c_{1,1}\alpha_1^2 + c_{2,2}\alpha_2^2 + c_{0,1}\alpha_0\alpha_1 + c_{0,2}\alpha_0\alpha_2 + c_{1,2}\alpha_1\alpha_2 \end{aligned}$$

with

$$\begin{aligned}
c_{0,0} &= -A_3^2(2-2A_1+2A_2+A_3+A_1^2)\ ,\\
c_{1,1} &= -40+80A_1+16A_2+4A_3\\
&\quad -64A_1^2-2A_2^2-A_3^2-28A_1A_2-20A_1A_3-2A_2A_3\\
&\quad +2A_1(2A_1+A_2+A_3)(6A_1+A_2+A_3)-A_1^2(2A_1+A_2+A_3)^2\ ,\\
c_{2,2} &= -10+12A_1+6A_2+8A_3-(2A_1+A_2+A_3)^2\ ,\\
c_{0,1} &= -A_3(16-28A_1+16A_1^2-2A_2^2-A_3^2+2A_1A_3-3A_2A_3\\
&\quad -2A_1^2(2A_1+A_2+A_3))\ ,\\
c_{0,2} &= -A_3(8-10A_1-3A_3+2A_1(2A_1+A_2+A_3))\ ,\\
c_{1,2} &= 2\big(-20+32A_1+10A_2+9A_3-18A_1^2-A_2^2-A_3^2\\
&\quad -11A_1A_2-12A_1A_3-2A_2A_3+A_1(2A_1+A_2+A_3)^2\big)
\end{aligned}$$

We conclude this section by mentioning the following more exotic example:

Example 4.2. (Central binomial coefficients) Consider the sequence

$$\alpha=(1,2,6,20,70,252,924,3432,12870,48620,184756,\dots,\alpha_k=\tbinom{2k}{k},\dots)$$

of central binomial coefficients. For $1\le n\le 36$ the values of $\det(P_{\alpha,\alpha}(n))$ are zero except if $n\equiv 1,3 \pmod 6$ and for $n\equiv 1,3 \pmod 6$ the values of $\det(P_{\alpha,\alpha}(n))$ have the following intriguing factorizations:

n :	$\det(P_{\alpha,\alpha}(n))$	n :	$\det(P_{\alpha,\alpha}(n))$
1 :	1	3 :	-2^2
7 :	-2^6	9 :	$2^8\ 3^2$
13 :	$2^{16}\ 3^6$	15 :	$-2^{26}\ 3^6$
19 :	$-2^{30}\ 3^2\ 103^4$	21 :	$2^{24}\ 103^4\ 4229^2$
25 :	$2^{24}\ 3^4\ 31^4\ 431^4\ 4229^2$	27 :	$-2^{26}\ 3^6\ 31^4\ 59^2\ 431^4\ 11701^2$
31 :	$-2^{30}\ 3^{10}\ 59^2\ 11701^2\ p^4$	33 :	$2^{32}\ 3^{12}\ 11^2\ 2017^2\ 28349^2\ p^4$

where $p=4893589$.

The matrix $P_{\alpha,\alpha}(n)$ seems to have rank n if $n\equiv 1,3 \pmod 6$, rank $n-1$ if $n\equiv 0 \pmod 2$ and rank $n-2$ if $n\equiv 5 \pmod 6$.

5. Skew-symmetric matrices

Given an arbitrary sequence $\alpha=(\alpha_0,\alpha_1,\dots)$ with $\alpha_0=0$, the matrices $P_{\alpha,-\alpha}(n)$ are skew-symmetric.

Determinants of integral skew-symmetric matrices are squares of integers and are zero in odd dimensions. We restrict hence ourself to even dimensions and consider sometimes also the (positive) square-roots of the determinants. Even if Conjecture 3.3 holds there is of course no reason that the square roots of the determinants satisfy a linear recursion.

The conjectural recurrence relation for skew-symmetric matrices has the form

$$\det(P_{\alpha,-\alpha}(2n)) = \sum_{i=1}^{d(\alpha)} D_i \det(P_{\alpha,-\alpha}(2n-2i)) \ .$$

However the coefficients $D_1, \dots, D_{d(\alpha)}$ seem no longer to be polynomial but rational for generic α. Moreover, the nice symmetry properties of the coefficients D_i present in the other cases seem to have disappeared too.

Proposition 5.1. (i) *The skew-symmetric matrices $P_{\alpha,-\alpha}(2n)$ associated to the sequence $\alpha = (0,1,1,1,1,\dots)$ have determinant* 1 *for every natural integer n.*
(ii) *The skew-symmetric matrices $P_{\alpha,-\alpha}(2n)$ associated to the sequence $\alpha = (0,1,2,3,4,5,\dots)$ have determinant* 1 *for every natural integer n.*

Both assertions follow of course from Theorem 3.1. We will however reprove them independently.

Proof. Consider the generalized Pascal triangle

$$P = P_{(1,1,1,\dots,1,\dots),(1,-1,-1,\dots,-1,\dots)}(\infty) = \begin{pmatrix} 1 & -1 & -1 & -1 & -1 & -1 & -1 & \dots \\ 1 & 0 & -1 & -2 & -3 & -4 & -5 & \dots \\ 1 & 1 & 0 & -2 & -5 & -9 & -14 & \dots \\ 1 & 2 & 2 & 0 & -5 & -14 & -28 & \dots \\ \vdots & & & & & & & \end{pmatrix} \ .$$

The matrices $P(m)$ given by retaining only the first m rows and columns of $P(\infty)$ are all of determinant 1 (compare the transposed matrix $P(m)^t$ with Example 2.1).

Expanding the determinant along the first row one gets

$$1 = \det(P(m))$$

$$= \det(P_{(0,1,1,1,\dots),-(0,1,1,\dots)}(m)) + \det(P_{(0,1,2,3,4,\dots),-(0,1,2,3,\dots)}(m-1)) \ .$$

The fact that skew-symmetric matrices of odd order have zero determinant proves now assertion (i) for even m and assertion (ii) for odd m. $\square$

Remark 5.2. The coefficients $p_{i,j}$ of the infinite skew-symmetric matrix

$$P_{(0,1,1,1,\dots),-(0,1,1,1,\dots)}(\infty)$$

have many interesting properties: One can for instance easily check that

$$p_{i,j} = \binom{i+j-1}{j} - \binom{i+j-1}{j-1}$$

(with the correct definition for $\binom{k}{-1}$ given by $\binom{-1}{-1} = \binom{-1}{0} = 1$ and $\binom{k}{-1} = 0$ for $k = 0,1,2,3,\dots$). These numbers are the orders of the irreducible matrix algebras in the Temperley-Lieb algebras (see for instance chapter 2.8, pages 86-101, in [GHJ]).

There are other matrices constructed using the numbers $\binom{i+j-1}{j}-\binom{i+j-1}{j-1}$ whose determinants have interesting properties: Let $A_k(n)$ and $B_k(n)$ be the $n\times n$ matrices with entries

$$a_{i,j}=\binom{2i+2j+k}{i}-\binom{2i+2j+k}{i-1} \text{ and } b_{i,j}=\binom{2i+2j+k}{i+1}-\binom{2i+2j+k}{i}$$

for $0\leq i,j<n$ and k a fixed integer. It follows from work of Krattenthaler ([K2] and Theorem 5 in [K3]) that

$$\det(A_k(n))=2^{\binom{n}{2}}$$

and

$$\det(B_k(n))=2^{\binom{n}{2}}\frac{\prod_{i=0}^{n-1}(k+2i-1)}{n!}\ .$$

Principal minors of $P_{(0,1,1,1,\dots),-(0,1,1,1,\dots)}(\infty)$ associated to submatrices consisting of $2n$ consecutive rows and columns and starting at rows and columns of index $k=0,1,2,\dots$ have interesting properties as given by the following result (cited without proof) which is an easy corollary of the computation of $\det((a+j-i)\Gamma(b+i+j))\ ,\ 0\leq i,j<n$ by Mehta and Wang (cf. [MW]).

Theorem 5.3. *Denote by $T_k(2n)$ the $2n\times 2n$ skew-symmetric matrix with coefficients*

$$t_{i,j}=\binom{2k+i+j-1}{k+j}-\binom{2k+i+j-1}{k+j-1}=\frac{(i-j)(2k+i+j-1)!}{(k+i)!(k+j)!}$$

for $0\leq i,j<2n$. One has

$$\sqrt{\det(T_k(2n))}=\prod_{t=1}^{k-1}\frac{\binom{2n+2t}{t}}{\binom{2t}{t}}\ ,n=0,1,2,\dots\ .$$

The first polynomials

$$D_k(n)=\prod_{t=1}^{k-1}\frac{\binom{2n+2t}{t}}{\binom{2t}{t}}=\sqrt{\det(T_k(2n))}$$

are given by

$$\begin{array}{rcl}
D_0(n) & = & 1\\
D_1(n) & = & 1\\
D_2(n) & = & (n+1)\\
D_3(n) & = & (2n+3)(n+1)(n+2)/6\\
D_4(n) & = & (2n+5)(2n+3)(n+3)(n+2)^2(n+1)/180
\end{array}$$

The sequences $(D_k(n))_{k=0,1,2,\dots}$ (for fixed n) seem also to be of interest since they have appeared elsewhere. They start as follows:

$(D_0(0), D_1(0), D_2(0), \dots) = (1, 1, 1, \dots)$
$(D_0(1), D_1(1), D_2(1), \dots) = (1, 1, 2, 5, 14, \dots)$ (Catalan numbers)
$(D_0(2), D_1(2), D_2(2), \dots) = (1, 1, 3, 14, 84, \dots)$ (cf. **A**005700 in [IS])
$(D_0(3), D_1(3), D_2(3), \dots) = (1, 1, 4, 30, 330, \dots)$ (cf. **A**006149 in [IS])
$(D_0(4), D_1(4), D_2(4), \dots) = (1, 1, 5, 55, 1001, \dots)$ (cf. **A**006150 in [IS])
$(D_0(5), D_1(5), D_2(5), \dots) = (1, 1, 6, 91, 2548, \dots)$ (cf. **A**006151 in [IS])

Geometric sequences provide other nice special cases of Theorem 3.1.

Example 5.4. (i) The sequence $\alpha = (0, 1, A, A^2, A^3, \dots)$ (for $A > 0$) yields $\det(P_{\alpha,-\alpha}(2n)) = A^{2(n-1)}$.
(ii) The slightly more general example $\alpha = (0, 1, A + B, \dots, \alpha_k = \frac{A^k - B^k}{A - B}, \dots)$ yields $\det(P_{\alpha,-\alpha}(2n)) = (A - AB + B)^{2(n-1)}$.

Finally, we would like to mention the following exotic example.

Example 5.5. The sequences

$$\alpha_C = (0, 1, 1, 2, 5, 14, 42, \dots)$$
$$\alpha_B = (0, 1, 2, 6, 20, 70, \dots)$$

related to Catalan numbers and central binomial coefficients yield the sequences $r_C(n) = \sqrt{\det(P_{\alpha_C,-\alpha_C}(2n))}$ and $r_B(n) = \sqrt{\det(P_{\alpha_B,-\alpha_B}(2n))}$:

$n =$	1	2	3	4	5	6	7
$r_C(n) =$	1	2	6	31	286	4600	130664
$r_B(n) =$	1	$2 \cdot 2$	$6 \cdot 2^2$	$31 \cdot 2^3$	$286 \cdot 2^4$	$4600 \cdot 2^5$	$130664 \cdot 2^6$

$n =$	8	9	10	$\dots$
$r_C(n) =$	6619840	591478944	93683332808	$\dots$
$r_B(n) =$	$6619840 \cdot 2^7$	$591478944 \cdot 2^8$	$93683332808 \cdot 2^9$	$\dots$

suggesting the conjecture $r_B(n) = 2^{n-1} r_C(n)$ for $n \geq 1$.

5.1. The even skew-symmetric construction and the even skew-symmetric unimodular tree. Given an arbitrary sequence $\beta = (\beta_0, \beta_1, \dots)$ we consider the sequence $\alpha = (0, \beta_0, 0, \beta_1, 0, \beta_2, \dots)$ defined by $\alpha_{2n} = 0$ and $\alpha_{2n+1} = \beta_n$. We call this way of constructing a skew-symmetric matrix $P_{\alpha,-\alpha}(2n)$ out of a sequence $\beta = (\beta_0, \beta_1, \dots)$ the *even skew-symmetric construction (of Pascal triangles)*.

Example 5.1.1. The skew-symmetric matrix of order 6 associated to the the sequence $\beta = (1, 1, -1, \dots)$ by the even skew-symmetric construction is the following determinant 1 matrix $\begin{pmatrix} 0 & -1 & 0 & -1 & 0 & 1 \\ 1 & 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & -1 & -2 & -2 \\ 1 & 1 & 1 & 0 & -2 & -4 \\ 0 & 1 & 2 & 2 & 0 & -4 \\ -1 & 0 & 2 & 4 & 4 & 0 \end{pmatrix}$.

The main feature of the even skew-symmetric construction is perhaps given by the following result.

Theorem 5.1.2. (i) *Let $(\beta_0, \beta_1, \dots, \beta_{n-1})$ be a sequence of integers such that*

$$\det(P_{(0,\beta_0,0,\beta_1,\dots,0,\beta_{n-1}),-(0,\beta_0,0,\beta_1,\dots,0,\beta_{n-1})}(2n)) = 1 \ .$$

Then there exists a unique even integer $\tilde\beta_n$ such that

$$\begin{aligned}
&\det(P_{(0,\beta_0,0,\beta_1,\dots,0,\beta_{n-1},0,\tilde\beta_n+1),-(0,\beta_0,0,\beta_1,\dots,0,\beta_{n-1},0,\tilde\beta_n+1)}(2n+2)) = 1 \\
&\det(P_{(0,\beta_0,0,\beta_1,\dots,0,\beta_{n-1},0,\tilde\beta_n),-(0,\beta_0,0,\beta_1,\dots,0,\beta_{n-1},0,\tilde\beta_n)}(2n+2)) = 0 \\
&\det(P_{(0,\beta_0,0,\beta_1,\dots,0,\beta_{n-1},0,\tilde\beta_n-1),-(0,\beta_0,0,\beta_1,\dots,0,\beta_{n-1},0,\tilde\beta_n-1)}(2n+2)) = 1 \ .
\end{aligned}$$

(ii) *If $\beta = (\beta_0, \beta_1, \beta_2, \dots)$ and $\beta' = (\beta'_0, \beta'_1, \beta'_2, \dots)$ are two infinite sequences of integers satisfying the assumption of assertion (i) above for all n, then there exists a unique integer m such that $\beta_i = \beta'_i$ for $i < m$ and $\beta_m = \tilde\beta_m + \epsilon$, $\beta'_m = \tilde\beta_m - \epsilon$ with $\tilde\beta_m$ as in assertion (i) above and $\epsilon \in \{\pm 1\}$.*

Proof. The determinant of the skew-symmetric matrix

$$P_{(0,\beta_0,0,\beta_1,\dots,0,\beta_{n-1},0,x),-(0,\beta_0,0,\beta_1,\dots,0,\beta_{n-1},0,x)}(2n+2)$$

is of the form $D(x) = (ax + b)^2$ for some suitable integers a and b (which are well defined up to multiplication by -1).

It is easy to see that it is enough to show that $a = \pm 1$ in order to prove the Theorem (the integer $\tilde\beta_m$ equals then $-ab$ and is even by a consideration (mod 2)). This is of course equivalent to showing that the polynomial $D(x)$ has degree 2 and leading term 1.

Consider now the skew-symmetric matrix M of order $2n + 2$ defined as follows: The entries of M except the last row and column are given by the odd-order (and hence degenerate) skew-symmetric matrix

$$P_{(0,\beta_0,0,\beta_1,\dots,0,\beta_{n-1},0),-(0,\beta_0,0,\beta_1,\dots,0,\beta_{n-1},0)}(2n+1) \ .$$

The last row (which determines by skew-symmetry the last column) of M is given by

$$(1, 1, 1, \dots, 1, 1, 0) \ .$$

It is obvious to check that $\det(M)$ is the coefficient of x^2 in the polynomial $D(x)$ introduced above.

Subtract now row number $2n - 1$ from row number $2n$ of M (with rows and columns of M indexed from 0 to $2n + 1$), subtract then row number $2n - 2$ from row number $2n - 1$, etc until subtracting row number 0 from row number 1. Do the same operations on columns thus producing a skew-symmetric matrix $\tilde{M}$ which is equivalent to M and whose last row is given by $(1,0,0,\dots,0,0)$. The determinant of M equals hence the determinant of the submatrix of $\tilde{M}$ obtained by deleting the first and last rows and columns in $\tilde{M}$. This submatrix is given by

$$P_{(0,\beta_0,0,\beta_1,\dots,0,\beta_{n-1}),-(0,\beta_0,0,\beta_1,\dots,0,\beta_{n-1})}(2n)$$

thus showing that $\det(M)=1=a^2$. $\square$

The set of sequences

$$\{\alpha=(0,\beta_0,0,\beta_1,0,\beta_2,\dots)\mid \det(P_{\alpha,-\alpha}(2n)=1,\ n=1,2,3,\dots\}$$

associated to unimodular skew-symmetric matrices $P_{\alpha,-\alpha}(2n)$ thus consists of integral sequences and has the structure of a tree. We call this tree the *even skew-symmetric unimodular tree.*

Table 5.1.3. (Part of the even skew-symmetric unimodular tree).

0_{+1}	0_{+1}	0_{+1}	0_{+1}	0_{+1}	$0_{\pm}$
0_{+1}	0_{+1}	0_{+1}	0_{+1}	0_{-1}	$-100_{\pm}$
0_{+1}	0_{+1}	0_{+1}	0_{-1}	-42_{+1}	$32658_{\pm}$
0_{+1}	0_{+1}	0_{+1}	0_{-1}	-42_{-1}	$-39754_{\pm}$
0_{+1}	0_{+1}	0_{-1}	-8_{+1}	68_{+1}	$434748_{\pm}$
0_{+1}	0_{+1}	0_{-1}	-8_{+1}	68_{-1}	$-400344_{\pm}$
0_{+1}	0_{+1}	0_{-1}	-8_{-1}	-254_{+1}	$12922350_{\pm}$
0_{+1}	0_{+1}	0_{-1}	-8_{-1}	-254_{-1}	$-13258926_{\pm}$
0_{+1}	0_{-1}	2_{+1}	0_{+1}	240_{+1}	$13257990_{\pm}$
0_{+1}	0_{-1}	2_{+1}	0_{+1}	240_{-1}	$-12923278_{\pm}$
0_{+1}	0_{-1}	2_{+1}	0_{-1}	-74_{+1}	$400664_{\pm}$
0_{+1}	0_{-1}	2_{+1}	0_{-1}	-74_{-1}	$-434420_{\pm}$
0_{+1}	0_{-1}	2_{-1}	0_{+1}	36_{+1}	$39594_{\pm}$
0_{+1}	0_{-1}	2_{-1}	0_{+1}	36_{-1}	$-32810_{\pm}$
0_{+1}	0_{-1}	2_{-1}	0_{-1}	2_{+1}	$92_{\pm}$
0_{+1}	0_{-1}	2_{-1}	0_{-1}	2_{-1}	$0_{\pm}$

The beginning of this tree is shown above and is to be understood as follows:

Column i displays the integer $\tilde{\beta}_i$ of the Theorem. Indices indicate if $\beta_i=\tilde{\beta}_i+1$ or $\tilde{\beta}_i-1$. Hence the row

$$0_{+1}\quad 0_{+1}\quad 0_{-1}\quad -8_{+1}\quad 68_{+1}\quad 434748_{\pm}$$

corresponds for instance to the sequence

$$(1,1,-1,-7,69)$$

implying $\tilde{\beta}_5 = 434748$ (the sequence $(1,1,-1,-7,69)$ can hence be extended either to $(1,1,-1,-7,69,434749)$ or to $(1,1,-1,-7,69,434747)$).

We have only displayed sequences starting with 1 since sequences starting with -1 are obtained by a global sign change.

6. A "skymmetric" tree?

The construction of a generalized Pascal triangle $P_{\alpha,\beta}(\infty)$ needs two sequences $\alpha = (\alpha_0, \alpha_1, \dots)$ and $\beta = (\beta_0, \beta_1, \dots)$. Starting with only one sequence $\alpha = (\alpha_0, \alpha_1, \dots)$ and considering $P_{\alpha,\alpha}(\infty)$ we get generalized symmetric Pascal triangles and considering $P_{\alpha,-\alpha}(\infty)$ we get generalized skew-symmetric Pascal triangles. Since the sequence $\tilde{\alpha} = (\tilde{\alpha}_0 = \alpha_0, \tilde{\alpha}_1 = -\alpha_1, \tilde{\alpha}_2 = \alpha_2, \dots, \tilde{\alpha}_i = (-1)^i \alpha_i, \dots)$ is half-way between α and $-\alpha$, we call the generalized Pascal triangle $P_{\alpha,\tilde{\alpha}}(\infty)$ the *generalized "skymmetric" Pascal triangle.*

The two sequences

$$\begin{array}{ll} \alpha = (0,1,1,2,3,5,8,13,\dots) & \text{Fibonacci} \\ \alpha = (0,1,1,0,-1,-1,0,1,1,0,-1,-1,\dots) & 6-\text{periodic} \end{array}$$

and the associated sequences $\tilde{\alpha}$ satisfy all linear recursions of order 2. Theorem 3.1 and a computation of the first few values show that both sequences $\det(P_{\alpha,\tilde{\alpha}}(n))$ equal $0, 1, 2, 2^2, 2^3, \dots, 2^{n-2}, \dots$. All the following finite sequences yield matrices $P_{\alpha\tilde{\alpha}}(n)$ with determinants $0, 1, 2, 4, 8, 16, 32, 64$ (for $n = 1, 2, 3, \dots$) too:

0	1	1	2	5	13	34	85 ± 4
0	1	1	2	5	13	28	79 ± 4
0	1	1	2	5	9	20	77 ± 38
0	1	1	2	5	9	2	-193 ± 110
0	1	1	2	3	5	10	19 ± 6
0	1	1	2	3	5	8	12 ± 1
0	1	1	2	3	3	10	-3 ± 6
0	1	1	2	3	3	-2	9 ± 30
0	1	1	0	1	3	4	-31 ± 38
0	1	1	0	1	3	-14	167 ± 110
0	1	1	0	1	-1	2	1 ± 4
0	1	1	0	1	-1	-4	-17 ± 4
0	1	1	0	-1	1	10	33 ± 6
0	1	1	0	-1	1	-2	-27 ± 30
0	1	1	0	-1	-1	2	3 ± 6
0	1	1	0	-1	-1	0	2 ± 1

Problem 6.1. Has the set of all infinite integral sequences $\alpha = (0, 1, 1, \alpha_3, \dots)$ such that $\det(P_{\alpha,\tilde{\alpha}}(n)) = (0, 1, 2, 4, \dots, 2^{n-2}, \dots)_{n=1,2,\dots}$ the structure of a tree (i.e. can every finite such sequence of length at least 3 be extended by one next term in exactly two ways)?

7. Periodic matrices

In this section we are interested in matrices coming from a kind of "periodic convolution with compact support on $\mathbf{N}$".

We say that an infinite matrix A with coefficients $a_{i,j},\ 0 \le i,j$ is $(s,t)-$*bounded* $(s,t \in \mathbf{N})$ if $a_{i,j} = 0$ for $(j-i) \notin [-s,t]$.

We call a matrix with coefficients $a_{i,j},\ 0 \le i,j$ $p-$*periodic* if $a_{i,j} = a_{i-p,j-p}$ for $i,j \ge p$.

An infinite matrix P with coefficients $p_{i,j},\ 0 \le i,j$ is a *finite perturbation* if it has only a finite number of non-zero coefficients.

As before, given an infinite matrix M with coefficients $m_{i,j},\ 0 \le i,j$ we denote by $M(n)$ the matrix with coefficients $m_{i,j},\ 0 \le i,j < n$ obtained by erasing all but the first n rows and columns of M.

Theorem 7.1. *Let $A = \tilde{A} + P$ be a matrix where $\tilde{A}$ is a $p-$periodic $(s,t)-$bounded matrix and where P is a finite perturbation. Then there exist constants $N, d \le \binom{s+t}{s}, C_1, \dots, C_d$ such that*

$$\det(A(n)) = \sum_{i=1}^{d} C_i \det(A(n-ip))$$

for $n > N$.

We will prove the theorem for $p = 1,\ s = t = 2$ and then describe the necessary modifications in the general case.

Proof in the case $p = 1,\ s = t = 2$. Suppose n huge. The matrix $A(n)$ has then the form

$$A(n) = \begin{pmatrix} \ddots & & & & \vdots \\ & c & d & e & 0 \\ & b & c & d & e \\ & a & b & c & d \\ \dots & 0 & a & b & c \end{pmatrix}.$$

Developing the determinant possibly several times along the last row one gets only matrices of the following six types

$$T_1 = \begin{pmatrix} \ddots & d & e & 0 \\ & c & d & e \\ & b & c & d \\ & a & b & c \end{pmatrix} \quad T_2 = \begin{pmatrix} \ddots & d & e & 0 \\ & c & d & 0 \\ & b & c & e \\ & a & b & d \end{pmatrix} \quad T_3 = \begin{pmatrix} \ddots & d & 0 & 0 \\ & c & e & 0 \\ & b & d & e \\ & a & c & d \end{pmatrix}$$

$$T_4 = \begin{pmatrix} \ddots & d & e & 0 \\ & c & d & 0 \\ & b & c & 0 \\ & a & b & e \end{pmatrix} \quad T_5 = \begin{pmatrix} \ddots & d & 0 & 0 \\ & c & e & 0 \\ & b & d & 0 \\ & a & c & e \end{pmatrix} \quad T_6 = \begin{pmatrix} \ddots & d & 0 & 0 \\ & c & 0 & 0 \\ & b & e & 0 \\ & a & d & e \end{pmatrix}$$

and writing $t_i(m) = \det(T_i(m))$ we have the identity

$$\begin{pmatrix} t_1(m) \\ t_2(m) \\ t_3(m) \\ t_4(m) \\ t_5(m) \\ t_6(m) \end{pmatrix} = \begin{pmatrix} c & -b & a & 0 & 0 & 0 \\ d & 0 & 0 & -b & a & 0 \\ 0 & d & 0 & -c & 0 & a \\ e & 0 & 0 & 0 & 0 & 0 \\ 0 & e & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & e & 0 & 0 \end{pmatrix} \begin{pmatrix} t_1(m-1) \\ t_2(m-1) \\ t_3(m-1) \\ t_4(m-1) \\ t_5(m-1) \\ t_6(m-1) \end{pmatrix}$$

for m huge enough. Writing R the above 6×6 matrix relating $t_i(m)$ to $t_j(m-1)$ we have $t(n) = R^{n-N}t(N)$ for $n \geq N$ huge enough and for $t(m)$ the vector with coordinates $t_1(m), \dots, t_6(m)$. Choosing a basis of a Jordan normal form of R and expressing the vector $t(N)$ with respect to this basis shows now that the determinants $t_i(n)$ (and hence $\det(A(n)) = t_1(n)$) satisfy for $n > N$ a linear recursion with characteristic polynomial dividing

$$\det(z \ \mathrm{Id}_6 - R) \ .$$

Proof of the general case. Let us first suppose $p = 1$. There are then $\binom{s+t}{s}$ (count the possibilities for the highest non-zero entry in the last s columns) different possible types T_i obtained by developing the determinant $\det(A(n))$ for huge n several times along the last row and one gets hence a square matrix R of order $\binom{s+t}{s}$ expressing the determinants $\det(T_i(n))$ linearly in $\det(T_j(n-1))$ for n huge enough. This shows that the determinants $\det(T_i(n))$ satisfy for n huge enough a linear recursion with characteristic polynomial dividing the characteristic polynomial of the square matrix R.

If $p > 1$, develop the determinant of $\det(A(n))$ a multiple of p times along the last row and proceed as above. One gets in this way matrices $R_0, \dots, R_{p-1}$ according to $n \pmod p$ with identical characteristic polynomials yielding recursion relations between $\det(A(n))$ and $\det(A(n-ip))$. $\square$

8. The diagonal construction

Let $\gamma = (\gamma_0, \gamma_1, \gamma_2, \dots,)$ be a sequence and let u_1, u_2, l_1, l_2 be four constants. The *diagonal-construction* is the (infinite) matrix $D_\gamma^{(u_1,u_2,l_1,l_2)}$ with entries

$$\begin{array}{ll} d_{i,i} = \gamma_i & 0 \leq i \\ d_{i,j} = u_1 d_{i,j-1} + u_2 d_{i+1,j} & 0 \leq i < j \\ d_{i,j} = l_1 d_{i-1,j} + l_2 d_{i,j+1} & 0 \leq j < i \end{array}$$

and we denote by $D(n) = D_\gamma^{(u_1,u_2,l_1,l_2)}(n)$ the $n \times n$ principal submatrix with coefficients $d_{i,j}$, $0 \leq i, j < n$ obtained by considering the first n rows and columns of $D_\gamma^{(u_1,u_2,l_1,l_2)}$.

The cases where $u_1u_2l_1l_2 = 0$ are degenerate. For instance, in the case $u_2 = 0$ one sees easily that the matrix $D_\gamma^{(u_1,u_2,l_1,l_2)}(n)$ has determinant

$$\gamma_0 \prod_{j=1}^{n-1}(\gamma_j - u_1(l_1\gamma_{j-1} + l_2\gamma_j)) \ .$$

The other cases are similar.

The following result shows that we loose almost nothing by assuming $u_1 = u_2 = 1$.

Proposition 8.1. *For λ, μ two invertible constants we have*

$$D_{\tilde\gamma}^{(\lambda u_1,\mu u_2,\mu^{-1}l_1,\lambda^{-1}l_2)}(n) = \left(\frac{\lambda}{\mu}\right)^{\binom{n}{2}} D_\gamma^{(u_1,u_2,l_1,l_2)}(n)$$

where

$$\tilde\gamma = (\gamma_0, \frac{\lambda}{\mu}\gamma_1, \frac{\lambda^2}{\mu^2}\gamma_2, \ldots, \tilde\gamma_k = \frac{\lambda^k}{\mu^k}\gamma_k, \ldots) \ .$$

Proof. Check that the coefficients $\tilde d_{i,j}$ of $D_{\tilde\gamma}^{(\lambda u_1,\mu u_2,\mu^{-1}l_1,\lambda^{-1}l_2)}(n)$ are given by $\tilde d_{i,j} = \mu^{-i}\lambda^j \ d_{i,j}$ where $d_{i,j}$ are the coefficients of $D_\gamma^{(u_1,u_2,l_1,l_2)}(n)$. This implies the result easily. □

Proposition 8.2. *For $n \geq 1$ the sequence*

$$d(n) = \det(D_\gamma^{(u_1,u_2,l_1,l_2)}(n))$$

associated to the geometric sequence $\gamma = (1, x, x^2, x^3, \ldots)$ is given by

$$d(n) = \left(-u_1l_1 + (1 - u_1l_2 - u_2l_1)x - u_2l_2x^2\right)^{n-1} \ x^{\binom{n-1}{2}} \ .$$

A nice special case is given by $u_1 = u_2 = l_1 = l_2 = 1$. The associated matrix $D_\gamma(4) = D_\gamma^{(1,1,1,1)}(4)$ for example is then given by

$$\begin{pmatrix} 1 & 1+x & 1+2x+x^2 & 1+3x+3x^2+x^3 \\ 1+x & x & x+x^2 & x+2x^2+x^3 \\ 1+2x+x^2 & x+x^2 & x^2 & x^2+x^3 \\ 1+3x+3x^2+x^3 & x+2x^2+x^3 & x^2+x^3 & x^3 \end{pmatrix}$$

and the reader can readily check that the coefficient $d_{i,j}$ of $D_\gamma(n)$ is given by

$$d_{i,j} = x^{\min(i,j)}(1+x)^{|i-j|} \ .$$

Proposition 8.2 shows that the determinant $\det(D_\gamma(n))$ is given by

$$\det(D_{(1,x,x^2,x^3,\ldots)}(n)) = \left(-1 - x - x^2\right)^{n-1} x^{\binom{n-1}{2}}$$

for $n \geq 1$.

Setting $x=1$ in this special case $u_1=u_2=l_1=l_2=1$, we get a matrix M with entries $m_{i,j}=2^{|i-j|}$ for $0\leq i,j<n$. Its determinant is $(-3)^{n-1}$. It is easy to show that the matrix M_a of order n with entries $m_{i,j}=a^{|i-j|}$ for $0\leq i,j<n$ has determinant $(1-a^2)^{n-1}$.

A similar example is the special case $-u_1=u_2=-l_1=l_2=1$ which yields for instance the matrix $D_\gamma(4)=D_\gamma^{(-1,1,-1,1)}(4)$ given by

$$\begin{pmatrix} 1 & -1+x & 1-2x+x^2 & -1+3x-3x^2+x^3 \\ -1+x & x & -x+x^2 & x-2x^2+x^3 \\ 1-2x+x^2 & -x+x^2 & x^2 & -x^2+x^3 \\ -1+3x-3x^2+x^3 & x-2x^2+x^3 & -x^2+x^3 & x^3 \end{pmatrix}$$

and the reader can readily check that the coefficient $f_{i,j}$ of $D_\gamma(n)$ is given by

$$d_{i,j}=x^{\min(i,j)}(x-1)^{|i-j|}\ .$$

The determinant $\det(D_{(1,x,x^2,x^3,\dots)}(n))$ is given by

$$\det(D_{(1,x,x^2,x^3,\dots)}(n))=\left(-x^2+3x-1\right)^{n-1}x^{\binom{n-1}{2}}$$

for $n\geq 1$.

Proof of Proposition 8.2. By continuity and Proposition 8.1 it is enough to prove the formula in the case $u_1=u_2=1$.

This implies $d_{i,j}=x^i(1+x)^{(j-i)}$ for $i\leq j$.

Subtracting $(1+x)$ times column number $(n-2)$ from column number $(n-1)$ (which is the last one), etc until subtracting $(1+x)$ times column number 0 from column number 1 transforms the matrix $D(n)$ into a lower triangular matrix with diagonal entries

$$1,x-(1+x)(l_1+l_2x),x\left(x-(1+x)(l_1+l_2x)\right),\\ \dots,x^{n-2}(-l_1+(1-l_1-l_2)x-l_2x^2)\ .$$

$\square$

Theorem 8.3. *Let $\gamma=(\gamma_0,\dots,\gamma_{p-1},\gamma_0,\dots,\gamma_{p-1},\dots)$ be a $p-$periodic sequence and let*

$$d(n)=\det(D_\gamma^{(u_1,u_2,l_1,l_2)}(n))$$

be the determinants of the associated matrices (for fixed (u_1,u_2,l_1,l_2)).

Then there exist an integer d and constants $C_1,\dots C_d$ such that

$$d(n)=\sum_{i=1}^{d}C_i\ d(n-ip)$$

for all n huge enough.

Remark 8.4. Generically, the coefficients C_i seem to display the symmetry

$$C_{d-i}=\rho^{(d-2i)/2}\ C_i$$

(with $C_0 = -1$) for some constant ρ which seems to be polynomial in $\gamma_0, \ldots, \gamma_{p-1}, u_1, u_2, l_1, l_2$.

Proof of Theorem 8.3. For $k \geq p$ add to the $k-$th row a linear combination (with coefficients depending only on (u_1, u_2, l_1, l_2)) of rows $k-1, k-2, \ldots, k-p$ such that $d_{k,k-i} = 0$ for $i \geq p$. Do the analogous operation on columns and apply Theorem 7.1 to the resulting matrices. □

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