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On the ultrametric Stone-Weierstrass theorem and Mahler's expansion

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Résumé. Nous explicitons une version ultramétrique du théorème de Stone-Weierstrass. Pour une partie E d'un anneau de valuation V de hauteur 1, nous montrons, sans aucune hypothèse sur le corps résiduel, que l'ensemble des fonctions polynomiales est dense dans l'anneau des fonctions continues de E dans V si et seulement si la clôture topologique $\widehat{E}$ de E dans le complété $\widehat{V}$ de V est compacte. Nous explicitons ainsi le développement d'une fonction continue en série de fonctions polynomiales.

Abstract. We describe an ultrametric version of the Stone-Weierstrass theorem, without any assumption on the residue field. If E is a subset of a rank-one valuation domain V, we show that the ring of polynomial functions is dense in the ring of continuous functions from E to V if and only if the topological closure $\widehat{E}$ of E in the completion $\widehat{V}$ of V is compact. We then show how to expand continuous functions in sums of polynomials.

Introduction

The classical Stone-Weierstrass theorem says that the ring $\mathbb{R}[X]$ of real polynomial functions is dense in the ring $\mathcal{C}(E, \mathbb{R})$ of real continuous functions on a compact subset E of $\mathbb{R}$ — endowed with the uniform convergence topology. Obviously, $\mathbb{Q}[X]$ is then also dense in $\mathcal{C}(E, \mathbb{R})$.

Dieudonné proved a similar p-adic result [9, Theorem 4], replacing $\mathbb{R}$ by the p-adic completion $\mathbb{Q}_p$ of $\mathbb{Q}$, and this was extended by Kaplansky [13] to a compact subset E of a valued field L for a rank-one valuation (with no assumption on the residue field) (see also [12, Theorem 32]). Since the ring of p-adic integers $\mathbb{Z}_p$ is compact, then every function $\phi \in \mathcal{C}(\mathbb{Z}_p, \mathbb{Z}_p)$ may be uniformly approximated by polynomials in $\mathbb{Q}[X]$ and, in particular, the ring $\mathrm{Int}(\mathbb{Z}) = \{f \in \mathbb{Q}[X] \mid f(\mathbb{Z}) \subseteq \mathbb{Z}\}$ of integer-valued polynomials is dense in the ring $\mathcal{C}(\mathbb{Z}_p, \mathbb{Z}_p)$.

Considering a domain D, with quotient field K, and a subset E of K, we more generally introduce the ring $\mathrm{Int}(E, D)$ of integer-valued polynomials

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on E:

$$\mathrm{Int}(E,D) = \{f \in K[X] \mid f(E) \subseteq D\}.$$

In particular, for $E = D$, we let $\mathrm{Int}(D) = \{f \in K[X] \mid f(D) \subseteq D\}$ be the ring of integer-valued polynomials on D. If V is a discrete valuation domain with finite residue field, its completion $\widehat{V}$ is compact, and hence, the polynomial ring $\mathrm{Int}(V)$ is dense in the ring of continuous functions $\mathcal{C}(\widehat{V},\widehat{V})$, for the uniform convergence topology [4, Theorem III.3.4] (in fact, one may even replace V by a local, one-dimensional Noetherian domain D, with finite residue field $D/\mathfrak{m}$, such that D is analytically irreducible, that is, its completion $\widehat{D}$ in the $\mathfrak{m}$-adic topology is a domain [4, Corollary III.5.4]).

Considering a subset E of a discrete valuation domain V such that $\widehat{E}$ is compact, it follows from Kaplansky's extension of the Stone-Weierstrass theorem that the polynomial ring $\mathrm{Int}(E,V)$ is dense in the ring $\mathcal{C}(\widehat{E},\widehat{V})$ of continuous functions from $\widehat{E}$ into $\widehat{V}$, with no assumption on the residue field (and we showed recently that, here again, we could replace V by a local, one-dimensional Noetherian domain which is analytically irreducible [5, Proposition 4.3]).

On the other hand, Mahler [14, Theorem 1] gave an explicit description of the expansion, of a continuous function $\phi \in \mathcal{C}(\mathbb{Z}_p,\mathbb{Z}_p)$, in series of the form $\phi = \sum_{n\geq 0} a_n \binom{X}{n}$, where

$$\binom{X}{n} = \frac{X(X-1)\ldots(X-n+1)}{n!}.$$

For the domain V of a discrete valuation v with finite residue field, replacing the sequence of integers by a 'very well distributed' sequence $\{u_n\}_{n\in\mathbb{N}}$, and the binomial polynomials $\binom{X}{n}$ by the integer-valued polynomials $f_n(X) = \prod_{k=0}^{n} \frac{X-u_k}{u_n-u_k}$, Amice [1, Chap. II] generalized this description to the expansion of a continuous function $\phi \in \mathcal{C}(\widehat{V},\widehat{V})$ (see also Wagner [21] for positive characteristic): ϕ can uniquely be expanded in the form

$$\phi = \sum_{n\geq 0} a_n f_n, \text{ with } a_n \in \widehat{V} \text{ and } \lim_{n\to\infty} a_n = 0.$$

Moreover

$$\inf_{x\in\widehat{V}} v\big(\phi(x)\big) = \inf_{n\in\mathbb{N}} v(a_n).$$

It is then said that the f_n's form a normal basis of the ultrametric Banach space $\mathcal{C}(\widehat{V},\widehat{K})$ [1].

In fact, Amice's result applies more generally to functions $\phi \in \mathcal{C}(E,\widehat{V})$ on 'regular' compact subspaces E of $\widehat{V}$. In a recent paper, Bhargava and Kedlaya [3] extended Amice's results to every compact subset E of $\widehat{V}$, using the notion of 'v-ordering', introduced in [2], which appears to be a fine generalization of Amice's very well distributed sequences. An alternate version of this result appeared also recently [18, Theorem 1.2], where the sequence $\{f_n\}$ is replaced by the sequence $\{F_n\}$ of the Fermat polynomials, which are known to form a basis of $\mathrm{Int}(V)$ ([10] or [4, Proposition II.2.12]).)

Despite their generalizations to subsets, these expansions in series concern only the case of a rank-one discrete valuation domain with finite residue field, although the Stone-Weierstrass theorem holds without this finiteness hypothesis, and, in fact, also for a non discrete rank-one valuation domain.

The aim of this paper is to give an extension of these results in this more general setting, and in the same time, to provide proofs which are significantly shorter than in [1] and [3] and also more elementary than in papers dealing with normal bases such as [20]. Moreover we conversely establish that the compactness of $\widehat{E}$ is necessary in the ultrametric Stone-Weierstrass theorem.

Hypothesis: We let V be the ring of a rank-one valuation v, with quotient field K, and we consider a subset E of K. We denote by $\widehat{V}$, $\widehat{K}$ and $\widehat{E}$ the completions of V, K and E (but simply denote by v the extension of the valuation to $\widehat{K}$).

For sake of completeness, we first establish an analogue of the Stone-Weierstrass theorem which, in fact, can be derived from Kaplansky's general results [12, Theorem 32]: if $\widehat{E}$ is compact, the ring $\mathrm{Int}(E,V)$ of integer-valued polynomials is dense in the ring $\mathcal{C}(\widehat{E},\widehat{V})$ of continuous functions from $\widehat{E}$ to $\widehat{V}$. In the second section, assuming moreover the subset E to be infinite, we use this result to give explicit series expansions of continuous functions. Finally, in the last section, we prove the converse: the Stone-Weierstrass theorem holds if and only if the completion $\widehat{E}$ of E is compact.

1. The Stone-Weierstrass theorem

Let us recall that the absolute value of x is defined by $|x| = e^{-v(x)}$; thus K is an ultrametric space, endowed with the distance

$$d(x,y) = |x-y| = e^{-v(x-y)}.$$

For $\alpha \in \mathbb{R}$ and $a \in V$, we denote by $B_\alpha(a)$ the closed ball with center a and radius $e^{-\alpha}$:

$$B_\alpha(a) = \{x \in V \mid v(x-a) \geq \alpha\}.$$

The balls with center a form a fundamental system of clopen neighborhoods of a.

If $f \in K[X]$ is a polynomial with coefficients in K, and b a common denominator of its coefficients, then $bf \in V[X]$, and hence, for $x, y \in V$, we have

$$v\big((f(y) - f(x)\big) \geq v(y-x) - v(b)$$

that is, f is uniformly continuous on V. Now, if E is a compact subset of K, then E is bounded, that is, $|x| \leq M$, for each $x \in E$, or equivalently, E is a *fractional subset* of V, that is, there is a nonzero element $d \in V$ such that $dE = \{dx \mid x \in E\} \subseteq V$.

Under the hypothesis that E is a fractional subset, letting $g = f(X/d)$, we have $f(x) = g(dx)$, and it follows that f is uniformly continuous on E. We can thus consider $K[X]$ as a subring of the ring $\mathcal{C}(E, K)$ of continuous functions from E to K, similarly we can consider the ring $\mathrm{Int}(E, V)$ of integer-valued polynomials as a subring of the ring $\mathcal{C}(E, V)$ of continuous functions from E to V.

We wish to prove that, if E is compact, then $\mathrm{Int}(E, V)$ is dense in $\mathcal{C}(E, V)$. First we establish that the polynomials in $\mathrm{Int}(E, V)$ separate the points of E (a property similar to that of *interpolation domains* [6]):

Lemma 1.1. *Let K be the quotient field of a rank-one valuation domain V and E be a compact subset of K. For each pair of points $a, b \in E$, there exists $f \in \mathrm{Int}(E, V)$ such that $f(a) = 1$ and $f(b) = 0$.*

Proof. Set $v(a - b) = \gamma$. Since E is compact, it is contained in the finite union of disjoint closed balls $B_\gamma(b_i)$, with centers $b_0, b_1, \ldots, b_n$, one of them containing a and b. With no loss of generality, we set $b_0 = b$. We prove, by induction on n, that there is a polynomial g, with coefficients in K, such that $g(b) = 0$ and $v\big(g(x)\big) \geq v\big(g(a)\big)$ for all $x \in E$. Hence, the lemma is proved, with $f(X) = g(X)/g(a)$.

— For one ball, that is, $E \subseteq B_\gamma(b)$, we have $v(x - b) \geq v(a - b)$, for each $x \in E$. We can thus take $g = X - b$.

— Assume the result to be true for n balls and consider the case of $n + 1$ balls: $E \subseteq \bigcup_{0 \leq i \leq n} B_\gamma(b_i)$. We set $\delta_i = v(b - b_i)$ and order $b_1, b_2, \ldots b_n$ in such a way that $\delta_1 \geq \delta_2 \geq \ldots \geq \delta_n$. We note that $\gamma > \delta_1$, and hence, $v(a - b_n) = \delta_n$; we note also that, for $1 \leq i \leq n$, $v(b_i - b_n) \geq \delta_n$. We let E_1 be the intersection of E with $\bigcup_{0 \leq i \leq n-1} B_\gamma(b_i)$. By induction hypothesis, there is a polynomial g_1 such that $g_1(b) = 0$ and $v\big(g_1(x)\big) \geq v\big(g_1(a)\big)$ for each $x \in E_1$. Multiplying g_1 by a constant, we may assume that $g_1 \in \mathrm{Int}(E, V)$, and hence, that $v\big(g_1(x)\big) \geq 0$, for each $x \in E$. Since the value group of the valuation is a subgroup of $\mathbb{R}$, and $(\gamma - \delta_n) > 0$, there is an integer m such that $m(\gamma - \delta_n) \geq v\big(g_1(a)\big)$. We claim that we can take g of the form $g = g_1(X - b_n)^m$. For this, we consider two cases:

• If $x \in E_1$, then $x \in B_\gamma(b_i)$ for some $i < n$. Then $v(x - b_n) = v(b_i - b_n) \geq \delta_n$. Thus, we have

$$v\big(g(x)\big) = v\big(g_1(x)\big) + mv(x - b_n) \geq v\big(g_1(a)\big) + m\delta_n = v\big(g(a)\big).$$

• If $x \in E$ but $x \notin E_1$, we have $x \in B_\gamma(b_n)$, that is, $v(x - b_n) \geq \gamma$. It follows that we have

$$v\big(g(x)\big) \geq v\big(g_1(x)\big) + m\gamma \geq v\big(g_1(x)\big) + v\big(g_1(a)\big) + m\delta_n \geq v\big(g(a)\big).$$

□

We are ready for a first version of the Stone-Weierstrass theorem (the proof is similar to [4, Exercise III.20]):

Lemma 1.2. *Let K be the quotient field of a rank-one valuation domain V and E be a compact subset of K. Then* $\mathrm{Int}(E,V)$ *is dense in the ring* $\mathcal{C}(E,V)$ *of continuous functions from E to V, endowed with the uniform convergence topology.*

Proof. Each continuous function can be arbitrarily approximated by a locally constant function. Thus it is enough to prove that, for each clopen set U, the characteristic function ϕ of U can be approximated by an integer-valued polynomial: for each $A > 0$, there exists a polynomial $f \in \mathrm{Int}(E,V)$, such that $v\big(f(x) - \phi(x)\big) \geq A$, for each $x \in E$. Choose $a \in U$. For each $b \notin U$, by the previous lemma, there is a polynomial $f_b \in \mathrm{Int}(E,V)$, such that $f_b(a) = 1$, and $f_b(b) = 0$. Since f_b is continuous, we have $v\big(f_b(x)\big) \geq A$ for each x in some neighborhood of b. Since U is a clopen set and E is compact, there is a product g_a of a finite number of such polynomials such that $g_a(a) = 1$, and $v\big(g_a(x)\big) \geq A$ for each x in the complement of U in E. Since g_a is continuous, we have $v\big(g_a(x) - 1\big) \geq A$, for each x in some neighborhood of a. Since U is compact, there is a finite product $\prod(1-g_a) = 1-g$ with such polynomials g_a such that $v\big(g(x) - 1\big) \geq A$, for each $x \in U$ and $v\big(g(x)\big) \geq A$, for each $x \in E \setminus U$. □

In fact, it is often the case that the topological closure $\widehat{E}$ of E in the completion $\widehat{V}$ of V is compact while E is not (as for the ring V itself, if V is a — non-complete — rank-one discrete valuation domain, with finite residue field). If $\widehat{E}$ is compact, then E is a fractional subset of V, the polynomials with coefficients in K are uniformly continuous on E, thus $\mathrm{Int}(E,V)$ can be considered as a subring of the ring $\mathcal{C}(\widehat{E},\widehat{V})$ of (uniformly) continuous functions from $\widehat{E}$ to $\widehat{V}$, and $K[X]$ as a subring of the ring $\mathcal{C}(\widehat{E},\widehat{K})$ of (uniformly) continuous functions from $\widehat{E}$ to $\widehat{K}$. We thus derive another version of the Stone-Weierstrass theorem:

Proposition 1.3. *Let K be the quotient field of a rank-one valuation domain V and E be a subset of K such that $\widehat{E}$ is compact. Then,*

(i) $\mathrm{Int}(E,V)$ *is dense in* $\mathcal{C}(\widehat{E},\widehat{V})$ *for the uniform convergence topology,*

(ii) $K[X]$ *is dense in* $\mathcal{C}(\widehat{E},\widehat{K})$ *for the uniform convergence topology.*

Proof. (i) From the previous lemma, $\mathrm{Int}(\widehat{E},\widehat{V})$ is dense in $\mathcal{C}(\widehat{E},\widehat{V})$: each $\phi \in \mathcal{C}(\widehat{E},\widehat{V})$ can arbitrarily be approximated by polynomials (with coefficients in $\widehat{K}$). In fact, as $K[X]$ is dense in $\widehat{K}[X]$, each $\phi \in \mathcal{C}(\widehat{E},\widehat{V})$ can be approximated by polynomials with coefficients in K. Finally, if $f \in K[X]$

is such that $v\big(f(x) - \phi(x)\big) \geq 0$ for every $x \in \widehat{E}$, then f is clearly an integer-valued polynomial.
(ii) Since $\widehat{E}$ is compact, each function $\phi \in \mathcal{C}(\widehat{E}, \widehat{K})$ is bounded. Multiplying ϕ by a constant, we obtain a function in $\mathcal{C}(\widehat{E}, \widehat{V})$. Hence the result follows from (i). □

2. Mahler's expansion

We assume from now on that E is an infinite subset of K. We first generalize the notion of *v-ordering*, introduced by Bhargava for a subset of a discrete valuation domain.

Definition 1. Let v be a valuation on a field K, and E be a infinite subset of K. We say that a sequence $\{u_n\}_{n\in\mathbb{N}}$ of elements of E is a *v-ordering* of E if, for each $n > 0$, and each $x \in E$, we have

$$v\Big(\prod_{k=0}^{n-1}(u_n - u_k)\Big) \leq v\Big(\prod_{k=0}^{n-1}(x - u_k)\Big).$$

It is immediate that such an (infinite) sequence exists in the case where v is rank-one discrete. We show that this is also the case, for a rank-one valuation, if $\widehat{E}$ is compact:

Lemma 2.1. *Let v be a rank-one valuation on a field K, and E be a infinite subset of K such that $\widehat{E}$ is compact. Then, for each $a \in E$, there exists a v-ordering $\{u_n\}_{n\in\mathbb{N}}$ of E such that $u_0 = a$.*

Proof. The sequence $\{u_n\}_{n\in\mathbb{N}}$ is obtained inductively. Supposing we have obtained the first n elements, we want to find u_n which is a minimum for the continuous function $\Psi(x) = v\Big(\prod_{k=0}^{n-1}(x - u_k)\Big)$. Since $\widehat{E}$ is compact, such a minimum is reached for some $y_n \in \widehat{E}$. On the other hand, if $u_n \in E$ is close enough to y_n (to be precise, if $v(u_n - y_n) > v(y_n - u_k)$, for $0 \leq k \leq n-1$), then $\Psi(u_n) = \Psi(y_n)$. □

We now suppose, as in the first section, that V is a rank-one valuation domain (corresponding to the valuation v). Given a v-ordering $\{u_n\}_n$ of the subset E of K, we set

$$f_n(X) = \prod_{k=0}^{n-1}\frac{X - u_k}{u_n - u_k} \quad (n \geq 0).$$

It is then easy to show that $\{f_n\}$ is a regular basis of the V-module $\mathrm{Int}(E, V)$, that is, a basis such that f_n is of degree n for each n [2, Theorem 1]. Indeed, it follows from the definition of a v-ordering that each f_n is in $\mathrm{Int}(E, V)$, and that they form a basis follows from the fact that, for each n, $f_n(u_n) = 1$ and $f_n(u_k) = 0$, for $k < n$.

By analogy with real analysis, the numerators of the f_n's, that is, the polynomials $g_n(X) = \prod_{k=0}^{n-1}(X - u_k)$ may be called *Tchebychev polynomials* [11]. Indeed, for the norm of the uniform convergence on E, one has:

$$||g_n||_E = \sup_{x\in E} |g_n(x)| = |g_n(u_n)|$$

and

$$||g_n||_E = \inf\{||g||_E \mid g \in K[X],\, \deg(g) = n,\, g\,\text{monic}\}\ [7].$$

But, contrarily to the real case, there is no uniqueness for the Tchebychev polynomials since there is no uniqueness for the v-orderings.

We shall prove that each continuous function can be expanded in series of the form $\phi = \sum_{n\geq 0} a_n f_n$. From the Stone-Weierstrass theorem, we first show that it is always possible to expand continuous functions in series of integer-valued polynomials. For sake of completeness we give the proof of the next lemma that we may find, for instance, in [19, Lemma 1.2].

Lemma 2.2. *Let K be the quotient field of a rank-one valuation domain V and E be a subset of K. Denote by t an element of V such that $v(t) > 0$. If $\widehat{E}$ is compact, then every continuous function $\phi \in \mathcal{C}(\widehat{E}, \widehat{V})$ can be expanded in the form $\phi = \sum_{n\geq 0} t^n g_n$ with $g_n \in \mathrm{Int}(E, V)$.*

Proof. Since $\mathrm{Int}(E, V)$ is dense in $\mathcal{C}(\widehat{E}, \widehat{V})$ [Proposition 1.3], there is a polynomial $g_0 \in \mathrm{Int}(E, V)$ such that, for all $x \in \widehat{E}, v\big(\phi(x) - g_0(x)\big) \geq v(t)$. That is, $\phi = g_0 + t\phi_1$, where $\phi_1 \in \mathcal{C}(\widehat{E}, \widehat{V})$. Similarly, one can find $g_1 \in \mathrm{Int}(E, V)$ and $\phi_2 \in \mathcal{C}(\widehat{E}, \widehat{V})$, such that $\phi_1 = g_1 + t\phi_2$, and hence, $\phi = g_0 + tg_1 + t^2\phi_2$. Proceeding by induction, we obtain the desired expansion. $\square$

We next show how to obtain a series expansion from another one.

Lemma 2.3. *Let $\phi = \sum_{n\geq 0} c_n g_n$, be a series expansion of a continuous function ϕ, such that each g_n is an integer-valued polynomial on E, and with coefficients $c_n \in \widehat{K}$ such that $\lim_{n\to\infty} c_n = 0$. Consider a set $\{\varphi_n\}_{n\in\mathbb{N}}$ of generators of the V-module $\mathrm{Int}(E, V)$, and decompose each g_n on these generators: $g_n = \sum_{k=0}^{i_n} b_{k,n}\varphi_k$, with $b_{k,n} \in V$. Then, for each $k \in \mathbb{N}$, the series $b_k = \sum_{n=0}^{\infty} c_n b_{k,n}$ converges in $\widehat{V}$, $\lim_{k\to\infty} b_k = 0$, and $\phi = \sum_{n\geq 0} b_n\varphi_n$.*

Proof. The series $\sum_{n=0}^{\infty} c_n b_{k,n}$ converges, since $\lim_{n\to\infty} c_n = 0$. Fix $N \in \mathbb{N}$, there is $j \in \mathbb{N}$ such that $n > j$ implies $v(c_n) \geq N$. Let $i(j) = \sup\{i_0, \ldots, i_j\}$. For $k > i(j)$, we then have $b_k = \sum_{n>j} c_n b_{k,n}$, and hence, $v(b_k) \geq N$. Therefore, $\lim_{k\to\infty} b_k = 0$, and the series $\sum_{n\geq 0} b_n\varphi_n$ converges. Now, for each j, consider the difference

$$\Psi_j = \phi - \sum_{k=0}^{i(j)} b_k\varphi_k = \sum_{n=0}^{\infty} c_n g_n - \sum_{k=0}^{i(j)} b_k\varphi_k.$$

From the definition of the coefficients b_k, we have

$$\Psi_j = \sum_{n=j+1}^{\infty} c_n \left[g_n - \sum_{k=0}^{i(j)} b_{k,n}\varphi_k \right].$$

And hence, $\inf_{x\in\widehat{V}} v\big(\Psi_j(x)\big) \geq N$, that is, $\lim_{n\to\infty} \Psi_j = 0$. In other words,

$$\phi = \sum_{n=0}^{\infty} c_n g_n = \sum_{k=0}^{\infty} b_k \varphi_k.$$

□

We then obtain the following, generalizing [3, Theorem 1]:

Theorem 2.4. *Let K be the quotient field of a rank-one valuation domain V and E be an infinite subset of K such that $\widehat{E}$ is compact. Considering a v-ordering $\{u_n\}_{n\in\mathbb{N}}$ of E, and letting $f_n(X) = \prod_{k=0}^{n-1} \frac{X-u_k}{u_n-u_k}$, then every continuous function $\phi \in \mathcal{C}(\widehat{E},\widehat{K})$ can be expanded in a series of the form $\phi = \sum_{n\geq 0} a_n f_n$, with $\lim_{n\to\infty} a_n = 0$. The coefficients $a_n \in \widehat{K}$ are uniquely determined by the recursive formulae:*

$$a_n = \phi(u_n) - \sum_{k=0}^{n-1} a_k f_k(u_n) \quad (n \geq 0).$$

Moreover

$$\sup_{x\in\widehat{E}} |\phi(x)| = \sup_{n\in\mathbb{N}} |\phi(u_n)| = \sup_{n\in\mathbb{N}} |a_n|.$$

Proof. The possibility to expand a function $\phi \in \mathcal{C}(\widehat{E},\widehat{V})$ in a series of the form $\phi = \sum_{n\geq 0} a_n f_n$, with $\lim_{n\to\infty} a_n = 0$, follows immediately from Lemma 2.2 and Lemma 2.3. This generalizes to a function $\phi \in \mathcal{C}(\widehat{E},\widehat{K})$: since $\widehat{E}$ is compact, ϕ is bounded, and hence, $a\phi \in \mathcal{C}(\widehat{E},\widehat{V})$ for some nonzero constant $a \in V$.

The recursive formulae, follow immediately from the fact that $f_n(u_n) = 1$ and $f_k(u_n) = 0$ for $k > n$. In particular, the a_k's are uniquely determined.

Finally, let $\alpha = \inf_{n\in\mathbb{N}} v(a_n)$; as $v(a_n) \to \infty$, this infimum is reached. Let n_0 be the smallest integer such that $v(a_{n_0}) = \alpha$. From the recursive formulae, we have

$$\phi(u_{n_0}) = \sum_{0\leq k<n_0} a_k f_k(u_{n_0}) + a_{n_0},$$

and hence, from the choice of n_0, $v\big(\phi(u_{n_0})\big) = v(a_{n_0}) = \alpha$. We obtain

$$\sup_{x\in\widehat{E}} |\phi(x)| \geq \sup_{n\in\mathbb{N}} |\phi(u_n)| = \sup_{n\in\mathbb{N}} |a_n|.$$

We conclude that we have equalities from the following (obvious) lemma. □

Lemma 2.5. *Let $\phi = \sum_{n\geq 0} c_n g_n$ be the sum of a series, where $\{g_n\}_{n\in\mathbb{N}}$ is a sequence of integer-valued polynomials, and where the coefficients $c_n \in \widehat{K}$ are such that $\lim\limits_{n\to\infty} c_n = 0$. Then we have $\inf_{x\in\widehat{E}} v\big(\phi(x)\big) \geq \inf_{n\in\mathbb{N}} v(c_n)$, that is, $\sup_{x\in\widehat{E}} |\phi(x)| \leq \sup_{n\in\mathbb{N}} |c_n|$.*

Remark 2.6. The previous result may be interpreted in the theory of normal bases. Let M be an ultrametric Banach space on $\widehat{K}$ and let M_0 be the unit ball, that is, the sub-$\widehat{V}$-module $M_0 = \{x \in M \mid ||x|| \leq 1\}$. Recall that a family $\{e_i \mid i \in I\}$ of elements of M_0 is called a *normal basis* [1] or an *orthonormal basis* ([16] or [17]) of M if each $x \in M$ has a unique representation of the form $x = \sum_{i\in I} x_i e_i$ with $x_i \in \widehat{K}$, $\lim_{\mathcal{F}} x_i = 0$ where $\mathcal{F}$ denotes the filter formed by the cofinite subsets of I, and $||x|| = \sum_{i\in I} |x_i|$.

(i) Of course, the polynomials $\{f_n\}_{n\in\mathbb{N}}$ in Theorem 2.4 form a normal basis of the ultrametric Banach space $\mathcal{C}(\widehat{E},\widehat{K})$.

(ii) The theoretical existence of such a normal basis of $\mathcal{C}(\widehat{E},\widehat{K})$ with a polynomial of each degree was already proved by Van der Put [20, Prop. 5.3], however these polynomials were not given explicitly.

(iii) One could give another proof of Theorem 2.4 using the theory of normal bases. Recall first a result of Coleman [8, Lemma A.1.2] (extending to a rank-one valuation a result of J.-P. Serre [17, lemme 1] dealing with the case where the valuation is discrete) which essentially says the following: a family $\{e_i \mid i \in I\}$ of elements of M_0 is a normal basis of M if and only if there exists $t \in K$ with $|t| < 1$ such that the classes of the e_i's form a basis of the V/tV-module M_0/tM_0.

In Theorem 2.4, $M = \mathcal{C}(\widehat{E},\widehat{K})$ and $M_0 = \mathcal{C}(\widehat{E},\widehat{V})$. It is then easy to see that the polynomials f_n satisfy Coleman's condition: it follows from Proposition 1.3 that $\mathcal{C}(\widehat{E},\widehat{V}) = \mathrm{Int}(E,V) + t\mathcal{C}(\widehat{E},\widehat{V})$, thus, the classes of the f_n's generate $\mathcal{C}(\widehat{E},\widehat{V})/t\mathcal{C}(\widehat{E},\widehat{V})$; on the other hand, the equalities $f_n(u_k) = \delta_{k,n}$ for $0 \leq k \leq n$ easily imply that these classes are linearly independent.

Finally, we generalize [3, Theorem 2] and [18, Theorem 3.3]:

Theorem 2.7. *Let K be the quotient field of a rank-one valuation domain V, and E be an infinite subset of K such that $\widehat{E}$ is compact. Then, for every set of generators $\{\varphi_n\}_{n\in\mathbb{N}}$ of the V-module $\mathrm{Int}(E,V)$ and every continuous function $\phi \in \mathcal{C}(\widehat{E},\widehat{K})$, there exists a sequence $\{b_n\}$ in $\widehat{K}$, such that $\phi = \sum_{n\geq 0} b_n\varphi_n$, with $\lim\limits_{n\to\infty} b_n = 0$. Moreover, if the φ_n's form a basis of the V-module $\mathrm{Int}(E,V)$, then the previous sequence $\{b_n\}$ is unique and one* *has:*

$$\sup_{x\in\widehat{E}} |\phi(x)| = \sup_{n\in\mathbb{N}} |b_n|.$$

In other words, if $\widehat{E}$ is compact, then every basis of the V-module $\mathrm{Int}(E,V)$ is a normal basis of the ultrametric Banach space $\mathcal{C}(\widehat{E},\widehat{K})$.

Proof. From the series expansion $\phi = \sum_{n\geq 0} a_n f_n$ [Theorem 2.4], we derive another one: $\phi = \sum_{k\geq 0} b_k\varphi_k$, where $b_k = \sum_{n=0}^{\infty} a_n b_{k,n}$, with $b_{k,n} \in V$ [Lemma 2.3]. Note that, for such a sequence $\{b_n\}$, we have the inequality

$$\inf_{n\in\mathbb{N}} v(b_n) \geq \inf_{n\in\mathbb{N}} v(a_n) = \inf_{x\in\widehat{E}} v\big(\phi(x)\big).$$

Finally, we assume that the φ_n's form a basis of $\mathrm{Int}(E,V)$, and prove that the expansion $\sum b_n\varphi_n$ is unique. Consider two series expansion $\sum_{n\geq 0} b_n\varphi_n$ and $\sum_{n\geq 0} b'_n\varphi_n$ of the same function ϕ and assume, by way of contradiction, that $b_{k_0} \neq b'_{k_0}$ for some k_0. Let $t \in V$ be such that $v(t) > v(b_{k_0} - b'_{k_0})$. As $v(b_k)$ and $v(b'_k) \to +\infty$, there is an integer n such that $v(b_k - b'_k) \geq v(t)$ for $k > n$ (note that $n \geq k_0$). Consider the polynomial $\psi = \sum_{k=0}^{n}(b_k - b'_k)\varphi_k$. As we can also write $\psi = \sum_{k=n+1}^{\infty}(b'_k - b_k)\varphi_k$, it follows that ψ belongs to $t\mathrm{Int}(E,V)$. Since the decomposition of ψ along the basis formed by the φ_n's is unique, we have in particular $v(b_{k_0} - b'_{k_0}) \geq v(t)$. We thus reach a contradiction.

The first part of the proof shows that $\inf_{n\in\mathbb{N}} v(b_n) \geq \inf_{x\in\widehat{E}} v\big(\phi(x)\big)$. It follows from Lemma 2.5 that we have an equality. $\square$

3. The compactness of $\widehat{E}$ is necessary

We have already noted that, if $\widehat{E}$ is compact, then E is a fractional subset of V. On the other hand, if E is not a fractional subset, then $\mathrm{Int}(E,V) = V$ [4, Proposition I.1.9], hence non-constant continuous functions cannot be approximated by polynomials in $\mathrm{Int}(E,V)$. We shall thus suppose that E is a fractional subset, and replacing E by dE (which is homeomorphic to it), that it is even a subset of V (see also [4, Remark I.1.11]).

We first give a characterization of the subsets E such that $\widehat{E}$ is compact, similar to [5, Lemma 4.4] (see also [4, Proposition III.1.2]). For this, recall that the non-zero ideals of V are either of the form $I_\alpha = \{x \in V \mid v(x) \geq \alpha\}$, or $I'_\alpha = \{x \in V \mid v(x) > \alpha\}$, for α real and positive. Correspondingly, the cosets modulo I_α are closed balls of the type $B_\alpha(a)$, and the cosets modulo I'_α are the corresponding open balls. In any case, since K is an ultrametric space, each such coset is a clopen set.

Lemma 3.1. *Let V be a rank-one valuation domain and E be a subset of V. The following assertions are equivalent.*

1. *The completion $\widehat{E}$ of E is compact.*

2. *For each non-zero ideal I of V, E meets only finitely many cosets of V modulo I.*
3. *For each positive integer n, E meets only finitely many cosets of V modulo the ideal $I_n = \{x \in V \mid v(x) \geq n\}$.*

Proof. $1 \Longrightarrow 2$. The condition is obviously necessary: if E meets infinitely many cosets modulo an ideal I, then E is covered by infinitely many disjoint clopen subsets, and its completion $\widehat{E}$ is then also covered by the disjoint union of the completions of these clopen subsets.

$2 \Longrightarrow 3$. This is obvious.

$3 \Longrightarrow 1$. By hypothesis, for each n, $\widehat{E}$ meets only finitely many cosets of $\widehat{V}$ modulo $\widehat{I_n}$. Let $\{x_n\}$ be a sequence in $\widehat{E}$. Infinitely many of its terms, forming a subset X_1 of $\widehat{E}$, are in the same class modulo $\widehat{I_1}$. Infinitely many terms of X_1, forming a subset X_2 of X_1, are in the same class modulo $\widehat{I_2}$. And so on. We thus define a decreasing sequence $\{X_n\}$ of subsets, the elements of X_n being in the same class modulo $\widehat{I_n}$. Let x_{k_n} be the first term of the sequence which is in X_n. The subsequence $\{x_{k_n}\}$ of $\{x_n\}$ is then a Cauchy sequence in $\widehat{E}$ and it converges. □

We shall write that E/I is infinite (resp. finite) if E meets infinitely many (resp. only finitely many) cosets modulo I.

Lemma 3.2. *Let V be a rank-one valuation domain, with quotient field K, and let E be a fractional subset of V. If $\mathrm{Int}(E, V)$ is dense in $\mathcal{C}(\widehat{E}, \widehat{V})$, then the completion $\widehat{E}$ of E is compact.*

Proof. Assume that E is a subset of V and, by way of contradiction, that $\widehat{E}$ is not compact, that is, E/I_α is infinite, for some $\alpha > 0$. The principle of the proof is the following:

We consider a sequence $\{x_n\}$ of elements of E in distinct classes modulo some I_α (or some I'_α). It is thus possible to define a function $\phi \in \mathcal{C}(\widehat{E}, \widehat{V})$ such that, alternatively, for n even and n odd, we have $\phi(x_n) = 0$ and $\phi(x_n) = 1$. Suppose that $\mathrm{Int}(E, V)$ is dense in $\mathcal{C}(\widehat{E}, \widehat{V})$: in particular, there exists $f \in K[X]$ such that, alternatively, $v\big(f(x_n)\big) > 1$ and $v\big(f(x_n)\big) = 0$. We reach a contradiction by choosing the sequence $\{x_n\}$ in a such a way that, for each $f \in K[X]$, the sequence $\big\{v\big(f(x_n)\big)\big\}$ converges. In fact, as we can write $f = a\prod_{1\leq k\leq d}(X - \xi_k)$ in an algebraic extension L of K, it is enough to make sure that, for $\xi \in L$, the sequence $\{v(x_n - \xi)\}$ converges (in $\mathbb{R}$, extending the valuation v to a rank-one valuation of L).

If E/I_α is infinite, then, a fortiori, E/I_β and E/I'_β are infinite for $\beta \geq \alpha$. We set

$$\gamma = \inf\{\alpha \mid E/I'_\alpha \text{ is infinite}\}.$$

If the valuation is discrete (with value group $\mathbb{Z}$), then γ is an integer and necessarily E/I_γ is finite while E/I'_γ is infinite. We then consider three cases:

1) E/I_γ is finite, and E/I'_γ is infinite: there is a sequence $\{x_n\}$ in E such that all the x_n's are in the same class modulo I_γ, but in distinct classes modulo I'_γ. For $n \neq m$, we then have $v(x_n - x_m) = \gamma$. Let ξ be an element of an algebraic extension L of K. Therefore
— either $v(x_{n_0} - \xi) < \gamma$ for some n_0, then $v(x_n - \xi) = v(x_{n_0} - \xi)$ for all n;
— or $v(x_{n_0} - \xi) > \gamma$ for some n_0, then $v(x_n - \xi) = \gamma$ for $n \neq n_0$;
— or $v(x_n - \xi) = \gamma$ for all n.
At any rate, the sequence $\{v(x_n - \xi)\}$ is eventually constant.

2) E/I'_γ is finite. In this case, the valuation is not discrete and E/I_β is infinite for each $\beta > \gamma$. Let $\{\beta_n\}$ be a strictly decreasing sequence in $\mathbb{R}$ converging to γ. Among the (finitely many) classes that E meets modulo I'_γ, there is one which, for each n, meets infinitely many classes modulo I_{β_n}. By induction, we can thus build a sequence $\{x_n\}$ such that all its terms are in the same class modulo I'_γ but x_n is not in the class of $x_1, x_2, \dots$ nor x_{n-1} modulo I_{β_n}. For $m > n$, we thus have $\gamma < v(x_m - x_n) < \beta_m$. In particular, all the x_n's are in distinct classes modulo I_{β_1}. Moreover
— either $v(x_{n_0} - \xi) < \gamma$ for some n_0, then $v(x_n - \xi) = v(x_{n_0} - \xi)$ for all n;
— or $v(x_n - \xi) \geq \gamma$ for all n and $v(x_{n_0} - \xi) > \beta_{n_0}$ for some n_0; for $n > n_0$ we then have $v(x_n - \xi) = v(x_n - x_{n_0})$, and hence $\gamma \leq v(x_n - \xi) < \beta_n$;
— or $\gamma \leq v(x_n - \xi) \leq \beta_n$ for all n.
Hence, the sequence $\{v(x_n - \xi)\}$ converges to γ or is eventually constant.

3) E/I_γ is infinite. In this case again the valuation is not discrete and E/I_β is finite for $\beta < \gamma$. Here, we let $\{\beta_n\}$ be a strictly increasing sequence in $\mathbb{R}$ converging to γ. Let X be an infinite subset of E, the elements of which are in distinct classes modulo I_γ. Infinitely many elements of X, forming a subset X_1 of X are in the same class modulo I_{β_1}. Infinitely many elements of X_1, forming a subset X_2 of X_1, are in the same class modulo I_{β_2}. And so on: we define a decreasing sequence $\{X_n\}$ of subsets, the elements of X_n being in the same class modulo I_{β_n}. For each n, choosing x_n in X_n, we thus build a sequence $\{x_n\}$ such that, for $m > n$, we have $\beta_n \leq v(x_m - x_n) < \gamma$. Therefore
— either $v(x_{n_0} - \xi) < \beta_{n_0}$ for some n_0, then $v(x_n - \xi) = v(x_{n_0} - \xi)$ for $n > n_0$;
— or $v(x_n - \xi) \geq \beta_n$ for all n and $v(x_{n_0} - \xi) \geq \gamma$ for some n_0; for $n \neq n_0$ we then have $v(x_n - \xi) = v(x_n - x_{n_0})$, thus $\beta_n \leq v(x_n - \xi) < \gamma$;
— or $\beta_n \leq v(x_n - \xi) < \gamma$ for all n.
Hence, the sequence $\{v(x_n - \xi)\}$ converges to γ or is eventually constant.

$\square$

We conclude with the following.

Theorem 3.3. *Let V be a rank-one valuation domain, with quotient field K, and let E be a fractional subset of V. The following assertions are equivalent.*

1. *The completion $\widehat{E}$ of E is compact.*
2. *$K[X]$ is dense in $\mathcal{C}(\widehat{E},\widehat{K})$, for the uniform convergence topology.*
3. *$\mathrm{Int}(E,V)$ is dense in $\mathcal{C}(\widehat{E},\widehat{V})$, for the uniform convergence topology.*
4. *Every continuous function $\phi \in \mathcal{C}(\widehat{E},\widehat{K})$ can be expanded in the form $\phi = \sum_{n=0}^{\infty} c_n g_n$, with $g_n \in \mathrm{Int}(E,V)$ and $\lim_{n\to\infty} c_n = 0$.*
5. *Every continuous function $\phi \in \mathcal{C}(\widehat{E},\widehat{K})$ is bounded.*

Proof. Replacing E by dE, we assume that E is a subset of the ring V. That 1 implies 2, 3 and 4 was proved in the previous sections [Proposition 1.3 & Proposition 2.4]. That 4 implies 5 follows, for instance, from Lemma 2.5. And that 2 implies 5 from the fact that the values of a polynomial are clearly bounded on a fractional subset.

Conversely, 5 implies 1. Indeed, assume by way of contradiction, that $\widehat{E}$ is not compact: then $\widehat{E}$ can be covered by infinitely many disjoint clopen subsets, and hence, it is easy to define a continuous (in fact, locally constant) function which is not bounded.

Finally, that 3 implies 1 is the previous lemma. □

Remarks 3.4. 1) Analogously to [3, Theorem 3], we derive an easy consequence of Theorem 2.7 (this is also a consequence of [17, Prop. 3]): Assume $K = \widehat{K}$ and $E = \widehat{E}$ is compact, and let $\{\varphi_n\}_{n\in\mathbb{N}}$ be a basis of $\mathrm{Int}(E,V)$. Then, a K-valued measure on E, that is, a continuous K-linear map $\mu : \mathcal{C}(E,K) \to K$, is uniquely determined by the sequence $\{\mu_n\}$ of its values on the basis $\{\varphi_n\}_{n\in\mathbb{N}}$ (that is, $\mu_n = \mu(\varphi_n)$): if $\phi = \sum_{n=0}^{\infty} a_n\varphi_n$, then $\mu(\phi) = \sum_{n=0}^{\infty} a_n\mu_n$. Conversely, given a sequence $\{\mu_n\}$ of K, we claim there is some measure μ such that $\mu_n = \mu(\varphi_n)$, if and only if this sequence is bounded. It remains to show that the boundedness is necessary:

Assume that for a measure μ, the sequence $\{\mu(\varphi_n)\}$ is not bounded. Let $t \in V$ be such that $v(t) > 0$. There is an increasing sequence $\{p_n\}$ of positive integers such that $v(\mu(\varphi_{p_n})\} \leq -2nv(t)$; but then, the value of μ cannot be defined on the continuous function $\phi = \sum_{n\geq 0} t^n\varphi_{p_n}$.

2) We considered only the case where E is infinite, although $\widehat{E}$ is compact when E is finite. In fact, it follows from Lagrange interpolation that each function from a finite set to $\widehat{K}$ is continuous and "polynomial": if $E = \{a_1,\dots,a_r\}$, if $\varphi_j = \prod_{i\neq j}\frac{X-a_i}{a_j-a_i}$, for $1 \leq j \leq r$, and if $\phi \in \mathcal{C}(E,\widehat{K})$, then $\phi = \sum_{j=1}^{r}\phi(a_j)\varphi_j$. However, this representation is not unique: for instance the zero function may be represented by the polynomial $f\prod_{i=0}^{r}(X-a_i)$, for every $f \in K[X]$. Finally, it follows from the fact that $\mathrm{Int}(E,V)$ contains the

ideal $\left(\prod_{i=0}^{r}(X-a_i)\right)K[X]$, that there is no regular basis of the V-module $\mathrm{Int}(E,V)$ (similar to the basis given by a v-ordering of an infinite subset).
3) Let D be a one-dimensional, local, Noetherian domain with maximal ideal $\mathfrak{n}$, such that the completion $\widehat{D}$ of D in the $\mathfrak{n}$-adic topology is an integral domain. It is known that, for a subset E of the quotient field of D, such that the topological closure $\widehat{E}$ of E is compact, $\mathrm{Int}(E,D)$ is dense in the ring $\mathcal{C}(\widehat{E},\widehat{D})$ of continuous functions from $\widehat{E}$ into $\widehat{D}$ [5, Proposition 4.3]. Conversely again, the compactness of $\widehat{E}$ is necessary. Indeed, assume that $\widehat{E}$ is not compact. Since $\widehat{D}$ is an integral domain, the integral closure V of D is a discrete valuation domain with maximal ideal $\mathfrak{m}$, and the $\mathfrak{m}$-adic topology induces the $\mathfrak{n}$-adic topology on D. Consequently, $\widehat{E}$ is not compact for the $\mathfrak{m}$-adic topology. If E is not a fractional subset of V, then $\mathrm{Int}(E,D) \subseteq \mathrm{Int}(E,V) = V$, and hence, $\mathrm{Int}(E,D)$ cannot be dense in $\mathcal{C}(\widehat{E},\widehat{D})$. If E is a fractional subset of V, it follows from the proof of Lemma 3.2 that some continuous function $\phi \in \mathcal{C}(\widehat{E},\widehat{V})$, taking only the values 0 and 1, cannot be approximated modulo $\widehat{\mathfrak{m}}$ by a polynomial $f \in \mathrm{Int}(E,V)$. Since its values are in $\widehat{D}$, then ϕ belongs to $\mathcal{C}(\widehat{E},\widehat{D})$, and a fortiori, it cannot be approximated modulo $\widehat{\mathfrak{n}}$ by a polynomial $f \in \mathrm{Int}(E,D)$.

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