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Best simultaneous diophantine approximations of some cubic algebraic numbers

par Nicolas CHEVALLIER

Résumé. Soit α un nombre algébrique réel de degré 3 dont les conjugués ne sont pas réels. Il existe une unité ζ de l'anneau des entiers de $K = \mathbb{Q}(\alpha)$ pour laquelle il est possible de décrire l'ensemble de tous les vecteurs meilleurs approximations de $\theta = (\zeta, \zeta^2)$.

Abstract. Let α be a real algebraic number of degree 3 over $\mathbb{Q}$ whose conjugates are not real. There exists an unit ζ of the ring of integer of $K = \mathbb{Q}(\alpha)$ for which it is possible to describe the set of all best approximation vectors of $\theta = (\zeta, \zeta^2)$.

1. Introduction

In his first paper ([10]) on best simultaneous diophantine approximations J. C. Lagarias gives an interesting result which, he said, is in essence a corollary of W. W. Adams' results ([1] and [2]):
Let $[1, \alpha_1, \alpha_2]$ be a $\mathbb{Q}$ basis to a non-totally real cubic field. Then the best simultaneous approximations of $\alpha = (\alpha_1, \alpha_2)$ (see definition below) with respect to a given norm N are a subset of

$$\{q_m^{(j)} : m \geq 0,\ 1 \leq j \leq p\}$$

where the $q_m^{(j)}$ satisfy a third-order linear recurrence (with constant coefficients).

$$q_{m+3} + a_2 q_{m+2} + a_1 q_{m+1} \pm q_m = 0$$

for a finite set of initial conditions $q_0^{(j)}$, $q_1^{(j)}$, $q_2^{(j)}$, for $1 \leq j \leq p$. The fundamental unit ξ of $K = \mathbb{Q}(\alpha_1, \alpha_2)$ satisfies

$$\xi^3 - a_2 \xi^2 - a_1 \xi \pm 1 = 0.$$

Now consider the particular case $X = (\zeta, \zeta^2) \in \mathbb{R}^2$ where ζ is the unique real root of $\zeta^3 + \zeta^2 + \zeta - 1 = 0$. The vector X can be seen as a two-dimensional golden number. N. Chekhova, P. Hubert and A. Messaoudi were able to precise Lagarias' result (cf. [7]):

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There exists a euclidean norm on $\mathbb{R}^2$ such that all best diophantine approximations of X are given by the 'Tribonacci' sequence $(q_n)_{n\in\mathbb{N}}$ defined by

$$q_0 = 1,\ q_2 = 2,\ q_3 = 4,\ q_{n+3} = q_{n+2} + q_{n+1} + q_n.$$

The aim of this work is to make precise Lagarias' result in the same way as N. Chekhova, P. Hubert and A. Messaoudi.

Definition ([10],[8]). *Let N be a norm on $\mathbb{R}^2$ and $\theta \in \mathbb{R}^2$.*
1) *A strictly positive integer q is a best approximation (denominator) of θ with respect to N if*

$$\forall k \in \{1, \dots, q-1\},\ \min_{P\in\mathbb{Z}^2} N(q\theta - P) < \min_{Q\in\mathbb{Z}^2} N(k\theta - Q)$$

2) *An element $q\theta - P$ of $\mathbb{Z}\theta + \mathbb{Z}^2$ is a best approximation vector of θ with respect to N if q is a best approximation of θ and if*

$$N(q\theta - P) = \min_{Q\in\mathbb{Z}^2} N(q\theta - Q)$$

We will call $\mathcal{M}(\theta)$ the set of all best approximation vectors of θ.

Using Dirichlet's theorem it is easy to show that there exists a positive constant C depending only on the norm N, such that for all θ in $\mathbb{R}^2$ and all best approximation vectors $q\theta - P$ of θ

$$N(q\theta - P) \leq \frac{C}{q^{1/2}}.$$

If $[1, \alpha_1, \alpha_2]$ is a $\mathbb{Q}$-basis of a real cubic field then $\theta = (\alpha_1, \alpha_2)$ is badly approximable (cf. [6] p. 79):
there exists $c > 0$ such that for all best approximation vectors $q\theta - P$ of θ

$$N(q\theta - P) \geq \frac{c}{q^{1/2}}.$$

Let $\theta \in \mathbb{R}^2 \backslash \mathbb{Q}^2$ and $\Lambda = \theta\mathbb{Z} + \mathbb{Z}^2$. Endow Λ with its natural $\mathbb{Z}$-basis θ, $e_1 = (1,0)$, $e_2 = (0,1)$. For a matrix $B \in M_3(\mathbb{Z})$ and $X = x_0\theta + x_1 e_1 + x_2 e_2 \in \Lambda$, the action $BX = Y$ of B on X is naturally defined: the coordinates vector of Y is the matrix product of B by the coordinates vector of X.

We shall prove the following results.

Proposition 1. *Let a_1, $a_2 \in \mathbb{N}^*$. Suppose $P(x) = x^3 + a_2 x^2 + a_1 x - 1$ has a unique real root ζ. Call $\theta = (\zeta, \zeta^2)$ and B the matrix*

$$B = \begin{pmatrix} a_1 & -a_2 & -1 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

There exist a norm N on $\mathbb{R}^2$ and a finite number of best approximation vectors $X_i = q_i\theta - P_i$, $i = 1, \ldots, m$ such that

$$\mathcal{M}(\theta)\backslash\{B^n X_i : n \in \mathbb{N} \text{ and } i = 1, \ldots, m\}$$

is a finite set.

Proposition 2. *Suppose α is a real algebraic number of degree 3 over $\mathbb{Q}$ whose conjugates are not real. There exist a unit ζ of the ring of integer of $K = \mathbb{Q}(\alpha)$, two positive integers a_1 and a_2 and euclidean norm on $\mathbb{R}^2$ such that the set of best approximation vectors of $\theta = (\zeta, \zeta^2)$, is*

$$\mathcal{M}\,(\theta) = \{B^n\theta : n \in \mathbb{N}\}$$

where

$$B = \begin{pmatrix} a_1 & -a_2 & -1 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

The proof of Proposition 1 is quite different from Chechkova, Hubert and Messaoudi's one. It is based on two simple facts:
Let a_1, $a_2 \in \mathbb{N}^*$. Suppose $P(x) = x^3 + a_2x^2 + a_1x - 1$ has a unique real root ζ and call $\theta = \big(\zeta, \zeta^2\big)$.
1) Following G. Rauzy ([14]) we construct a euclidean norm N on $\mathbb{R}^2$ and a linear contracting similarity F on $\mathbb{R}^2$ (i.e. $N(F(x)) = rN(x)$ for all x in $\mathbb{R}^2$ where the ratio $r \in]0,1[$) which is one to one on $\Lambda = \mathbb{Z}\theta + \mathbb{Z}^2$.
2) Since $a_1, a_2 > 0$ the map F preserves the positive cone $\Lambda^+ = \mathbb{N}\theta - \mathbb{N}^2$.
We deduce from these observations that F send best approximation vectors of θ to best approximation vectors of θ (see lemma 2) and proposition 1 follow easily. Our method cannot be extended to higher dimension, because for F to be a similarity, it is necessary that P has one dominant root, all other roots being of the same modulus, and H. Minkowski proved that this can only occur for polynomials of degree 2 or 3 ([12]).
The sequence of best approximation vectors of $\theta \in \mathbb{R}^2$ may be seen as a two-dimensional continued fraction 'algorithm'. In this case Proposition 1 means that the 'development' of $\big(\zeta, \zeta^2\big)$ becomes periodic when ζ is the unique real root of the polynomial $x^3 + a_2x^2 + a_1x - 1$ with a_1, $a_2 \in \mathbb{N}$. This may be compared to the following results about Jacobi-Perron's algorithm:
(O. Perron [13]) *Let ζ be the root of $P \in \mathbb{Z}[X]$, $\deg P = 3$. If the development of $\big(\zeta, \zeta^2\big)$ by Jacobi-Perron's algorithm becomes periodic and if this development gives good approximations, i.e.*

$$\max(|q_n\zeta - p_{1,n}|, |q_n\zeta^2 - p_{2,n}|) \leq \frac{C}{q_n^{1/2}}$$

where $(p_{1,n}, p_{2,n}, q_n)_{n\in\mathbb{N}}$ are given by Jacobi-Perron's algorithm, then the conjugates of ζ are complex (see [4] p.7).

(P. Bachman [1]) *Let* $\zeta = d^{\frac{1}{3}}$ *where* d *is a cube-free integer greater than 1. If the development by Jacobi- Perron's algorithm of* (ζ, ζ^2) *turns out to be periodic it gives good approximations as above.*

(E. Dubois - R. Paysant [9]) *If* K *is a cubic extension of* $\mathbb{Q}$ *then there exist* β_1, β_2 *in* K, *linearly independent with* 1, *such that the development of* (β_1, β_2) *by Jacobi-Perron's algorithm is periodic.*

O. Perron (see [13] Theorem VII or Brentjes [5] Theorem 3.4.) also gives some numbers with a purely periodic development of length 1.
We should also note that A. J. Brentjes gives a two-dimensional continued fraction algorithm which finds all best approximations of a certain kind and he uses it to find the coordinates of the fundamental unit in a basis of the ring of integers of a non-totally real cubic field.(see Brentjes' book on multi-dimensional continued fraction algorithms [5] section 5F).
Finally, we shall give a proof of Chechkova, Hubert and Messaoudi's result using proposition 1 together with the set of best approximations corresponding to the equation $\zeta^3 + 2\zeta^2 + \zeta = 1$.

2. The Rauzy norm

Fix a_1, $a_2 \in \mathbb{N}^*$ and suppose that the polynomial $P(x) = -x^3 + a_1x^2 + a_2x + 1$ has a unique real root. Endow $\mathbb{R}^3$ with its standard basis e_1, e_2, e_3. Let M be the matrix

$$M = \begin{pmatrix} a_1 & a_2 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

The characteristic polynomial of M is $-x^3 + a_1x^2 + a_2x + 1$, the unique positive eigenvalue of M is $\lambda = \frac{1}{\zeta}$ and $\Theta = \left(\zeta, \zeta^2, \zeta^3\right)$ is the eigenvector associated with λ. Let l be the linear form on $\mathbb{R}^3$ with coefficients a_1, a_2, 1; we have $l(\Theta) = l(e_3) = 1$. Put $\Delta(X) = X - l(X)\Theta$. $\Delta \circ M$ map $\ker l$ into itself and $\mathbb{R}\Theta \subseteq \ker \Delta \circ M$. The eigenvalues of the restriction of $\Delta \circ M$ to $\ker l$, are λ_1 and $\lambda_2 = \overline{\lambda_1}$, the two other eigenvalues of M. In fact, if Z is an eigenvector of M associated to λ_1 then $\Delta(Z) \in \ker l$ and

$$\Delta \circ M \circ \Delta(Z) = \Delta(\lambda_1 Z - l(Z)\lambda\Theta) = \lambda_1 \Delta(Z).$$

Call p the projection $\mathbb{R}^3$ onto $\mathbb{R}^2$. p is one to one from $\ker l$ onto $\mathbb{R}^2$, call i its inverse map and consider the linear map

$$F : X \in \mathbb{R}^2 \to p \circ \Delta \circ M \circ i(X) \in \mathbb{R}^2.$$

The linear maps F and $\Delta \circ M$ are conjugate, therefore the eigenvalues of F are λ_1 and λ_2.

Lemma 3. *F is one to one of $\Lambda = \mathbb{Z}\theta + \mathbb{Z}^2$ on itself, where $\theta = (\zeta, \zeta^2)$.*

Proof. Since $i(\theta) = \Theta - e_3$ we have

$$F(\theta) = p \circ \Delta(\lambda\Theta - e_1) = p(l(e_1)\Theta - e_1) = a_1\theta - e_1 \in \Lambda.$$

Similarly $i(e_k) = e_k - l(e_k)e_3$, $k = 1, 2$, then $X_k = M \circ i(e_k) \in \mathbb{Z}^3$ and

$$F(e_k) = p(X_k - l(X_k)\Theta) = p(X_k) - l(X_k)\theta \in \Lambda.$$

Since F maps Λ into itself, it remains to show that F is one to one. Call B the matrix of F with respect to the basis (θ, e_1, e_2). We have

$$\begin{aligned} X_1 &= M(e_1 - l(e_1)e_3) = a_1e_1 + e_2 - l(e_1)e_1 = e_2, \\ X_2 &= M(e_2 - l(e_2)e_3) = a_2e_1 + e_3 - l(e_2)e_1 = e_3 \end{aligned}$$

so that

$$B = \begin{pmatrix} a_1 & -a_2 & -1 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

and

$$\det B = -1.$$

□

Call $\Lambda^+ = \{q\theta - P : q \in \mathbb{N} \textit{ and } P \in \mathbb{N}^2\}$. Since a_1 and a_2 are positive we have:

Corollary 4. $F(\Lambda^+) \subseteq \Lambda^+$.

Since $\lambda_2 = \overline{\lambda_1}$ there exists a euclidean norm N on $\mathbb{R}^2$ such that F is a linear similar map for this norm (i.e. $N(F(x)) = rN(x)$ for all x in $\mathbb{R}^2$, where r in $\mathbb{R}^+$ is call the ratio of F). The ratio of F is $r = |\lambda_1| = \frac{1}{\sqrt{\lambda}} = \sqrt{\zeta} < 1$. Now let us determine the matrix M of the bilinear form $\langle x, y\rangle$ associated with N, this is necessary for Proposition 2 but not for Proposition 1. M is unique up to a multiplicative constant. Since the ratio of F is $\sqrt{\zeta}$,

$$\begin{aligned} \langle F(e_1), F(e_2)\rangle &= \zeta\langle e_1, e_2\rangle, \\ \langle F(e_2), F(e_2)\rangle &= \zeta\langle e_2, e_2\rangle, \end{aligned}$$

and computing $F(e_1)$ and $F(e_2)$, we find that $\langle e_1, e_1\rangle$, $\langle e_1, e_2\rangle$ and $\langle e_2, e_2\rangle$ satisfy

$$\begin{cases} a_2\zeta\langle e_1, e_1\rangle + (-2 + 2a_2\zeta^2)\langle e_1, e_2\rangle + (-\zeta + a_2\zeta^3)\langle e_2, e_2\rangle = 0 \\ \zeta\langle e_1, e_1\rangle + 2\zeta^2\langle e_1, e_2\rangle + (\zeta^3 - 1)\langle e_2, e_2\rangle = 0. \end{cases}$$

Since $1 = a_1\zeta + a_2\zeta^2 + \zeta^3$, we find

$$\langle e_1, e_1\rangle = 2(a_1 + \zeta^2),\ \langle e_1, e_2\rangle = a_2 - \zeta,\ \langle e_2, e_2\rangle = 2.$$

3. Best diophantine approximations

We suppose $\mathbb{R}^2$ is endowed with the norm N defined in the previous section.

Notations. 1) $\rho_0 = d(0,\{(x_1,x_2) \in \mathbb{R}^2 : \sup(|x_1|,|x_2|) \geq 1\})$.

2) For $x \in \mathbb{R}$ we denote the nearest integer to x by $I(x)$ (it is well-defined for all irrational number x).

We will often use the simple fact:
Let $X = (x_1,x_2) \in \mathbb{R}^2$ and $P = (p_1,p_2) \in \mathbb{Z}^2$. If $N(X-P) < \frac{1}{2}\rho_0$ then $p_1 = I(x_1)$, $p_2 = I(x_2)$ and P is the nearest point of $\mathbb{Z}^2$ to X (for the norm N).

We will say that two best approximation vectors $q_1\theta - P_1$ and $q_2\theta - P_2$ are consecutive if q_1 and q_2 are consecutive best approximations.

Lemma 5. 1) *If $q\theta - P$ is a best approximation vector such that $N(q\theta - P) < \frac{1}{2}\rho_0$ then $q'\theta - P' = F(q\theta - P)$ is a best approximation vector of θ.*
2) *Let q_1 and q_2 be two consecutive best approximations of θ and $q_1\theta - P_1$ and $q_2\theta - P_2$ be two corresponding best approximation vectors. If $N(q_2\theta - P_2) < \frac{1}{2}\rho_0$ and if $F(q_1\theta - P_1)$ is a best approximation vector then $F(q_1\theta - P_1)$ and $F(q_2\theta - P_2)$ are consecutive best approximation vectors.*

Proof. 1) Let $Y = k'\theta - R' \in \Lambda\backslash\{(0,0)\}$ be such that $N(Y) \leq N(q'\theta - P')$. We have to prove that $|k^{'}| > q'$ or that $k'\theta - R' = \pm(q'\theta - P')$. By Lemma 1, we have $Y = F(X)$ with $X = k\theta - R \in \Lambda$. Since F is a similar map, we have $N(X) \leq N(q\theta - P)$ and by the definition of best approximations $|k| \geq q$. If $k < 0$ we can replace Y by $-Y$ so we can suppose that $k \geq q$. Since $N(X) \leq N(q\theta - P) < \frac{1}{2}\rho_0$, $R = (I(k\zeta), I(k\zeta^2))$ and $P = (I(q\zeta), I(q\zeta^2))$. The nearest integer function $x \to I(x)$ is nondecreasing so $I(k\zeta) \geq I(q\zeta)$ and $I(k\zeta^2) \geq I(q\zeta^2)$. This shows that $(k\theta - R) - (q\theta - P) \in \Lambda^+$ and by corollary 4, $F(k\theta - R) - F(q\theta - P) \in \Lambda^+$. Therefore $k' \geq q'$. If $k' = q'$, we have $R' = (I(k'\zeta), I(k'\zeta^2)) = (I(q'\zeta), I(q'\zeta^2)) = P^{'}$.
2) Put $F(q_i\theta - P_i) = k_i\theta - R_i$, $i = 1,2$. Suppose $k\theta - R$ is a best approximation vector with $k_1 < k \leq k_2$. We want to prove that $k\theta - R = k_2\theta - R_2$. Put $F^{-1}(k\theta - R) = q\theta - P$. On the one hand, since F is similar, we have $N(q\theta - P) < N(q_1\theta - P_1)$, so $q > q_1$. Furthermore q_1 and q_2 are consecutive best approximations, so $q \geq q_2$.
On the other hand, $k_1\theta - R_1 = F(q_1\theta - P_1)$ is a best approximation with $N(k_1\theta - R_1) = N(F(q_1\theta - P_1) < N(q_1\theta - P_1)$, then $k_1 \geq q_2$ and $N(k_1\theta - P_1) \leq N(q_2\theta - P_2) < \frac{1}{2}\rho_0$. Therefore $N(k_2\theta - R_2)$ and $N(k\theta - R) < \frac{1}{2}\rho_0$. It follows that

$$R = (I(k\zeta), I(k\zeta^2)),\ R_2 = (I(k_2\zeta), I(k_2\zeta^2)).$$

We have $I(k\zeta) \leq I(k_2\zeta)$ for $k \leq k_2$. Using the matrix B we see that $R = (q, .)$ and $R_2 = (q_2, .)$. This shows $q \leq q_2$ and $q = q_2$, which implies $q\theta - P = q_2\theta - P_2$ and $k\theta - R = k_2\theta - R_2$. □

The increasing sequence of all best approximations of θ will be denoted by $(q_n)_{n\in\mathbb{N}}$ $(q_0 = 1)$.

Proposition 6. *If $q_{n_0}\theta - P_{n_0}, \ldots, q_{n_0+m}\theta - P_{n_0+m}$ are (consecutive) best approximation vectors such that $F(q_{n_0}\theta - P_{n_0}) = q_{n_0+m}\theta - P_{n_0+m}$ and $N(q_{n_0+1}\theta - P_{n_0+1}) < \frac{1}{2}\rho_0$, then for all $j \geq 0$ and all $k \in 0, \ldots, m-1$,*

$$q_{n_0+jm+k}\theta - P_{n_0+jm+k} = F^j(q_{n_0+k}\theta - P_{n_0+k}).$$

Proof. Put $V_n = q_n\theta - P_n$. The previous lemma shows that $F(V_{n_0+k})$, $k = 0, \ldots, m$, are consecutive best approximation vectors. By induction on $j \geq 0$, we see that $F^j(V_{n_0+k}) = V_{n_0+jm+k}$, $k = 0, \ldots, m$ are consecutive best approximation vectors and $F(V_{n_0+jm}) = V_{n_0+(j+1)m}$. □

Proof of Proposition 1. Since $\lim_{n\to\infty} \min_{P\in\mathbb{Z}^2} N(q_n\theta - P) = 0$, there exists an integer n_0 such that for each $n \geq n_0$, $N(q_n\theta - P_n) < \frac{1}{2}\rho_0$. By Lemma 4, 1), $F(q_{n_0}\theta - P_{n_0})$ is a best approximation vector and Proposition 1 follows of Proposition 6.

4. Proof of Proposition 2

Lemma 7. *Let $P \in \mathbb{Q}$ be an irreducible polynomial of degree 3 with a unique real root α and $K = \mathbb{Q}(\alpha)$. There exist infinitely many $\lambda \in K$ such that*

i) $\lambda > 1$

ii) *λ is a root of $Q(x) = x^3 - a_1x^2 - a_2x - 1$*

iii) *$a_1,\ a_2 \in \mathbb{N}$ and $3a_1 \geq a_2^2$.*

Proof. Since P has a unique real root, Dirichlet's theorem shows that the group of unit of the integral ring of K contains an abelian free sub-group G of rank 1. Let $\xi \neq 1$ be in G. We can suppose $\xi > 1$ and the norm $N_K(\xi) = 1$. The conjugates of ξ are not real because those of α are not. Call γ and $\overline{\gamma}$ these conjugates. We have $\xi\gamma\overline{\gamma} = 1$ and $|\gamma| < 1$ since the norm of ξ is 1 and $\xi > 1$. We will show that $\lambda = \xi^m$ satisfy i), ii) and iii) for infinitely many $m \in \mathbb{N}$.

The minimal polynomial of λ is $Q(x) = x^3 - a_1x^2 - a_2x - 1$ with

$$\begin{array}{rclcl} a_1 & = & a_1(m) & = & \xi^m + \gamma^m + \overline{\gamma}^m \\ a_2 & = & a_2(m) & = & -[\xi^m(\gamma^m + \overline{\gamma}^m) + |\gamma|^{2m}] \end{array}$$

Since $\xi > 1 > |\gamma|$, a_1 is positive for m large and a_2 will be positive if the argument of γ is well chosen. Call α the argument of γ and $\rho = \frac{1}{\sqrt{\xi}}$ its modulus.

First case. $\frac{\alpha}{2\pi} \notin \mathbb{Q}$.
There exist infinitely many $m \in \mathbb{N}$ such that $m\alpha \in [\frac{2\pi}{3}, \frac{4\pi}{5}] \bmod 2\pi$. Call I the set of such m. For $m \in I$

$$\begin{aligned} a_1(m) &= \xi^m + \frac{2}{\xi^{\frac{m}{2}}}\cos m\alpha \\ a_2(m) &= -2\xi^{\frac{m}{2}}\cos m\alpha - \frac{1}{\xi^m} \geq -2\xi^{\frac{m}{2}}\cos\frac{2\pi}{3} - \frac{1}{\xi^m} \end{aligned}$$

then

$$\lim_{m\to\infty,\, m\in I} a_1(m) = \lim_{m\to\infty,\, m\in I} a_2(m) = +\infty.$$

Moreover,

$$a_2(m) \leq -2\xi^{\frac{m}{2}}\cos\frac{4\pi}{5} - \frac{1}{\xi^m}$$

then

$$\liminf_{m\to\infty,\, m\in I} \frac{a_1(m)}{a_2^2(m)} \geq \frac{1}{4\cos^2\frac{4\pi}{5}} > \frac{1}{3}.$$

Therefore the conditions i), ii) and iii) are satisfied for large m in I.
Second case. $\frac{\alpha}{2\pi} = \frac{p}{q} \in \mathbb{Q}$.
Since $\gamma \notin \mathbb{R}$, $q > 2$. First note that $q \neq 4$ for, if $q = 4$, we have

$$\begin{aligned} 0 &= \mathrm{Re}(\gamma^3 - a_1\gamma^2 - a_2\gamma - 1) &= a_1\rho^2 - 1 \\ 0 &= \mathrm{Im}(\gamma^3 - a_1\gamma^2 - a_2\gamma - 1) &= \pm\rho(\rho^2 + a_2) \end{aligned}$$

so $a_1 = -a_2 = \rho = 1$ and $\gamma = \pm i$. This is impossible because the degree of the minimal polynomial of γ is 3. So $q \in \{3\} \cup \{5, 6, \dots\}$. If $q = 3, 5$ or 6, it is easy to see that there exist infinitely many $m \in \mathbb{N}$ such that $m\alpha \in [\frac{4\pi}{5} - \frac{2\pi}{7}, \frac{4\pi}{5}] \bmod 2\pi$ while a similar conclusion is obvious if $q \geq 7$. Now, we can conclude as in the previous case for $\frac{4\pi}{5} - \frac{2\pi}{7} > \frac{\pi}{2}$. □

From now on, $a_1, a_2 \geq 1$ are two integers such that $P(x) = -1 + a_1x + a_2x^2 + x^3$ has a unique real root ζ. We use the notations of Sections 2 and 3, the norm N as defined in Section 2 and ρ_0 as defined at the beginning of Section 3.

Lemma 8.

$$\rho_0^2 \geq \frac{4a_1 - a_2^2 + 2a_2\zeta + 3\zeta^2}{2(a_1 + \zeta^2)}$$

Proof. By definition

$$\rho_0^2 \geq \min(\min_{x\in\mathbb{R}} N^2(e_1 + xe_2), \min_{x\in\mathbb{R}} N^2(e_2 + xe_1)).$$

We have

$$N^2(e_1 + xe_2) = \langle e_1, e_1\rangle + 2x\langle e_1, e_2\rangle + x^2\langle e_2, e_2\rangle$$

then

$$\min_{x\in\mathbb{R}} N^2(e_1 + xe_2) = \langle e_1, e_1\rangle - \frac{\langle e_1, e_2\rangle^2}{\langle e_2, e_2\rangle} = \frac{4(a_1+\zeta^2) - (a_2-\zeta)^2}{2}$$

similarly

$$\min_{x\in\mathbb{R}} N^2(e_2 + xe_1) = \langle e_2, e_2\rangle - \frac{\langle e_1, e_2\rangle^2}{\langle e_1, e_1\rangle} = \frac{4(a_1+\zeta^2) - (a_2-\zeta)^2}{2(a_1+\zeta^2)},$$

and since $a_1 \geq 1$,

$$\rho_0^2 \geq \frac{4a_1 - a_2^2 + 2a_2\zeta + 3\zeta^2}{2(a_1+\zeta^2)}.$$

□

Lemma 9. *Suppose a_1 and a_2 satisfy condition iii) of Lemma 7. For a_1 sufficiently large, $N(\theta) \leq \frac{1}{2}\rho_0$ and θ is a best approximation vector of θ .*

Proof. Put $\phi(a_1, a_2) = \frac{4a_1 - a_2^2 + 2a_2\zeta + 3\zeta^2}{2(a_1+\zeta^2)}$. We have

$$\lim_{a_1\to\infty} \zeta(a_1, a_2) = 0$$

whereby

$$\lim_{\substack{a_1\to\infty \\ 3a_1\geq a_2^2}} \phi(a_1, a_2) \geq \frac{1}{2}$$

and so

$$N^2(\theta) = N^2(F(e_2)) = 2\zeta < \frac{1}{4}\phi(a_1, a_2) \leq \frac{1}{4}\rho_0^2$$

for a_1 sufficiently large. Now if $P \in \mathbb{Z}^2\backslash\{(0,0)\}$, then $N(\theta - P) \geq N(P) - N(\theta) \geq \frac{1}{2}\rho_0$. □

Lemma 10. *If $q \in \{0, \ldots, a_1 - 1\}$ then $N(q\theta - e_1) > N(\theta)$.*

Proof.

$$N^2(q\theta - e_1) > N^2(\theta)$$

$$\begin{aligned}
&\Leftrightarrow (q^2-1)\langle\theta,\theta\rangle - 2q\langle\theta, e_1\rangle + \langle e_1, e_1\rangle > 0\\
&\Leftrightarrow (q^2-1)\langle F(e_2), F(e_2)\rangle - 2q[2(a_1+\zeta^2)\zeta + (a_2-\zeta)\zeta^2] + 2(a_1+\zeta^2) > 0\\
&\Leftrightarrow 2(q^2-1)\zeta - 2q(a_1\zeta + 1) + 2(a_1+\zeta^2) > 0\\
&\Leftrightarrow a_1 - q + (q^2 - 1 - a_1 q)\zeta + \zeta^2 > 0\\
&\Leftrightarrow (a_1 - q)(a_1\zeta + a_2\zeta^2 + \zeta^3) + (q^2 - 1 - a_1 q)\zeta + \zeta^2 > 0\\
&\Leftrightarrow q^2 + a_1^2 - 2a_1 q - 1 + a_2(a_1 - q)\zeta + (a_1 - q)\zeta^2 > 0.
\end{aligned}$$

□

Lemma 11. *Suppose a_1 and a_2 satisfy condition iii) of Lemma 7. For a_1 sufficiently large, θ and $a_1\theta - e_1$ are the first two best approximation vectors.*

Proof. Since $a_1\theta - e_1 = F(\theta)$, the only thing to prove is

$$\inf_{q\in\{2,\dots,a_1-1\}} \inf_{P\in\mathbb{Z}^2} N(q\theta - P) > N(\theta).$$

If $N(q\theta - P) \leq \frac{1}{2}\rho_0$, then by definition of ρ_0

$$\begin{aligned} |q\zeta - p_1| &\leq \frac{1}{2} \\ |q\zeta^2 - p_2| &\leq \frac{1}{2} \end{aligned}$$

where $P = (p_1, p_2)$. Furthermore, if $q < a_1$ and if a_1 is large, then $q\zeta \leq 1$ and $q\zeta^2 \leq \frac{1}{2}$. Therefore ,

$$\begin{aligned} \inf_{P\in\mathbb{Z}^2} N(q\theta - P) &= \inf(N(q\theta), N(q\theta - e_1)) \\ &\geq \inf(qN(\theta), N(q\theta - e_1)) > N(\theta) \end{aligned}$$

for $q \in \{2, \dots, a_1 - 1\}$. $\square$

End of proof of Proposition 2. By Lemma 7 there exists a unit $\lambda \in \mathbb{Q}(\alpha)$ which satisfies conditions i), ii) and iii) with a_1 large. $\zeta = \frac{1}{\lambda}$ is also unit. By Lemma 9, $\theta = (\zeta, \zeta^2)$ is a best approximation vector and by Lemma 11, $F(\theta) = a_1\theta - e_1$ is the next best approximation vector. Since $N(a_1\theta - e_1) < N(\theta) < \frac{1}{2}\rho_0$, by Proposition 6 we have $\mathcal{M}(\theta) = \{F^n(\theta) : n \in \mathbb{N}\}$.

5. The equations $1 = x^3 + a_2x^2 + x$

The polynomial $P(x) = x^3 + a_2x^2 + x - 1$ has only one real root if $a_2 = 1$ or 2.

5.1. $a_2 = 1$. Call ζ the positive root of $1 = x^3 + x^2 + x$ and $\theta = (\zeta, \zeta^2)$. N. Chekhova, P. Hubert, A. Messaoudi have proved that $\mathcal{M}(\theta) = \{F^n(\theta - e_1) : n \in \mathbb{N}\}$. If we want to recover this result with Proposition 6, we just have to show:
i) $\theta - e_1$ is a best approximation vector,
ii) $F(\theta - e_1)$ is the next best approximation vector,
iii) $N(F(\theta - e_1)) < \frac{1}{2}\rho_0$.
First note that $F(\theta - e_1) = 2\theta - e_1 - e_2$ and $N(F(\theta - e_1)) = \zeta N(\theta - e_1) < N(\theta - e_1)$, so if i) is true then 2 is the next best approximation and if iii) is also true, then $2\theta - e_1 - e_2$ is a best approximation vector. Let us now prove iii) and afterward i):

$$N^2(F(\theta - e_1)) = N^2(F^3(e_2)) = 2\zeta^3 < \frac{3 + 2\zeta + 3\zeta^2}{8(1 + \zeta^2)} \leq \frac{1}{4}\rho_0^2$$

for

$$
\begin{aligned}
2\zeta^3 \quad &< \quad \frac{3+2\zeta+3\zeta^2}{8(1+\zeta^2)} \\
&\Leftrightarrow \quad 3+2\zeta+3\zeta^2-16\zeta^3(1+\zeta^2)>0 \\
&\Leftrightarrow \quad 3(\zeta+\zeta^2+\zeta^3)+2\zeta+3\zeta^2-16\zeta^3(1+\zeta^2)>0 \\
&\Leftrightarrow \quad 5+6\zeta-13\zeta^2-16\zeta^4>0 \\
&\Leftrightarrow \quad 11-8\zeta+5\zeta^2-16\zeta^3>0 \\
&\Leftrightarrow \quad 3+16\zeta-5\zeta^2>0
\end{aligned}
$$

and the last inequality is obvious. Since $\zeta > \frac{1}{2}$, $2\zeta^3 < \frac{1}{4}\rho_0^2 \Rightarrow N^2(\theta - e_1) = 2\zeta^2 < \frac{1}{2}\rho_0^2 \le \rho_0^2$. Then the point $P = (p_1, p_2) \in \mathbb{Z}^2$ which is the nearest to θ, is one of $(0,0)$, e_1, e_2 or $e_1 + e_2$. We have

$$N^2(\theta - e_1) = \zeta N^2(\theta) < N^2(\theta)$$

and

$$
\begin{aligned}
N^2(\theta-e_2) &= N^2(\theta) - 2\langle\theta, e_2\rangle + 2 = 2\zeta - 2\zeta(1-\zeta) - 4\zeta^2 + 2 \\
&= 2(1-\zeta^2) > 2\zeta^2 = N(\theta - e_1), \\
N^2(\theta-e_1-e_2) &= N^2(\theta-e_1) - 2\langle\theta-e_1, e_2\rangle + 2 \\
&= 2\zeta^2 - 2\zeta(1-\zeta) - 4\zeta^2 + 2\langle e_1, e_2\rangle + 2 \\
&= \zeta^2 - 2\zeta(1-\zeta) - 4\zeta^2 + 2(1-\zeta) + 2 = 4 - 4\zeta > 2\zeta^2,
\end{aligned}
$$

so P must be e_1 and this completes the proof of i).

5.2. $a_2 = 2$. Call ζ the positive root of $1 = x^3 + 2x^2 + x$ and $\theta = \big(\zeta, \zeta^2\big)$. The set of all best approximations is given by two initial points

$$\mathcal{M}(\theta) = \{B^n X_i : n \in \mathbb{N},\ i = 1, 2\}$$

where $X_1 = \theta$ and $X_2 = 2\theta - e_1$. To prove this result, by Proposition 6, we have to check the following properties:
i) $\theta - e_1$ is the best approximation vector,
ii) $2\theta - e_1$ is the next best approximation vector,
iii) $F(\theta - e_1) = 3\theta - e_1$, $F(2\theta - e_1) = 4\theta - 2e_1 - e_2$,
iv) $N(3\theta - e_1) < \frac{1}{2}\rho_0$.
This requires some tedious calculations very similar to the case $a_2 = 1$.

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Nicolas CHEVALLIER
Université de Haute-Alsace
4, rue des frères Lumière
68093 Mulhouse Cedex, France
E-mail : n.chevallier@uha.fr