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Prime geodesic Theorem

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Résumé. Soit $\Gamma = PSL(2, Z)$. On démontre que $\pi_\Gamma(x) = \mathrm{li}x + O(x^{\frac{71}{102}+\varepsilon})$, $\varepsilon > 0$, où l'exposant $\frac{71}{102}$ améliore l'exposant $\frac{7}{10}$ précédemment obtenu par W. Z. Luo et P. Sarnak.

Abstract. Let Γ denote the modular group $PSL(2, Z)$. In this paper it is proved that $\pi_\Gamma(x) = \mathrm{li}x + O(x^{\frac{71}{102}+\varepsilon})$, $\varepsilon > 0$. The exponent $\frac{71}{102}$ improves the exponent $\frac{7}{10}$ obtained by W. Z. Luo and P. Sarnak.

1. Introduction

Let Γ denote the modular group $PSL(2, Z)$. By definition an element $P \in \Gamma$ is hyperbolic if as a linear fractional transformation

$$Pz = \frac{az+b}{cz+d}, \qquad P = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

it has two distinct real fixed points. By a conjugation any hyperbolic element P can be given in a form $P = \sigma^{-1}P'\sigma$ with $\sigma \in SL(2, R)$ and $P' = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$, $t > 1$. Here P' acts as multiplication by t^2, $P'z = t^2 z$. The factor t^2 is called the norm of P, let us denote it by NP, it depends only on the class $\{P\}$ of elements conjugate to P, $NP = N\{P\} = t^2$. P and $\{P\}$ are called primitive if they are not essential powers of other hyperbolic elements and classes respectively. For primitive P the norms can be viewed as "pseudoprimes", they have the same asymptotic distribution as the rational primes,

$$\pi_\Gamma(x) = \#\{\{P\} - \text{primitive}; NP \le x\} \sim \mathrm{li}x,$$

where

$$\mathrm{li}x = \int_2^x \frac{dt}{\log t}.$$

The problem of finding a formula with good error term was intensively studied by many mathematicians, first of all by H. Huber [4], D. Hejhal [2,

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3], A. B. Venkov [13] and N. V. Kuznetzov [7] before the eighties, not always for the same group. The result were of the type

$$\pi_\Gamma(x) = \mathrm{li}x + O(x^{\frac{3}{4}} \log^\alpha x), \qquad \alpha > 0,$$

this result was also known to A. Selberg and P. Sarnak [12] gave a "direct" proof of it.

In order to investigate the asymptotic distribution of $\pi_\Gamma(x)$ A. Selberg introduced the Selberg zeta-function which mimics the classical zeta-function of Riemann in various aspects. The Selberg zeta-function is defined by

$$Z(s) = \prod_{\{P\}} \prod_{k=0}^{\infty} (1 - (NP)^{-s-k})$$

for $Re(s) > 1$ where $\{P\}$ runs over the set of all primitive hyperbolic classes of conjugate elements in Γ. The most fascinating property of the Selberg zeta-function $Z(s)$ is that the analogue of the Riemann hypothesis is true. In view of this property one should expect an error term $O(x^{\frac{1}{2}+\varepsilon})$. Let us explain why this result is not obvious. It is convenient to speak of the allied sum

$$\Psi_\Gamma(x) = \sum_{N\{P\} \leq x} \Lambda P$$

where $\Lambda P = \log NP$ if $\{P\}$ is a power of a primitive hyperbolic class and $\Lambda P = 0$ otherwise. Then like in the theory of rational primes, we have the following explicit formula:

Lemma 1 ([5]).

$$(1.1) \quad \Psi_\Gamma(x) = x + \sum_{|t_j| \leq T} \frac{x^{s_j}}{s_j} + O\left(\frac{x \log^2 x}{T}\right), \qquad 1 \leq T \leq x^{\frac{1}{2}} \log^{-2} x$$

where $s_j = \frac{1}{2} + t_j$ runs over the zeroes of $Z(x)$ on $Re(s) = \frac{1}{2}$ counted with their multiplicities.

Here, in the sum each term $\frac{x^{s_j}}{s_j}$ has the order $\frac{x^{\frac{1}{2}}}{|t_j|}$ but the number of terms is (refer to [2])

$$\#\{j; |t_j| \leq T\} \sim \frac{T^2}{12},$$

therefore, treat (1.1) trivially yields

$$\Psi_\Gamma(x) = x + O\left(x^{\frac{1}{2}} T + \frac{x \log^2 x}{T}\right).$$

On taking the optimal value $T = x^{\frac{1}{4}} \log x$ we obtain the error term $O(x^{\frac{3}{4}} \log x)$. At this point the situation differs very much from the one concerning rational primes because the Selberg zeta-function $Z(x)$ has much more zeroes than does the Riemann $\zeta(s)$.

From the above arguments one sees that in order to reduce the exponent $\frac{3}{4}$ one cannot simply handle the sum over the zeroes in (1.1) by summing up the terms with absolute values but a significant concellation of terms must be taken into account. Only after N. V. Kuznetzov [8] published his summation formula does this suggestion become realizable. In 1983 H. Iwaniec [5] realized such a treatment and proved that

$$\pi_\Gamma(x) = \mathrm{li}x + O(x^{\frac{35}{48}+\varepsilon}), \qquad \varepsilon > 0$$

by means of Kuznetzov trace formula incorporated with estimates for sums of real character of a special type. More precisely, the basic ingredients in Iwaniec's arguments is the following mean value estimate for the Rankin zeta-function:

$$\sum_{|t_j| \le T} \frac{|R_j(s)|}{\cosh \pi t_j} \ll T^{\frac{5}{2}} |s|^A \log^2 T \tag{1.2}$$

where $R_j(s)$ is the Rankin zeta-function and $Re(s) = \frac{1}{2}$; and the estimate for the following sum

$$\sum_{B < a \le B+A} \sum_{R < r \le 2R} \left(\frac{a^2 - 4}{r} \right) \tag{1.3}$$

where $\left(\frac{\cdot}{\cdot}\right)$ denotes the Jacobi's symbol.

In the meantime P. G. Gallagher obtained

$$\pi_\Gamma(x) = \mathrm{li}x + O(x^{\frac{3}{4}} \log^{-1} x).$$

In 1994 W. Z. Luo and P. Sarnak [9] proved Iwaniec's mean value conjecture: (1.2) holds with the exponent $\frac{5}{2}$ replaced by $2 + \varepsilon$. By this mean value estimate and A. Weil's upper bound for Kloosterman sum,

$$\Bigg| \sum_{\substack{a (\mathrm{mod} c) \\ a\bar{a} \equiv 1 (\mathrm{mod} c)}} e\left(\frac{ma + n\bar{a}}{c} \right) \Bigg| \le 2(m,n,c)^{\frac{1}{2}} c^{\frac{1}{2}} d(c)$$

they proved

$$\pi_\Gamma(x) = \mathrm{li}x + O(x^{\frac{7}{10}+\varepsilon}), \qquad \varepsilon > 0$$

for the modular group $\Gamma = PSL(2, Z)$.

In 1994 W. Z. Luo, Z. Rudnick and P. Sarnak [10] made considerable advance in the research of Selberg's eigenvalue conjecture. They showed that for any congruence subgroup $\Gamma \subset SL(2,Z)$ the least nonzero eigenvalue $\lambda_1(\Gamma) \geq \frac{21}{100}$. As a by-product they obtained

$$\pi_\Gamma(x) = \mathrm{li}x + O(x^{\frac{7}{10}})$$

for any congruence subgroup $\Gamma \subset SL(2,Z)$.

In this paper we insert Burgess' bound for the character sum estimate and the mean value estimate (1.2) for the Rankin zeta-function into the arguments of Iwaniec and obtain the following result.

Theorem. *For the modular group $\Gamma = PSL(2,Z)$,*

$$\pi_\Gamma(x) = li x + O(x^{\frac{71}{102}+\varepsilon}), \qquad \varepsilon > 0.$$

2. Some preliminary lemmas

Lemma 2 ([9]). *Let $\rho_j(n)$ denote the n-th Fourier coefficients for the Fourier expansion at ∞ of the j-th Maass cusp form, and*

$$R_j(s) = \sum_{n=1}^{\infty} \frac{|\rho_j(n)|^2}{n^s}$$

denote the Rankin-Selberg L-function for the j-th Maass cusp form. Then

$$\sum_{|t_j| \leq T} \frac{|R_j(s)|}{\cosh \pi t_j} \ll |s|^4 T^{2+\varepsilon}.$$

Lemma 3 ([1]). *If q is not a square then*

$$\sum_{r \leq H} \left(\frac{q}{r}\right) \ll H^{\frac{2}{3}} q^{\frac{1}{9}+\varepsilon},$$

where $\left(\frac{q}{r}\right)$ denotes the Jacobi's symbol and the constant implied in $\ll$ depends on ε at most.

We infer from Lemma 3 a simple corollary.

Lemma 4. *Let $1 \leq R_1 < R_2, a \geq 1, q \geq 1$. Then*

$$\sum_{\substack{R_1 < r \leq R_2 \\ (r,q)=1}} \mu^2(r) \left(\frac{a^2-4}{r}\right) \ll a^{\frac{2}{9}} R_2^{\frac{2}{3}} (aqR_2)^{\varepsilon}.$$

Lemma 5 ([8]). *Let $\varphi(x)$ be a smooth function on $[0,\infty]$ such that*

$$\varphi(0) = \varphi'(0), \varphi^{(l)}(x) \ll x^{-3}, x \to \infty, (l = 0,1,2,3).$$

Define

$$\varphi_0 = \frac{1}{2\pi}\int_0^\infty J_0(y)\varphi(y)dy,$$

$$\varphi_B(x) = \int_0^1\int_0^\infty \xi x J_0(\xi x)J_0(\xi y)\varphi(y)dyd\xi,$$

$$\varphi_H(x) = \int_1^\infty\int_0^\infty \xi x J_0(\xi x)J_0(\xi y)\varphi(y)dyd\xi,$$

$$\hat{\varphi}(t) = \frac{\pi}{2i\sinh\pi t}\int_0^\infty (J_{2it}(x) - J_{-2it}(x))\frac{\varphi(x)}{x}dx,$$

where $J_v(x)$ is the Bessel function. Then

$$\sum_{t_j}\frac{\rho(m)\overline{\rho_j(n)}}{\cosh\pi t_j}\hat{\varphi}(t_j) + \frac{2}{\pi}\int_0^\infty \frac{\hat{\varphi}(t)}{|\zeta(1+2it)|^2}d_{it}(m)d_{-it}(n)dt$$

$$= \delta_{mn}\varphi_0 + \sum_{c=1}^\infty \frac{S(m,n,c)}{c}\varphi_H\left(\frac{4\pi\sqrt{mn}}{c}\right),$$

where

$$d_{it}(n) = \sum_{d|n} d^{it}, S(m,n,c) = \sum_{\substack{a(\mathrm{mod} c)\\ a\bar{a}\equiv 1(\mathrm{mod} c}} e\left(\frac{ma+n\bar{a}}{c}\right),$$

δ_{mn} is the Kronecker's delta symbol.

Proof. This is Kuznetzov Trace Formula. See Theorem 9.5 in [6]. □

3. A mean value theorem for Fourier coefficients

Let

$$\Lambda_j(N) = \sum_n \frac{h(n)|\rho_j(n)|^2}{\cosh\pi t_j}$$

where $h(\xi)$ is a smooth function supported in $[N, 2N]$ such that

$$|h^{(l)}(\xi)| \ll N^{-l}, l = 0, 1, 2, \cdots$$
$$\int h(\xi)d\xi = N.$$

By Rankin [11] we know that $R_j^\star(s) = \zeta(2s)R_j(s)$ has meromorphic continuation onto the whole complex with only a simple pole at $s = 1$ with residue $2\cosh\pi t_j$. And by [11] we know that $R_j(s)$ is of polynomial growth in $|s|$.

Lemma 6. *We have*

$$\Lambda_j(N) = \frac{12}{\pi^2}N + r(t_j, N)$$

with

$$\sum_{|t_j|\le T} |r(t_j, N)| \ll T^2 N^{\frac{1}{2}} \log^2 NT.$$

Proof. Consider the Mellin transformation

$$\Omega(s) = \int h(\xi)\xi^{s-1}d\xi \ll (1+|\tau|)^{-1999}N^{\sigma}$$

for $s = \sigma + i\tau$ by partial integration 1999 times. By the inverse Mellin transformation and Cauchy's theorem we have

$$\begin{aligned}\Lambda_j(N)\cosh \pi t_j &= \frac{1}{2\pi i}\int_{(2)} \Omega(s)R_j(s)ds \\ &= 2\zeta^{-1}(2)\Omega(1)\cosh \pi t_j + O\left(N^{\frac{1}{2}}\int_0^{\infty} \frac{|R(\frac{1}{2}+i\tau)|}{(1+\tau)^{1999}}d\tau\right),\end{aligned}$$

and Lemma 6 follows from Lemma 2. □

4. A mean value theorem for $\rho(c, a)$

Let $\rho(c, a)$ stand for the number of solutions $d(\bmod c)$ of

$$d^2 - ad + 1 \equiv 0(\bmod c),$$

and

$$F(A, B, C) = \sum_{B<a\le A+B}\sum_{c\le C}\rho(c, a)$$

for $1 \le A \le B$ and $C \ge 1$. We have

$$\sum_{a(\bmod c)} \rho(c, a) = \varphi(c)$$

where $\varphi(c)$ is the Euler's function. Now

$$F(A, B, C) = \sum_{c\le C}\left(\frac{A}{c} + O(1)\right)\varphi(c) = \frac{6}{\pi^2}AC + O(A + C^2).$$

Lemma 7. *For* $1 \le A \le B$, $C \ge 1$ *and any* $\varepsilon > 0$,

$$F(A, B, C) = \frac{6}{\pi^2}AC + O((A^{\frac{3}{5}}B^{\frac{2}{15}}C + AC^{\frac{1}{2}})(BC)^{\varepsilon}).$$

Proof. Write $c = kl$ where k is a squarefree odd number and $4l$ is a squarefull number coprime with k. By the multiplicativity of $\rho(c, a)$ in c we get

$$\rho(c, a) = \rho(k, a)\rho(l, a).$$

$d^2 - ad + 1 \equiv 0(\mathrm{mod} c)$ in $d(\mathrm{mod} k)$ is equivalent to $x^2 \equiv a^2 - 4(\mathrm{mod} k)$ in $x(\mathrm{mod} k)$ and the number of incongruent solutions of the latter is

$$\rho(k, a) = \prod_{p|k}\left(1 + \left(\frac{a^2-4}{p}\right)\right) = \sum_{r|k}\left(\frac{a^2-4}{r}\right).$$

Let Q stands for the set of squarefull numbers. Then

$$\begin{aligned} (4.1) \qquad F(A, B, C) &= \sum_{\substack{lrs \le C \\ l \in Q, (rs, 4l) = 1}} \mu^2(rs) \sum_{B < a \le A+B} \rho(l, a)\left(\frac{a^2-4}{r}\right) \\ &= \sum_{lr \le R} + \sum_{lr > R} = F_0(A, B, C) + F_\infty(A, B, C). \end{aligned}$$

By (44) in [5] we have

$$(4.2) \qquad F_0(A, B, C) = \lambda AC + O((CR^{\frac{1}{2}} + ACR^{-\frac{1}{2}} + AC^{\frac{1}{2}})(BC)^\varepsilon).$$

with some absolute constant λ.

By Lemma 4, after splitting up the summation over r in $F_\infty(A, B, C)$ into intervals of the form (R_1, R_2) with $l^{-1}R \le R_1 < R_2 \le 2R_1$ we deduce that

$$\begin{aligned} F_\infty(A, B, C) &\ll \sum_{\substack{l \in Q \\ l \le C}} \sum_{B < a \le A+B} \rho(l, a) \sum_{s \le \frac{C}{lR_1}} R_1^{\frac{2}{3}} B^{\frac{2}{9}} (BC)^\varepsilon \\ &\ll \sum_{\substack{l \le C \\ l \in Q}} \sum_{B < a \le A+B} \rho(l, a) \frac{CR_1^{\frac{2}{3}} B^{\frac{2}{9}}}{lR_1} (BC)^\varepsilon \\ &\ll R^{-\frac{1}{3}} B^{\frac{2}{9}} C (BC)^\varepsilon \sum_{B < a \le A+B} \sum_{\substack{l \le C \\ l \in Q}} \rho(l, a) l^{-\frac{2}{3}}. \end{aligned}$$

Since for $p>2, \alpha \geq 2$ we have $\rho(p^\alpha, a) \leq 2p^{[\frac{\beta}{2}]}$ where $p^\beta = (a^2-4, p^\alpha)$ it follows that

$$\begin{aligned}\sum_{\substack{l\leq C\\ l\in Q}} \frac{\rho(l,a)}{l^{\frac{2}{3}}} &\ll \prod_{p\leq C}\left(1+\sum_{\alpha\geq 2}\frac{\rho(p^\alpha,a)}{p^{\frac{2\alpha}{3}}}\right)\\ &\ll \prod_{\substack{1000<p\leq C\\ (p,a^2-4)=1}}\left(1+\frac{2}{p^{\frac{4}{3}}}\right)\prod_{1000<p\mid(a^2-4)}\left(1+\frac{10}{p^{\frac{1}{3}}}\right)\\ &\ll 2^{d(a^2-4)} \ll (BC)^\varepsilon,\end{aligned}$$

and finally

$$F_\infty(A,B,C) \ll AR^{-\frac{1}{3}}B^{\frac{2}{9}}C(BC)^\varepsilon. \tag{4.3}$$

By (4.3) and (4.4) with $R = A^{\frac{6}{5}}B^{\frac{4}{15}}$ we get that

$$F(A,B,C) = \lambda AC + O((A^{\frac{3}{5}}B^{\frac{2}{15}}C + AC^{\frac{1}{2}})(BC)^\varepsilon). \tag{4.4}$$

Comparing (4.1) and (4.5) for $B=A^2, C=A^{\frac{1}{2}}, A\to\infty$ one finds $\lambda = \frac{6}{\pi^2}$, which completes the proof of Lemma 7. □

5. An application of Kuznetzov trace formula

Let

$$\varphi(x) = -\frac{\sinh\beta}{\pi}x\exp(ix\cosh\beta),$$

$$2\beta = \log X + \frac{i}{T}.$$

Then (cf. [9])

$$\hat{\varphi}(t) = \frac{\sinh(\pi+2i\beta)t}{\sinh\pi t},$$

$$\varphi_0 = -\frac{\cosh\beta}{2\pi^2\sinh^2\beta},$$

$$\varphi_B(x) = -\frac{\sinh 2\beta}{2\pi}\int_0^1 \xi x J_0(\xi x)(\cosh^2\beta - \xi^2)^{-\frac{3}{2}}d\xi,$$

$$\varphi_H(x) = -\frac{\sinh 2\beta}{2\pi}\int_1^\infty \xi x J_0(\xi x)(\cosh^2\beta - \xi^2)^{-\frac{3}{2}}d\xi.$$

Then we have $\varphi = \varphi_B + \varphi_H$. It is easy to show that

$$\hat{\varphi}(t) = X^{it} \exp\left(-\frac{t}{T}\right) + O(e^{-\pi t}), \quad \varphi_0 = O(X^{-\frac{1}{2}}),$$
$$\int_0^\infty \frac{\hat{\varphi}(t)|d_{it}(n)|^2}{|\zeta(1+2it)|^2} dt = O(T \log^2 T d^2(n)).$$

Let

$$S_n(\varphi) = \sum_{t_j>0} \frac{|\rho_j(n)|^2}{\cosh \pi t_j} \hat{\varphi}(t_j),$$
$$T_n(\varphi) = \frac{2}{\pi} \int_0^\infty \frac{\hat{\varphi}(t)|d_{it}(n)|^2}{|\zeta(1+2it)|^2} dt,$$
$$W_n(\varphi_H) = \sum_{c=1}^\infty \frac{S(n,c)}{c} \varphi_H\left(\frac{4\pi n}{c}\right),$$

where $S(n,c) = S(n,n;c)$. Then by Lemma 6 we have

$$\sum_n h(n) S_n(\varphi) = \sum_{t_j>0} \hat{\varphi}(t_j)\Lambda_j(N) = \frac{12N}{\pi^2} \sum_{t_j>0} \hat{\varphi}(t_j) + \sum_{t_j>0} \hat{\varphi}(t_j) r(t_j, N)$$
$$= \frac{12N}{\pi^2} \sum_{t_j>0} X^{it_j} \exp\left(-\frac{t_j}{T}\right) + O(N^{\frac{1}{2}} T^2 \log^2 T),$$
$$\sum_n h(n) T_n(\varphi) \ll NT \log^{10} NT.$$

By $J_0(y) \ll \min(1, y^{-\frac{1}{2}})$ and $|S(n,c)| \le 2(n,c)^{\frac{1}{2}} c^{\frac{1}{2}} d(c)$ we get

$$W_n(\varphi_B) \ll N^{\frac{1}{2}} X^{-\frac{1}{2}} \log^2 N.$$

By the above arguments and Kuznetzov Trace formula we get

$$\text{(5.1)} \quad \frac{12}{\pi^2} \sum_{t_j>0} X^{it_j} \exp\left(-\frac{t_j}{T}\right)$$
$$= \frac{1}{N} \sum_n h(n) W_n(\varphi) + O(T \log^{10} T + N^{-\frac{1}{2}} T^2 \log^2 T)$$
$$= \frac{1}{N} \sum_n h(n) W_n(\varphi) + O(T^{\frac{3}{2}} \log^2 T),$$

where and below, we take

$$X^{\frac{1}{10}} \le T \le X^{\frac{1}{3}}, N = T \log^{10} X.$$

6. An estimation for $\sum_n h(n)W_n(\varphi)$

Since

$$S(n,c)=\sum_{a(\bmod c)}^{\star}\rho(c,a)e\left(\frac{na}{c}\right)$$

we have

$$\sum_n h(n)W_n(\varphi)=-4\sinh\beta\sum_1^\infty\frac{1}{c^2}\sum_{-c<2a\le c}\rho(c,a)\times\sum_n h(n)ne\left(\frac{n(2\cosh\beta-a)}{c}\right). \tag{6.1}$$

Notice that

$$\begin{aligned}2\cosh\beta &= (X^{\frac12}+X^{-\frac12})\cos\frac{1}{2T}+i(X^{\frac12}-X^{-\frac12})\sin\frac{1}{2T}\\ &= B+iE,\ \text{say}\end{aligned}$$

thus

$$B=X^{\frac12}+O(X^{\frac12}T^{-2}),\qquad E=\frac{X^{\frac12}}{2T}+O(X^{\frac12}T^{-3}).$$

Moreover we have

$$\left|e\Big(\frac{n(2\cosh\beta-a)}{c}\Big)\right|=\exp(-\frac{2\pi n}{c}E)\le\exp\Big(-\frac{NX^{\frac12}}{cT}\Big).$$

Let $C_1=18X^{\frac12}, C_2=X$. Then

$$\sum_{c=1}^{C_1}\frac{1}{c^2}\sum_{-c<2a\le c}\rho(c,a)\sum_n h(n)ne\left(\frac{n(2\cosh\beta-a)}{c}\right)\ll N^2X^{-10}. \tag{6.2}$$

And Weil's bound $|S(n,c)|\le 2(n,c)^{\frac12}c^{\frac12}d(c)$ implies

$$\begin{aligned}&\sum_{c=C_2}^{\infty}\frac{1}{c^2}\sum_{-c<2a\le c}\rho(c,a)\sum_n h(n)ne\left(\frac{n(2\cosh\beta-a)}{c}\right)\\ &\ll\sum_n h(n)n\sum_{c=C_2}^{\infty}\frac{|S(n,c)|}{c^2}\\ &\ll\sum_n h(n)n\sum_{c=C_2}^{\infty}\frac{(n,c)^{\frac12}d(c)}{c^{\frac32}}\\ &\ll C_2^{-\frac12}N^2\log^{10}C_2\ll X^{-\frac12}N^2\log^{10}X.\end{aligned} \tag{6.3}$$

Now

$$\sum_{C_1}^{C_2}\frac{1}{c^2}\sum_{-c<2a\le c}\rho(c,a)\sum_n h(n)ne\left(\frac{n(2\cosh\beta-a)}{c}\right)$$

falls into one of at most $O(\log X)$ partial sums of the form

$$\sum_{C<c\le 2C}\frac{1}{c^2}\sum_{-c<2a\le c}\rho(c,a)\sum_n h(n)ne\left(\frac{n(2\cosh\beta-a)}{c}\right)$$

with $C_1\le C<2C\le C_2$. We have

$$\left|\frac{2\cosh\beta-a}{c}\right|\le\frac{5}{8}.$$

By Poisson summation formula we have

$$\sum_n h(n)ne\left(\tfrac{n(2\cosh\beta-a)}{c}\right)=\sum_{k=-\infty}^{\infty}\int h(x)xe\left(\tfrac{x(2\cosh\beta-a)}{c}-kx\right)dx$$
$$=\int h(x)xe\left(\tfrac{x(2\cosh\beta-a)}{c}\right)dx+\sum_{k\ne0}\int h(x)xe\left(\tfrac{x(2\cosh\beta-a)}{c}-kx\right)dx.$$

For $k\ne0$ we have

$$\int h(x)xe\left(\frac{x(2\cosh\beta-a)}{c}-kx\right)dx=-\frac{1}{4\pi^2\left(\frac{2\cosh\beta-a}{c}-k\right)^2}$$
$$\times\int(h(x)x)''e\left(\frac{x(2\cosh\beta-a)}{c}-kx\right)dx\ll\frac{1}{k^2}.$$

Hence

$$\text{(6.4)}\quad\sum_{C<c\le 2C}\frac{1}{c^2}\sum_{-c<2a\le c}\rho(c,a)\sum_n h(n)ne\left(\frac{n(2\cosh\beta-a)}{c}\right)$$
$$=\sum_{C<c\le 2C}\frac{1}{c^2}\sum_{-c<2a\le c}\rho(c,a)\int h(x)xe\left(\frac{x(2\cosh\beta-a)}{c}\right)dx+O(1).$$

For $|B-a|>CN^{-1+\varepsilon}=A$ by a multiple partial integration we have

$$\int h(x)xe\Big(\frac{x(2\cosh\beta-a)}{c}\Big)dx$$
$$=\frac{(-1)^l}{\left(\frac{2\pi i(2\cosh\beta-a)}{c}\right)^l}\int(h(x)x)^{(l)}e\left(\frac{x(2\cosh\beta-a)}{c}\right)dx$$
$$\ll\left(\frac{C}{|B-a|}\right)^l N^{2-l}\ll1,$$

where $l = [\frac{2}{\varepsilon}] + 2$. Thus

$$(6.5)\qquad \sum_{C<c\le 2C} \frac{1}{c^2} \sum_{-c<2a\le c} \rho(c,a) \sum_n h(n) n e\left(\frac{n(2\cosh\beta - a)}{c}\right)$$

$$= \sum_{C<c\le 2C} \frac{1}{c^2} \sum_{|B-a|\le A} \rho(c,a) \int h(x) x e\left(\frac{x(2\cosh\beta-a)}{c}\right) dx + O(1)$$

$$= \int \sum_{C<c\le 2C} \frac{1}{c^2} \sum_{|B-a|\le A} \rho(c,a) e\left(\frac{(B-a)}{c}x\right) \exp\left(-\frac{2\pi E x}{c}\right) h(x) x dx + O(1)$$

$$= \int \sum_{C<c\le 2C} \sum_{|B-a|\le A} \rho(c,a) e\left((B-a)x\right) \exp\left(-2\pi E x\right) h(cx) x dx + O(1).$$

Let

$$\begin{aligned} F_x(A,B,C) &= \sum_{C<c\le 2C} \sum_{|B-a|\le A} \rho(c,a) e\left((B-a)x\right) \\ &= \int_{-A}^{A} e(\alpha x) d_\alpha F_x(A+\alpha, B-A, C) \\ &= \frac{6C\sin(2\pi A x)}{\pi^3 x} + O((1+xA)(A^{\frac{3}{5}} B^{\frac{2}{15}} C + AC^{\frac{1}{2}})(BC)^{\varepsilon}), \end{aligned}$$

then

$$\begin{aligned} &\sum_{C<c\le 2C} \sum_{|B-a|\le A} \rho(c,a) e\left((B-a)x\right) h(cx) \\ &\quad = \int_C^{2C} h(yx) d_y F_x(A,B,y) \\ &\quad = \frac{6\sin(2\pi A x)}{\pi^3 x} \int_C^{2C} h(yx) dy + O((A^{\frac{3}{5}} B^{\frac{2}{15}} C + AC^{\frac{1}{2}})(BC)^{\varepsilon}) \\ &\quad = M + R, \text{say} \end{aligned}$$

the integration of M over x is

$$(6.6)\qquad \frac{6}{\pi^3} \int_C^{2C} \left(\int \frac{\sin(2\pi A x)}{\exp(2\pi E x)} h(yx) dx \right) dy$$

$$= \frac{6}{\pi^3} \int_C^{2C} \left(\int \frac{\sin(2\pi A x y^{-1})}{\exp(2\pi E x y^{-1})} h(x) dx \right) \frac{dy}{y} \ll \frac{1}{N^2},$$

where we have used the multiple integration by parts

$$\begin{aligned} \int \frac{\sin(2\pi A x y^{-1})}{\exp(2\pi E x y^{-1})} h(x) dx &= \mathrm{Im} \int \frac{e(A x y^{-1})}{\exp(2\pi E x y^{-1})} h(x) dx \\ &= \mathrm{Im}\Big(\frac{(-1)^l}{(2\pi i A y^{-1})^l} \int \Big(\frac{h(x)}{\exp(2\pi E x y^{-1})} \Big)^{(l)} e(A x y^{-1}) dx \Big) \\ &\ll \frac{N}{(ANC^{-1})^l} \ll \frac{1}{N^2}, \end{aligned}$$

with $l = [\frac{4}{\varepsilon}] + 2$, since $ANC^{-1} \geq X^{\varepsilon}$.

Integration over x from $(\frac{N}{2C}, \frac{2N}{C})$ of R yields

$$\ll N^2C^{-2}(A^{\frac{3}{5}}B^{\frac{2}{15}}C + AC^{\frac{1}{2}})(BC)^{\varepsilon}) \ll X^{\frac{1}{15}+\varepsilon}N^{\frac{7}{5}}C^{-\frac{2}{5}}. \tag{6.7}$$

Combining all the above arguments (6.1)—(6.7) we get

$$\begin{aligned} \sum_n h(n)W_n(\varphi) &\ll X^{\frac{17}{30}+\varepsilon}N^{\frac{7}{5}}C^{-\frac{2}{5}} + X^{\frac{1}{2}} + N^2\log^{10} X \\ &\ll X^{\frac{17}{30}+\varepsilon}N^{\frac{7}{5}}C^{-\frac{2}{5}}. \end{aligned} \tag{6.8}$$

7. Proof of the theorem

By (5.1) and (6.8) we get

$$\sum_{t_j>0} X^{it_j} \exp\left(-\frac{t_j}{T}\right) \ll T^{\frac{2}{5}}X^{\frac{11}{30}+\varepsilon} + T^{\frac{3}{2}}\log^2 T \ll T^{\frac{2}{5}}X^{\frac{11}{30}+\varepsilon}. \tag{7.1}$$

By (7.1) and the Fourier Technique used in [9] we get

$$\sum_{|t_j|\leq T} X^{it_j} \ll T^{\frac{2}{5}}X^{\frac{11}{30}+\varepsilon}. \tag{7.2}$$

By (58) in [9]

$$\sum_{|t_j|\leq T} X^{it_j} \ll T^{\frac{5}{4}}X^{\frac{1}{8}}\log^2 T. \tag{7.3}$$

By (7.2) and (7.3) and the inequality $\min(A,B) \leq A^{\alpha}B^{\beta}$ for $A > 0, B > 0, \alpha > 0, \beta > 0, \alpha+\beta = 1$ we get

$$\begin{aligned} \sum_{|t_j|\leq T} X^{it_j} &\ll \min(T^{\frac{2}{5}}X^{\frac{11}{30}+\varepsilon}, T^{\frac{5}{4}}X^{\frac{1}{8}}\log^2 T) \\ &\ll (T^{\frac{2}{5}}X^{\frac{11}{30}+\varepsilon})^{\frac{5}{17}}(T^{\frac{5}{4}}X^{\frac{1}{8}}\log^2 T)^{\frac{12}{17}} \ll TX^{\frac{10}{51}+\varepsilon}, \end{aligned}$$

and the theorem follows from (7.4), Lemma 1 and summation by parts.

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