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Kummer congruences for expressions involving generalized Bernoulli polynomials

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Résumé. Nous illustrons le fait qu'une expression particulière, impliquant les polynômes de Bernoulli généralisés, satisfait un système de congruences si et seulement si une expression semblable, impliquant les nombres de Bernoulli généralisés, satisfait les mêmes relations de congruence. Parmi ces relations se trouvent les congruences de Kummer ainsi que des généralisations fournies par Gunaratne.

Abstract. We illustrate how a particular expression, involving the generalized Bernoulli polynomials, satisfies systems of congruence relations if and only if a similar expression, involving the generalized Bernoulli numbers, satisfies the same congruence relations. These congruence relations include the Kummer congruences, and recent extensions of the Kummer congruences provided by Gunaratne.

1. Introduction

Let χ be a primitive Dirichlet character having conductor f_χ. The generalized Bernoulli polynomials associated with χ, $B_{n,\chi}(t)$, are defined by the generating function

$$\sum_{a=1}^{f_\chi} \frac{\chi(a)xe^{(a+t)x}}{e^{f_\chi x}-1} = \sum_{n=0}^{\infty} B_{n,\chi}(t)\frac{x^n}{n!}. \tag{1}$$

The corresponding generalized Bernoulli numbers can then be defined by $B_{n,\chi} = B_{n,\chi}(0)$. If we let $\mathbf{Z}[\chi]$ denote the ring generated over $\mathbf{Z}$ by all of the values $\chi(a)$, $a \in \mathbf{Z}$, then it can be shown that $f_\chi B_{n,\chi}$ must be in $\mathbf{Z}[\chi]$ for each $n \geq 0$, whenever $\chi \neq 1$. Thus, as elements of $\mathbf{Q}[\chi]$, each such $B_{n,\chi}$ can be described as having a denominator that is a divisor of f_χ. The case $\chi = 1$ is best considered in terms of the classical Bernoulli polynomials.

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The generating function for the classical Bernoulli polynomials, $B_n(t)$, is given by

$$\frac{xe^{tx}}{e^x-1}=\sum_{n=0}^{\infty}B_n(t)\frac{x^n}{n!}, \tag{2}$$

and from the classical Bernoulli polynomials we obtain the classical Bernoulli numbers, $B_n = B_n(0)$. The Bernoulli numbers, B_n, are rational numbers, and, by the von Staudt-Clausen theorem, whenever $B_n \neq 0$, they satisfy

$$B_n+\sum_{\substack{p \text{ prime}\\(p-1)|n}}\frac{1}{p}\in \mathbf{Z}.$$

Therefore, whenever $B_n \neq 0$, the denominator of B_n consists of the product of those primes p such that $(p-1) \mid n$.

The classical Bernoulli polynomials are related to the generalized Bernoulli polynomials in that, for $\chi = 1$, we have $B_{n,1}(t) = (-1)^n B_n(-t)$ for all n, so also $B_{n,1} = (-1)^n B_n$. Therefore, from the above discussion, we have a description of the possible contents of the denominators of the classical and the generalized Bernoulli numbers.

By considering congruences on the classical and the generalized Bernoulli numbers, we gain information about their numerators. One of the more notable examples of this is Kummer's congruence for the classical Bernoulli numbers, which states that

$$p^{-1}\Delta_c\frac{1}{n}B_n\in \mathbf{Z}_p,$$

where $c \in \mathbf{Z}$ is positive, with $c \equiv 0 \pmod{p-1}$, and $n \in \mathbf{Z}$ is positive, even, and $n \not\equiv 0 \pmod{p-1}$ (see [11], p. 61). Note that we are using Δ_c to denote the forward difference operator, $\Delta_c x_n = x_{n+c} - x_n$. More generally, it can be shown that

$$p^{-k}\Delta_c^k\frac{1}{n}B_n\in \mathbf{Z}_p,$$

where $k \in \mathbf{Z}$ is positive, and c and n are as above, but with $n > k$.

The application of Kummer's congruence to the generalized Bernoulli numbers was first treated by Carlitz in [1], with the result that

$$p^{-k}\Delta_c^k\frac{1}{n}B_{n,\chi}\in \mathbf{Z}_p[\chi],$$

for positive $c \in \mathbf{Z}$, with $c \equiv 0 \pmod{p-1}$, $n,k \in \mathbf{Z}$ with $n > k \geq 1$, and χ such that $f_\chi \neq p^\mu$, where $\mu \in \mathbf{Z}$, $\mu \geq 0$. From [5] (see also [10]), we see that, if the operator Δ_c^k is applied to the quantity

$$\beta_{n,\chi}=-\frac{1}{n}\left(1-\chi_n(p)p^{n-1}\right)B_{n,\chi_n},$$

then the congruence will still hold if the restriction $n > k$ is dropped, requiring only that $n \geq 1$. In addition to this, the divisibility requirements on c can be removed, yielding a congruence of the form $q^{-k}\Delta_c^k \beta_{n,\chi} \in \mathbf{Z}_p[\chi]$, for $c, n, k \in \mathbf{Z}$, each positive, and χ such that $f_\chi \neq p^\mu$, $\mu \in \mathbf{Z}$, $\mu \geq 0$. Here, we are using χ_n to represent the primitive Dirichlet character $\chi\omega^{-n}$, with ω the Teichmüller character, and we are taking $q = 4$ if $p = 2$, and $q = p$ otherwise. The values $\beta_{n,\chi}$ arise in connection with the p-adic L-function $L_p(s;\chi)$ of Kubota and Leopoldt [8], in that they yield the values of this function at the nonpositive integers, $L_p(1-n;\chi) = \beta_{n,\chi}$. With respect to this expression, the restriction $c \equiv 0 \pmod{p-1}$, as in the previous versions of Kummer's congruence, causes each of $\chi_n = \chi_{n+c} = \cdots = \chi_{n+kc}$, when $p \geq 3$. Thus, the expression $\Delta_c^k \beta_{n,\chi}$ involves generalized Bernoulli numbers associated with only one Dirichlet character, χ_n.

As an extension of the Kummer congruences, Gunaratne (see [6], [7]) has shown that if $p \geq 5$, $c, n, k \in \mathbf{Z}$ are positive, and $\chi = \omega^h$, where $h \in \mathbf{Z}$ and $h \not\equiv 0 \pmod{p-1}$, then $p^{-k}\Delta_c^k \beta_{n,\chi} \in \mathbf{Z}_p$, with a value modulo $p\mathbf{Z}_p$ that is independent of n, and

$$p^{-k}\Delta_c^k \beta_{n,\chi} \equiv p^{-k'}\Delta_c^{k'} \beta_{n,\chi} \pmod{p\mathbf{Z}_p}$$

for positive $k' \in \mathbf{Z}$ with $k \equiv k' \pmod{p-1}$. Additionally, by means of the binomial coefficient operator

$$\binom{p^{-1}\Delta_c}{r} x_n = \frac{1}{r!}\left(\prod_{j=0}^{r-1}\left(p^{-1}\Delta_c - j\right)\right) x_n,$$

where $r \in \mathbf{Z}$ is positive, for these χ, the $\beta_{n,\chi}$ satisfy $\binom{p^{-1}\Delta_c}{r}\beta_{n,\chi} \in \mathbf{Z}_p$, also with a value, modulo $p\mathbf{Z}_p$, that is independent of n. Young [12] has extended these congruences to show that whenever $c \equiv 0 \pmod{\phi(p^N)}$, for N a positive integer, we must have $\binom{p^{-N}\Delta_c}{r}\beta_{n,\chi} \in \mathbf{Z}_p[\chi]$ for all primes p and for all nontrivial primitive characters χ, with $(f_\chi, p) = 1$.

Extending work in [4], we derive a collection of congruences concerning the generalized Bernoulli polynomials. These congruences are similar to the results of Gunaratne, but they are without any restriction on either χ or p. Given χ a primitive Dirichlet character and p a prime, we consider congruences for the polynomials that satisfy the generating function

$$\sum_{\substack{a=1\\(a,p)=1}}^{pf_\chi} \frac{\chi(a)xe^{(a+pt)x}}{e^{pf_\chi x}-1} = \sum_{n=0}^{\infty}\left(B_{n,\chi}(pt) - \chi(p)p^{n-1}B_{n,\chi}(t)\right)\frac{x^n}{n!},$$

which follows from (1). For positive $n \in \mathbf{Z}$, denote

$$\beta_{n,\chi}(t) = -\frac{1}{n}\left(B_{n,\chi_n}(qt) - \chi_n(p)p^{n-1}B_{n,\chi_n}\left(p^{-1}qt\right)\right),$$

so that $\beta_{n,\chi}(0) = \beta_{n,\chi}$. The $\beta_{n,\chi}(t)$ make up the set of values taken on by a particular two-variable p-adic L-function, $L_p(s,t;\chi)$, found in [4], as the variable s ranges over the nonpositive integers: $L_p(1-n,t;\chi) = \beta_{n,\chi}(t)$. Since $\beta_{n,\chi}(t)$ is a polynomial for each n, it follows that $\Delta_c^k\beta_{n,\chi}(t) \to \Delta_c^k\beta_{n,\chi}(0)$ as $t \to 0$, in the p-adic metric. Therefore, given some positive $\alpha \in \mathbf{Z}$, there must exist a region $R \subset \mathbf{C}_p$, containing 0, such that for any $t \in R$ we have $|\Delta_c^k\beta_{n,\chi}(t) - \Delta_c^k\beta_{n,\chi}(0)|_p \leq |p|_p^\alpha$. Thus, for certain values of t, we can expect $\Delta_c^k\beta_{n,\chi}(t)$ to have the same magnitude, p-adically, as that of $\Delta_c^k\beta_{n,\chi}(0)$. Our main result relates congruences on values of the $\beta_{n,\chi}(t)$ in the following manner:

Theorem 1. *Let $n, c, k, r \in \mathbf{Z}$, with n, c positive and k, r nonnegative, and let $F_0 \in \mathbf{Z}$ be the smallest positive multiple of each of $pq^{-1}f_{\chi_n}$, $pq^{-1}f_{\chi_{n+c}}$, $\ldots$, $pq^{-1}f_{\chi_{n+kc}}$. Furthermore, let $t \in \mathbf{C}_p$, with $|t|_p \leq 1$, and $\tau \in \mathbf{Z}_p$ such that $|\tau|_p \leq |F_0|_p$. Then*
(a) *the quantity $c^{-k}q^{-k}\Delta_c^k(\beta_{n,\chi}(t+\tau) - \beta_{n,\chi}(t)) \in \mathbf{Z}_p[\chi,t]$;*
(b) *if $n' \in \mathbf{Z}$ such that $n' > n$, then*

$$\begin{aligned}
&c^{-k}q^{-k}\Delta_c^k\left(\beta_{n,\chi}(t+\tau) - \beta_{n,\chi}(t)\right) \\
&\qquad \equiv c^{-k}q^{-k}\Delta_c^k\left(\beta_{n',\chi}(t+\tau) - \beta_{n',\chi}(t)\right) \pmod{(n'-n)q\mathbf{Z}_p[\chi,t]};
\end{aligned}$$

(c) *if k is positive, and if $c' \in \mathbf{Z}$ is positive with $c' \neq c$, then*

$$\begin{aligned}
&c^{-k}q^{-k}\Delta_c^k\left(\beta_{n,\chi}(t+\tau) - \beta_{n,\chi}(t)\right) \\
&\quad \equiv c'^{-k}q^{-k}\Delta_{c'}^k\left(\beta_{n,\chi}(t+\tau) - \beta_{n,\chi}(t)\right) \pmod{(c-1,c'-1)kp\mathbf{Z}_p[\chi,t]};
\end{aligned}$$

(d) *if $k' \in \mathbf{Z}$ such that $k' \geq k$ and $k' \equiv k \pmod{\phi(p^N)}$, for some positive $N \in \mathbf{Z}$, and if $t \in \mathbf{Z}_p$, then*

$$\begin{aligned}
&c^{-k}q^{-k}\Delta_c^k\left(\beta_{n,\chi}(t+\tau) - \beta_{n,\chi}(t)\right) \\
&\qquad \equiv c^{-k'}q^{-k'}\Delta_c^{k'}\left(\beta_{n,\chi}(t+\tau) - \beta_{n,\chi}(t)\right) \pmod{(p^k,p^N)\mathbf{Z}_p[\chi]};
\end{aligned}$$

(e) *if $t \in \mathbf{Z}_p$, then $\binom{c^{-k}q^{-k}\Delta_c^k}{r}(\beta_{n,\chi}(t+\tau) - \beta_{n,\chi}(t)) \in \mathbf{Z}_p[\chi]$;*
(f) *if $n' \in \mathbf{Z}$ such that $n' > n$, and if $t \in \mathbf{Z}_p$, then*

$$\begin{aligned}
&\binom{c^{-k}q^{-k}\Delta_c^k}{r}\left(\beta_{n,\chi}(t+\tau) - \beta_{n,\chi}(t)\right) \\
&\qquad \equiv \binom{c^{-k}q^{-k}\Delta_c^k}{r}\left(\beta_{n',\chi}(t+\tau) - \beta_{n',\chi}(t)\right) \pmod{(n'-n)q\mathbf{Z}_p[\chi]}.
\end{aligned}$$

Most of these congruences are extensions of the work of Guneratne. The independence of $c^{-k}q^{-k}\Delta_c^k(\beta_{n,\chi}(t+\tau) - \beta_{n,\chi}(t))$, with repect to the value of n, when considered modulo $q\mathbf{Z}_p[\chi,t]$, is implied by (b). Similarly, (f) implies that the quantity $\binom{c^{-k}q^{-k}\Delta_c^k}{r}(\beta_{n,\chi}(t+\tau)-\beta_{n,\chi}(t))$ is independent of n when considered modulo $q\mathbf{Z}_p[\chi]$. On the other hand, (c) implies that we have independence with respect to the value of c when $c^{-k}q^{-k}\Delta_c^k(\beta_{n,\chi}(t+\tau)-\beta_{n,\chi}(t))$ is considered modulo $kp\mathbf{Z}_p[\chi,t]$. There are no results involving independence with respect to c in any of the earlier works [6], [7], [12], or [13], so that this appears to be a new direction of investigation with respect to Kummer's congruence when applied to the generalized Bernoulli polynomials and numbers.

Both (e) and (f) in the above can be generalized quite nicely. As we have seen, the binomial coefficient operator $\binom{c^{-k}q^{-k}\Delta_c^k}{r}$ yields a polynomial in $c^{-k}q^{-k}\Delta_c^k$, having coefficients in $\mathbf{Q}$. Thus, to generalize (e), one may wish to consider what other polynomials in $c^{-k}q^{-k}\Delta_c^k$, when applied to $\beta_{n,\chi}(t+\tau)-\beta_{n,\chi}(t)$, yield values in $\mathbf{Z}_p[\chi]$. The following result, which can be found in [9], enables such a generalization:

Theorem 2. *A polynomial $f(x)\in\mathbf{Q}_p[x]$ of degree N maps $\mathbf{Z}_p$ into $\mathbf{Z}_p$ if and only if it can be written in the form*

$$f(x)=\sum_{r=0}^{N} b_r\binom{x}{r},$$

where $b_0,b_1,\ldots,b_N\in\mathbf{Z}_p$.

Assuming the hypotheses of Theorem 1, along with the additional restriction that $t\in\mathbf{Z}_p$, since $\binom{c^{-k}q^{-k}\Delta_c^k}{r}(\beta_{n,\chi}(t+\tau)-\beta_{n,\chi}(t))\in\mathbf{Z}_p[\chi]$ for each nonnegative $r\in\mathbf{Z}$, any sum of the form

$$\sum_{r=0}^{N} b_r\binom{c^{-k}q^{-k}\Delta_c^k}{r}\left(\beta_{n,\chi}(t+\tau)-\beta_{n,\chi}(t)\right),$$

where $b_0,b_1,\ldots,b_N\in\mathbf{Z}_p$, must also be in $\mathbf{Z}_p[\chi]$. Thus, Theorem 2 implies that every polynomial $f(x)\in\mathbf{Q}_p[x]$ that maps $\mathbf{Z}_p$ into $\mathbf{Z}_p$ yields an operator $f(c^{-k}q^{-k}\Delta_c^k)$ such that

$$f\left(c^{-k}q^{-k}\Delta_c^k\right)\left(\beta_{n,\chi}(t+\tau)-\beta_{n,\chi}(t)\right)\in\mathbf{Z}_p[\chi].$$

It then follows, by (f), that

$$f\left(c^{-k}q^{-k}\Delta_c^k\right)\left(\beta_{n,\chi}(t+\tau)-\beta_{n,\chi}(t)\right)$$

$$\equiv f\left(c^{-k}q^{-k}\Delta_c^k\right)\left(\beta_{n',\chi}(t+\tau)-\beta_{n',\chi}(t)\right)\pmod{(n'-n)q\mathbf{Z}_p[\chi]}$$

whenever $n'\in\mathbf{Z}$ with $n'>n$.

Letting $t = 0$ in Theorem 1 yields congruences on $\beta_{n,\chi}(\tau) - \beta_{n,\chi}(0)$, and, thus, provides equivalences between congruences on $\beta_{n,\chi}(\tau)$, involving generalized Bernoulli polynomials, and congruences on $\beta_{n,\chi}(0)$, involving generalized Bernoulli numbers. Furthermore, if n is even and $\chi(-1) = -1$, or if n is odd and $\chi(-1) = 1$, $\chi \neq 1$, then $B_{n,\chi} = 0$, with the only exception for the case $\chi = 1$ being when $n = 1$, for which we have $B_{1,1} = 1/2$. Thus, given χ, we can always find n and c such that $\Delta_c^k \beta_{n,\chi}(0) = 0$, implying that each of the above congruences hold for the corresponding $\Delta_c^k \beta_{n,\chi}(\tau)$, provided we have the appropriate restrictions on τ.

A polynomial structure for the classical Bernoulli polynomials, similar to that of $\beta_{n,\chi}(t)$, is incorporated in a result of Eie and Ong. In [3], they apply Kummer's congruence to show that, for $p \geq 5$, if $n, c \in \mathbf{Z}$ are each positive with $n \not\equiv 0 \pmod{\phi(p)}$ and $c \equiv 0 \pmod{\phi(p)}$, then

$$c^{-1} p^{-1} \Delta_c \frac{1}{n} \left(B_n(p\tau - j) - p^{n-1} B_n(\tau) \right) \in \mathbf{Z}_p,$$

where $\tau \geq 0$ is an element of $\mathbf{Q}$ having denominator relatively prime to p, and $j \in \mathbf{Z}$ with both $0 \leq j \leq p-1$ and $j \leq p\tau$. This has been extended to applications of Δ_c^k, with $k \geq 1$, by Young. Furthermore, Young shows that similar congruences hold for the generalized Bernoulli polynomials associated with certain primitive Dirichlet characters [13]. We continue our work by showing how the congruences of Eie and Ong with $1 \leq j \leq p-1$ relate to the same congruences with $j = 0$.

Finally, we consider the expression

$$\beta_{n,\chi}\left(q^{-1} p t\right) = -\frac{1}{n} \left(B_{n,\chi_n}(pt) - \chi_n(p) p^{n-1} B_{n,\chi_n}(t) \right),$$

where $t \in \mathbf{C}_p$, $|t|_p \leq 1$, and discuss the strength of the congruences that can be obtained with regard to it.

2. Preliminaries

One of the more well-known properties of the generalized Bernoulli polynomials is that, for positive $m \in \mathbf{Z}$,

$$B_{n,\chi}(t + m f_\chi) - B_{n,\chi}(t) = n \sum_{a=1}^{m f_\chi} \chi(a)(t+a)^{n-1} \tag{3}$$

for all $n \geq 0$. This can be derived from (1). Similarly, for the classical Bernoulli polynomials, (2) implies that, whenever $m \in \mathbf{Z}$ is positive,

$$B_n(t+m) - B_n(t) = n \sum_{a=1}^{m} (t+a-1)^{n-1} \tag{4}$$

for all $n \geq 0$. These properties are key to the derivation of our congruences, as we will see.

Let p be a fixed prime. We will use $\mathbf{Z}_p$ to represent the set of p-adic integers, and $\mathbf{Q}_p$ the set of p-adic rationals. Let $\mathbf{C}_p$ denote the completion of the algebraic closure of $\mathbf{Q}_p$ under the p-adic absolute value $|\cdot|_p$, normalized so that $|p|_p = p^{-1}$. Note that, for each $t \in \mathbf{C}_p$, there exists some $\alpha \in \mathbf{Q}$ such that $|t|_p = |p|_p^{\alpha}$. Fix an embedding of the algebraic closure of $\mathbf{Q}$ into $\mathbf{C}_p$. Each value of a Dirichlet character χ is either 0 or a root of unity. Thus, we may consider the values of χ as lying in $\mathbf{C}_p$.

Denote $q = 4$ if $p = 2$ and $q = p$ if $p \geq 3$. Let ω represent the Dirichlet character having conductor $f_\omega = q$, and whose values $\omega(a) \in \mathbf{Z}_p$ satisfy $\omega(a)^{\phi(q)} = 1$ and $\omega(a) \equiv a \pmod{q\mathbf{Z}_p}$ for $(a,p) = 1$, with $\omega(a) = 0$ otherwise. For an arbitrary character χ, we then define the character $\chi_n = \chi\omega^{-n}$, where $n \in \mathbf{Z}$, in the sense of the product of characters.

Let $\langle a \rangle = \omega^{-1}(a)a$ whenever $(a,p) = 1$. We then have $\langle a \rangle \equiv 1 \pmod{q\mathbf{Z}_p}$ for these values of a. For our purposes we extend this notation by defining $\langle a+qt \rangle = \omega^{-1}(a)(a+qt)$ for all $a \in \mathbf{Z}$, with $(a,p) = 1$, and $t \in \mathbf{C}_p$ such that $|t|_p \leq 1$. Thus, $\langle a+qt \rangle = \langle a \rangle + q\omega^{-1}(a)t$, so that $\langle a+qt \rangle \equiv 1 \pmod{q\mathbf{Z}_p[t]}$.

We now use these quantities to express the difference $\beta_{n,\chi}(t+F) - \beta_{n,\chi}(t)$, for $F \in \mathbf{Z}$ a positive multiple of $pq^{-1}f_{\chi_n}$, in a manner that will enable the proof of the main result.

Lemma 3. *Let $t \in \mathbf{C}_p$, $|t|_p \leq 1$, and let $n \in \mathbf{Z}$ be positive. Then, for $F \in \mathbf{Z}$ a positive multiple of $pq^{-1}f_{\chi_n}$,*

$$\beta_{n,\chi}(t+F) - \beta_{n,\chi}(t) = -\sum_{\substack{a=1 \\ (a,p)=1}}^{qF} \chi_1(a)\langle a+qt \rangle^{n-1}.$$

Proof. By definition,

$$\begin{aligned}
&\beta_{n,\chi}(t+F) - \beta_{n,\chi}(t) \\
&\quad = -\frac{1}{n}\left(B_{n,\chi_n}(q(t+F)) - \chi_n(p)p^{n-1}B_{n,\chi_n}\left(p^{-1}q(t+F)\right)\right) \\
&\qquad\qquad + \frac{1}{n}\left(B_{n,\chi_n}(qt) - \chi_n(p)p^{n-1}B_{n,\chi_n}\left(p^{-1}qt\right)\right) \\
&\quad = -\frac{1}{n}\left(B_{n,\chi_n}(q(t+F)) - B_{n,\chi_n}(qt)\right) \\
&\qquad\qquad + \frac{1}{n}\chi_n(p)p^{n-1}\left(B_{n,\chi_n}\left(p^{-1}q(t+F)\right) - B_{n,\chi_n}\left(p^{-1}qt\right)\right).
\end{aligned}$$

Thus, by (3), we can write

$$\begin{aligned}
&\beta_{n,\chi}(t+F)-\beta_{n,\chi}(t)\\
&\quad=-\sum_{a=1}^{qF}\chi_n(a)(a+qt)^{n-1}+\chi_n(p)p^{n-1}\sum_{a=1}^{p^{-1}qF}\chi_n(a)(a+p^{-1}qt)^{n-1}\\
&\quad=-\sum_{a=1}^{qF}\chi_n(a)(a+qt)^{n-1}+\sum_{\substack{a=1\\ p\mid a}}^{qF}\chi_n(a)(a+qt)^{n-1}.
\end{aligned}$$

Since $\chi_n(a)=\chi_1(a)\omega^{-(n-1)}(a)$ and $\omega^{-1}(a)(a+qt)=\langle a+qt\rangle$, whenever $(a,p)=1$, the result follows. $\square$

Lemma 4. *Let i and m be integers such that $2\le i\le m$. Then $\binom{m}{i}qp^{i-2}\equiv 0 \pmod{m(m-1)p\mathbf{Z}_p}$.*

Proof. For $2\le i\le m$,

$$\binom{m}{i}=\frac{m(m-1)}{i(i-1)}\binom{m-2}{i-2}.$$

Since $(i,i-1)=1$, the lemma will follow if we show that $qp^{i-2}/i\equiv 0 \pmod{p\mathbf{Z}_p}$ and $qp^{i-2}/(i-1)\equiv 0 \pmod{p\mathbf{Z}_p}$, whenever $i\ge 2$.

If $p=2$, then the power of p that divides any $j\in\mathbf{Z}$, $1\le j\le i$, is at most $i/2$. Thus, the power of p that divides qp^{i-2}/j is at least $i-i/2=i/2$. For $i\ge 2$, this implies that the power of p that divides qp^{i-2}/j is at least 1. Therefore, $qp^{i-2}/j\equiv 0 \pmod{p\mathbf{Z}_p}$.

If $p>2$, then the power of p that divides $j\in\mathbf{Z}$, $1\le j\le i$, is at most $i/3$, so the power of p that divides qp^{i-2}/j is at least $i-1-i/3=2i/3-1$. For $i\ge 2$, the power of p that divides qp^{i-2}/j is then at least $1/3$, and, thus, at least 1. Therefore, $qp^{i-2}/j\equiv 0 \pmod{p\mathbf{Z}_p}$.

Therefore, for any prime p, the quantity $qp^{i-2}/j\equiv 0 \pmod{p\mathbf{Z}_p}$ whenever $j\in\mathbf{Z}$ with $1\le j\le i$, implying the result. $\square$

In the following lemma, we derive congruence properties of the quantity $\langle a+qt\rangle^m$ that will be utilized to prove the main result.

Lemma 5. *Let $a,m\in\mathbf{Z}$, with $(a,p)=1$ and $m\ge 2$, and let $t\in\mathbf{C}_p$ such that $|t|_p\le 1$. Then*

$$\langle a+qt\rangle^m-1\equiv m\left(\langle a\rangle-1+q\omega^{-1}(a)t\right) \pmod{m(m-1)pq\mathbf{Z}_p[t]}.$$

Proof. Recall that $\langle a+qt\rangle=\langle a\rangle+q\omega^{-1}(a)t$. Since $\langle a\rangle\in\mathbf{Z}_p$ and $\langle a\rangle\equiv 1 \pmod{q\mathbf{Z}_p}$, there must be some $b(a)\in\mathbf{Z}_p$ such that $\langle a\rangle=1+qb(a)$.

Thus,

$$\begin{aligned}\langle a+qt\rangle^m &= \left(1+q\left(b(a)+\omega^{-1}(a)t\right)\right)^m \\ &= \sum_{i=0}^{m}\binom{m}{i}q^i\left(b(a)+\omega^{-1}(a)t\right)^i.\end{aligned}$$

Lemma 4 then implies that

$$\langle a+qt\rangle^m \equiv 1+mq\left(b(a)+\omega^{-1}(a)t\right) \pmod{m(m-1)pq\mathbf{Z}_p[t]}.$$

The result follows since $qb(a)=\langle a\rangle-1$. □

As a weaker form of this lemma, we have the congruence

$$\langle a+qt\rangle^m \equiv 1 \pmod{mq\mathbf{Z}_p[t]},$$

with the same restrictions on a and t as in the lemma, but for all positive $m\in\mathbf{Z}$.

For each $r\in\mathbf{Z}$, $r\geq 0$, the quantity $\binom{x}{r}$ is defined in like manner as the binomial coefficients, denoting $\binom{x}{0}=1$ and

$$\binom{x}{r}=\frac{1}{r!}x(x-1)\cdots(x-(r-1))$$

for $r>0$. Note that each such quantity is a polynomial in x, and has the expansion

$$\binom{x}{r}=\frac{1}{r!}\sum_{m=0}^{r}s(r,m)x^m, \tag{5}$$

where the values $s(r,m)$ are Stirling numbers of the first kind, defined by the generating function

$$\sum_{r=0}^{\infty}s(r,m)\frac{t^r}{r!}=\frac{1}{m!}\left(\log(1+t)\right)^m.$$

The $s(r,m)$ are integers, where $r,m\in\mathbf{Z}$, $r\geq 0$, $m\geq 0$, satisfying $s(r,m)=0$ whenever $0\leq r<m$, and $s(r,r)=1$ for all $r\geq 0$. For additional information on Stirling numbers of the first kind, we refer the reader to [2], pp. 214–217.

3. Congruences for generalized Bernoulli polynomials

From Lemma 3 we have an indication that the divisibility properties of $\beta_{n,\chi}(t+\tau)-\beta_{n,\chi}(t)$ must be similar to the divisibility properties of $\langle a+qt\rangle^n$, whenever $a\in\mathbf{Z}$ with $(a,p)=1$. We will now consider how we can utilize this to derive a collection of congruences concerning the generalized Bernoulli polynomials. Recall that Δ_c denotes the forward difference operator, $\Delta_c x_n = x_{n+c} - x_n$. Repeated application of this operator can be expressed in the form

$$\Delta_c^k x_n = \sum_{m=0}^{k} \binom{k}{m} (-1)^{k-m} x_{n+mc},$$

for some positive $k \in \mathbf{Z}$.

We can now prove our main result:

Proof of Theorem 1. The set of positive integers in $F_0\mathbf{Z}$ is dense in $F_0\mathbf{Z}_p$. Thus, for $\tau \in F_0\mathbf{Z}_p$, there exists a sequence $\{\tau_i\}_{i=1}^{\infty}$ in $F_0\mathbf{Z}$, with $\tau_i > 0$ for each i, such that $\tau_i \to \tau$. This implies that any $f(x) \in \mathbf{C}_p[x]$ must satisfy $f(\tau_i) \to f(\tau)$, and, thus, any congruence satisfied by each $f(\tau_i)$ must also be satisfied by $f(\tau)$. Since each of $\Delta_c^k(\beta_{n,\chi}(t+x) - \beta_{n,\chi}(t))$ and $\binom{c^{-k}q^{-k}\Delta_c^k}{r}(\beta_{n,\chi}(t+x) - \beta_{n,\chi}(t))$ are in $\mathbf{C}_p[x]$ for t in $\mathbf{C}_p$, any of the stated congruences that hold for all positive $\tau \in F_0\mathbf{Z}$ must also hold for arbitrary $\tau \in F_0\mathbf{Z}_p$.

(a) Since Δ_c is a linear operator, Lemma 3 implies that

$$\Delta_c^k \left(\beta_{n,\chi}(t+F) - \beta_{n,\chi}(t)\right) = - \sum_{\substack{a=1 \\ (a,p)=1}}^{qF} \chi_1(a) \langle a+qt \rangle^{-1} \Delta_c^k \langle a+qt \rangle^n,$$

where F is a positive multiple of F_0. Note that

$$\begin{aligned} \Delta_c^k \langle a+qt \rangle^n &= \sum_{m=0}^{k} \binom{k}{m} (-1)^{k-m} \langle a+qt \rangle^{n+mc} \\ &= \langle a+qt \rangle^n \left(\langle a+qt \rangle^c - 1\right)^k, \end{aligned} \tag{6}$$

so that,

$$\begin{aligned} \Delta_c^k \left(\beta_{n,\chi}(t+F) - \beta_{n,\chi}(t)\right) = - \sum_{\substack{a=1 \\ (a,p)=1}}^{qF} \chi_1(a) \langle a+qt \rangle^{n-1} \\ \times \left(\langle a+qt \rangle^c - 1\right)^k. \end{aligned} \tag{7}$$

By Lemma 5, $(\langle a+qt \rangle^c - 1)^k \equiv 0 \pmod{c^k q^k \mathbf{Z}_p[t]}$, implying the result.

(b) Let F be a positive multiple of F_0. By (7) we can write

$$\begin{aligned} &\Delta_c^k \left(\beta_{n,\chi}(t+F) - \beta_{n,\chi}(t)\right) - \Delta_c^k \left(\beta_{n',\chi}(t+F) - \beta_{n',\chi}(t)\right) \\ &\quad = \sum_{\substack{a=1 \\ (a,p)=1}}^{qF} \chi_1(a) \left(\langle a+qt \rangle^c - 1\right)^k \langle a+qt \rangle^{n-1} \left(\langle a+qt \rangle^{n'-n} - 1\right). \end{aligned}$$

Lemma 5 implies that

$$(\langle a+qt\rangle^c-1)^k(\langle a+qt\rangle^{n'-n}-1)\equiv 0 \pmod{(n'-n)c^kq^{k+1}\mathbf{Z}_p[t]},$$

and the result follows.

(c) By utilizing (7), we obtain

$$\begin{aligned}c^{-k}q^{-k}\Delta_c^k(\beta_{n,\chi}(t+\tau)-\beta_{n,\chi}(t))-c'^{-k}q^{-k}\Delta_{c'}^k(\beta_{n,\chi}(t+\tau)-\beta_{n,\chi}(t))\\ =-\sum_{\substack{a=1\\(a,p)=1}}^{qF}\chi_1(a)\langle a+qt\rangle^{n-1}\Big(c^{-k}q^{-k}\left(\langle a+qt\rangle^c-1\right)^k\\ -c'^{-k}q^{-k}\big(\langle a+qt\rangle^{c'}-1\big)^k\Big),\end{aligned}$$

where F is a positive multiple of F_0.

For each $a\in\mathbf{Z}$, with $(a,p)=1$, Lemma 5 implies that

$$\langle a+qt\rangle^c=1+cqu+c(c-1)pqv,$$

where $u,v\in\mathbf{Z}_p[t]$, each depending on a. In particular, $u=q^{-1}(\langle a\rangle-1)+\omega^{-1}(a)t$. Therefore,

$$(\langle a+qt\rangle^c-1)^k=c^kq^k(u+(c-1)pv)^k.$$

Similarly, there exists $v'\in\mathbf{Z}_p[t]$ such that

$$\left(\langle a+qt\rangle^{c'}-1\right)^k=c'^kq^k(u+(c'-1)pv')^k.$$

Now consider

$$\begin{aligned}c^{-k}q^{-k}\left(\langle a+qt\rangle^c-1\right)^k-c'^{-k}q^{-k}\left(\langle a+qt\rangle^{c'}-1\right)^k\\ =(u+(c-1)pv)^k-(u+(c'-1)pv')^k.\end{aligned}$$

This can be expanded to yield

$$\begin{aligned}c^{-k}q^{-k}\left(\langle a+qt\rangle^c-1\right)^k-c'^{-k}q^{-k}\left(\langle a+qt\rangle^{c'}-1\right)^k\\ =\sum_{m=1}^{k}\binom{k}{m}u^{k-m}p^m\left((c-1)^mv^m-(c'-1)^mv'^m\right).\end{aligned}$$

By Lemma 4, $\binom{k}{m}p^m\equiv 0 \pmod{kp}$ for each $m\geq 1$. Also, for these same values of m, the quantity $(c-1)^mv^m-(c'-1)^mv'^m$ is 0 modulo $(c-1,c'-1)\mathbf{Z}_p[t]$. The result then follows.

(d) From (7) we can write

$$c^{-k}q^{-k}\Delta_c^k\left(\beta_{n,\chi}(t+F)-\beta_{n,\chi}(t)\right)-c^{-k'}q^{-k'}\Delta_c^{k'}\left(\beta_{n,\chi}(t+F)-\beta_{n,\chi}(t)\right)$$
$$=\sum_{\substack{a=1\\(a,p)=1}}^{qF}\chi_1(a)\langle a+qt\rangle^{n-1}\left(\frac{\langle a+qt\rangle^c-1}{cq}\right)^k\left(\left(\frac{\langle a+qt\rangle^c-1}{cq}\right)^{k'-k}-1\right),$$

where F is a positive multiple of F_0. Now, Lemma 5 implies that $\langle a+qt\rangle^c-1\equiv 0 \pmod{cq\mathbf{Z}_p}$. Thus, if a is such that $\langle a+qt\rangle^c-1\equiv 0 \pmod{cpq\mathbf{Z}_p}$, then

$$\left(\frac{\langle a+qt\rangle^c-1}{cq}\right)^k\equiv 0 \pmod{p^k\mathbf{Z}_p}.$$

However, if a is such that $\langle a+qt\rangle^c-1\not\equiv 0 \pmod{cpq\mathbf{Z}_p}$, then

$$\left(\frac{\langle a+qt\rangle^c-1}{cq}\right)^{k'-k}-1\equiv 0 \pmod{p^N\mathbf{Z}_p},$$

since $k'-k\equiv 0 \pmod{\phi(p^N)}$. Therefore, the result.

(e) We are once again working with a linear operator, so Lemma 3 implies that

$$\binom{c^{-k}q^{-k}\Delta_c^k}{r}\left(\beta_{n,\chi}(t+F)-\beta_{n,\chi}(t)\right)=-\sum_{\substack{a=1\\(a,p)=1}}^{qF}\chi_1(a)\langle a+qt\rangle^{-1}$$
$$\times\binom{c^{-k}q^{-k}\Delta_c^k}{r}\langle a+qt\rangle^n,$$

where F is a positive multiple of F_0. Utilizing (5), and then (6), we can write

$$\binom{c^{-k}q^{-k}\Delta_c^k}{r}\langle a+qt\rangle^n=\frac{1}{r!}\sum_{m=0}^{r}s(r,m)c^{-mk}q^{-mk}\Delta_c^{mk}\langle a+qt\rangle^n$$
$$=\frac{1}{r!}\sum_{m=0}^{r}s(r,m)c^{-mk}q^{-mk}\langle a+qt\rangle^n\left(\langle a+qt\rangle^c-1\right)^{mk}.$$

This can then be rewritten as

$$\binom{c^{-k}q^{-k}\Delta_c^k}{r}\langle a+qt\rangle^n=\langle a+qt\rangle^n\binom{c^{-k}q^{-k}(\langle a+qt\rangle^c-1)^k}{r}.$$

Therefore,

$$\binom{c^{-k}q^{-k}\Delta_c^k}{r}\big(\beta_{n,\chi}(t+F)-\beta_{n,\chi}(t)\big)$$
$$= -\sum_{\substack{a=1\\(a,p)=1}}^{qF}\chi_1(a)\langle a+qt\rangle^{n-1}\binom{c^{-k}q^{-k}(\langle a+qt\rangle^c-1)^k}{r}. \tag{8}$$

From Lemma 5, $c^{-k}q^{-k}(\langle a+qt\rangle^c-1)^k\in\mathbf{Z}_p$ for each $a\in\mathbf{Z}$ with $(a,p)=1$, implying the result.

(f) Whenever F is a positive multiple of F_0, (8) implies that

$$\binom{c^{-k}q^{-k}\Delta_c^k}{r}(\beta_{n,\chi}(t+F)-\beta_{n,\chi}(t))$$
$$-\binom{c^{-k}q^{-k}\Delta_c^k}{r}\big(\beta_{n',\chi}(t+F)-\beta_{n',\chi}(t)\big)$$
$$=\sum_{\substack{a=1\\(a,p)=1}}^{qF}\chi_1(a)\langle a+qt\rangle^{n-1}\left(\langle a+qt\rangle^{n'-n}-1\right)\binom{c^{-k}q^{-k}(\langle a+qt\rangle^c-1)^k}{r}.$$

The quantity $\langle a+qt\rangle^{n'-n}-1$ must be 0 modulo $(n'-n)q\mathbf{Z}_p$, by Lemma 5, so the result follows. $\square$

Note that both (e) and (f) require that $t\in\mathbf{Z}_p$. This is because the quantity $c^{-k}q^{-k}(\langle a+qt\rangle^c-1)^k\in\mathbf{Z}_p$ if this is true, and, hence,

$$\binom{c^{-k}q^{-k}(\langle a+qt\rangle^c-1)^k}{r}\in\mathbf{Z}_p$$

for all integers $r\geq 0$, whereas, this will not hold for all values of r for certain $t\in\mathbf{C}_p$. However, if $r<p$, then $(p,r!)=1$, so that

$$\binom{c^{-k}q^{-k}(\langle a+qt\rangle^c-1)^k}{r}\in\mathbf{Z}_p[t]$$

for any $t\in\mathbf{C}_p$ with $|t|_p\leq 1$. Thus, we can still obtain congruences that correspond to those of (e) and (f) for such values of t whenever $r<p$.

4. The classical Bernoulli polynomials and the congruences of Eie and Ong

We will now show how the congruences that we derived for the generalized Bernoulli polynomials can be utilized to derive congruences for the classical Bernoulli polynomials. For each positive $n \in \mathbf{Z}$, denote

$$\beta_n(t) = -\frac{1}{n}\left(B_n(qt) - p^{n-1}B_n\left(p^{-1}qt\right)\right).$$

Note that this structure involves only the classical Bernoulli polynomials. Consider how to relate $\beta_{n,\chi}(t)$ with $\beta_n(t)$. Given $n \in \mathbf{Z}$, we need $\chi = \omega^n$ in order for $\chi_n = 1$. Thus,

$$\begin{aligned}\beta_{n,\omega^n}(t) &= -\frac{1}{n}\left(B_{n,1}(qt) - p^{n-1}B_{n,1}\left(p^{-1}qt\right)\right) \\ &= -\frac{(-1)^n}{n}\left(B_n(-qt) - p^{n-1}B_n\left(-p^{-1}qt\right)\right).\end{aligned}$$

Since $B_n(-t) = (-1)^n B_n(t) + (-1)^n n t^{n-1}$ for all $n \geq 0$, the above expression simplifies to

$$\beta_{n,\omega^n}(t) = -\frac{1}{n}\left(B_n(qt) - p^{n-1}B_n\left(p^{-1}qt\right)\right) = \beta_n(t),$$

so that, from Lemma 3,

$$\beta_n(t+F) - \beta_n(t) = -\sum_{\substack{a=1 \\ (a,p)=1}}^{qF} \omega^{n-1}(a)\langle a+qt\rangle^{n-1} = -\sum_{\substack{a=1 \\ (a,p)=1}}^{qF} (a+qt)^{n-1},$$

where $F \in \mathbf{Z}$ is positive. For positive $c \in \mathbf{Z}$ such that $\phi(q) \mid c$, we must have $\omega^{n+c} = \omega^n$, so that

$$\Delta_c^k \beta_{n,\omega^n}(t) = \Delta_c^k \beta_n(t),$$

which is a quantity expressed solely in terms of the classical Bernoulli polynomials. Therefore, some of the congruences of Theorem 1, involving $\beta_{n,\chi}(t)$ with $\chi = \omega^n$, will also hold for $\beta_n(t)$, provided $\phi(q) \mid c$. In fact, since our character ω^n does not vary with c or k, each of (a), (c), (d), and (e) of Theorem 1 will hold for congruences on $\beta_n(t+\tau) - \beta_n(t)$, with the same restrictions on c, k, n, t, and τ, except also with $\phi(q) \mid c$. Note that our value $F_0 = 1$. However, in the cases (b) and (f), the corresponding characters, ω^n and $\omega^{n'}$, are not the same unless $n' \equiv n \pmod{\phi(q)}$. Thus, under the hypotheses of each of (b) and (f), and the resulting generalization of (f), along with the restriction $\phi(q) \mid c$, the resulting congruences of these cases can be obtained for the difference $\beta_n(t+\tau) - \beta_n(t)$ provided we include the additional restriction that $n' \equiv n \pmod{\phi(q)}$.

Now let $t \in \mathbf{C}_p$, $|t|_p \leq 1$, and let $j \in \mathbf{Z}$ with $1 \leq j \leq p-1$. Eie and Ong obtained congruences on an expression of the form

$$\beta_n^{(j)}(t) = -\frac{1}{n}\left(B_n(qt-j) - p^{n-1}B_n\left(p^{-1}qt\right)\right).$$

We will see how congruences on $\beta_n^{(j)}(t)$ relate to congruences on $\beta_n(t)$. By utilizing (4), we can write

$$B_n(qt-j) = B_n(qt) - n\sum_{a=1}^{j}(qt-a)^{n-1}.$$

Since $1 \le j \le p-1$, we must also have $(a,p)=1$ for $1 \le a \le j$, so that

$$\beta_n^{(j)}(t) = \beta_n(t) + \sum_{a=1}^{j}\omega^{n-1}(-a)\langle a-qt\rangle^{n-1}.$$

The congruences on the difference $\beta_n(t+\tau)-\beta_n(t)$ were derived because of certain congruence properties that are satisfied by $\omega^n(a)\langle a+qt\rangle^n$. This quantity is very similar to $\omega^n(-a)\langle a-qt\rangle^n$, so that the congruences that were obtained with regard to this difference should find parallels in congruences on $\beta_n^{(j)}(t)-\beta_n(t)$, involving the expression of Eie and Ong, provided $\phi(q)\mid c$. The following result, concerning the difference $\beta_n^{(j)}(t)-\beta_n(t)$, can then be obtained:

Theorem 6. *Let $n,c,k,r\in\mathbf{Z}$, with n, c positive, $c\equiv 0 \pmod{\phi(q)}$, and k, r nonnegative. Furthermore, let $t\in\mathbf{C}_p$, with $|t|_p\le 1$. Then for $j\in\mathbf{Z}$, with $1\le j\le p-1$,*

(a) *the quantity $c^{-k}q^{-k}\Delta_c^k(\beta_n^{(j)}(t)-\beta_n(t))\in\mathbf{Z}_p[t]$;*

(b) *if $n'\in\mathbf{Z}$ such that $n'>n$ and $n'\equiv n \pmod{\phi(q)}$, then*

$$c^{-k}q^{-k}\Delta_c^k\left(\beta_n^{(j)}(t)-\beta_n(t)\right) \equiv c^{-k}q^{-k}\Delta_c^k\left(\beta_{n'}^{(j)}(t)-\beta_{n'}(t)\right) \pmod{(n'-n)q\mathbf{Z}_p[t]};$$

(c) *if k is positive, and if $c'\in\mathbf{Z}$ is positive with $c'\ne c$ and $c'\equiv 0 \pmod{\phi(q)}$, then*

$$c^{-k}q^{-k}\Delta_c^k\left(\beta_n^{(j)}(t)-\beta_n(t)\right) \equiv c'^{-k}q^{-k}\Delta_{c'}^k\left(\beta_n^{(j)}(t)-\beta_n(t)\right) \pmod{(c-1,c'-1)kp\mathbf{Z}_p[t]};$$

(d) *if $k'\in\mathbf{Z}$ such that $k'\ge k$ and $k'\equiv k \pmod{\phi(p^N)}$, for some positive $N\in\mathbf{Z}$, and if $t\in\mathbf{Z}_p$, then*

$$c^{-k}q^{-k}\Delta_c^k\left(\beta_n^{(j)}(t)-\beta_n(t)\right) \equiv c^{-k'}q^{-k'}\Delta_c^{k'}\left(\beta_n^{(j)}(t)-\beta_n(t)\right) \pmod{(p^k,p^N)\mathbf{Z}_p};$$

(e) *if $t\in\mathbf{Z}_p$, then $\binom{c^{-k}q^{-k}\Delta_c^k}{r}(\beta_n^{(j)}(t)-\beta_n(t))\in\mathbf{Z}_p$;*

(f) *if* $n' \in \mathbf{Z}$ *such that* $n' > n$ *and* $n' \equiv n \pmod{\phi(q)}$, *and if* $t \in \mathbf{Z}_p$, *then*

$$\binom{c^{-k}q^{-k}\Delta_c^k}{r}\left(\beta_n^{(j)}(t)-\beta_n(t)\right) \equiv \binom{c^{-k}q^{-k}\Delta_c^k}{r}\left(\beta_{n'}^{(j)}(t)-\beta_{n'}(t)\right) \quad (\text{mod } (n'-n)q\mathbf{Z}_p).$$

Proof. We derive the identity

$$\Delta_c^k\left(\beta_n^{(j)}(t)-\beta_n(t)\right)=\sum_{a=1}^{j}\omega^{n-1}(-a)\langle a-qt\rangle^{n-1}\left(\langle a-qt\rangle^c-1\right)^k,$$

for the application of the forward difference operator, and for the binomial coefficient operator

$$\binom{c^{-k}q^{-k}\Delta_c^k}{r}\left(\beta_n^{(j)}(t)-\beta_n(t)\right)=\sum_{a=1}^{j}\omega^{n-1}(-a)\langle a-qt\rangle^{n-1} \times\binom{c^{-k}q^{-k}\left(\langle a-qt\rangle^c-1\right)^k}{r}.$$

The congruences then follow from the analysis of these expressions by the same means used in the proof of Theorem 1. □

Assuming the hypotheses of Theorem 6, along with the restriction that $t \in \mathbf{Z}_p$, Theorem 2 can also be applied to the quantity $\beta_n^{(j)}(t)-\beta_n(t)$ to yield generalizations of (e) and (f). If $f(x) \in \mathbf{Q}_p[x]$ maps $\mathbf{Z}_p$ into $\mathbf{Z}_p$, then $f(c^{-k}q^{-k}\Delta_c^k)$ is an operator for which

$$f\left(c^{-k}q^{-k}\Delta_c^k\right)\left(\beta_n^{(j)}(t)-\beta_n(t)\right) \in \mathbf{Z}_p.$$

Furthermore,

$$f\left(c^{-k}q^{-k}\Delta_c^k\right)\left(\beta_n^{(j)}(t)-\beta_n(t)\right) \equiv f\left(c^{-k}q^{-k}\Delta_c^k\right)\left(\beta_{n'}^{(j)}(t)-\beta_{n'}(t)\right) \quad (\text{mod } (n'-n)q\mathbf{Z}_p),$$

where $n' \in \mathbf{Z}$ such that $n' > n$ and $n' \equiv n \pmod{\phi(q)}$.

5. Additional congruences for the case $p = 2$

We now wish to consider congruences on the expression

$$\beta_{n,\chi}\left(q^{-1}pt\right)=-\frac{1}{n}\left(B_{n,\chi_n}(pt)-\chi_n(p)p^{n-1}B_{n,\chi_n}(t)\right),$$

taken over $\mathbf{C}_p$, with $t \in \mathbf{C}_p$, $|t|_p \le 1$. Note that, for primes $p \ge 3$, this expression is the same as $\beta_{n,\chi}(t)$, which we have already considered. Thus, the only additional interest in congruences on this expression occurs when $p=2$, the case for which we now restrict the remaining discussion.

In the case $p=2$, we write the difference

$$\beta_{n,\chi}\left(2^{-1}t+F\right)-\beta_{n,\chi}\left(2^{-1}t\right)=-\sum_{\substack{a=1\\(a,2)=1}}^{4F}\chi_1(a)\left(\langle a\rangle+2\omega^{-1}(a)t\right)^{n-1},$$

where $n\in\mathbf{Z}$ is positive, $F\in\mathbf{Z}$ is a positive multiple of $2^{-1}f_{\chi_n}$, and $t\in\mathbf{C}_2$ with $|t|_2\le 1$. The quantity $\langle a\rangle+2\omega^{-1}(a)t$ does not have quite the same congruence properties as $\langle a+4t\rangle=\langle a\rangle+4\omega^{-1}(a)t$, and our congruences on this expression do not, in general, attain the same strength as those on $\langle a+4t\rangle$, as in Lemma 5. Instead, we have the following:

Lemma 7. *Let $a,m\in\mathbf{Z}$, with $(a,2)=1$ and $m\ge 2$, and let $t\in\mathbf{C}_2$ such that $|t|_2\le 1$. Furthermore, let $t_0\in\mathbf{C}_2$, $|t_0|_2=1$, and $\alpha\in\mathbf{Q}$, $\alpha\ge 0$, such that $t=2^{\alpha}t_0$. Then*

$$\begin{aligned}&\left(\langle a\rangle+2\omega^{-1}(a)t\right)^m-1\\&\quad\equiv m\left(\langle a\rangle-1+2^{1+\alpha}\omega^{-1}(a)t_0\right)\pmod{m(m-1)2^{1+2\min\{1,\alpha\}}\mathbf{Z}_2[t_0]}.\end{aligned}$$

The proof of this result can be obtained by the same means that we utilized to derive the proof of Lemma 5.

As a weaker form of this lemma, we have

$$\left(\langle a\rangle+2\omega^{-1}(a)t\right)^m\equiv 1\pmod{m2^{1+\min\{1,\alpha\}}\mathbf{Z}_2[t_0]},$$

with the same restrictions on a and t, but for all positive $m\in\mathbf{Z}$.

From Lemma 7 we can then derive congruences similar to those found in Theorem 1, having the same strength as those results if $|t|_2\le|2|_2$, but weaker if otherwise. Results similar to those of Theorem 6 can also be derived. The methods given in the proof of Theorem 1 can be employed in these derivations, which we leave to the reader.

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