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Digital expansion of exponential sequences

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Résumé. On s'intéresse au développement en base q des N premiers termes de la suite exponentielle a^n. En utilisant un résultat dû à Kiss et Tichy, nous montrons que le nombre moyen d'occurrences d'un bloc de chiffres donné est égal asymptotiquement à sa valeur supposée. Sous une hypothèse plus forte nous montrons un résultat similaire en ne considérant seulement les $(\log N)^{\frac{3}{2}-\epsilon}$, avec $\epsilon > 0$, premiers termes de la suite a^n.

Abstract. We consider the q-ary digital expansion of the first N terms of an exponential sequence a^n. Using a result due to Kiss and Tichy [8], we prove that the average number of occurrences of an arbitrary digital block in the last $c \log N$ digits is asymptotically equal to the expected value. Under stronger assumptions we get a similar result for the first $(\log N)^{\frac{3}{2}-\epsilon}$ digits, where ϵ is a positive constant. In both methods, we use estimations of exponential sums and the concept of discrepancy of real sequences modulo 1 plays an important role.

1. Introduction

In this paper, we write $\mathbb{N}, \mathbb{Z}, \mathbb{R}$ for the sets of positive integers, integers, and real numbers. With $\mathbb{P}$, we denote the set of primes and for an element of $\mathbb{P}$ we usually write p. For a real number x, we use the standard notations $e(x) = e^{2\pi i x}$, $\{x\}$ for the fractional part of x, and $\|x\|$ for the distance from x to the nearest integer.

Let $q \geq 2$ be an integer. We consider for $n \in \mathbb{N}$ the q-ary digital expansion

$$n = \sum_{i \geq 0} d_i(n) q^i, \quad 0 \leq d_i(n) \leq q-1, \forall i. \tag{1}$$

We are going to introduce further notations, which we use throughout this paper. We start with

$$B_q(n) := \#\{i \geq 1 \mid d_i(n) \neq d_{i-1}(n)\}, \tag{2}$$

which is the number of changes of digits (or the number of blocks) in the digital expansion of n. Furthermore, we write for arbitrary digits $e_0, e_1, \cdots, e_s$

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with $s \geq 0, 0 \leq e_i \leq q-1, 0 \leq i \leq s$, not all digits equal to 0 and integers $a, b \geq 0$

$$B_{q,a,b}(n; e_s e_{s-1} \cdots e_0) := \#\{a \leq i \leq b \mid d_{i-s+j}(n) = e_j, 0 \leq j \leq s\} \tag{3}$$

for the number of occurrences of the digital block $e_s e_{s-1} \cdots e_0$ in the digital expansion of n between a and b. (If $a < s$ then we start with $i = s$ and if we omit a and b then we assume $i \geq s$.) If we use the word digital block, we always assume that at least one digit is not equal to zero. Finally, we use the well known notation of

$$S_q(n) := \sum_{i \geq 0} d_i(n) \tag{4}$$

for the *sum-of-digits* function.

In this paper, we consider the q-ary expansion of an exponential sequence a^n, where $a \geq 2$ is an integer. In a recent work Blecksmith, Filaseta, and Nicol [5] proved the following result:

$$\log_a q \in \mathbb{R}\backslash\mathbb{Q} \Longrightarrow \lim_{n \longrightarrow \infty} B_q(a^n) = \infty.$$

Later Barat, Tichy, and Tijdeman [3] gave a quantitative version of the above result, by applying Baker's theorem on linear forms in logarithm (see for instance [1] or [2]). They proved the following result:

Theorem 1. *Let a and q be integers both ≥ 2. Assume that $\log_a q$ is irrational. Then there exist effectively computable constants c_0 and n_0, where c_0 is a positive real number and n_0 is an integer, such that*

$$B_q(a^n) > c_0 \frac{\log n}{\log \log n}$$

for all $n \geq n_0$.

Clearly, as a consequence of this result, we obtain the same lower bound for the sum-of-digits function $S_q(a^n)$ and for the *mean value* of the sum-of-digits function of an exponential sequence.

Corollary 1. *Let q, a be as in Theorem 1. Then we have, as $N \longrightarrow \infty$,*

$$\frac{1}{N} \sum_{n=1}^{N} S_q(a^n) \gg \frac{\log N}{\log \log N}.$$

One aim of this paper is to improve this lower bound. More generally we are interested in the behaviour of the following mean value

$$\frac{1}{N} \sum_{n=1}^{N} B_q(a^n; e_s e_{s-1} \cdots e_0) \tag{5}$$

where $e_s e_{s-1} \cdots e_0$ is an arbitrary digital block. Of course, results about the behaviour of (5) imply results about other interesting mean values, e.g., the mean value of the sum-of-digits function and the mean value of the number of changes of digits.

First, we consider only the last digits in the digital expansion of the exponential sequence. By using a result due to Kiss and Tichy [8], we can prove that the average number of occurrences of an arbitrary digital block is, except of a bounded error term, asymptotically equal to the expected value. In detail the following theorem holds:

Theorem 2. *Let a, q be integer both ≥ 2 such that $\log_a q$ is irrational. We consider a digital block $e_s e_{s-1} \cdots e_0$ with $s \geq 0, 0 \leq e_i \leq q-1, 0 \leq i \leq s$. There exists a positive real constant γ, such that we have, as $N \longrightarrow \infty$,*

$$\frac{1}{N}\sum_{n=1}^{N} B_{q,u(n),v(n)}(a^n; e_s e_{s-1} \cdots e_0) = \frac{\gamma}{q^{s+1}} \log_q N + O(1),$$

with

$$u(n) = \left[n \log_q a - \gamma \log_q N\right], \quad v(n) = \left[n \log_q a\right].$$

As an easy consequence, we can remove the $\log\log N$ factor in the lower bound of Corollary 1.

Corollary 2. *Let a, q and $e_s e_{s-1} \cdots e_0$ be as in Theorem 2. Then we have, as $N \longrightarrow \infty$,*

$$\frac{1}{N}\sum_{n=1}^{N} B_q(a^n; e_s e_{s-1} \cdots e_0) \gg \log N$$

and consequently

$$\frac{1}{N}\sum_{n=1}^{N} B_q(a^n) \gg \log N$$

and

$$\frac{1}{N}\sum_{n=1}^{N} S_q(a^n) \gg \log N.$$

Next, we consider the first digits. Here it seems to be more convenient to use the stronger assumption $(a, q) = 1$, instead of $\log_a q \in \mathbb{R}\backslash\mathbb{Q}$. Then, we are able to prove a result similar to Theorem 2 for the first $\log N$ digits, but such a result yields no improvement of the lower bounds of the mean values considered in Corollary 2. Therefore, we don't state it, but we are going to state a stronger result, which follows similarly but under stronger assumptions, namely that q is a prime:

Theorem 3. *Let $a \geq 2$ be an integer and $p \in \mathbb{P}$ a prime with $(a, p) = 1$. We consider a digital block $e_s e_{s-1} \cdots e_0$ with $s \geq 0$ and $0 \leq e_i \leq p -$*

$1, 0 \le i \le s$. *Furthermore let* ϵ, η *be arbitrary positive real numbers and* $A_1(N), A_2(N)$ *positive integer-valued functions with*

$$[(\log_p N)^\eta] \le A_1(N) < A_2(N) \le [(\log_p N)^{\frac{3}{2}-\epsilon}].$$

Then we have for a positive real number λ, *as* $N \longrightarrow \infty$,

$$\frac{1}{N}\sum_{n=1}^{N} B_{p,A_1(N)+1,A_2(N)}(a^n; e_s e_{s-1}\cdots e_0) = \frac{1}{p^{s+1}}(A_2(N) - A_1(N)) + O\left(\frac{1}{\log^\lambda N}\right).$$

Again we have the following simple consequence:

Corollary 3. *Let* a, p, *and* $e_s e_{s-1}\cdots e_0$ *be as in Theorem 3 and* ϵ *an arbitrary positive real number. Then we have, as* $N \longrightarrow \infty$,

$$\frac{1}{N}\sum_{n=1}^{N} B_p(a^n; e_s e_{s-1}\cdots e_0) \gg (\log N)^{\frac{3}{2}-\epsilon}$$

and consequently

$$\frac{1}{N}\sum_{n=1}^{N} B_p(a^n) \gg (\log N)^{\frac{3}{2}-\epsilon}$$

and

$$\frac{1}{N}\sum_{n=1}^{N} S_p(a^n) \gg (\log N)^{\frac{3}{2}-\epsilon}.$$

The paper is organized as follows: in Section 2, we prove Theorem 2 and in Section 3 Theorem 3. In the final section, we make some remarks.

2. Proof of the Theorem 2

In this section, we use the following notation: with a and q we denote two integers both ≥ 2. We define $\alpha := \log_a q$ and assume that α is irrational.

First, we need the well-known concept of discrepancy (see [6]):

Definition 1. *Let* $(x_n)_{n\ge 1}$ *be a sequence of real numbers and* $N \ge 1$. *Then the* N*-th discrepancy of the sequence* x_n *is defined by*

$$D_N(x_n) = \sup_{[a,b)\subseteq[0,1)} \left| \frac{1}{N}\sum_{n=1}^{N} \chi_{[a,b)}(\{x_n\}) - (b-a) \right| \tag{6}$$

where $\chi_{[a,b)}$ *is the characteristic function of the set* $[a,b)$.

Our first Lemma is a famous inequality for the discrepancy, which is due to Erdős and Turán [7].

Lemma 1. *Let $(x_n)_{n\geq 1}$ be a sequence of real numbers and $N \geq 1$. Then we have*

$$D_N(x_n) \leq c\left(\sum_{h=1}^{K}\frac{1}{h}\left|\frac{1}{N}\sum_{n=1}^{N}e(hx_n)\right| + \frac{1}{K}\right) \tag{7}$$

for any positive integer K. The constant c is absolute.

Next, we need a result, which is a special case of a more general result due to Kiss and Tichy [8]. The proof follows by using the Erdős-Turán inequality together with Baker's theorem on linear forms in logarithm.

Lemma 2. *There exists a positive real constant γ such that*

$$D_N(-\alpha n) \ll N^{-\gamma}.$$

The last ingredient is a very simple fact, but it is one of the key ideas of the proofs of Theorem 2 and Theorem 3.

Lemma 3. *Let $n \in \mathbb{N}$ and we consider the q-ary digital expansion (1) of n. Let $e_s e_{s-1}\cdots e_0$ be a digital block and put $m = \sum_{i=0}^{s} e_i q^i$. Then for all $k \geq s$ we have*

$$d_{k-s+j}(n) = e_j, 0 \leq j \leq s \Longleftrightarrow \left\{\frac{n}{q^{k+1}}\right\} \in \left[\frac{m}{q^{s+1}}, \frac{m+1}{q^{s+1}}\right[. \tag{8}$$

Now, we are able to prove Theorem 2.

Proof of Theorem 2. Let $K = \left[\frac{N}{\alpha}\right]$, $l \leq [N^\gamma]$ be a positive integer, where γ is the constant in Lemma 3, and put $m = \sum_{i=0}^{s} e_i q^i$. We consider

$$A_l = \#\left\{(n,k)|1 \leq n \leq N, s \leq k \leq K : l + \frac{m}{q^{s+1}} \leq \frac{a^n}{q^{k+1}} < l + \frac{m+1}{q^{s+1}}\right\}.$$

It is easy to see that

$$\begin{aligned} A_l &= \#\Big\{1 \leq k \leq K - s + 1 | \exists n : 1 \leq n \leq N \\ &\qquad \frac{\log\left(l + \frac{m}{q^{s+1}}\right)}{\log a} \leq n - (k+s)\alpha < \frac{\log\left(l + \frac{m+1}{q^{s+1}}\right)}{\log a}\Big\} \\ &= \#\{1 \leq k \leq \tilde{K} | \{-(k+s)\alpha\} \in I\} + \mathrm{O}(u_2^{(l)} - u_1^{(l)}), \end{aligned}$$

where $\tilde{K} = K - c\log l - s + 1$ with a suitable constant c and I is either $[\{u_1^{(l)}\}, \{u_2^{(l)}\}[$ or $[0, \{u_2^{(l)}\}[\cup[\{u_1^{(l)}\}, 1[$, where

$$u_1^{(l)} = \frac{\log\left(l + \frac{m}{q^{s+1}}\right)}{\log a}, \quad u_2^{(l)} = \frac{\log\left(l + \frac{m+1}{q^{s+1}}\right)}{\log a}.$$

We use now the definition of discrepancy (6) and it follows

$$A_l = \tilde{K}(u_2^{(l)} - u_1^{(l)}) + \mathrm{O}(\tilde{K}D_{\tilde{K}}(-(k+s)\alpha)) + \mathrm{O}(u_2^{(l)} - u_1^{(l)}).$$

Applying Lemma 3 yields

$$A_l = \tilde{K}(u_2^{(l)} - u_1^{(l)}) + \mathrm{O}(N^{-\gamma+1}) + \mathrm{O}(u_2^{(l)} - u_1^{(l)}).$$

Next, we consider

$$\sum_{l=1}^{[N^\gamma]} A_l = \sum_{l=1}^{[N^\gamma]} \tilde{K}(u_2^{(l)} - u_1^{(l)}) + \mathrm{O}(N) + \sum_{l=1}^{[N^\gamma]} \mathrm{O}(u_2^{(l)} - u_1^{(l)})$$

and it is an easy calculation that

$$\sum_{l=1}^{[N^\gamma]} (u_2^{(l)} - u_1^{(l)}) = \frac{\gamma \log N}{q^{s+1} \log a} + \mathrm{O}(1),$$

and

$$\sum_{l=1}^{[N^\gamma]} \log l (u_2^{(l)} - u_1^{(l)}) = \frac{(\gamma \log N)^2}{q^{s+1} \log a} + \mathrm{O}(1).$$

Therefore, we have

$$\sum_{l=1}^{[N^\gamma]} A_l = \frac{\gamma N}{q^{s+1}} \log_q N + \mathrm{O}(N) \tag{9}$$

In the sum on the left hand side of (9), we count all tuples $(n,k), 1 \leq n \leq N, s \leq k \leq K$, such that the following condition holds

$$l + \frac{m}{q^{s+1}} \leq \frac{a^n}{q^{k+1}} < l + \frac{m+1}{q^{s+1}},$$

where l is an integer with $1 \leq l \leq [N^\gamma]$. If we fix n, then, the above inequality implies

$$\max\Big\{\Big[n \log_q a - \log_q\Big([N^\gamma] + \frac{m+1}{q^{s+1}}\Big)\Big], s\Big\} \leq k$$
$$\leq \Big[n \log_q a - \log_q\Big(1 + \frac{m}{q^{s+1}}\Big)\Big] - 1$$

and Theorem 2 follows from (9). □

3. Proof of Theorem 3

In this section $a \geq 2$ is an integer and $p \in \mathbb{P}$ denotes a prime with $(a,p) = 1$.

Let k be a positive integer. With $\tau(p^k)$, we denote the multiplicative order of a mod p^k. For $\tau(p)$ we write just τ. If p is odd then, we denote by β the smallest number such that $p^\beta | a^\tau - 1$. If $p = 2$ then, we set $\delta = 1$ if $a \equiv 1 \bmod 4$ and $\delta = 2$ if $a \equiv 3 \bmod 4$. In this case β is the smallest number such that $2^\beta | a^\delta - 1$. This number β has the following property:

Lemma 4. *Let a, p and β be as above. For all integers $n > \beta$ we have*

$$\tau(p^n) = p\tau(p^{n-1}).$$

Proof. See [9]. $\square$

For the proof of Theorem 3, we need estimations for special exponential sums. The first Lemma is a special case of a result, which is due to Niederreiter [13].

Lemma 5. *Let $k \geq 2, h$ be integers and $(h, p) = 1$. Assume that $\tau(p^k) = p\tau(p^{k-1})$. Then it follows*

$$\sum_{n=1}^{\tau(p^k)} e\left(h\frac{a^n}{p^k}\right) = 0.$$

The next result is due to Korobov (see [10] or [11]).

Lemma 6. *Let $m \geq 2, h$ be integers with $(a, m) = 1$ and $(h, m) = 1$. Let τ be the multiplicative order of $a \mod m$. Then we have for $1 \leq N \leq \tau$*

$$\left|\sum_{n=1}^{N} e\left(h\frac{a^n}{m}\right)\right| \leq \sqrt{m}(1 + \log\tau).$$

We will apply this Lemma for the special case $m = p^k$. Notice that this lemma provides only a good estimation when N is not too small. We also need good estimations for very small N. The best known result in this direction is again due to Korobov (see [10] or [11]).

Lemma 7. *Let $k \geq 1, h$ be integers and $(h, p) = 1$. Then for all integers N with $N \leq \tau(p^k)$ we have*

$$\left|\sum_{n=1}^{N} e\left(h\frac{a^n}{p^k}\right)\right| \ll N\exp\left(-\gamma\frac{\log^3 N}{\log^2 p^k}\right), \tag{10}$$

where $\gamma > 0$ is an absolute constant and the implied constant depends only on a and p.

If n is a positive integer then we write in the following for the p-ary digital expansion of a^n:

$$a^n = \sum_{i\geq 0} d_i(a^n)p^i.$$

We prove now the following Lemma:

Lemma 8. *Let $e_s e_{s-1} \cdots e_0$ be a digital block to base p. Let $\epsilon, \eta > 0$ be given and N, k be positive integers such that*

$$(\log_p N)^\eta < k \leq (\log_p N)^{\frac{3}{2}-\epsilon}, \tag{11}$$

We consider

$$A_k = \#\{1 \le n \le N | d_{k-s+j}(a^n) = e_j, 0 \le j \le s\}.$$

Then we have for an arbitrary positive real number λ, as $N \longrightarrow \infty$,

$$A_k = \frac{N}{p^{s+1}} + O\left(\frac{N}{\log^\lambda N}\right)$$

and this holds uniformly for k with (11).

Proof. Put $m = \sum_{i=0}^s e_i p^i$. We use (8) and obtain

$$A_k = \#\left\{1 \le n \le N | \left\{\frac{a^n}{p^{k+1}}\right\} \in \left[\frac{m}{p^{s+1}}, \frac{m+1}{p^{s+1}}\right[\right\}.$$

With the definition of discrepancy (6) it follows

$$A_k = \frac{N}{p^{s+1}} + \mathrm{O}\left(ND_N\left(\frac{a^n}{p^{k+1}}\right)\right), \tag{12}$$

where the implied O-constant is 1. In order to get the desired result, we have to estimate the discrepancy on the right hand side. Therefore, we use once more inequality (7).

Let $1 < \delta < 2$ be a real number. We distinguish between two cases.

First we consider k with

$$\delta \log_p N < k \le (\log_p N)^{\frac{3}{2}-\epsilon} \tag{13}$$

Let $\lambda > 0$ be a real number and $h \le \log^\lambda N$ be a positive integer. First, we observe for large enough N

$$\frac{p^{k+1}}{h} \ge \frac{N^\delta}{\log^\lambda N} \ge p^{\beta+1}N = p^{\log_p N+1+\beta} \ge p^{[\log_p N]+1+\beta},$$

where β is the integer introduced in the beginning of the section. We use Lemma 5 and it follows

$$\tau(p^{[\log_p N]+1+\beta}) = p^{[\log_p N]+1}\tau(p^\beta) \ge N \tag{14}$$

if N is large enough. Because of (14) we can estimate the exponential sum in inequality (7) with help of Lemma 8 for $h \le \log^\lambda N$. It follows

$$\left|\sum_{n=1}^N e\left(h\frac{a^n}{p^{k+1}}\right)\right| \le cN\exp\left(-\gamma\frac{\log^3 N}{\log^2 p^{k+1}}\right),$$

where c depends only on a, p and γ is absolute. With (13) we can estimate the right hand side of the above inequality

$$\exp\left(-\gamma\frac{\log^3 N}{(k+1)^2\log^2 p}\right) \le \exp\left(-\bar{\gamma}(\log_p N)^{2\epsilon}\right),$$

where $\bar{\gamma}$ is a suitable constant.

Now we can finish the proof of the first case. We consider

$$D_N\left(\frac{a^n}{p^{k+1}}\right) \ll \frac{1}{N}\sum_{h=1}^{K}\frac{1}{h}\left|\sum_{n=1}^{N} e\left(h\frac{a^n}{p^{k+1}}\right)\right| + \frac{1}{K}$$

and choose $K = [\log^\lambda N]$. Then, with the estimation of the exponential sum, we have

$$D_N\left(\frac{a^n}{p^{k+1}}\right) \ll \frac{1}{\log^\lambda N},$$

where the implied constant does not depend on k with (13). By (12) this completes the proof of the first case.

Next, we consider

$$(\log_p N)^\eta < k \le \delta \log_p N. \tag{15}$$

Let λ and h be as in the first case. With the notations of Lemma 5 and because of (15) we have for large enough N

$$\frac{p^{k+1}}{h} \ge p^{\beta+1}.$$

It follows from Lemma 6 that the exponential sum in the inequality (7) is 0, if we sum over a period. Hence, we can use the estimation of Lemma 7:

$$\left|\sum_{n=1}^{N} e\left(h\frac{a^n}{p^{k+1}}\right)\right| \le \sqrt{p^{k+1}}(1+\log\tau(p^{k+1})).$$

Using (15) it is an easy calculation to show that

$$\frac{1}{N}\left|\sum_{n=1}^{N} e\left(h\frac{a^n}{p^{k+1}}\right)\right| \ll \frac{1}{N^{\bar\delta}},$$

where $\bar\delta$ is a suitable constant. Notice that the implied constant does not depend on k.

The rest of the proof of the second case is similar to the first case. If we combine the two cases, then we get the claimed result. $\square$

Theorem 3 is an easy consequence of this Lemma:

Proof of Theorem 3. Of course the following equality is true

$$\frac{1}{N}\sum_{n=1}^{N} B_{p,A_1(N)+1,A_2(N)}(a^n; e_s e_{s-1}\cdots e_0) = \sum_{A_1(N)+1\le k\le A_2(N)} \frac{1}{N}A_k,$$

where A_k is as in Lemma 9.

We use now Lemma 9 and the claimed result follows. $\square$

4. Remarks

Remark 1. In Theorem 2, we consider the last digits of the digital expansion of the exponential sequence. Notice that the leading term $\frac{\gamma}{q^{s+1}} \log_q N$ is exactly the expected term, if one assumes that the digits are equidistributed.

A similar result should hold for more digits. However, with the method of proof, it doesn't seem to be possible to extend the range of digits in order to prove a stronger result.

Remark 2. In Theorem 3, we are interested in the first digits of the digital expansion of the exponential sequence. The result is of the same type as Theorem 2, especially we have the expected order of magnitude. Truncation of the first digits is necessary, because the multiplicative order of $a \bmod p^k$ can be very small, for small k and therefore, it is possible that not all digits occur at the $k-th$ position. However, the lower bound for the digit range could be reduced to $c \log_p \log_p N + d$, where c and d are suitable constants, but then λ in the error term would not be arbitrary any more.

If we assume that p is not necessary a prime, then the method of proof could be used to get a result for the first $\log N$ digits of the digital expansion. In this situation only the simpler estimation of Lemma 7 for the involved exponential sum of the form

$$\sum_{n=1}^{N} e\left(h \frac{a^n}{p^k}\right), \quad (a,p) = 1, (h,p) = 1,$$

is needed.

These exponential sums have been very frequently studied, because they are important in the theory of generating pseudo-random numbers with the linear congruential generator (see for instance [12] or [13]).

The proof of Theorem 3 heavily depends on estimations of these exponential sums, especially one needs estimations for very short intervals. Of course, better estimations would yield a better result, however, to obtain good estimations for very short intervals seems to be a hard problem.

Remark 3. In the proof of Theorem 1, all digits of the digital expansions are considered. One can adopt this idea to get a lower bound for the number of digits, which are not zero and therefore a lower bound for the mean value of the sum-of-digits function. However, we have not been able to obtain a lower bound better than the one in Theorem 1 with such ideas. It seems that for better results by taking all digits into account, a totally new method is needed.

We end with a conjecture, which seems to be far away from what can be obtained with the methods introduced in this paper.

Conjecture 1. *Let $a, q \geq 2$ be integers and assume that $\log_a q$ is irrational. Let $e_s e_{s-1} \cdots e_0$ be a digital block. Then we have*

$$\frac{1}{N}\sum_{n=1}^{N} B_q(a^n; e_s e_{s-1} \cdots e_0) \sim \frac{N \log a}{2q^{s+1} \log q}.$$

As a consequence one would have N as lower bound for the mean values in Corollary 2 and Corollary 3.

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