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Products and quotients of numbers with small partial quotients

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Résumé. On note $F(m)$ l'ensemble des nombres dont tous les quotients partiels (autres que le premier) sont inférieurs à m. Dans cet article, nous nous intéressons aux produits et quotients d'ensembles du type $F(m)$.

Abstract. For any positive integer m let $F(m)$ denote the set of numbers with all partial quotients (except possibly the first) not exceeding m. In this paper we characterize most products and quotients of sets of the form $F(m)$.

1. Introduction

Let x be a real number and n a positive integer. We say that x is *n-badly approximable* if for every rational number p/q,

$$\left|x-\frac{p}{q}\right|>\frac{1}{nq^2}.$$

It can be shown that the set of such numbers is of Lebesgue measure zero; however, in the following sense it is still quite large. For any positive integer m let $F(m)$ be the set of numbers

$$F(m)=\{[t,a_1,a_2,\dots]\,;\,t\in\mathbb{Z},\,1\le a_i\le m\text{ for }i\ge 1\}$$

where by $[a_0,a_1,a_2,\dots]$ we denote the *continued fraction*

$$a_0+\cfrac{1}{a_1+\cfrac{1}{a_2+\cfrac{1}{\cdots}}}$$

with *partial quotients* a_0, a_1, a_2 and so on. It can be shown that for every $x\in F(m)$ and every $p/q\in\mathbb{Q}$,

$$\left|x-\frac{p}{q}\right|>\frac{1}{(m+2)q^2}$$

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so that $F(m)$ is a set of $(m+2)$-badly approximable numbers. Note that

$$F(1) = \mathbb{Z} + \frac{1+\sqrt{5}}{2}$$

and hence is of Hausdorff dimension zero, and is of little interest to us.

In 1947 Marshall Hall, Jr. proved [6] that

$$[1, \infty) \subseteq F(4) \cdot F(4)$$

where for two sets A and B of real numbers we denote by $A \cdot B$ the set

$$A \cdot B = \{a \cdot b\,; a \in A \text{ and } b \in B\}.$$

In fact, we shall show in Theorem 1.2 that

$$F(4) \cdot F(4) = \Big(-\infty, \frac{7}{2} - \frac{5}{2}\sqrt{2}\Big] \bigcup \Big[17 - 12\sqrt{2}, \infty\Big).$$

More generally we will examine products and quotients of sets of the form $F(m)$. For any set A of real numbers we define A^{-1} to be the set

$$A^{-1} = \{a^{-1}\,; a \in A \text{ and } a \neq 0\}$$

and let A/B denote the set $A \cdot B^{-1}$. We would like to completely describe

$$\frac{F(m_1) \cdots F(m_k)}{F(n_1) \cdots F(n_l)}$$

where k and l are non-negative integers and $m_i, n_j \geq 2$ for $1 \leq i \leq k$ and $1 \leq j \leq l$. We can do this except in a few cases. Define $\mathcal{E}$ by

$$\mathcal{E} = \{(t, 2)\ ;\ 2 \leq t \leq 6\} \cup \{(3,3), (2,2,2), (3,2,2)\}$$

where we consider the components of each (b_i) in $\mathcal{E}$ to be unordered. We have the following results.

Theorem 1.1. *Let $m_1, \ldots, m_k \geq 2$ and $n_1, \ldots, n_l \geq 2$ be integers for some $k, l \geq 1$. If $(m_1, \ldots, m_k, n_1, \ldots, n_l) \notin \mathcal{E}$ then*

$$\frac{F(m_1) \cdots F(m_k)}{F(n_1) \cdots F(n_l)} = \mathbb{R} \backslash \{0\}.$$

For positive integers m we define $g(m)$ by

$$g(m) = \frac{-m + \sqrt{m^2 + 4m}}{2}. \tag{1}$$

Theorem 1.2. *Let $m_1 \geq m_2 \geq \cdots \geq m_k \geq 2$ be integers for some $k \geq 2$. If $(m_1, \ldots, m_k) \notin \mathcal{E}$ then*

$$F(m_1) \cdots F(m_k) = (\infty, -L] \cup [U, \infty)$$

for some L and U depending on $m_1, \ldots, m_k$. More precisely, we have

$$L = \begin{cases} \frac{g(m_1)}{m_1}\,(1 - g(m_2)) \cdots (1 - g(m_k))\,, & \text{if } k \text{ is even,} \\ (1 - g(m_1)) \cdots (1 - g(m_k))\,, & \text{if } k \text{ is odd} \end{cases}$$

and

$$U = \begin{cases} (1-g(m_1))\cdots(1-g(m_k)), & \text{if } k \text{ is even,} \\ \frac{g(m_1)}{m_1}(1-g(m_2))\cdots(1-g(m_k)), & \text{if } k \text{ is odd.} \end{cases}$$

We shall prove partial results along the lines of Theorems 1.1 and 1.2 for a few of the exceptional cases.

Theorem 1.3. *$F(3)\cdot F(3)$ and $F(5)\cdot F(2)$ both contain $(-\infty,-c]\cup[c,\infty)$ for some constant c.*

Theorem 1.4. *Let x be a non-zero real number. Then x is a member of $F(3)/F(3)$ and $F(5)/F(2)$ except possibly if $x = r/s$ for some relatively prime integers r and s with either r or s less than 8.*

Note that since $F(5) \subseteq F(6)$ Theorems 1.3 and 1.4 also yield partial descriptions of $F(6)\cdot F(2)$ and $F(6)/F(2)$.

Finally, we establish the multiplicative analog of results contained in [3].

Theorem 1.5. *For some constant c, $F(3)F(2)F(2) \supseteq (-\infty,-c]\cup[c,\infty)$. Further,*

$$F(3)F(2)/F(2) = F(2)F(2)/F(3) = R\backslash\{0\}.$$

Our basic approach in establishing these theorems will be similar in spirit to that of Hall in that for integers m we will characterize $[0,1]\cap F(m)$ as a Cantor set and use Cantor set techniques to prove our results.

2. Background

Let T be a connected directed graph. We say that T is a *tree* if every vertex V of T has at most one edge terminating at V, and one vertex V_R has no edges terminating at V_R. We call V_R the *root* of T. If there is an edge connecting V_1 to V_2, then we say that V_2 is a *subvertex* of V_1. A tree where each vertex has at most 2 subvertices is called a *binary* tree.

We define a *generalized Cantor set* (henceforth known as a *Cantor set*) to be any set C of real numbers of the form

$$C = I \setminus \bigcup_{i\geq 1} O_i$$

where I is a finite closed interval and $\{O_i\ ;\ i\geq 1\}$ is a countable (finite or infinite) collection of disjoint open intervals contained in I. We may inductively define a binary tree $\mathcal{D}$ that will represent C. Let the root of the tree be the interval I. We say that $\{I\}$ is the *zeroth level* of the tree. Now say that we have defined our tree up to the n^{th} level. We define the $(n+1)^{th}$ level of the tree as follows. Let I^w be an n^{th} level vertex of our tree for some binary word w (i.e. w is a finite string of zeros and ones). Assume first that

$$I^w \bigcap \left(\bigcup_{i \geq 1} O_i \right) \neq \emptyset$$

(so that $I^w \not\subseteq C$). Let O_{I^w} be the interval in the set $\{O_i \; ; \; i \geq 1\}$ of least index which is contained in I^w and let I^{w0} and I^{w1} be closed intervals with

$$I^w = I^{w0} \cup O_{I^w} \cup I^{w1}$$

We let I^{w0} and I^{w1} be subvertices of I^w in $\mathcal{D}$ and define the *thickness* of I^w to be

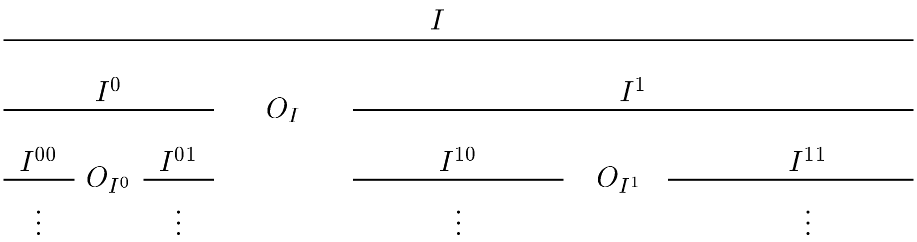
$$\tau_{\mathcal{D}}(I^w) = \frac{\min\{|I^{w0}|, |I^{w1}|\}}{|O_{I^w}|}$$

where for any interval J we let $|J|$ denote the length of J. If

$$I^w \bigcap \left(\bigcup_{i \geq 1} O_i \right) = \emptyset$$

then we set $I^{w0} = I^w$, let I^{w0} be the subvertex of I^w in $\mathcal{D}$ and put $\tau_{\mathcal{D}}(I^w) = \infty$.

We repeat this process for every vertex I^w in the n^{th} level of $\mathcal{D}$. The $(n+1)^{th}$ level of the tree is the set of vertices I^v in $\mathcal{D}$ with $|v| = n+1$, where $|v|$ denotes the length of the word v. We continue this process inductively, creating the infinite tree $\mathcal{D}$. For example, we might construct the following tree.



Now,

$$\{O_{I^w} \; ; \; I^w \text{ is a vertex of } \mathcal{D} \text{ with } I^w \not\subseteq C\} = \{O_i \; ; \; i \geq 1\}$$

hence

$$C = \bigcap_{n=0}^{\infty} \left(\bigcup_{|w|=n} I^w \right).$$

Any tree with this property is said to be a *derivation* of the Cantor set C from I. The intervals $I, I^0, \ldots$ are called *bridges* of the derivation, while the open intervals $O_I, O_{I^0}, \ldots$ are called *gaps* of C. We define the *thickness* of the derivation $\mathcal{D}$ to be

$$\tau(\mathcal{D}) = \inf_w \tau_{\mathcal{D}}(I^w).$$

For example, if C is the usual middle–third Cantor set we may take a derivation of C to be the tree $\mathcal{D}$ with root $I = [0,1]$ and bridges

$$I^{d_1 d_2 \cdots d_t} = \left[2\sum_{i=1}^{t} \frac{d_i}{3^i}, \frac{1}{3^t} + 2\sum_{i=1}^{t} \frac{d_i}{3^i} \right]$$

for all finite binary words $d_1 d_2 \ldots d_t$. We find that in this case $\tau(\mathcal{D}) = 1$.

Of course we may reorder the set of O_i's and hence obtain many different derivations of the same Cantor set. These derivations may have different thicknesses, so we define the *thickness* of the Cantor set C to be

$$\tau(C) = \sup_{\mathcal{D}} \tau(\mathcal{D})$$

where the supremum is over all derivations $\mathcal{D}$ of C. It is not difficult to show that the supremum is attained if the sequence $\{|O_i|\}_i$ is non-decreasing (see Lemma 3.1 of [2]). We also define the *normalized thickness* of C, $\gamma(C)$, to be

$$\gamma(C) = \frac{\tau(C)}{\tau(C)+1}.$$

The problem of characterizing products and quotients of $F(m)$'s will involve finding sums of certain Cantor sets. Let k be an integer which is at least 2, and assume that for $1 \le j \le k$, C_j is a Cantor set derived from I_j. We would like to determine when

$$C_1 + \cdots + C_k = I_1 + \cdots + I_k \tag{2}$$

where we are considering the pointwise sum of the sets. If $I_1, \ldots, I_{k-1}$ are all much smaller than one of the gaps in C_k then (2) cannot hold. Hence in our approach to finding sums of Cantor sets we will only consider sets that are approximately the same size, as follows. Let k be an integer which is at least 2, and assume that for $1 \le j \le k$, A_j is a bridge of the Cantor set C_j, with O_j a gap of C_j of maximal size contained in A_j. We say that the sequence of bridges $(A_1, \ldots, A_k)$ is *compatible* if

$$|A_{r+1}| \ge |O_j| \quad \text{and} \quad |A_1| + \cdots + |A_r| \ge |O_{r+1}|$$

for $r = 1, \ldots, k-1$ and $j = 1, \ldots, r$. Note that if $k = 2$ than this is equivalent to the condition

$$|A_1| \ge |O_2| \quad \text{and} \quad |A_2| \ge |O_1|.$$

In [2] the author derived a result concerning the sum of a finite number of Cantor sets.

Theorem 2.1. *Let k be a positive integer and for $j = 1, 2, \ldots, k$ let C_j be a Cantor set derived from I_j. Put $S_\gamma = \gamma(C_1) + \cdots + \gamma(C_k)$ and assume that $(I_1, \ldots, I_k)$ is compatible. If $S_\gamma \geq 1$ then*

$$C_1 + \cdots + C_k = I_1 + \cdots + I_k.$$

Otherwise

$$\gamma(C_1 + \cdots + C_k) \geq S_\gamma \quad \textit{and} \quad \dim_H(C_1 + \cdots + C_k) \geq \frac{\log 2}{\log(1 + 1/S_\gamma)}.$$

For positive integers $a_1, a_2, \ldots$ we denote by $\langle a_1, a_2, \ldots \rangle$ the continued fraction $[0, a_1, a_2, \ldots]$. For any positive integer $m \geq 2$ let $C(m) = [0,1] \cap F(m)$ and put

$$\begin{aligned} I(m) = [\langle \overline{m,1} \rangle, \langle \overline{1,m} \rangle] &= \left[\frac{g(m)}{m}, g(m) \right] \\ &= \left[\frac{-1 + \sqrt{1 + 4/m}}{2}, \frac{-m + \sqrt{m^2 + 4m}}{2} \right] \end{aligned}$$

where $g(m)$ is defined as in (1). We may characterize $C(m)$ as a Cantor set derived from $I(m)$ in the following manner. For any real a and b, we denote by $[[a, b]]$ and $((a, b))$ the intervals

$$[[a, b]] = [\min\{a, b\}, \max\{a, b\}]$$

and

$$((a, b)) = (\min\{a, b\}, \max\{a, b\}).$$

Assume that

$$A = [[\langle a_1, \ldots, a_r, b, \overline{m,1} \rangle, \langle a_1, \ldots, a_r, m, \overline{1,m} \rangle]] \tag{3}$$

is a bridge of $C(m)$ with $b < m$. We form the subvertices of A by setting

$$\begin{aligned} A^0 &= [[\langle a_1, \ldots, a_r, b, \overline{m,1} \rangle, \langle a_1, \ldots, a_r, b, \overline{1,m} \rangle]], \\ O_A &= ((\langle a_1, \ldots, a_r, b, \overline{1,m} \rangle, \langle a_1, \ldots, a_r, b+1, \overline{m,1} \rangle)) \end{aligned}$$

and

$$A^1 = [[\langle a_1, \ldots, a_r, b+1, \overline{m,1} \rangle, \langle a_1, \ldots, a_r, m, \overline{1,m} \rangle]].$$

Note that A^0 is of the form (3) with $a_{r+1} = b$ and b replaced by 1. Similarly A^1 is also of the form (3). Since $I(m)$ is of the form (3) with $r = 0$ and $b = 1$, by induction we obtain a derivation $\mathcal{D}(m)$ of $C(m)$ from $I(m)$.

By calculation it can be shown (see Lemma 4.2 of [2]) that

$$\tau(C(m)) = \frac{g(m)(m-1)}{m - g(m)(m-1)} \cdot \frac{m + g(m) - 1}{m + g(m)}.$$

TABLE 1. Values of $\tau(C(m))$ and $\gamma(C(m))$ to three decimal places

m	2	3	4	5	6	7
$\tau(C(m))$	0.366	0.822	1.300	1.788	2.281	2.775
$\gamma(C(m))$	0.267	0.451	0.565	0.641	0.695	0.735

Hence we may easily calculate $\tau(C(m))$ and $\gamma(C(m))$ (See Table 1). In [2] the author used these calculations along with Theorem 2.1 to derive results concerning sums and differences of numbers with small partial quotients. We may also use this approach to examine products and quotients of $F(m)$'s. For any set E of positive, non-zero numbers we denote by E^* the set

$$E^* = \{\log e;\ e \in E\}.$$

It is not difficult to see that $C(m)^*$ is also a Cantor set. We have the following result.

Lemma 2.2. *Let A and B be intervals contained in $[x, x+w]$, where x and w are real numbers with $0 < 2w \leq x$. Then*

$$\left|\frac{|A^*|}{|B^*|} - \frac{|A|}{|B|}\right| < \frac{7w}{x} \cdot \frac{|A|}{|B|}.$$

Proof. We may assume that A and B are closed. Let $A = [x + a_0, x + a_1]$ and $B = [x + b_0, x + b_1]$. Then

$$\frac{|A^*|}{|B^*|} = \frac{\log\left(\frac{x+a_1}{x+a_0}\right)}{\log\left(\frac{x+b_1}{x+b_0}\right)} = \frac{\log\left(1 + \frac{|A|}{x+a_0}\right)}{\log\left(1 + \frac{|B|}{x+b_0}\right)}$$

and the result follows from the application of the power series expansion of $\log(1+y)$. $\square$

Note that by Lemma 2.2 we can make the thickness of $(n + C(m))^*$ as close as desired to that of $C(m)$ by choosing n large. For example, it follows from Theorem 2.1, Lemma 2.2 and Table 1 that for n sufficiently large,

$$(n + C(2))(n + C(7)) = \left[(n + \langle\overline{2,1}\rangle)(n + \langle\overline{7,1}\rangle), (n + \langle\overline{1,2}\rangle)(n + \langle\overline{1,7}\rangle)\right].$$

3. Proofs of Theorems 1.1 and 1.2

For any set B of positive integers we let $F(B)$ denote the set of real numbers x such that all partial quotients of x, except possibly the first, are members of B. We also let $\gamma(B)$ denote the normalized thickness of the Cantor set $F(B) \cap [0,1]$. In this manuscript we are concerned with

$$F(\{1,2,\ldots,m\}) = F(m) \quad \text{and} \quad \gamma(\{1,2,\ldots,m\}) = \gamma(C(m)).$$

In [2] the author derived the following general result.

Theorem 3.1. *Let k be a positive integer. For $j = 1, \dots, k$ let B_j be a set of positive integers and let $\epsilon_j \in \{1, -1\}$. If*

$$\gamma(B_1) + \cdots + \gamma(B_k) > 1 \quad \textit{and} \quad |\epsilon_1 + \cdots + \epsilon_k| < k$$

then

$$(-\infty, 0) \cup (0, \infty) \subseteq F(B_1)^{\epsilon_1} \cdots F(B_k)^{\epsilon_k}.$$

Proof. See [2], Theorem 1.7, part 2. □

Proof of Theorem 1.1. Since zero is not contained in $F(m)$ for any m, Theorem 1.1 follows from Table 1 and Theorem 3.1. □

The proof of Theorem 1.2 will require several lemmas. For positive integers N and $m \geq 2$ we put

$$F^+(m) = F(m) \bigcap (0, \infty), \qquad C_N^+(m) = \bigcup_{n=0}^{N} (n + C(m)),$$

$$F^-(m) = F(m) \bigcap (-\infty, 0) \quad \text{and} \quad C_N^-(m) = \bigcup_{n=1}^{N} (n - C(m)).$$

Note that for all positive integers N and $m \geq 2$ the sets $C_N^+(m)$ and $C_N^-(m)$ are Cantor sets. We have the following results.

Lemma 3.2. *If N is sufficiently large then*

$$\begin{array}{ll}
\gamma(C_N^+(2)^*) > 0.2679, & \gamma(C_N^-(2)^*) > 0.2307, \\
\gamma(C_N^+(3)^*) > 0.4511, & \gamma(C_N^-(3)^*) > 0.4342, \\
\gamma(C_N^+(4)^*) > 0.5653, & \gamma(C_N^-(4)^*) > 0.5566, \\
\gamma(C_N^+(5)^*) > 0.6414, & \gamma(C_N^-(5)^*) > 0.6364, \\
\gamma(C_N^+(6)^*) > 0.6952, & \gamma(C_N^-(6)^*) > 0.6920, \\
\gamma(C_N^+(7)^*) > 0.7351, & \gamma(C_N^-(7)^*) > 0.7330, \\
\gamma(C_N^+(8)^*) > 0.7659, & \gamma(C_N^-(8)^*) > 0.7644.
\end{array}$$

For $m \geq 9$ and N sufficiently large,

$$\gamma(C_N^+(m)^*) > 0.7727 \quad \textit{and} \quad \gamma(C_N^-(m)^*) > 0.7727\ .$$

Proof. The result is attained by calculation, using Lemma 2.2 to limit the number of calculations that must be performed in each case. □

Lemma 3.3. *If $0 < m \leq n$ then*

$$(1 - \langle \overline{1, m} \rangle) \langle \overline{n, 1} \rangle \leq (1 - \langle \overline{1, n} \rangle) \langle \overline{m, 1} \rangle \quad \textit{and} \quad 1 - \langle \overline{1, m} \rangle < \langle \overline{m, 1} \rangle.$$

Proof. See [1], Lemmas 6.4.1 and 6.4.2. □

Lemma 3.4. *We have*

$$F(4)F(3) = \big(-\infty, -\langle\overline{4,1}\rangle(1-\langle\overline{1,3}\rangle)\big] \cup \big[(1-\langle\overline{1,4}\rangle)(1-\langle\overline{1,3}\rangle), \infty\big)$$

and

$$F(7)F(2) = \big(-\infty, -\langle\overline{7,1}\rangle(1-\langle\overline{1,2}\rangle)\big] \cup \big[(1-\langle\overline{1,7}\rangle)(1-\langle\overline{1,2}\rangle), \infty\big)\,.$$

Proof. The first result is Theorem 1.8 of [2], so we shall only prove the second. By Lemma 3.2 we have

$$\gamma(C_N^-(7)^*) + \gamma(C_N^+(2)^*) > 1 \quad \text{and} \quad \gamma(C_N^+(7)^*) + \gamma(C_N^+(2)^*) > 1$$

for N sufficiently large. Therefore by Theorem 2.1

$$\begin{aligned} F(7)F(2) \supseteq \big(&-(N-\langle\overline{7,1}\rangle)(N+\langle\overline{1,2}\rangle), -(1-\langle\overline{1,7}\rangle)\langle\overline{2,1}\rangle\big] \\ &\cup \big[\langle\overline{7,1}\rangle\langle\overline{2,1}\rangle, (N+\langle\overline{1,7}\rangle)(N+\langle\overline{1,2}\rangle)\big) \end{aligned}$$

and letting N tend to infinity we find

$$\begin{aligned} F(7)F(2) &\supseteq \big(-\infty, -(1-\langle\overline{1,7}\rangle)\langle\overline{2,1}\rangle\big] \cup \big[\langle\overline{7,1}\rangle\langle\overline{2,1}\rangle, \infty\big) \\ &= (-\infty, -0.0411\ldots] \cup [0.0464\ldots, \infty)\,. \end{aligned} \tag{4}$$

To complete the proof of the lemma we must slightly enlarge the set on the right-hand side of (4). Let

$$I^1 = [\langle 1, \overline{1,2}\rangle, \langle\overline{1,2}\rangle] \quad \text{and} \quad C^1 = C(2) \cap I^1,$$

then we find by calculation that

$$\gamma((1-C^1)^*) \geq 0.2679.$$

Let O^- and O^+ denote the largest gap in $(1-C(7))^*$ and $C(7)^*$ respectively. Then

$$O^- = (1-\langle 1, \overline{1,7}\rangle, 1-\langle 2, \overline{7,1}\rangle)^* \quad \text{and} \quad O^+ = (\langle 2, \overline{7,1}\rangle, \langle 1, \overline{1,7}\rangle)^*.$$

Further,

$$|O^-| = 0.119\ldots\,, \qquad |O^+| = 0.119\ldots\,,$$
$$|(1-I^1)^*| = 0.4557\ldots\,, \qquad |I(7)^*| = 1.94\ldots$$

and

$$|(1-I(7))^*| = 2.04\ldots$$

so $((1-I(7))^*, (1-I^1)^*)$ and $(I(7)^*, (1-I^1)^*)$ are compatible pairs of bridges. Therefore by Theorem 2.1 we have

$$\begin{aligned} (1-C(7))(1-C^1) &= \big[(1-\langle\overline{1,7}\rangle)(1-\langle\overline{1,2}\rangle), \\ &\qquad (1-\langle\overline{7,1}\rangle)(1-\langle 1, \overline{1,2}\rangle)\big] \\ &= [0.0301\ldots, 0.369\ldots] \end{aligned} \tag{5}$$

and

$$-C(7)(1-C^1) = \big[-\langle\overline{1,7}\rangle(1-\langle 1,\overline{1,2}\rangle), -\langle\overline{7,1}\rangle(1-\langle\overline{1,2}\rangle)\big] \\ = [-0.375\ldots, -0.0339\ldots]. \tag{6}$$

Since

$$F(7)\cdot F(2) \subseteq \big(-\infty, -\langle\overline{7,1}\rangle(1-\langle\overline{1,2}\rangle)\big] \cup \big[(1-\langle\overline{1,7}\rangle)(1-\langle\overline{1,2}\rangle), \infty\big)$$

by Lemma 3.3, our result follows from (4), (5) and (6). □

Lemma 3.5. *We have*

$$F(2)F(2)F(2)F(2) = \big(-\infty, -\langle\overline{2,1}\rangle(1-\langle\overline{1,2}\rangle)^3\big] \cup \big[(1-\langle\overline{1,2}\rangle)^4, \infty\big)\,.$$

Proof. By Lemma 3.2 we have, for N sufficiently large,

$$\gamma(C_N^-(2)^*) > 0.2307 \quad \text{and} \quad \gamma(C_N^+(2)^*) > 0.2679.$$

Therefore by Theorem 2.1 and letting N tend to infinity we have

$$F(2)^4 \supseteq \big(-\infty, -\langle\overline{2,1}\rangle^3(1-\langle\overline{1,2}\rangle)\big] \cup \big[\langle\overline{2,1}\rangle^4, \infty\big) \\ = (-\infty, -0.0131\ldots] \cup [0.0179\ldots, \infty). \tag{7}$$

As in the proof of Lemma 3.4 we must slightly enlarge this set to achieve the desired result. In particular we put

$$I^+ = [\langle\overline{2,1}\rangle, \langle 2,\overline{2,1}\rangle], \qquad I^- = [1-\langle 1,\overline{1,2}\rangle, 1-\langle\overline{1,2}\rangle],$$
$$C^+ = C(2)\cap I^+ \qquad \text{and} \quad C^- = (1-C(2))\cap I^-.$$

By calculation we find that

$$\gamma(C^{+*}), \gamma(C^{-*}) > 0.2679.$$

and by Theorem 2.1 we have

$$C^+(C^-)^3 = I^+(I^-)^3 = \big[\langle\overline{2,1}\rangle(1-\langle\overline{1,2}\rangle)^3, (1-\langle 1,\overline{1,2}\rangle)^3\langle 2,\overline{2,1}\rangle\big] \\ = [0.00704\ldots, 0.0319\ldots] \tag{8}$$

and

$$(C^-)^4 = (I^-)^4 = \big[(1-\langle\overline{1,2}\rangle)^4, (1-\langle 1,\overline{1,2}\rangle)^4\big] \\ = [0.00514\ldots, 0.0319\ldots]. \tag{9}$$

The lemma follows from (7), (8) and (9). □

Proof of Theorem 1.2. Assume first that $k=2$. If (m_1, m_2) equals $(4,3)$ or $(7,2)$ then our result is a consequence of Lemma 3.4. Otherwise by Lemma 3.3 we have

$$F(m_1)\cdot F(m_2) \subseteq \big(-\infty, -\langle\overline{m_1,1}\rangle(1-\langle\overline{1,m_2}\rangle)\big] \\ \bigcup \big[(1-\langle\overline{1,m_1}\rangle)(1-\langle\overline{1,m_2}\rangle), \infty\big)\,.$$

By Lemmas 3.2 it follows that for N sufficiently large

$$\gamma(C_N^+(m_1)^*) + \gamma(C_N^-(m_2)^*) > 1 \quad \text{and} \quad \gamma(C_N^-(m_1)^*) + \gamma(C_N^-(m_2)^*) > 1$$

so by Theorem 2.1

$$\big(-(N + \langle\overline{1,m_1}\rangle)(N - \langle\overline{m_2,1}\rangle), -\langle\overline{m_1,1}\rangle(1 - \langle\overline{1,m_2}\rangle)\big] \subseteq F(m_1)\cdot F(m_2)$$

and

$$\big[(1 - \langle\overline{1,m_1}\rangle)(1 - \langle\overline{1,m_2}\rangle), (N - \langle\overline{m_1,1}\rangle)(N - \langle\overline{m_2,1}\rangle)\big) \subseteq F(m_1)\cdot F(m_2).$$

Since $g(m) = \langle\overline{1,m}\rangle$ our result follows upon letting N tend to infinity.

Assume next that $k > 2$. If $k = 4$ and $(m_1, m_2, m_3, m_4) = (2,2,2,2)$ then the theorem is a consequence of Lemma 3.5. Otherwise our result follows from Lemma 3.2, Lemma 3.3 and Theorem 2.1. □

4. $F(5)\cdot F(2)$ and $F(3)\cdot F(3)$

Unfortunately both $\gamma(C(5)) + \gamma(C(2))$ and $\gamma(C(3)) + \gamma(C(3))$ are less than one, so we cannot use Theorem 2.1 to find intervals in $F(5)\cdot F(2)$ or $F(3)\cdot F(3)$. Instead we must use a more complicated approach, which is similar in spirit to the approaches used independently by both Hanno Schecker [8] and Gregory Freiman [5] in their examinations of $F(3)+F(3)$.

Assume that m is an integer with $m \geq 3$, and that A is a bridge of $C(m)$ with

$$A = [[\langle a_1, \ldots, a_r, \overline{m,1}\rangle, \langle a_1, \ldots, a_r, \overline{1,m}\rangle]]$$

for some positive integers $a_1, a_2, \ldots, a_r$. Then we denote by $\overline{A}$ and $\overline{O_A}$ the intervals

$$\overline{A} = [[\langle a_1, \ldots, a_r, \overline{m-1,1}\rangle, \langle a_1, \ldots, a_r, \overline{1,m-1}\rangle]]$$

and

$$\overline{O_A} = ((\langle a_1, \ldots, a_r, 1, \overline{1,m-1}\rangle, \langle a_1, \ldots, a_r, 2, \overline{m-1,1}\rangle)).$$

We will derive results concerning sets of the form $\overline{A}+B$. To do so we must modify our concept of compatibility as follows. If A is a bridge of $C(m)$ and B is a bridge of $C(n)$ for some $m \geq 3$ and $n \geq 2$ then we say that $\overline{A}$ and B are *P-compatible*, written $\overline{A} \approx B$, if

$$|\overline{A}| \geq |O_B| \quad \text{and} \quad |B| \geq |\overline{O_A}|.$$

For integers $m \geq 2$ let $\mathcal{W}^m$ denote the set of finite words with digits between 1 and m inclusive. For any $w \in \mathcal{W}^m$ with $w = a_1 a_2 \cdots a_r$ we put

$$I(m;w) = [[\langle a_1, a_2, \ldots, a_r, \overline{m,1}\rangle, \langle a_1, a_2, \ldots, a_r, \overline{1,m}\rangle]]$$

and

$$C(m;w) = C(m) \cap I(m;w).$$

In [4] the author proved the following result.

Theorem 4.1. *Let $I(5;w)$ and $I(2;v)$ be bridges of $C(5)$ and $C(2)$ respectively. Assume that $\overline{I(5;w)}$ and $I(2;v)$ are P-compatible, and that $w, v \notin \{\emptyset, 1, 12\}$. Then*

$$\overline{I(5;w)} + I(2;v) \subseteq C(5;w) + C(2;v).$$

The proof of Theorem 4.1 is similar is spirit to the proof of Theorem 2.1 (as given in [2]), the key difference being that instead of relying on two constants (the thicknesses) we must calculate 25 constants. However, all of these constants are infimums of ratios of lengths of intervals (similar to thickness), and so by Lemma 2.2 and the proof of Theorem 4.1 we have the following result.

Lemma 4.2. *If n and m are sufficiently large then*

$$(n \pm \overline{I(5;w)})^* + (m + I(2;v))^* \subseteq (n \pm C(5))^* + (m + C(2))^*$$

and

$$(n \pm \overline{I(5;w)})^* - (m + I(2;v))^* \subseteq (n \pm C(5))^* - (m + C(2))^*$$

whenever $w, v \notin \{\emptyset, 1, 12\}$ and $(n + \overline{I(5;w)})^*$ and $(m + I(2;v))^*$ are P-compatible.

Using the same technique as in the proofs of Theorem 4.1 and Lemma 4.2 (in this case using 91 constants) we may establish the following result concerning $F(3) \cdot F(3)$.

Lemma 4.3. *If n and m are sufficiently large then*

$$(n \pm \overline{I(3;w)})^* + (m + I(3;v))^* \subseteq (n \pm C(3))^* + (m + C(3))^*$$

and

$$(n \pm \overline{I(3;w)})^* - (m + I(3;v))^* \subseteq (n \pm C(3))^* - (m + C(3))^*$$

whenever $w, v \notin \{\emptyset, 1, 13\}$ and $(n + \overline{I(3;w)})^*$ and $(m + I(3;v))^*$ are P-compatible.

The reader is directed to Lemma 7.5.1 of [1] for more information regarding the calculations.

To prove Theorem 1.3 we will tile our infinite rays with intervals of the form $(n \pm \overline{I(k;w)})(m + I(l;v))$, for (k,l) equal to $(5,2)$ or $(3,3)$. By Lemmas 4.2 and 4.3 the infinite rays will be contained in $F(k) \cdot F(l)$. To construct the tilings we will use the following technical lemmas.

Lemma 4.4. *Let $I(k;w)$ and $I(l;v)$ be bridges of $C(k)$ and $C(l)$ respectively, for some words w and v. Assume that $\overline{I(k;w)} \approx I(l;v)$ and $|I(l;v)| > 2|\overline{O_{I(k;w)}}|$. Then for n sufficiently large and $n \leq m \leq 2n$,*

$$(n + \overline{I(k;w)})^* \approx (m + I(l;v))^*.$$

Proof. See [1], Lemma 7.6.1. □

Lemma 4.5. *Let α_1, α_2, β_1, β_2 be real numbers with $0 \le \alpha_1 < \beta_1 \le 1$ and $0 \le \alpha_2 < \beta_2 \le 1$. Let δ be any number in the range*

$$1 - (\beta_1 - \alpha_1) - (\beta_2 - \alpha_2) < \delta < 1.$$

For any integers r and s let $J(r,s)$ denote the interval

$$J(r,s) = (r + [\alpha_1, \beta_1])(s + [\alpha_2, \beta_2]).$$

Then for N_1 sufficiently large,

$$\bigcup_{n \ge N_1} \bigcup_{\lfloor \frac{\delta n}{2} \rfloor \le k \le \lfloor \frac{n}{2} \rfloor} J(n+k, 2n-k)$$

is an interval.

Proof. This is equivalent to Lemma 7.6.2 of [1]. □

Proof of Theorem 1.3. We first consider $F(5) \cdot F(2)$. Let

$$I_5 = [\langle 1,1,\overline{1,5}\rangle, \langle 1,1,\overline{5,1}\rangle] \quad \text{and} \quad I_2 = [\langle 1,1,\overline{1,2}\rangle, \langle 1,1,\overline{2,1}\rangle].$$

Then by Lemma 4.4 there exists N_0 such that

$$(n + \overline{I_5})^* \approx (m + I_2)^*$$

for any $n \ge N_0$ and $n \le m \le 2n$. By Lemma 4.2 we have

$$[(n+\alpha_1)(m+\alpha_2), (n+\beta_1)(m+\beta_2)] \subseteq F(5) \cdot F(2) \tag{10}$$

where

$$\alpha_1 = \langle 1,1,\overline{1,4}\rangle, \qquad \beta_1 = \langle 1,1,\overline{4,1}\rangle,$$
$$\alpha_2 = \langle 1,1,\overline{1,2}\rangle \quad \text{and} \quad \beta_2 = \langle 1,1,\overline{2,1}\rangle.$$

Put

$$J(n,m) = [(n+\alpha_1)(m+\alpha_2), (n+\beta_1)(m+\beta_2)].$$

By Lemma 4.5, for N_1 sufficiently large,

$$\bigcup_{n \ge N_1} \bigcup_{\lfloor \frac{\delta n}{2} \rfloor \le k \le \lfloor \frac{n}{2} \rfloor} J(n+k, 2n-k)$$

is an interval. Therefore by (10) there exists a constant c_1 such that

$$[c_1, \infty) \subseteq F(5)F(2).$$

By a similar process we find that

$$[c_2, \infty) \subseteq -F(5) \cdot F(2)$$

for some constant c_2. Now we let

$$I_3 = [\langle 1,1,\overline{1,3}\rangle, \langle 1,1,\overline{3,1}\rangle].$$

and consider $(n+\overline{I_3})(m+I_3)$. Using the same technique as above we have

$$[c_4,\infty)\subseteq F(3)\cdot F(3) \quad \text{and} \quad [c_5,\infty)\subseteq -F(3)\cdot F(3)$$

for some sufficiently large constants c_4 and c_5, and the theorem follows. $\square$

5. $F(3)/F(3)$ and $F(5)/F(2)$

To prove Theorem 1.4 we will add and subtract scaled versions of our $C(m)$ Cantor sets. We require a few preliminary results.

Lemma 5.1. *Let (m,n) equal $(5,2)$ or $(3,3)$ and let a and b be non-zero real numbers. Let $I(m;w)$ and $I(n;v)$ be bridges of $C(5)$ and $C(2)$ respectively for some words w and v. Assume that $a\overline{I(5;w)}$ and $bI(2;v)$ are P-compatible, and that $w,v\notin\{\emptyset,1,12,13\}$. Then*

$$a\overline{I(m;w)}+bI(n;v)\subseteq aC(m;w)+bC(n;v).$$

Proof. This lemma is a consequence of the proofs of Theorem 4.1 and Lemma 4.3 , since ratios of lengths of intervals remain unchanged when the intervals are scaled. $\square$

Lemma 5.2. *Let a and b be real numbers with $a>b$ and suppose that (m,n) equals $(3,3)$, $(5,2)$ or $(2,5)$. Then there exists an integer t such that*

$$a\overline{I(n;11(1)^t)}\approx bI(m;11) \quad \textit{and} \quad a|\overline{I(n;11(1)^t)}|>3.3b|O(m;11)|$$

if (m,n) equals $(3,3)$ or $(5,2)$, and

$$aI(n;11(1)^t)\approx b\overline{I(m;11)} \quad \textit{and} \quad a|I(n;11(1)^t)|>3.3b|\overline{O(m;11)}|$$

otherwise.

Proof. We will prove the case $(m,n)=(3,3)$. The other cases are similar. If $b|I(3;11)|>a|\overline{O(3;11)}|$ then we may take $t=0$. Otherwise let t be chosen such that

$$a|\overline{I(3;11(1)^{t+1})}|\leq 3.3b|O(3;11)|<a|\overline{I(3;11(1)^t)}|.$$

Then

$$\frac{I(3;11)}{O(3;11)}\cdot\frac{\overline{I(3;11(1)^{t+1})}}{\overline{O(3;11(1)^t)}}>4.4\cdot 0.85>3.3$$

by calculation, so $b|I(3;11)|>a|\overline{O(3;11(1)^t)}|$ and the result follows. $\square$

We will also need the following following version of Kronecker's Theorem.

Lemma 5.3. *Let θ and x be real numbers with θ irrational. Then for any number N there exists integers q and p with $p,q>N$ and*

$$|q\theta-p-x|<\frac{3}{q}.$$

Proof. See [7], Theorem 440. $\square$

Proof of Theorem 1.4. Assume that (m,n) equals (3,3). The other cases are similar. Let $\theta \in \mathbb{R}$. Assume first that $\theta > 1$ and that $\theta \notin \mathbb{Q}$. By Lemmas 5.1 and 5.2 the set $C(3) - \theta C(3)$ contains an interval $[x-\delta, x+\delta]$ for some real x and δ. By Lemma 5.3 there exist integers p and q such that $q\theta - p \in [x-\delta, x+\delta]$, so

$$\theta \in \frac{p + C(3)}{q + C(3)}$$

as required.

Next assume that $\theta = r/s$ for some relatively prime integers r and s with $r > s \geq 8$. By Lemma 5.2 there exists an integer t such that $r\overline{I(3;11(1)^t)} \approx sI(3;11)$ and $r|\overline{I(3;11(1)^t)}| > 3.3s|O(3;11)|$. By Lemma 5.1 we have

$$s\overline{I(3;11)} - rI(3;11(1)^t) \subseteq sC(3) - rC(3).$$

Further,

$$s|\overline{I(3;11)}| + r|I(3;11(1)^t)| > s\left(|\overline{I(3;11)}| + 3.3|O(3;11)|\right) > 0.14s > 1$$

hence there exists real numbers $\alpha, \beta \in C(3)$ and integers p and q such that $s\alpha - r\beta = rq - sp$ (since r and s are relatively prime). Therefore

$$\frac{r}{s} \in \frac{p + C(3)}{q + C(3)}$$

as required.

Assume next that $\theta \in (0,1)$ and either $\theta \notin \mathbb{Q}$ or $\theta = r/s$ with both r and s at least 8. Then by the above argument we have $1/\theta \in F(3)/F(3)$, hence $\theta \in F(3)/F(3)$.

Finally, we may extend our result to the negative portion of the real line. Assume that $\theta < 0$ and either $\theta \notin \mathbb{Q}$ or $\theta = -r/s$ with $r, s \geq 8$. By a process similar to the above it follows that $-\theta \in -F(3)/F(3)$, and the theorem follows. □

6. Proof of Theorem 1.5

In [3] the author used Theorem 2.1 to established the following result.

Theorem 6.1. *We have*

$$F(3) \pm F(2) \pm F(2) = \mathbb{R}.$$

We can use this result to find intervals in the product and quotients of $F(3)$, $F(2)$ and $F(2)$.

Proof of Theorem 1.5. The first part of the theorem may be proved using the techniques employed in the proofs of Theorems 6.1 and 1.3. Consider the quotient $F(3)F(2)/F(2)$, and let $\alpha = \langle\overline{1}\rangle$ and $\beta = \langle 2, \overline{1}\rangle$. By a process similar to the irrational case of the proof of Theorem 1.4 it follows that for all $x \notin \mathbb{Q}$, αx and βx are both in $F(3)F(2)/F(2)$. Let $\theta \in \mathbb{R}\backslash\{0\}$. Then since $\alpha/\beta \notin \mathbb{Q}$ either θ/α or θ/β is irrational, so $\theta \in F(3)F(2)/F(2)$ as required.

The proof is similar for the set $F(2)F(2)/F(3)$. $\square$

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