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Algebraic and ergodic properties of a new continued fraction algorithm with non-decreasing partial quotients

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Résumé. On introduit la notion de développement en fractions continues de Engel. Nous étudions notamment les propriétés ergodiques de ce développement et le lien avec celui introduit par F. Ryde *monotonen, nicht-abnehmenden Kettenbruch.*

Abstract. In this paper the Engel continued fraction (ECF) expansion of any $x \in (0,1)$ is introduced. Basic and ergodic properties of this expansion are studied. Also the relation between the ECF and F. Ryde's *monotonen, nicht-abnehmenden Kettenbruch* (MNK) is studied.

1. Introduction

Over the last 15 years the ergodic properties of several continued fraction expansions have been studied for which the underlying dynamical system is ergodic, but for which no finite invariant measure equivalent to Lebesgue measure exists. Examples of such continued fraction expansions are the 'backward' continued fraction (see [AF]), the 'continued fraction with even partial quotients' (see [S3]) and the Farey-shift (see [Leh]). All these (and other) continued fraction expansions are ergodic, and have a σ-finite, infinite invariant measure. Since these continued fractions are closely related to the regular continued fraction (RCF) expansion, their ergodic properties follow from those of the RCF by using standard techniques in ergodic theory (see [DK]).

In this paper we introduce a new continued fraction expansion, which we call—for reasons which will become apparent shortly—the *Engel continued fraction* (ECF) expansion. We will show that this ECF has an underlying dynamical system which is ergodic, but that no finite invariant measure equivalent to Lebesgue measure exists for the ECF.
As the name suggests, the ECF is a generalization of the classical *Engel*

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Series expansion, which is generated by the map $S : [0,1) \to [0,1)$, given by

$$S(x) := \left(\left\lfloor \frac{1}{x} \right\rfloor + 1 \right) \left(x - \frac{1}{\lfloor \frac{1}{x} \rfloor + 1} \right), \, x \neq 0; \;\; S(0) := 0,$$

where $\lfloor \xi \rfloor$ is the largest integer not exceeding ξ, see also Figure 1.

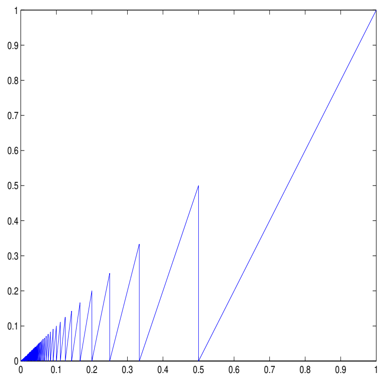

FIGURE 1. Engel Series map S

Using S, one can find a (unique) series expansion of every $x \in (0,1)$, given by

$$x = \frac{1}{q_1} + \frac{1}{q_1 q_2} + \cdots + \frac{1}{q_1 q_1 \cdots q_n} + \cdots ,$$

where $q_n = q_n(x) = \lfloor 1/S^{n-1}(x) \rfloor + 1$, $n \geq 1$. In fact it was W. Sierpiński [Si] in 1911 who first studied these series expansions.

The metric properties of the Engel Series expansion have been studied in a series of papers by É. Borel [B], P. Lévy [L], P. Erdös, A. Rényi and P. Szüsz [ERS] and Rényi [R]. In [ERS] it is shown that the random variables $X_1 = \log q_1$, $X_n = \log(q_n/q_{n-1})$ are 'almost independent' and 'almost identically distributed'. From this first the central limit theorem is derived for $\log q_n$, then the strong law of large numbers and finally the law of the iterated logarithm are obtained. The second result has been announced earlier without proof by E. Borel [B]. The first and third results are due to P. Lévy [L]. In [R], Rényi finds new (and more elegant) proofs to these and other results. Later F. Schweiger [S1] showed that S is ergodic, and M. Thaler [T] found a whole family of σ-finite, infinite measures for S. Further information on the Engel Series (and the related Sylvester Series) can be found in the books by J. Galambos [G] and Schweiger [S3].

Clearly one can generalize the Engel Series expansion by changing the time-zero partition, or by 'flipping' the map S on each partition element (thus obtaining an alternating Engel Series expansion, see also [K2K]).

Let $(r_n)_{n\geq 1}$ be a monotonically decreasing sequence of numbers in $(0,1)$, with $r_1 = 1$, $r_n > 0$ for $n \geq 1$ and $\lim_{n\to\infty} r_n = 0$. Furthermore, let $\ell_n = r_{n+1}$, $n \geq 1$, and let the time-zero partition $\mathcal{P}$ be given by

$$\mathcal{P} := \{[\ell_n, r_n);\, n \in \mathbb{N}\}.$$

Then

$$S_E(x) := \frac{r_n}{r_n - \ell_n}(x - \ell_n),\ x \neq 0;\ S_E(0) := 0,$$

where $n \in \mathbb{N}$ is such that $x \in [\ell_n, r_n)$ generalizes the 'Engel-map' S. With some effort the ideas from the above mentioned papers can be carried over to this generalization, also see W. Vervaat's thesis [V].

In this paper we study a different variation of the Engel Series expansion. Let the *Engel continued fraction* (ECF) map $T_E : [0,1) \to [0,1)$ be given by

$$T_E(x) := \frac{1}{\lfloor \frac{1}{x} \rfloor}\left(\frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor\right),\ x \neq 0;\ T_E(0) := 0, \tag{1}$$

see also Figure 2. Notice that

$$T_E(x) = \frac{1}{a_1(x)} T(x), \qquad 0 \leq x < 1,$$

where $T : [0,1] \to [0,1)$ is the regular continued fraction map, given by

$$T(x) := \frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor, \qquad 0 < x < 1; \qquad T(0) := 0,$$

and $a_1(x) = \lfloor 1/x \rfloor$. For any $x \in (0,1)$, the ECF-map 'generates' (in a way

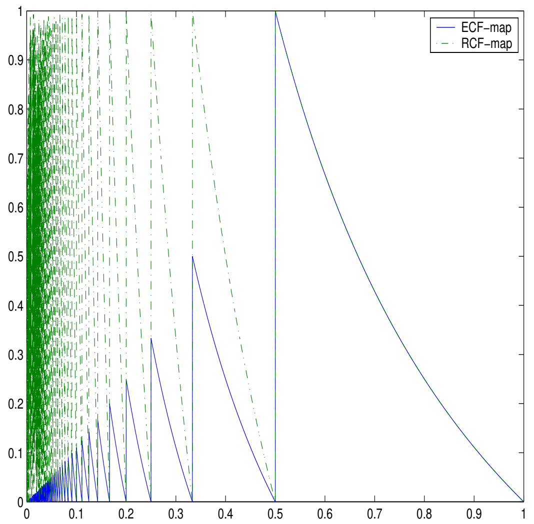

FIGURE 2. The ECF-map T_E and the RCF-map T

similar to the way the RCF-map T 'generates' the RCF-expansion of x) a new continued fraction expansion of x of the form

$$\frac{1}{b_1 + \cfrac{b_1}{b_2 + \cfrac{b_2}{b_3 + \ddots + \cfrac{b_{n-1}}{b_n + \ddots}}}}, \quad b_n \in \mathbb{N},\ b_n \le b_{n+1}. \tag{2}$$

In this paper we will study in Section 3 the ergodic properties of this new continued fraction expansion, the so-called *Engel continued fraction* (ECF) expansion. However, since the ECF is new, we will first show in the next section that the ECF 'behaves' in many ways like any other (semi-regular) continued fraction. In the last section we will study the relation between the ECF, and a continued fraction expansion introduced by F. Ryde [Ry1] in 1951, the so-called *monotonen, nicht-abnehmenden Kettenbruch* (MNK). We will show that the ECF and a minor modification of Ryde's MNK are metrically isomorphic, and due to this many properties of the ECF—such as ergodicity, the existence of σ-finite, infinite measures—can be carried over to the MNK. Conversely, the fact that not every quadratic irrational x has an ultimately periodic MNK-expansion can be carried over to the ECF via our isomorphism.

2. Basic properties

In this section we will study the basic properties of the ECF. In many ways it resembles the RCF, but there are also some open questions which suggest that there are fundamental differences.

Let $x \in (0,1)$, and define

$$\begin{aligned} &b_1 = b_1(x) := \lfloor 1/x \rfloor \\ &b_n = b_n(x) := b_1(T_E^{n-1}(x)), \qquad n \ge 2, \qquad T_E^{n-1}(x) \ne 0. \end{aligned} \tag{3}$$

From definition (1) of T_E it follows that

$$x = \frac{1}{b_1 + b_1 T_E(x)} = \cdots = \frac{1}{b_1 + \cfrac{b_1}{b_2 + \cfrac{b_2}{b_3 + \ddots + \cfrac{b_{n-1}}{b_n + b_n T_E^n(x)}}}},$$

where $T_E^0(x) = x$ and $T_E^n(x) = T_E(T_E^{n-1}(x))$ for $n \ge 1$, and one has—similar to the Engel Series case—that

$$1 \le b_1 \le b_2 \le \cdots \le b_n \le \cdots . \tag{4}$$

As usual the convergents are obtained via finite truncation;

$$\frac{1}{b_1 + \dfrac{b_1}{b_2 + \ddots + \dfrac{b_{n-1}}{b_n}}} = C_n, \qquad n \geq 1. \tag{5}$$

The finite continued fraction in the left-hand side of (5) is denoted by $[[\,0;\, b_1, \cdots, b_n]]$.

It is clear that the left-hand side of (5) is a rational number, so that we have the following result.

Theorem 2.1. *Let $x \in (0,1)$, then x has a finite ECF-expansion (i.e., $T_E^n(x) = 0$ for some $n \geq 1$) if and only if $x \in \mathbb{Q}$.*

Proof. The necessary condition is obvious since x in (5) is a rational number. For the sufficient condition, let $x = P/Q$ with $P, Q \in \mathbb{N}$ and $0 < P < Q$. Then

$$T_E(x) = \frac{\dfrac{Q}{P} - \left\lfloor \dfrac{Q}{P} \right\rfloor}{\left\lfloor \dfrac{Q}{P} \right\rfloor} = \frac{Q - b_1 P}{b_1 P},$$

where $b_1 = \lfloor Q/P \rfloor$. It is clear that $0 \leq Q - b_1 P < P$ and $1 \leq b_1 P \leq Q$ because $Q = b_1 P + r$ with $0 \leq r < P$ by the Euclidean algorithm. Now let

$$T_E^n(x) = \frac{P^{(n)}}{Q^{(n)}}, \quad n = 0, 1, 2, \ldots$$

then

$$0 \leq \cdots < P^{(n+1)} < P^{(n)} < \cdots < P^{(0)}.$$

Since $P^{(N)} \in \mathbb{N} \cup \{0\}$ there exists an $n \geq 1$ such that $T_E^n(x) = 0$. Notice that one has to apply T_E to x at most P times to get $T_E^n(x) = 0$. $\square$

The proof of the following theorem is omitted, since it is quite straightforward. For a proof the interested reader is referred to Section 1 in [K], where a similar result has been obtained for a general class of continued fractions.

Proposition 2.1. *Let the sequences $(P_n)_{n \geq 0}$ and $(Q_n)_{n \geq 0}$ recursively be defined by*

$$\begin{aligned} &P_0 = 0, \quad P_1 = 1, \quad P_n = b_n P_{n-1} + b_{n-1} P_{n-2} \quad \textit{for } n \geq 2, \\ &Q_0 = 1, \quad Q_1 = b_1, \quad Q_n = b_n Q_{n-1} + b_{n-1} Q_{n-2} \quad \textit{for } n \geq 2. \end{aligned} \tag{6}$$

Then $(P_n)_{n\geq 0}$ *and* $(Q_n)_{n\geq 0}$ *are both increasing sequences. Furthermore, one has for* $n \geq 1$ *that*

$$C_n = \frac{P_n}{Q_n}, \quad n \geq 1,$$

and

$$P_n Q_{n-1} - P_{n-1} Q_n = (-1)^{n-1} \prod_{j=1}^{n-1} b_j. \tag{7}$$

For the RCF-expansion it is known that even-numbered convergents are strictly increasing and odd-numbered strictly decreasing and that every even-numbered convergent is less than every odd-numbered one. The same result also holds for ECF convergents as stated in the following proposition.

Proposition 2.2. *Let* $C_n = P_n/Q_n$ *be the* n*-th ECF-convergent of* $x \in [0,1) \setminus \mathbb{Q}$. *Then*

$$0 = C_0 < C_2 < \cdots < C_3 < C_1 \leq 1.$$

Moreover, $C_{2j} < C_{2k+1}$ *for any nonnegative* j *and* k.

Proof. It follows from Proposition 2.1, using (6), that

$$P_n Q_{n-2} - P_{n-2} Q_n = b_n (-1)^{n-2} \prod_{j=1}^{n-2} b_j.$$

Dividing both sides by $Q_n Q_{n-2}$ gives

$$C_n - C_{n-2} = (-1)^{n-2} b_n \frac{\prod_{j=1}^{n-2} b_j}{Q_n Q_{n-2}}.$$

Since $b_j \geq 1$ for all j (and so are the Q_j's), $C_n - C_{n-2} > 0$ if n is even. Hence, $C_{2m-2} < C_{2m}$ for $m \geq 1$. Similarly, $C_{2m+1} < C_{2m-1}$. Now upon division on both sides of (7) by $Q_n Q_{n-1}$ one obtains $C_n - C_{n-1} = (-1)^{n-1}(\prod_{j=1}^{n-1} b_j)/Q_n Q_{n-1}$ which gives $C_{n-1} < C_n$ if n is odd, that is, $C_{2m} < C_{2m+1}$. Thus for any nonnegative j and k

$$C_{2j} < C_{2j+2k} < C_{2j+2k+1} < C_{2k+1},$$

as desired. □

In the next proposition, we will see that the sequence of ECF-convergents converges to the number from which it is generated.

Proposition 2.3. *For* $x \in [0,1)$, *let* $(P_n/Q_n)_{n\geq 1}$ *be the sequence of ECF-convergents of* x. *Then* $\lim_{n\to\infty} P_n/Q_n = x$.

Proof. In case x is rational, the result is clear; see Theorem 2.1. Now suppose that x is irrational. By induction one has that

$$x = \frac{P_n + b_n T_E^n(x) P_{n-1}}{Q_n + b_n T_E^n(x) Q_{n-1}}, \qquad n \geq 1. \tag{8}$$

In fact, if x is rational the special case $T_E^n(x) = 0$ gives $x = P_n/Q_n$. From (7) and (8) it follows that

$$x - \frac{P_{n-1}}{Q_{n-1}} = \frac{(-1)^{n-1}\prod_{j=1}^{n-1} b_j}{Q_{n-1}(Q_n + b_n T_E^n(x) Q_{n-1})}, \qquad n \geq 2, \tag{9}$$

which trivially yields that

$$\left| x - \frac{P_n}{Q_n} \right| \leq \frac{\prod_{j=1}^{n} b_j}{Q_n Q_{n+1}}, \qquad n \geq 1. \tag{10}$$

Letting $\phi_n = \prod_{j=1}^n b_j$ and using (6), one can see that ϕ_n^2 is one of the terms in $Q_n Q_{n+1}$ so that the right-most side of (10) goes to zero as $n \to \infty$. This completes the proof. □

Notice that (9) yields that $C_{2n} < x < C_{2n+1}$, for $n \geq 1$, where $C_n = P_n/Q_n$. We now will show that the ECF-expansion is unique.

Proposition 2.4. *Let $(b_n)_{n\geq 1}$ be a sequence of positive integers satisfying (4), and let the sequences $(P_n)_{n\geq 0}$ and $(Q_n)_{n\geq 0}$ be given by (6). Then the limit*

$$\lim_{n\to\infty} \frac{P_n}{Q_n}$$

exists. Say this limit equals x, then $x \in (0,1)$. Furthermore, $b_n = b_n(x)$ for $n \geq 1$.

Proof. Let $C_n = P_n/Q_n$. From the proof of Proposition 2.2 it follows that (C_{2k}) is an increasing sequence that is bounded from above; therefore, $\lim_{k\to\infty} C_{2k}$ exists. Similarly, (C_{2k+1}) is a decreasing sequence that is bounded from below, and so $\lim_{k\to\infty} C_{2k+1}$ also exists. It remains to show that these two limits are equal. To this end, note that, in the proof of Proposition 2.2, $C_{2k+1} - C_{2k} \to 0$ as $k \to \infty$ (by the same argument as in the proof of Proposition 2.3) so that $\lim_{k\to\infty} C_{2k+1} = \lim_{k\to\infty} C_{2k}$. Let $x = \lim_{k\to\infty} C_{2k}$; then $\lim_{k\to\infty} C_k$ exists and equals x. This completes the first part of the proof.

Since $0 < C_2 < x < C_1 \leq 1$, the second statement that $x \in (0,1)$ follows trivially.

For the third part, recall that by definition of C_n we have for $n \geq 2$

$$C_n = [[\,0;\, b_1, \cdots, b_n]] = \frac{1}{b_1 + b_1[[\,0;\, b_2, \cdots, b_n]]}. \tag{11}$$

Setting for $n \geq 2$

$$\tilde{C}_n = [[\,0;\, b_2, \cdots, b_n]],$$

it follows from the first part of the proof that there exists a number $\tilde{x} \in (0,1)$ such that $\lim_{n\to\infty} \tilde{C}_n = \tilde{x}$. By letting $n \to \infty$ in (11) we find that

$$x = \frac{1}{b_1 + b_1 \tilde{x}},$$

from which

$$\frac{\frac{1}{x}-b_1}{b_1} = \tilde{x} \in (0,1)\,. \tag{12}$$

Since $b_1(x)$ is the unique positive integer for which $\frac{1}{x}-b_1(x)\in[0,1)$, it follows that $b_1(x)$ is also the unique positive integer for which

$$T_E(x) = \frac{\frac{1}{x}-b_1(x)}{b_1(x)} \in [0,1).$$

But then it follows from (12) that $b_1(x)=b_1$, and moreover we see that $T_E(x)=\tilde{x}$. Repeating the above argument, now applied to $T_E(x)$, yields that $b_2(x)=b_2$. By induction one finds that $b_n(x)=b_n$ for $n\in\mathbb{N}$. $\square$

Let $x\in[0,1)\setminus\mathbb{Q}$. We denote the (infinite) ECF-expansion (2) of x by

$$x = [[\,0;b_1,b_2,\cdots,b_n,\cdots]]\,.$$

For the RCF-expansion one has the theorem of Lagrange, which states that x is a quadratic irrational (i.e., $x\in\mathbb{R}\setminus\mathbb{Q}$ and is a root of $ax^2+bx+c=0$, $a,b,c\in\mathbb{Z}$) if and only if x has a RCF-expansion which is ultimately (that is, from some moment on) periodic. For the ECF the situation is more complicated. Suppose $x\in[0,1)$ has a periodic ECF-expansion, say

$$x = [[0;b_1,\ldots,b_p,\overline{b_{p+1},\ldots,b_{p+\ell}}]],$$

where the bar indicates the period. Clearly one has that x is a quadratic irrational, and from (4) it follows that the period-length ℓ is always equal to 1.

Due to this, purely periodic expansions can easily be characterized; for $n\in\mathbb{N}$ one has

$$\tau_n := \frac{-n+\sqrt{n^2+4n}}{2n} = [[\,0;\bar{n}]]\,. \tag{13}$$

One could wonder whether every quadratic irrational x has an eventually periodic ECF-expansion. In [Ry2], Ryde showed that a quadratic irrational x has an eventually periodic MNK-expansion if and only if a certain set of conditions are satisfied. In Section 4 we will see that the ECF-map T_E and a modified version of the MNK-map are isomorphic, and due to this we will obtain that not every quadratic irrational x has an eventually periodic ECF-expansion.

An important question is the relation between the convergents of the RCF and those of the ECF. Let $x\in[0,1)\setminus\mathbb{Q}$, with RCF-expansion $x=[\,0;\,a_1,a_2,\ldots]$, with RCF-convergents $(p_n/q_n)_{n\geq1}$ and ECF-convergents $(P_n/Q_n)_{n\geq1}$. Moreover, define the *mediant* convergents of x by

$$\frac{ap_n+p_{n-1}}{aq_n+q_{n-1}}\,,\qquad \text{for } 1\leq a\leq a_{n+1}-1\,,$$

then the question arises whether infinitely many RCF-convergents and/or mediants are among the ECF-convergents, and conversely.

Example. Let $x = \dfrac{37+5\sqrt{15}}{142} = 0.396936\ldots$, then

$$x = [0;\, 2,\, 1,\, 1,\, \overline{12,\, 2,\, 5,\, 19,\, 5,\, 2}\,] = [[\,0;\, 2,\, 3,\, 3,\, 5,\, \bar{6}\,]]$$

and $\frac{P_1}{Q_1} = \frac{1}{2} = \frac{p_1}{q_1}$, $\frac{P_2}{Q_2} = \frac{3}{8}$ (a mediant), $\frac{P_3}{Q_3} = \frac{2}{5} = \frac{p_3}{q_3}$, $\frac{P_4}{Q_4} = \frac{23}{28}$ (a mediant), $\frac{P_5}{Q_5}$ is neither a RCF-convergent nor a mediant, $\frac{P_6}{Q_6} = \frac{181}{456}$ (a mediant), $\frac{P_7}{Q_7} = \frac{622}{1567}$ (a mediant), $\frac{P_8}{Q_8} = \frac{285}{718} = \frac{p_6}{q_6}$, etc. Among the first 15 ECF-convergents $\frac{P_5}{Q_5} = \frac{79}{199}$ and $\frac{P_{11}}{Q_{11}} = \frac{38554}{97129}$ are neither RCF-convergents nor mediants.

A related question is the value of the first point in a 'Hurwitz spectrum' for the ECF. Let $x \in [0,1)\setminus\mathbb{Q}$, again with RCF-convergents $(p_n/q_n)_{n\geq 0}$ and ECF-convergents $(P_n/Q_n)_{n\geq 0}$, where we moreover assume that $(P_n, Q_n) = 1$ for $n \geq 0$. Setting for $n \geq 0$

$$\theta_n = \theta_n(x) := q_n|q_n x - p_n| \quad \text{and} \quad \Theta_n = \Theta_n(x) := Q_n|Q_n x - P_n|\,,$$

one has the classical results that

$$\theta_n < 1 \quad \text{and} \quad \min(\theta_{n-1}, \theta_n, \theta_{n+1}) < 1/\sqrt{a_{n+1}^2 + 4} \quad \text{for } n \geq 1,$$

which trivially implies that

$$\theta_n < \frac{1}{\sqrt{5}} \quad \text{infinitely often for all } x \in [0,1) \setminus \mathbb{Q}.$$

If infinitely many RCF-convergents of x are also ECF-convergents of x, then one has that

$$\Theta_n < C \quad \text{infinitely often,} \tag{14}$$

with $C = 1$. Consider the number x having a purely periodic expansion with $b_n = 2$. Then $x = \frac{1}{2}(-1+\sqrt{3})$. The difference equation $A_n = 2A_{n-1} + 2A_{n-2}$ controls the growth of $\Theta_n(x) = Q_n|Q_n x - P_n|$. Its eigenvalues are $1-\sqrt{3}$ and $1+\sqrt{3}$, and therefore we see that $\Theta_n(x) = Q_n|Q_n x - P_n|$ is asymptotically equal to 2^n. Furthermore from (9) one can see that $\frac{\prod_{j=1}^n b_j}{b_{n+1}+2} \leq \Theta_n(x) = Q_n|Q_n x - P_n| \leq \frac{\prod_{j=1}^n b_j}{b_{n+1}}$, which shows that such a constant C cannot exist for all x.

3. Ergodic properties

In this section we will show that T_E has no finite invariant measure, equivalent to the Lebesgue measure λ, but that T_E has infinitely many σ-finite, infinite invariant measures. Furthermore it is shown that T_E is ergodic with respect to λ.

Let

$$B(n) := \left[\frac{1}{n+1}, \frac{1}{n}\right) \quad \text{for} \quad n \in \mathbb{N}, \tag{15}$$

and define for $n \in \mathbb{N}$, $b_1, \ldots, b_n \in \mathbb{N}$ with $b_1 \leq \ldots \leq b_n$ the cylinder sets (or: fundamental intervals) $B(b_1, \ldots, b_n)$ by

$$B(b_1, \ldots, b_n) = \left\{x \in [0,1);\, T_E^{i-1}x \in B(b_i),\, i = 1, \ldots, n\right\}.$$

Then it is clear that, see also Figure 2,

$$T_E^{-1}B(n) = \bigcup_{k=1}^{n} \left(\frac{n}{k(n+1)}, \frac{n+1}{k(n+2)}\right]. \tag{16}$$

Now let μ be a finite T_E-invariant measure, that is,

$$\mu(T_E^{-1}A) = \mu(A)$$

for any Borel set $A \in [0,1)$.

Since μ is a measure, we have that

$$\mu([1/2, \tau_1]) = \mu([1/2, 3/5)) + \mu([3/5, \tau_1]),$$

where $\tau_n \in B(n)$ denotes the invariant point under T_E, see also (13). On the other hand,

$$\begin{aligned} \mu([1/2, \tau_1]) &= \mu(T_E^{-1}[1/2, \tau_1]) \\ &= \mu([\tau_1, 2/3]) \\ &= \mu(T_E^{-1}[\tau_1, 2/3]) \\ &= \mu([3/5, \tau_1]) \end{aligned}$$

because μ is T_E-invariant. Hence, since we assumed that μ is a finite measure,

$$\mu([1/2, 3/5)) = 0.$$

Furthermore, we also have

$$\mu((2/3, 1)) = \mu(T_E^{-1}(2/3, 1)) = \mu((1/2, 3/5)) = 0.$$

Similar arguments yield that $\mu([3/5, \tau_1]) = \mu([8/13, \tau_1])$, and therefore

$$\mu((5/8, 2/3]) = \mu(T_E^{-1}(5/8, 2/3]) = \mu([3/5, 8/13)) = 0$$

so that we have

$$\mu([1/2, 8/13]) = 0 \quad \text{and} \quad \mu([5/8, 1]) = 0.$$

Continue this iteration to see that $B(1)$ must have its mass $\mu(B(1))$ concentrated at τ_1. Next, it follows from (16) that $T_E^{-1}B(2) = (1/3, 3/8] \cup (2/3, 3/4]$. But $\mu(2/3, 3/4] = 0$ so that $\mu(T_E^{-1}B(2)) = \mu((1/3, 3/8])$ and following the same arguments as above gives that $B(2)$ has its mass $\mu(B(2))$ concentrated at τ_2.

Applied to other values of n, induction yields that on $(0,1)$ the measure μ has mass $\mu(B(n))$ concentrated at τ_n for $n \geq 1$. Consequently, we have proved the following result.

Theorem 3.1. *There does not exist a non-atomic finite T_E-invariant measure.*

Next, we will prove ergodicity of T_E with respect to Lebesgue measure.

Theorem 3.2. *T_E is ergodic with respect to Lebesgue measure λ.*

Proof. Let $T_E^{-1}A = A$ be an invariant Borel set. Define

$$d(b) := b\int_0^{\frac{1}{b}} c_A(y)dy,$$

where c_A denotes the indicator function of A. Then we calculate

$$\begin{aligned} d(b) - d(b+1) &= b\int_0^{\frac{1}{b}} c_A(y)dy - (b+1)\int_0^{\frac{1}{b+1}} c_A(y)dy \\ &= b\int_{\frac{1}{b+1}}^{\frac{1}{b}} c_A(y)dy - \int_0^{\frac{1}{b+1}} c_A(y)dy \\ &= \frac{1}{b+1}(b(b+1)\int_{\frac{1}{b+1}}^{\frac{1}{b}} c_A(y)dy - (b+1)\int_0^{\frac{1}{b+1}} c_A(y)dy). \end{aligned}$$

We put

$$\begin{aligned} \delta(b_1,\ldots,b_n) &:= \frac{\lambda(A\cap B(b_1,\ldots,b_n))}{\lambda(B(b_1,\ldots,b_n))} = \frac{\lambda(T^{-n}A\cap B(b_1,\ldots,b_n))}{\lambda(B(b_1,\ldots,b_n))} \\ &= \frac{\int_0^{\frac{1}{b_n}} \omega(b_1,\ldots,b_n;y)c_A(y)dy}{\int_0^{\frac{1}{b_n}} \omega(b_1,\ldots,b_n;y)dy}. \end{aligned}$$

Note that $T_E^n B(b_1,\ldots,b_n) = [0,\frac{1}{b_n})$, and that it follows from (8) and (7) that

$$\omega(b_1,\ldots,b_n;y) = \frac{\prod_{i=1}^n b_i}{(Q_n + Q_{n-1}b_n y)^2}, \quad \text{with } 0 \leq y \leq \frac{1}{b_n}.$$

From this it follows that

$$\lambda(B(b_1,\ldots,b_n)) = \frac{\prod_{j=1}^{n-1} b_j}{Q_n(Q_n+Q_{n-1})}. \tag{17}$$

Moreover we find that

$$\delta(b) = b(b+1)\int_{\frac{1}{b+1}}^{\frac{1}{b}} c_A(y)dy, \tag{18}$$

and that
$$\frac{\prod_{i=1}^n b_i}{4Q_n^2} \le \omega(b_1, \dots, b_n; y) \le \frac{\prod_{i=1}^n b_i}{Q_n^2},$$
which shows that

(A) $\frac{d(b_n)}{4} \le \delta(b_1, \dots, b_n) \le 4d(b_n)$.

For $n = 1$ we have a more precise estimate. Since $\omega(b; y) = \dfrac{1}{b(1+y)^2}$ for $0 \le y \le \frac{1}{b}$, we get
$$\frac{b}{(b+1)^2} \le \omega(b; y) \le \frac{1}{b}, \quad \text{for } 0 \le y \le \frac{1}{b}.$$
Together with (18) this yields that

(B) $\frac{b}{(b+1)}\, d(b) \le \delta(b) \le \frac{(b+1)}{b}\, d(b)$.

Furthermore we have that

(C) $d(b) - d(b+1) = \dfrac{\delta(b) - d(b+1)}{b+1}$.

Note that for Engel's series $\delta(b) = d(b)$ which fact makes the proof easier.

Together with (B) we get the estimate $d(b+1) \le \dfrac{b^2+b+1}{b^2+b}\, d(b)$.

Setting $\prod_{b=1}^\infty \frac{b^2+b+1}{b^2+b} =: \gamma$ we get

(D) $d(c) \le \gamma d(b)$ for all $c \ge b$.

The Martingale Convergence Theorem shows that
$$\lim_{n\to\infty} \delta(b_1(x), \dots, b_n(x)) \;=\; c_A(x) \quad \text{almost everywhere},$$
see also Theorem 9.3.3 in [S3]. If $c_A(x) = 1$ a.e. there is nothing to show. Let us therefore assume that $\lambda(A) < 1$. Suppose that there is some $z \in A^c$ such that $\lim_{n\to\infty} \delta(b_1(z), \dots, b_n(z)) = 0$. Then for n sufficiently large $\delta(b_1(z), \dots, b_n(z)) \le \frac{\varepsilon}{16\gamma}$ for any given $\varepsilon > 0$, and by (A) for $b = b_n(z)$ we find $d(b) \le \frac{\varepsilon}{4\gamma}$. Applying (D) this yields that $d(c) \le \frac{\varepsilon}{4}$ for all $c \ge b$.

Since the set $F_N = \{x : b_j(x) \le N, j \ge 1\}$ is countable (since every $x \in F_N$ ends in a periodic ECF-expansion with digit $b \le N$), it follows that F_N has measure 0 and we clearly have $\lim_{n\to\infty} b_n = \infty$ a.e. Therefore by (A) we see that $\delta(b_1(x), \dots, b_n(x)) \le \varepsilon$ for almost all points x. Assuming that $\varepsilon < 1$ this shows that $c_A(x) = 0$ almost everywhere, i.e., $\lambda(A) = 0$. $\square$

For the Engel's series Rényi [R] showed that for almost all x the sequence of digits is monotonically increasing from some moment $n_0(x)$ on. For the ECF a similar result holds.

Theorem 3.3. *For almost all $x \in [0, 1)$ the sequence of digits $(b_n(x))_{n=1}^\infty$ is strictly increasing for some $n \ge n_0(x)$.*

Proof. Setting $y := \frac{Q_{n-1}}{Q_n}$, it follows from (17) and (6) that

$$\frac{\lambda(B(b_1,\ldots,b_n,b_{n+1}))}{\lambda(B(b_1,\ldots,b_n))} = \frac{b_n(1+y)}{(b_{n+1}+b_n y)(b_{n+1}+1+b_n y)}\,, \tag{19}$$

and that $0 \le y \le \frac{1}{b_n}$.

If we put $b_n = b_{n+1}$ we immediately get

$$\lambda(\{x : b_n(x) = b_{n+1}(x)\}) \le \sum_{b_1 \le \cdots \le b_n} \frac{\lambda(B(b_1,\ldots,b_n))}{b_n+1}\,,\ n \ge 1.$$

Lemma 3.1. $\sum_{b_1 \le \cdots \le b_n} \dfrac{\lambda(B(b_1,\ldots,b_n))}{b_n+1} \le (\frac{313}{324})^n$.

Proof (of the lemma, see also [S3], p. 68-69). It follows from (19) that

$$\sum_{b_1 \le \cdots \le b_{n+1}} \frac{\lambda(B(b_1,\ldots,b_{n+1}))}{b_{n+1}+1}$$

equals

$$\sum_{b_1 \le \cdots \le b_n} \frac{\lambda(B(b_1,\ldots,b_n))}{b_n+1} \sum_{b_n \le b_{n+1}} \frac{(b_n+1)b_n(1+y)}{(b_{n+1}+1)(b_{n+1}+b_n y)(b_{n+1}+1+b_n y)}\,.$$

Therefore we have to estimate the sum

$$\sum_{b=a}^{\infty} \frac{(a+1)a(1+y)}{(b+1)(b+ay)(b+1+ay)}\,.$$

For $b = a$ the first term gives $\frac{1}{a+1+ay} \le \frac{1}{a+1}$, while for $b \ge a+1$ we use $0 \le y \le \frac{1}{a}$ to obtain the estimate

$$\frac{(a+1)a(1+y)}{(b+1)(b+ay)(b+1+ay)} \le \frac{(a+1)^2}{b(b+1)^2}\,,$$

and it follows that

$$\sum_{b=a}^{\infty} \frac{(a+1)a(1+y)}{(b+1)(b+ay)(b+1+ay)} \le \frac{1}{a+1} + \sum_{b=a+1}^{\infty} \frac{(a+1)^2}{b(b+1)^2}$$

$$\le \frac{1}{a+1} + (a+1)^2 \left(\frac{1}{(a+1)(a+2)^2} + \frac{1}{(a+2)(a+3)^2} + \int_{a+2}^{\infty} \frac{dz}{z(z+1)^2} \right)\,.$$

Using

$$\begin{aligned}\int_{a+2}^{\infty} \frac{dz}{z(z+1)^2} &= \log\frac{a+3}{a+2} - \frac{1}{a+3} \\ &\le \frac{1}{(a+2)(a+3)} - \frac{1}{2(a+2)^2} + \frac{1}{3(a+2)^3}\,,\end{aligned}$$

we eventually get

$$\sum_{b=a}^{\infty} \frac{(a+1)a(1+y)}{(b+1)(b+ay)(b+1+ay)} \le \frac{1}{a+1} + (a+1)^2\Big(\frac{1}{(a+1)(a+2)^2}$$
$$+\frac{1}{(a+2)(a+3)^2} + \frac{1}{(a+2)(a+3)} - \frac{1}{2(a+2)^2} + \frac{1}{3(a+2)^3}\Big)\,.$$

This expression has its maximum for $a = 1$ which gives the value $\frac{313}{324}$.

The claim on the maximal value can be seen as follows.
The sum of the four terms

$$\frac{1}{a+1} + \frac{a+1}{(a+2)^2} + \frac{(a+1)^2}{(a+2)(a+3)^2} + \frac{(a+1)^2}{3(a+2)^3}$$

decreases to 0 as $a \to \infty$ and becomes smaller than $\frac{1}{2}$ for $a \ge 3$. On the other hand the remaining term

$$(a+1)^2\Big(\frac{1}{(a+2)(a+3)} - \frac{1}{2(a+2)^2}\Big) = \frac{(a+1)^3}{2(a+2)^2(a+3)}$$

is increasing and is bounded by $\frac{1}{2}$. Therefore numerical calculations suffice for $n = 1, 2, 3$. This proves the Lemma.

Now the Borel-Cantelli lemma yields that the set of all points x for which $b_n(x) = b_{n+1}(x)$ for infinitely many values of n has measure 0. $\square$

Now we give two constructions of σ-finite, infinite invariant measures for T_E. The first construction follows to some extent Thaler's construction from [T] of σ-finite, infinite invariant measures for the Engel's series map S.

First, let $B(n)$ be as in (15). Define

$$A(n,k) = \{x;\, b_j = n, 1 \le j \le k, b_{k+1} > n\}\,.$$

Obviously one has that $A(n,k) \cap A(m,\ell) = \emptyset$ for $(n,k) \ne (m,\ell)$ and that, apart from a set of Lebesgue measure zero

$$B(n) = \bigcup_{k=1}^{\infty} A(n,k).$$

We now choose a monotonically increasing sequence of positive real numbers $(a_n)_{n=1}^{\infty}$, satisfying

$$a_n > \frac{n}{n-1}\sum_{j=1}^{n-1} \frac{a_j}{j}, \quad n \ge 2.$$

For any non-purely periodic $x \in B = \bigcup_{n=1}^{\infty} B(n)$, there exist positive integers n and ℓ such that $x \in A(n,\ell)$. For any non-eventually periodic $x \in B \setminus \mathbb{Q}$ we inductively define a sequence $(\alpha_k(x))_{k=1}^{\infty}$ by

$$\alpha_1(x) = a_n \quad \text{if} \quad x \in A(n,\ell) \quad \text{for some } \ell \ge 1,$$

and

$$\alpha_{k+1}(x) = n\,(T_E(x)+1)^2\,\alpha_k(T_E(x)) - n\sum_{j=1}^{n-1}\frac{a_j}{j}, \quad k \geq 1.$$

For $x \in [0,1) \setminus \mathbb{Q}$ given one has, since $\alpha_1(x) = a_n \leq \alpha_1(T_E(x))$

$$\begin{aligned}\alpha_2(x) - \alpha_1(x) &\geq (n(T_E(x)+1)^2 - 1)a_n - n\sum_{j=1}^{n-1}\frac{a_j}{j} \\ &> n\left(\frac{n\,(T_E(x)+1)^2 - 1}{n-1} - 1\right)\sum_{j=1}^{n-1}\frac{a_j}{j} \\ &> 0,\end{aligned}$$

and for $k \geq 2$ one has

$$\alpha_{k+1}(x) - \alpha_k(x) = n(T_E(x)+1)^2\left(\alpha_k\left(T_E(x)\right) - \alpha_{k-1}\left(T_E(x)\right)\right).$$

So by induction it follows that the sequence $(\alpha_k(x))_{k=1}^{\infty}$ is a positive monotonically non-decreasing sequence for each $x \in [0,1)$. We will show that

$$h(x) := \alpha_k(x) \quad \text{for} \quad x \in A(n,k)$$

and $h(x) := 0$ for $x \notin \cup_{n=1}^{\infty} B(n)$, is a density of a measure equivalent to Lebesgue measure, that is, it satisfies Kuzmin's equation, see also Chapter 13 in [S3]. First note that for $x \in B(n)$ Kuzmin's equation reduces to

$$h(x) = \sum_{j=1}^{n}\frac{h(V_j(x))}{j(x+1)^2},$$

where V_j is the local inverse of T_E on $B(j)$, i.e.,

$$V_j(x) = \frac{1}{j(1+x)}.$$

Note also that $V_n(x) \in A(n,k+1)$ and $V_j(x) \in A(j,1), 1 \leq j \leq n-1$, when $x \in A(n,k)$. Therefore, $h(x) = \alpha_k(x)$, $h(V_n(x)) = \alpha_{k+1}(V_n(x))$ and $h(V_j(x)) = \alpha_1(V_j(x)) = a_j, 1 \leq j \leq n-1$. We now see that

$$\sum_{j=1}^{n}\frac{h(V_j(x))}{j(x+1)^2} = \frac{\alpha_{k+1}(V_n(x))}{n(x+1)^2} + \sum_{j=1}^{n-1}\frac{a_j}{j(x+1)^2} = \alpha_k(x) = h(x).$$

As in Thaler's case for the Engel Series expansion, this construction yields infinitely many different σ-finite, infinite invariant measures which are not multiples of one-another.

For the second construction, let $w_t^s(x) = \left|\dfrac{dV_t^s(x)}{dx}\right|$. Note that $w_t^1(x) = \dfrac{1}{t(1+x)^2}$. Furthermore, let $G > 1$ be the *golden mean*, defined by $G^2 =$

$G+1$, $g=\frac{1}{G}$ and

$$g_0(x) := \left|\frac{1}{(x-g)(x+G)}\right|, \quad 0 \le x < 1, \quad x \ne g.$$

Here $f^0(x) = x, f^1(x) = f(x), f^{s+1}(x) = f^s(f(x))$. Note that g_0 is a solution of the functional equation $f(x) = f(V_1(x))w_1^1(x)$.

Setting for $t = 2, 3, \ldots$

$$g_{t-1}(x) := \sum_{s=0}^{\infty} g_{t-2}(V_t^s(x))w_t^s(x), \qquad 0 \le x < 1/t,$$

then $h(x) := g_{t-1}(x)$ on $B(t)$ is an invariant density. To see this note that Kuzmin's equation reads

$$h(x) = \sum_{s=1}^{k} h(V_s(x))w_s^1(x)$$

on $B(k)$. Then for $x \in B(k)$ we have $g_{k-1}(x)$ on the left hand, while expanding the right hand side gives

$$\begin{aligned} g_{k-1}(x) &= h(V_1(x))w_1^1(x) + h(V_2(x))w_2^1(x) + \cdots + h(V_k(x))w_k^1(x) \\ &= g_0(V_1(x))w_1^1(x) + g_1(V_2(x))w_2^1(x) + \cdots + g_{k-1}(V_k(x))w_k^1(x) \\ &= g_0(x) + \sum_{s=1}^{\infty} g_0(V_2^s(x))w_2^s(x) + \cdots + \sum_{s=1}^{\infty} g_{k-2}(V_k^s(x))w_t^s(x) \\ &= g_{k-1}(x). \end{aligned}$$

In the calculation we used that $w_t^{s+1}(x) = w_t^s(V_t(x))w_t^1(x)$.

We end this section with a theorem on the renormalization of the ECF-map T_E. See also Hubert and Lacroix [HL] for a recent survey of the ideas behind the renormalization of algorithms.

Setting

$$z_n(x) = b_n(x)T_E^n x,$$

then clearly $0 \le z_n(x) \le 1$. We have the following theorem.

Theorem 3.4. *Let* $\gamma = \frac{313}{324}$, *then*

$$\lambda(\{x : z_n(x) < t\}) = t(1 + O(\gamma^n)).$$

Proof. We introduce for $t \in [0, 1]$ the map

$$S(b_1, \ldots, b_n)(t) := V(b_1, \ldots, b_n)(\frac{t}{b_n}).$$

Applying the chain-rule we find, see also (8)

$$S'(b_1, \ldots, b_n)(t) = (-1)^n \frac{\prod_{j=1}^{n-1} b_j}{(Q_n + Q_{n-1}t)^2}.$$

Then $\lambda(\{x : z_n(x) < t\}) = \int_0^t \rho_n(s)ds$, where

$$\rho_n(s) \;=\; (-1)^n \sum_{b_1 \le \cdots \le b_n} S'(b_1, \ldots, b_n)(s) \;=\; \sum_{b_1 \le \cdots \le b_n} \frac{\prod_{j=1}^{n-1} b_j}{(Q_n + Q_{n-1}s)^2} .$$

On the other hand we have, see also (17)

$$1 \;=\; \sum_{b_1 \le \cdots \le b_n} \lambda(B(b_1, \ldots, b_n)) \;=\; \sum_{b_1 \le \cdots \le b_n} \frac{\prod_{j=1}^{n-1} b_j}{Q_n(Q_n + Q_{n-1})} .$$

Therefore

$$\begin{aligned} |\rho_n(s) - 1| &\le \sum_{b_1 \le \cdots \le b_n} \frac{\prod_{j=1}^{n-1} b_j}{Q_n(Q_n + Q_{n-1})} \left| \frac{Q_n Q_{n-1} - 2sQ_nQ_{n-1} - s^2 Q_{n-1}^2}{(Q_n + Q_{n-1}s)^2} \right| \\ &\le 4 \sum_{b_1 \le \cdots \le b_n} \frac{\lambda(B(b_1, \ldots, b_n))}{b_n} , \end{aligned}$$

which yields that

$$\rho_n(s) = 1 + O(\gamma^n) \quad \text{and} \quad \int_0^t \rho_n(s)ds = t + tO(\gamma^n).$$

□

Remark. Following the ideas in Schweiger [S2] it should be easy to prove that the sequence $(z_n(x))$ is uniformly distributed for almost all points x.

4. On Ryde's continued fraction with non-decreasing digits

In 1951, Ryde [Ry1] showed that every $x \in (0,1)$ can be written as a *monotonen, nicht-abnehmenden Kettenbruch* (MNK) of the form

$$\text{(20)} \qquad \cfrac{c_1}{sc_1 + \cfrac{c_2}{c_2 + \cfrac{c_3}{c_3 + \ddots + \cfrac{c_n}{c_n + \ddots}}}} , \quad c_n \in \mathbb{N},\ c_n \le c_{n+1},\ s = \left\lfloor \frac{1}{x} \right\rfloor ,$$

which is finite if and only if x is rational.

Underlying this expansion is the map $S_R : (0,1) \to (0,1)$, defined by

$$S_R(x) \;:=\; \frac{\left\lfloor \dfrac{1}{\frac{1}{x} - \lfloor \frac{1}{x} \rfloor} \right\rfloor}{x} - \left\lfloor \frac{1}{x} \left\lfloor \frac{1}{\frac{1}{x} - \lfloor \frac{1}{x} \rfloor} \right\rfloor \right\rfloor , \quad x \in (0,1).$$

However, and this was already observed by Ryde, one has that

$$S_R\left((0,\frac{1}{2})\right) = \left(\frac{1}{2},1\right) = S_R\left((\frac{1}{2},1)\right) \pmod 0,$$

and therefore we might as well restrict our attention to the interval $(\frac{1}{2},1)$, and just consider the map $T_R : (\frac{1}{2},1) \to (\frac{1}{2},1)$, given by

$$T_R(x) = S_R(x) = \frac{k}{x} - k, \quad \text{for } x \in R(k) := \left(\frac{k}{k+1}, \frac{k+1}{k+2}\right), \ k \in \mathbb{N},$$

see also [S3], p. 26. Now every $x \in (\frac{1}{2},1)$ has a unique NMK of the form (20) (with $s = 1$), which we abbreviate by

$$x = < 0; c_1, c_2, \cdots, c_n, \cdots > .$$

The following theorem establishes the relation between the ECF and the MNK. We omitted the proof, since it follows by direct verification.

Theorem 4.1. *Let the bijection $\phi : (0,1) \to (\frac{1}{2},1)$ be defined by*

$$\phi(x) := \frac{1}{1+x}, \quad \textit{for } x \in (0,1).$$

Then

$$T_R(\phi(x)) = \phi(T_E(x)), \quad \textit{for } x \in (0,1). \tag{21}$$

Furthermore, for $b_1 \le b_2 \le \cdots \le b_n \le \cdots$

$$\phi([[0; b_1, b_2, \cdots, b_n, \cdots]]) = < 0; b_1, b_2, \cdots, b_n, \cdots >,$$

and if we define for $n \ge 1$ the cylinders of T_R by

$$R(c_1, \cdots, c_n) := \{x \in (\frac{1}{2},1);\, T_R^{i-1}(x) \in R(c_i),\, i = 1, \ldots, n\}, \tag{22}$$

then

$$\phi(B(b_1, \cdots, b_n)) = R(b_1, \cdots, b_n).$$

Due to Theorem 4.1 we can 'carry-over' the whole 'metrical structure' of the ECF to the MNK. To be more precise, letting $\mathcal{B}$ be the collection of Borel sets of $(\frac{1}{2},1)$, and setting

$$\nu(A) := \rho(\phi^{-1}(A)), \quad A \in \mathcal{B}, \tag{23}$$

where ρ is a σ-finite, infinite T_E-invariant measure on $(0,1)$ with density h (with h from Section 3), then we have the following corollary.

Corollary 4.1. *The map T_R is ergodic with respect to Lebesgue measure λ, but no finite T_R-invariant measure exist equivalent to λ. Each of the measures ν from* (23) *is a σ-finite, infinite T_R-invariant measure on $(\frac{1}{2},1)$.*

Proof. We only give a proof of the first statement. Suppose that there exists a Borel set $A \subset (\frac{1}{2}, 1)$ for which $0 < \lambda(A) < \lambda(\frac{1}{2}, 1) = \frac{1}{2}$, such that $T_R^{-1}(A) = A$. From the fact that $\phi : (0,1) \to (\frac{1}{2}, 1)$ is a bijection, and due to (21) one has that $\phi^{-1}(A)$ is a T_E-invariant set, and hence $\lambda(\phi^{-1}(A)) \in \{0,1\}$, which is impossible. $\square$

To conclude this paper, let us return to the question of periodicity of the ECF-expansion of a quadratic irrational x. Due to (22) we have that the ECF-expansion of $x \in (0,1)$ is (ultimately) periodic if and only if the NMK-expansion of $\xi = \phi(x) \in (\frac{1}{2}, 1)$ is (ultimately) periodic. The main result of Ryde's second 1951 paper [Ry2] now states that a quadratic irrational $\xi \in (0,1)$ has an (ultimately) periodic NMK-expansion if and only if a (rather large) set of constraints—too large to be mentioned here; the statement of his theorem covers almost 2 pages!—has been satisfied. Due to Theorem 4.1 these constraints can trivially be translated into a set of constraints for the ECF.

As a consequence there exist (infinitely many) quadratic irrationals x for which the ECF-expansion is **not** ultimately periodic. We end this paper with an example.

Example. Let $x = \frac{1}{5}(-1 + 2\sqrt{5}) = 0.6944271\ldots$. Then the RCF-expansion of x is $x = [\,0; \overline{1, 2, 3, 1, 2, 44, 2, 1, 3, 2, 1, 1, 10, 1}\,]$. Using MAPLE we obtained the first 22 partial quotients of the ECF-expansion of x:

$$x = [[\,0; 1, 2, 7, 20, 28, 30, 187, 541, 711, 989, 2280, 7630, 8683, 13941, 26110, 32685, 199856, 866227, 5897902, 8834278, 24269774, 96660239, \ldots\,]]\,.$$

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