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The Diophantine equation $ax^2 + bxy + cy^2 = N$, $D = b^2 - 4ac > 0$

par Keith MATTHEWS

Résumé. Nous revisitons un algorithme dû à Lagrange, basé sur le développement en fraction continue, pour résoudre l'équation $ax^2 + bxy + cy^2 = N$ en les entiers x, y premiers entre eux, où $N \neq 0$, $\mathrm{pgcd}(a, b, c) = \mathrm{pgcd}(a, N) = 1$ et $D = b^2 - 4ac > 0$ n'est pas un carré.

Abstract. We make more accessible a neglected simple continued fraction based algorithm due to Lagrange, for deciding the solubility of $ax^2 + bxy + cy^2 = N$ in relatively prime integers x, y, where $N \neq 0, \gcd(a, b, c) = \gcd(a, N) = 1$ and $D = b^2 - 4ac > 0$ is not a perfect square. In the case of solubility, solutions with least positive y, from each equivalence class, are also constructed.

Our paper is a generalisation of an earlier paper by the author on the equation $x^2 - Dy^2 = N$. As in that paper, we use a lemma on unimodular matrices that gives a much simpler proof than Lagrange's for the necessity of the existence of a solution.

Lagrange did not discuss an exceptional case which can arise when $D = 5$. This was done by M. Pavone in 1986, when $N = \pm\mu$, where $\mu = \min_{(x,y)\neq(0,0)} |ax^2 + bxy + cy^2|$. We only need the special case $\mu = 1$ of his result and give a self–contained proof, using our unimodular matrix approach.

Keith Matthews
Department of Mathematics
University of Queensland
4072 Brisbane, Australia
E-mail : `krm@maths.uq.edu.au`

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