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A group law on smooth real quartics having at least 3 real branches

par Johan HUISMAN

Résumé. Soit C une courbe quartique réelle lisse dans $\mathbb{P}^2$, qui admet au moins trois branches réelles B_1, B_2, B_3. On pose $B = B_1 \times B_2 \times B_3$ et soit $O \in B$. On note τ_O l'application de B sur la composante neutre $\mathrm{Jac}(C)(\mathbb{R})^0$ de l'ensemble des points réels de la jacobienne de C, définie en posant $\tau_O(P)$ comme étant la classe du diviseur $\sum P_i - O_i$. Alors, τ_O est bijective. On montre que cela permet une description géométrique explicite de la loi de groupe sur $\mathrm{Jac}(C)(\mathbb{R})^0$. Cela généralise la description géométrique classique de la loi de groupe sur la composante neutre de l'ensemble des points réels de la jacobienne d'une cubique. Si la quartique est définie sur un corps de nombres réel, alors on obtient une description géométrique d'un sous-groupe du groupe de Mordell-Weil d'indice un diviseur de 8.

Abstract. Let C be a smooth real quartic curve in $\mathbb{P}^2$. Suppose that C has at least 3 real branches B_1, B_2, B_3. Let $B = B_1 \times B_2 \times B_3$ and let $O \in B$. Let τ_O be the map from B into the neutral component $\mathrm{Jac}(C)(\mathbb{R})^0$ of the set of real points of the Jacobian of C, defined by letting $\tau_O(P)$ be the divisor class of the divisor $\sum P_i - O_i$. Then, τ_O is a bijection. We show that this allows an explicit geometric description of the group law on $\mathrm{Jac}(C)(\mathbb{R})^0$. It generalizes the classical geometric description of the group law on the neutral component of the set of real points of the Jacobian of a cubic curve. If the quartic curve is defined over a real number field then one gets a geometric description of a subgroup of its Mordell-Weil group of index a divisor of 8.

1. Introduction

The group law on a cubic curve is a famous classical geometric construction [9]. It gives a geometric description of the Jacobian of the curve. For curves of higher—or lower—degree such a geometric description is nonexistent. If one is only interested in real points of a curve then the present paper may be of interest: We show that there is a group law on the Cartesian product of 3 real branches of a smooth real quartic curve in $\mathbb{P}^2$, in

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case, of course, it does have at least 3 real branches. It gives rise to a geometric description of the neutral component of the set of real points of the Jacobian of the quartic. This generalizes the group law on a real branch of a smooth real cubic curve.

Other generalizations of the group law on a real branch of a cubic curve can be found in [7, 2, 3, 4]. All present generalizations are based on an explicit class of nonspecial divisors on real algebraic curves [6] (see also Section 4).

The paper is organized as follows. In Section 2, we will be more precise about the object of the paper. In Section 3, we give the geometric description of the neutral component of the set of real points of the Jacobian of a quartic. It relies on a statement concerning points in general position whose proof is given in Section 5. In Section 3, we also give an application to the Mordell-Weil group if the quartic is defined over a real number field. This application is proved in Section 6. Section 4 recalls some useful facts on nonspecial divisors on real curves.

2. The object

Let C be smooth real quartic curve in $\mathbb{P}^2$. By the genus formula, the genus of C is equal to 3. A connected component of the set $C(\mathbb{R})$ of real points of C is called a *real branch* of C. By Harnack's Inequality [5], the number of real branches of C is less than or equal to 4. Smooth real quartic curves having at least 3 real branches abound.

We assume throughout the paper that C has at least 3 real branches. Choose, once and for all, 3 mutually distinct real branches B_1, B_2, B_3 of C and put

$$B = B_1 \times B_2 \times B_3.$$

The Jacobian $\mathrm{Jac}(C)$ of C is a real Abelian variety of dimension 3. Its set of real points $\mathrm{Jac}(C)(\mathbb{R})$ is a compact commutative real Lie group. We denote by $\mathrm{Jac}(C)(\mathbb{R})^0$ the connected component of $\mathrm{Jac}(C)(\mathbb{R})$ that contains 0.

Let $O \in B$ be a base point. Define a map

$$\tau_O \colon B \longrightarrow \mathrm{Jac}(C)(\mathbb{R})$$

by

$$\tau_O(P) = \mathrm{cl}(\sum_{i=1}^{3}(P_i - O_i)),$$

for all $P \in B$, where cl denotes the class of a divisor. Since $\tau_O(O) = 0$ and since B is connected, $\tau_O(P)$ belongs to $\mathrm{Jac}(C)(\mathbb{R})^0$, for all $P \in B$. Hence, τ_O maps B into $\mathrm{Jac}(C)(\mathbb{R})^0$. Applying the general result of [7] to the present situation, one has the following statement.

Theorem 2.1. *The map $\tau_O \colon B \to \mathrm{Jac}(C)(\mathbb{R})^0$ is a bijection.* $\square$

In particular, one gets, by transport of structure, a group law on B. The object of this paper is to give a geometric description of this group law.

3. The geometric description

The following statement is the main ingredient in the geometric description of group law on the neutral component of the set of real points of the Jacobian of C. It states, among other things, that any 9 real points of C which are uniformly distributed over the real branches B_1, B_2, B_3, are in general position with respect to cubics.

Theorem 3.1. *Let $P, Q, R \in B$. Then, there is a unique real cubic F passing through the 9 points P_i, Q_i, R_i, for $i = 1, 2, 3$, i.e.,*

$$F \cdot C \geq \sum_{i=1}^{3} P_i + Q_i + R_i.$$

Moreover, there is a unique $S \in B$ such that

$$F \cdot C = \sum_{i=1}^{3} P_i + Q_i + R_i + S_i.$$

We postpone its proof to Section 5 and explain first how Theorem 3.1 gives rise to the geometric description of the group law we are looking for.

Choose $O \in B$. According to Theorem 3.1, there is a real cubic G such that

$$G \cdot C \geq 3O_1 + 3O_2 + 3O_3.$$

Moreover, there is an element $X \in B$ such that

$$G \cdot C = 3O_1 + X_1 + 3O_2 + X_2 + 3O_3 + X_3.$$

Now, the geometric description of the group law of $\mathrm{Jac}(C)(\mathbb{R})^0$ is given in the following statement. Recall (Theorem 2.1) that τ_O is a bijective map from B onto $\mathrm{Jac}(C)(\mathbb{R})^0$, and is defined by sending $P \in B$ to $\mathrm{cl}(\sum P_i - O_i)$.

Theorem 3.2. *Let $P, Q, R \in B$. Then,*

$$\tau_O(P) + \tau_O(Q) + \tau_O(R) = 0$$

in $\mathrm{Jac}(C)(\mathbb{R})$ if and only if there is a real cubic F such that

$$F \cdot C = \sum_{i=1}^{3} P_i + Q_i + R_i + X_i.$$

Proof. Suppose that $\tau_O(P) + \tau_O(Q) + \tau_O(R) = 0$ in $\mathrm{Jac}(C)(\mathbb{R})$. By Theorem 3.1, there is a real cubic F such that

$$F \cdot C \geq \sum_{i=1}^{3} P_i + Q_i + R_i.$$

Moreover, there is an element $S \in B$ such that

$$F \cdot C = \sum_{i=1}^{3} P_i + Q_i + R_i + S_i.$$

Then,

$$(F - G) \cdot C = \sum_{i=1}^{3} (P_i - O_i) + (Q_i - O_i) + (R_i - O_i) + (S_i - X_i).$$

Taking divisor classes and using the hypothesis, one gets that

$$\mathrm{cl}(\sum_{i=1}^{3} (S_i - X_i)) = 0$$

in $\mathrm{Jac}(C)(\mathbb{R})$, i.e., $\tau_X(S) = 0$. By Theorem 2.1, $S = X$ and it follows that there is a real cubic F such that

$$F \cdot C = \sum_{i=1}^{3} P_i + Q_i + R_i + X_i.$$

In order to show the converse, suppose that there is a real cubic F satisfying the above equation. Then,

$$(F - G) \cdot C = \sum_{i=1}^{3} (P_i - O_i) + (Q_i - O_i) + (R_i - O_i),$$

i.e., $\tau_O(P) + \tau_O(Q) + \tau_O(R) = 0$ in $\mathrm{Jac}(C)(\mathbb{R})$. $\square$

Remark 3.3. Theorem 3.2 allows to do calculations in the neutral component of the set of real points of the Jacobian of C, just as in the case of elliptic curves [9]:

The inverse of an element $P \in B$ is obtained as follows. By Theorem 3.1, there is a unique real cubic F such that $F \cdot C \geq \sum O_i + P_i + X_i$. Moreover, there is a unique element $Q \in B$ such that $F \cdot C = \sum O_i + P_i + Q_i + X_i$. Then, Q is the inverse of P.

The sum of two elements $P, Q \in B$ is obtained as follows. By Theorem 3.1, there is a unique real cubic F such that $F \cdot C \geq \sum P_i + Q_i + X_i$. Moreover, there is a unique element $R \in B$ such that $F \cdot C = \sum P_i + Q_i + R_i + X_i$. Then, the sum of P and Q is equal to the inverse of R.

I would like to conclude this section with the following application of Theorem 3.2 to the Mordell-Weil group of a smooth quartic defined over a real number field.

Let K be a real number field and let C be a smooth quartic in $\mathbb{P}^2$ defined over K. Assume that $C_{\mathbb{R}} = C \times_K \mathbb{R}$ has at least 3 real branches B_1, B_2, B_3. Let again $B = B_1 \times B_2 \times B_3$. An element $P \in B$ is a *K-rational point* of B if the divisor $P_1 + P_2 + P_3$ on $C_{\mathbb{R}}$ comes from a divisor on C. Equivalently, $P \in B$ is K-rational if and only if the divisor $P_1 + P_2 + P_3$ is invariant for the action of the Galois group $\mathrm{Gal}(\mathbb{C}/K)$. The set of K-rational points of B is denoted by $B(K)$.

Denote by $\mathrm{Jac}(C)(K)^0$ the inverse image of $\mathrm{Jac}(C)(\mathbb{R})^0$ by the natural map from $\mathrm{Jac}(C)(K)$ into $\mathrm{Jac}(C)(\mathbb{R})$. Since the subgroup $\mathrm{Jac}(C)(\mathbb{R})^0$ is of index 4 or 8 in $\mathrm{Jac}(C)(\mathbb{R})$ [1, Theorem 4.1.7], the subgroup $\mathrm{Jac}(C)(K)^0$ has index 2^i in the Mordell-Weil group $\mathrm{Jac}(C)(K)$ of C, for some $i \in \{0, 1, 2, 3\}$.

Theorem 3.4. *Suppose that O is a K-rational point of B. Then, the map τ_O maps the subset $B(K)$ of B bijectively onto $\mathrm{Jac}(C)(K)^0$. In particular, $B(K)$ is a subgroup of B and the restriction of τ_O to $B(K)$ is an isomorphism onto $\mathrm{Jac}(C)(K)^0$.*

A proof is given in Section 6.

4. Nonspecial divisors

The following proposition [1, Corollary 4.2.2] (cf. [6, Proposition 2.1]) is at the basis of the results. For the convenience of the reader, a proof is given.

Proposition 4.1. *Let C be a geometrically integral proper smooth real algebraic curve and let ω be a nonzero rational differential form on C. Then, for all real branches B of C, the degree of the divisor of ω on B is even.*

Proof. The restriction $\omega|_B$ of ω to a real branch B of C is a nonzero real meromorphic differential on B. Since C is proper and smooth, B is real analytically isomorphic to the real projective line $\mathbb{P}^1(\mathbb{R})$. Therefore, we may assume that $B = \mathbb{P}^1(\mathbb{R})$. Then, there is a nonzero real meromorphic function f on $\mathbb{P}^1(\mathbb{R})$ such that

$$\omega|_B = f \cdot \frac{dx}{x^2 + 1}.$$

In particular, the divisor $\mathrm{div}(\omega|_B)$ of $\omega|_B$ is equal to the divisor $\mathrm{div}(f)$ of f. Note that $\mathrm{div}(f) = f^\star 0 - f^\star \infty$. Since $\deg(f^\star P)$ is constant mod 2 [8], for $P \in \mathbb{P}^1(\mathbb{R})$, the degree of $\mathrm{div}(f)$ is even. □

Proposition 4.1 has the following statement [6, Theorem 2.3] as a consequence. We present here a different proof.

Theorem 4.2. *Let C be a geometrically integral proper smooth real algebraic curve and let g be its genus. Let D be a divisor on C and let d be its degree. Let k be the number of real branches of C on which D has odd degree. If $d+k>2g-2$ then D is nonspecial.*

We need the following lemma.

Lemma 4.3. *Let C be a geometrically integral proper smooth real algebraic curve and let g be its genus. Let D be a divisor on C and let d be its degree. Let k be the number of real branches of C on which D has odd degree. If $d<k$ then $h^0(D)=0$.*

Proof. Suppose $h^0(D)\neq 0$. Then there is an effective divisor E on C which is linearly equivalent to D. Let f be a nonzero rational function on C such that $\operatorname{div}(f)=D-E$. As we have seen before, the degree of $\operatorname{div}(f)$ is even on any real branch of C. Hence, the degree of E is odd on k real branches of C. Since E is effective, there are real points $P_1,\ldots,P_k$ of C, each on a different real branch, such that $E\geq P_1+\cdots+P_k$. In particular, $\deg(E)\geq k$. But, $\deg(E)=\deg(D)=d<k$. Contradiction. □

Proof of Theorem 4.2. Let K be a canonical divisor on C. Consider the divisor $K-D$. It is a divisor of degree $2g-2-d$. By Proposition 4.1, $K-D$ is of odd degree on k real branches of C. By hypothesis, $2g-2-d<k$. Hence, $h^0(K-D)=0$ by Lemma 4.3. □

5. Points in general position on a real quartic

In this section, we prove Theorem 3.1. Let us fix ourselves again a smooth real quartic C in $\mathbb{P}^2$. Let H be a hyperplane section of $\mathbb{P}^2$ and let $K=H\cdot C$. Then, K is a canonical divisor on C. The statement that will make things work out is the following.

Proposition 5.1. *The map from the linear system $|3H|$ on $\mathbb{P}^2$ into the linear system $|3K|$ on C defined by sending an element $F\in|3H|$ onto $F\cdot C\in|3K|$ is an isomorphism of projective spaces.*

Proof. Observe that the map $|3H|\to|3K|$ is well defined since no divisor in $|3H|$ can contain the curve C as a component. Since this map is a linear map of projective spaces, it is injective. Now, $\dim|3H|=9$ and, by Riemann-Roch, $\dim|3K|=9$ as well. Therefore, the map $|3H|\to|3K|$ is an isomorphism. □

From now on, we assume again that C has at least 3 real branches and we choose 3 of them, B_1,B_2,B_3. As observed above, each real branch B_i is real analytically isomorphic to the real projective line. Moreover, each B_i

is homologically trivial in $\mathbb{P}^2(\mathbb{R})$. Indeed, this is a consequence of Proposition 4.1, since C is a canonical curve. Put

$$B = B_1 \times B_2 \times B_3$$

and choose $O \in B$, as before.

Proof of Theorem 3.1. Let D be the divisor $3K - \sum P_i + Q_i + R_i$ on C. By Proposition 5.1, it suffices—for the first statement of Theorem 3.1—to show that the linear system $|D|$ on C is 0-dimensional. By Proposition 4.1, the divisor D has odd degree on all real branches B_1, B_2, B_3. Moreover, its degree is equal to 3. Since $3 + 3 > 2 \cdot 3 - 2$, the divisor D is nonspecial, by Theorem 4.2. By Riemann-Roch, $\dim |D| = 0$. This shows the first statement of Theorem 3.1.

In order to show the last statement of Theorem 3.1, let F be the unique real cubic passing through the points P_i, Q_i, R_i, for $i = 1, 2, 3$. Since each real branch B_i of C is homologically trivial in $\mathbb{P}^2(\mathbb{R})$, the degree of the intersection product $F \cdot C$ on B_i is even. Therefore, there is $S \in B$, such that

$$F \cdot C \geq \sum_{i=1}^{3} P_i + Q_i + R_i + S_i.$$

Now, both divisors that intervene in this inequality are of degree 12. Hence, the inequality is an equality and the points S_i are unique. □

6. The Mordell-Weil group

Proof of Theorem 3.4. It is clear that τ_O maps $B(K)$ into $\mathrm{Jac}(C)(K)^0$. In order to show that the image of $B(K)$ is all of $\mathrm{Jac}(C)(K)^0$, let δ be an element of $\mathrm{Jac}(C)(K)^0$. By Theorem 2.1, there is an element $P \in B$ such that $\tau_O(P) = \delta$. Let D be the divisor $\sum P_i$ on $C_{\mathbb{R}}$. By Theorem 4.2, D is nonspecial. Hence, the linear system $|D|$ is 0-dimensional and consists of D only. Since $\mathrm{cl}(D) = \delta + \mathrm{cl}(\sum O_i)$ is a K-rational point of the Picard scheme of C, the divisor D on $C_{\mathbb{R}}$ is equal to $D'_{\mathbb{R}}$ for some divisor D' on C. Hence, $P \in B(K)$ and δ is in the image of $B(K)$ by τ_O. □

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Johan HUISMAN
Institut de Recherche Mathématique de Rennes
Université de Rennes 1
Campus de Beaulieu
35042 Rennes Cedex, France
E-mail : huisman@univ-rennes1.fr