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Polynomial growth of sumsets in abelian semigroups

par Melvyn B. NATHANSON* et Imre Z. RUZSA**

To Michel Mendès France

Résumé. Soit S un semi-groupe abélien et A un sous-ensemble fini de S. On désigne par hA l'ensemble de toutes les sommes de h éléments de A, et par $|hA|$ son cardinal. On montre, par des arguments élémentaires de comptage de points dans les réseaux, qu'il existe un polynôme $p(t)$ tel que pour tout entier h assez grand $|hA| = p(h)$. Plus généralement, on étend ce résultat aux ensembles $h_1A_1+\cdots+h_rA_r$ en obtenant la croissance polynomiale du cardinal en termes des variables $h_1, h_2, \ldots, h_r$.

Abstract. Let S be an abelian semigroup, and A a finite subset of S. The sumset hA consists of all sums of h elements of A, with repetitions allowed. Let $|hA|$ denote the cardinality of hA. Elementary lattice point arguments are used to prove that an arbitrary abelian semigroup has polynomial growth, that is, there exists a polynomial $p(t)$ such that $|hA| = p(h)$ for all sufficiently large h. Lattice point counting is also used to prove that sumsets of the form $h_1A_1 + \cdots + h_rA_r$ have multivariate polynomial growth.

1. Introduction

Let $\mathbf{N}_0$ denote the set of nonnegative integers, and $\mathbf{N}_0^k$ the set of all k-tuples of nonnegative integers. Geometrically, $\mathbf{N}_0^k$ is the set of lattice points in the euclidean space $\mathbf{R}^k$ that lie in the nonnegative octant.

If A is a finite, nonempty subset of $\mathbf{N}_0$, then the *sumset* hA is the set of all integers that can be represented as the sum of h elements of A, with repetitions allowed. A classical problem in additive number theory concerns the growth of a finite set of nonnegative integers. For h sufficiently large,

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the structure of the sumset hA is completely determined (Nathanson [5]), and its cardinality $|hA|$ is a linear function of h.

If $A_1, \dots, A_r$ are finite, nonempty subsets of $\mathbf{N}_0$ and if $h_1, \dots, h_r$ are positive integers, then $h_1A_1 + \cdots + h_rA_r$ is the sumset consisting of all integers of the form $b_1 + \cdots + b_r$, where $b_j \in h_jA_j$ for $j = 1, \dots, r$. For $h_1, \dots, h_r$ sufficiently large, the structure of this "linear form" has also been completely determined (Han, Kirfel, and Nathanson [2]), and its cardinality is a linear function of $h_1, \dots, h_r$.

If A is a finite, nonempty subset of $\mathbf{N}_0^k$, the geometrical structure of the sumset hA is complicated, but the cardinality of hA is a polynomial in h of degree at most k for h sufficiently large (Khovanskii [3]). If the set A is not contained in a hyperplane of dimension $k-1$, then the degree of this polynomial is exactly equal to k.

The sets $\mathbf{N}_0$ and $\mathbf{N}_0^k$ are abelian semigroups, that is, sets with a binary operation, called addition, that is associative and commutative. Let S be an arbitrary abelian semigroup. Without loss of generality, we can assume that S contains an additive identity 0. If A is a finite, nonempty subset of S and h a positive integer, we again define the sumset hA as the set of all sums of h elements of A, with repetitions allowed. Khovanskii [3, 4] made the remarkable observation that the cardinality of hA is a polynomial in h for all sufficiently large h, that is, there exists a polynomial $p(t)$ and an integer h_0 such that $|hA| = p(h)$ for $h \geq h_0$. Khovanskii proved this result by constructing a finitely generated graded module $M = \sum_{h=0}^{\infty} M_h$ over the polynomial ring $\mathbf{C}[t_1, \dots, t_k]$, where $|A| = k$, with the property that the homogeneous component M_h is a vector space over $\mathbf{C}$ of dimension exactly $|hA|$ for all $h \geq 1$. A theorem of Hilbert asserts that $\dim_{\mathbf{C}} M_h$ is a polynomial in h for all sufficiently large h, and this gives the result.

If $A_1, \dots, A_r$ are finite, nonempty subsets of an abelian semigroup S, and if $h_1, \dots, h_r$ are positive integers, then the "linear form" $h_1A_1 + \cdots + h_rA_r$ is the sumset consisting of all elements of S of the form $b_1 + \cdots + b_r$, where $b_j \in h_jA_j$ for $j = 1, \dots, r$. Using a generalization of Hilbert's theorem to finitely generated modules graded by the semigroup $\mathbf{N}_0^r$, Nathanson [6] proved that there exists a polynomial $p(t_1, \dots, t_r)$ such that $|h_1A_1 + \cdots + h_rA_r| = p(h_1, \dots, h_r)$ for all sufficiently large integers $h_1, \dots, h_r$.

The purpose of this note is to give elementary combinatorial proofs of the theorems of Khovanskii and Nathanson that avoid the use of Hilbert polynomials. Our arguments reduce to an easy computation about lattice points in euclidean space.

2. Growth of sumsets

We begin with some geometrical lemmas about lattice points. Let $x = (x_1, \dots, x_k)$ and $y = (y_1, \dots, y_k)$ be elements of $\mathbf{N}_0^k$. Define the *height* of

x by $\mathrm{ht}(x) = \sum_{i=1}^{n} x_i$. Let

$$\begin{aligned} \sigma(h) &= \{x \in \mathbf{N}_0^k : \mathrm{ht}(x) = h\} \\ &= \{(x_1, \ldots, x_k) \in \mathbf{N}_0^k : x_1 + \cdots + x_k = h\}. \end{aligned}$$

The set $\sigma(h)$ is a finite set of lattice points whose cardinality is the number of ordered partitions of h as a sum of k nonnegative integers, and so

$$|\sigma(h)| = \binom{h+k-1}{k-1} = \frac{h^{k-1}}{(k-1)!} + \frac{kh^{k-2}}{2(k-2)!} + \cdots + 1,$$

which is a polynomial in h for fixed k.

We define a partial order on $\mathbf{N}_0^k$ by

$$x \leq y \qquad \text{if } x_i \leq y_i \text{ for all } i = 1, \ldots, k.$$

In $\mathbf{N}_0^2$, for example, $(2,5) \leq (4,6)$ and $(4,3) \leq (4,6)$, but the lattice points $(2,5)$ and $(4,3)$ are incomparable. Thus, the relation $x \leq y$ is a partial order but not a total order. We write $x < y$ if $x \leq y$ and $x \neq y$. If $x \leq y$, then $x + t \leq y + t$ for all $t \in \mathbf{N}_0^k$.

Lemma 1. *Let W be a finite subset of $\mathbf{N}_0^k$, and let $B(h, W)$ be the set of all lattice points $x \in \sigma(h)$ such that $x \geq w$ for all $w \in W$. Then $|B(h, W)|$ is a polynomial in h for all sufficiently large h.*

Proof. Let $x = (x_1, \ldots, x_k) \in \sigma(h)$. Let $W = \{w_1, \ldots, w_m\}$, where $w_j = (w_{1,j}, w_{2,j}, \ldots, w_{k,j}) \in \mathbf{N}_0^k$ for $j = 1, \ldots, m$. Then $x \geq w_j$ for $j = 1, \ldots, m$ if and only if, for all $i = 1, \ldots, k$, we have $x_i \geq w_{i,j}$ for $j = 1, \ldots, m$, that is, $x_i \geq \max\{w_{i,j} : j = 1, \ldots, m\} = w_i^*$ for $i = 1, \ldots, k$. Define $w^* \in \mathbf{N}_0^k$ by $w^* = (w_1^*, \ldots, w_k^*)$. Then

$$\begin{aligned} B(h, W) &= B(h, \{w^*\}) \\ &= \{x \in \mathbf{N}_0^k : \mathrm{ht}(x) = h \text{ and } x \geq w^*\} \\ &= \{x \in \mathbf{N}_0^k : \mathrm{ht}(x - w^*) = h - \mathrm{ht}(w^*) \text{ and } x - w^* \geq 0\} \\ &= \{y + w^* \in \mathbf{N}_0^k : \mathrm{ht}(y) = h - \mathrm{ht}(w^*) \text{ and } y \geq 0\} \\ &= \{w^*\} + \sigma(h - \mathrm{ht}(w^*)), \end{aligned}$$

and so

$$|B(h, W)| = |\sigma(h - \mathrm{ht}(w^*))| = \binom{h - \mathrm{ht}(w^*) + k - 1}{k-1}$$

for $h \geq \mathrm{ht}(w^*)$. This completes the proof. $\square$

An *ideal* in an abelian semigroup is a nonempty set I such that if $x \in I$, then $x + t \in I$ for every element t in the semigroup. In the partially ordered semigroup $\mathbf{N}_0^k$, a nonempty set I is an ideal if and only if $x \in I$ and $y \geq x$ imply $y \in I$. The following result about lattice points and partial orders is known as *Dickson's lemma* [1]. We include a proof for completeness.

Lemma 2. *If I is a ideal in the abelian semigroup $\mathbf{N}_0^k$, then there exists a finite set W^* of lattice points in $\mathbf{N}_0^k$ such that*

$$I = \{x \in \mathbf{N}_0^k : x \geq w \text{ for some } w \in W^*\}.$$

Proof. The proof is by induction on the dimension k. If $k = 1$, then I is a nonempty set of nonnegative integers, hence contains a least integer w. If $x \geq w$, then $x \in I$ since I is an ideal, and so $I = \{x \in \mathbf{N}_0 : x \geq w\}$.

Let $k \geq 2$, and assume that the result holds for dimension $k-1$. We shall write the lattice point $x = (x_1, \ldots, x_{k-1}, x_k) \in \mathbf{N}_0^k$ in the form $x = (x', x_k)$, where $x' = (x_1, \ldots, x_{k-1}) \in \mathbf{N}_0^{k-1}$. Define the projection map $\pi : \mathbf{N}_0^k \to \mathbf{N}_0^{k-1}$ by $\pi(x) = x'$. Let $I' = \pi(I)$ be the image of the ideal I, that is,

$$I' = \{x' \in \mathbf{N}_0^{k-1} : \ (x', x_k) \in I \text{ for some } x_k \in \mathbf{N}_0\}.$$

We have $I' \neq \emptyset$ since $I \neq \emptyset$. Let $x' \in I'$ and $y' \in \mathbf{N}_0^{k-1}$. Since $x' \in I'$, there is a nonnegative integer x_k such that $(x', x_k) \in I$. If $y' \geq x'$, then $(y', x_k) \geq (x', x_k)$ in $\mathbf{N}_0^k$, and so $(y', x_k) \in I$, hence $y' \in I'$. Thus, I' is an ideal in $\mathbf{N}_0^{k-1}$. Since the Lemma holds in dimension $k-1$, there is a finite set $W' \subseteq I'$ such that $x' \in I'$ if and only if $x' \geq w'$ for some $w' \in W'$. Associated to each lattice point $w' \in W'$ is a nonnegative integer $x_k(w')$ such that $(w', x_k(w')) \in I$. Let $m = \max\{x_k(w') : w' \in W'\}$ and $W_m = \{(w', m) : w' \in W'\}$. If $w' \in W'$, then $(w', m) \geq (w', x_k(w'))$ and so $(w', m) \in I$. Therefore, $W_m \subseteq I$.

For $\ell = 0, 1, \ldots, m-1$, we consider the set

$$I'_\ell = \{x' \in \mathbf{N}_0^{k-1} : (x', \ell) \in I\}.$$

If $I'_\ell = \emptyset$, let $W_\ell = \emptyset$. If $I'_\ell \neq \emptyset$, then I'_ℓ is an ideal in $\mathbf{N}_0^{k-1}$, and there is a finite set W'_ℓ such that $x' \in I'_\ell$ if and only if $x' \geq w'$ for some $w' \in W'_\ell$. Let $W_\ell = \{(w', \ell) : w' \in W'_\ell\}$. Then $W_\ell \subseteq I$. We consider the set

$$W^* = \bigcup_{\ell=0}^{m} W_\ell,$$

which is a finite subset of the ideal I.

We shall prove that $x \in I$ if and only if $x \geq w$ for some $w \in W^*$. If $x = (x', x_k) \in I$ and $x_k \geq m$, then $x' \in I'$, hence $x' \geq w'$ for some $w' \in W'$. It follows that

$$x = (x', x_k) \geq (x', m) \geq (w', m),$$

and $(w', m) \in W_m \subseteq W^*$.

If $x = (x', \ell) \in I$ and $0 \leq \ell < m$, then $x' \in I'_\ell$, and so $x' \geq w'$ for some $w' \in W'_\ell$. It follows that

$$x = (x', \ell) \geq (w', \ell),$$

and $(w', \ell) \in W_\ell \subseteq W^*$. This completes the proof. $\square$

Let $x = (x_1, \ldots, x_k)$ and $y = (y_1, \ldots, y_k)$ be lattice points in $\mathbf{N}_0^k$. We define the *lexicographical order* $x \leq_{lex} y$ on $\mathbf{N}_0^k$ as follows: $x \leq_{lex} y$ if either $x = y$ or there exists $j \in \{1, 2, \ldots, k\}$ such that $x_i = y_i$ for $i = 1, \ldots, j-1$ and $x_j < y_j$. This is a total order, so every finite, nonempty set of lattice points contains a smallest lattice point. For example, $(2,5) \leq_{lex} (4,3) \leq_{lex} (4,6)$. If $x \leq_{lex} y$, then $x + t \leq_{lex} y + t$ for all $t \in \mathbf{N}_0^k$. We write $x <_{lex} y$ if $x \leq_{lex} y$ and $x \neq y$

Theorem 1. *Let S be an abelian semigroup, and let A be a finite nonempty subset of S. There exists a polynomial $p(t)$ such that $|hA| = p(h)$ for all sufficiently large h.*

Proof. Let $A = \{a_1, \ldots, a_k\}$, where $|A| = k$. We define a map $f : \mathbf{N}_0^k \longrightarrow S$ as follows: If $x = (x_1, \ldots, x_k) \in \mathbf{N}_0^k$, then

$$f(x) = \sum_{i=1}^{k} x_i a_i.$$

This is well-defined, since each x_i is a nonnegative integer and we can add the semigroup element a_i to itself x_i times. The map f is a homomorphism of semigroups: If $x, y \in \mathbf{N}_0^k$, then $f(x+y) = f(x) + f(y)$. We consider the set

$$\sigma(h) = \{x \in \mathbf{N}_0^k : \mathrm{ht}(x) = h\}.$$

If $x \in \sigma(h)$, then $f(x) \in hA$ and $f(\sigma(h)) = hA$. The map f is not necessarily one-to-one on the set $\sigma(h)$. For any $s \in hA$, there can be many lattice points $x \in \sigma(h)$ such that $f(x) = s$. However, for each $s \in hA$, there is a unique lattice point $u_h(s) \in f^{-1}(s) \cap \sigma(h)$ that is lexicographically smallest, that is, $u_h(s) \leq_{lex} x$ for all $x \in f^{-1}(s) \cap \sigma(h)$. Then

$$|hA| = \left|\{u_h(s) : s \in hA\}\right|.$$

The lattice point $x \in \mathbf{N}_0^k$ will be called *useless* if, for $h = \mathrm{ht}(x)$, we have $x \neq u_h(s)$ for all $s \in hA$. Equivalently, $x \in \mathbf{N}_0^k$ is useless if there exists a lattice point $u \in \sigma(\mathrm{ht}(x))$ such that $f(u) = f(x)$ and $u <_{lex} x$. Let I be the set of all useless lattice points in $\mathbf{N}_0^k$.

We shall prove that I is an ideal in the semigroup $\mathbf{N}_0^k$. Let $x \in I$, $\mathrm{ht}(x) = h$, and $t \in \mathbf{N}_0^k$. Since $x \in I$, there exists a lattice point $u \in \sigma(h)$ such that $f(u) = f(x)$ and $u <_{lex} x$. Then

$$f(u+t) = f(u) + f(t) = f(x) + f(t) = f(x+t),$$

$$u + t <_{lex} x + t,$$

and

$$\mathrm{ht}(u+t) = \mathrm{ht}(u) + \mathrm{ht}(t) = \mathrm{ht}(x) + \mathrm{ht}(t) = \mathrm{ht}(x+t),$$

hence

$$u + t \in \sigma(ht(x + t)).$$

It follows that $x + t$ is useless, hence $x + t \in I$ and I is an ideal of the semigroup $\mathbf{N}_0^k$. We call I the *useless ideal.*

By Dickson's lemma (Lemma 2), there is a finite set W^* of lattice points in $\mathbf{N}_0^k$ such that $x \in \mathbf{N}_0^k$ is useless if and only if $x \geq w$ for some $w \in W^*$. The cardinality of the sumset hA is the number of lattice points in $\sigma(h)$ that are not in the useless ideal I. For every subset $W \subseteq W^*$, we define the set

$$B(h, W) = \{x \in \sigma(h) : x \geq w \text{ for all } w \in W\}.$$

By the principle of inclusion-exclusion,

$$|hA| = \sum_{W \subseteq W^*} (-1)^{|W|} |B(h, W)|.$$

By Lemma 1, for every $W \subseteq W^*$ there is an integer $h_0(W)$ such that $|B(h, W)|$ is a polynomial in h for $h \geq h_0(W)$. Therefore, $|hA|$ is a polynomial in h for all sufficiently large h. This completes the proof. $\square$

3. Growth of linear forms

Let $k_1, \ldots, k_r$ be positive integers, and let $k = k_1 + \cdots + k_r$. We shall write the semigroup $\mathbf{N}_0^k$ in the form

$$\mathbf{N}_0^k = \mathbf{N}_0^{k_1} \times \cdots \times \mathbf{N}_0^{k_r},$$

and denote the lattice point $x \in \mathbf{N}_0^k$ by $x = (x_1, \ldots, x_r)$, where $x_j \in \mathbf{N}_0^{k_j}$ for $j = 1, \ldots, r$. Let $h_j = \operatorname{ht}(x_j)$ for $j = 1, \ldots, r$. We define the *r-height* of x by $\operatorname{ht}_r(x) = (h_1, \ldots, h_r)$. For any positive integers $h_1, \ldots, h_r$, we consider the set

$$\begin{aligned} \sigma(h_1, \ldots, h_r) &= \{x \in \mathbf{N}_0^k : \operatorname{ht}_r(x) = (h_1, \ldots, h_r)\} \\ &= \{(x_1, \ldots, x_r) \in \mathbf{N}_0^k : \operatorname{ht}(x_j) = h_j \text{ for } j = 1, \ldots, r\}. \end{aligned}$$

Then

$$|\sigma(h_1, \ldots, h_r)| = \prod_{j=1}^{r} |\sigma(h_j)| = \prod_{j=1}^{r} \binom{h_j + k_j - 1}{k_j - 1}$$

is a polynomial in the r variables $h_1, \ldots, h_r$ for fixed integers $k_1, \ldots, k_r$.

Lemma 3. *Let $k_1, \ldots, k_r$ be positive integers, and $k = k_1 + \cdots + k_r$. Let W be a finite subset of $\mathbf{N}_0^k = \mathbf{N}_0^{k_1} \times \cdots \times \mathbf{N}_0^{k_r}$, and let $B(h_1, \ldots, h_r, W)$ be the set of all lattice points $x \in \mathbf{N}_0^k$ such that $x \in \sigma(h_1, \ldots, h_r)$ and $x_j \geq w_j$ for all $w = (w_1, \ldots, w_j, \ldots, w_r) \in W$ and $j = 1, \ldots, r$. Then*

$|B(h_1, \ldots, h_r, W)|$ is a polynomial in $h_1, \ldots, h_r$ for all sufficiently large integers $h_1, \ldots, h_r$.

Proof. Let $x = (x_1, \ldots, x_r) \in \mathbf{N}_0^k$. Let W_j be the set of all lattice points $w_j \in \mathbf{N}_0^{k_j}$ such that there exists a lattice point $w \in W$ of the form $w = (w_1, \ldots, w_j, \ldots, w_r)$. Since $x \geq w$ for all $w \in W$ if and only if $x_j \geq w_j$ for all $w_j \in W_j$, it follows that the set $B(h_1, \ldots, h_r, W)$ consists of all lattice points $x = (x_1, \ldots, x_r) \in \mathbf{N}_0^k$ such that $x_j \in B(h_j, W_j)$ for all $j = 1, \ldots, r$. Therefore,

$$|B(h_1, \ldots, h_r, W)| = \prod_{j=1}^{r} |B(h_j, W_j)|.$$

It follows from Lemma 1 that $|B(h_1, \ldots, h_r, W)|$ is a polynomial in the r variables $h_1, \ldots, h_r$ for all sufficiently large integers $h_1, \ldots, h_r$. This completes the proof. □

Theorem 2. *Let S be an abelian semigroup, and let $A_1, \ldots, A_r$ be finite, nonempty subsets of S. There exists a polynomial $p(t_1, \ldots, t_r)$ such that $|h_1A_1 + \cdots + h_rA_r| = p(h_1, \ldots, h_r)$ for all sufficiently large integers $h_1, \ldots, h_r$.*

Proof. For $j = 1, \ldots, r$, let $|A_j| = k_j$ and

$$A_j = \{a_{1,j}, \ldots, a_{k_j,j}\}.$$

Let $k = k_1 + \cdots + k_r$. We consider lattice points

$$x = (x_1, \ldots, x_r) \in \mathbf{N}_0^k = \mathbf{N}_0^{k_1} \times \cdots \times \mathbf{N}_0^{k_r},$$

where

$$x_j = (x_{1,j}, \ldots, x_{k_j,j}) \in \mathbf{N}_0^{k_j}.$$

Define the semigroup homomorphism $f : \mathbf{N}_0^k \to S$ as follows: If $x = (x_1, \ldots, x_r) \in \mathbf{N}_0^k$, then

$$f(x) = \sum_{j=1}^{r} \sum_{i=1}^{k_j} x_{i,j} a_{i,j}.$$

A lattice point $x \in \mathbf{N}_0^k$ will be called *r-useless* if there exists a lattice point $u \in \sigma(\mathrm{ht}_r(x))$ such that $f(u) = f(x)$ and $u <_{lex} x$. As in the proof of Theorem 1, the set I_r of useless lattice points in $\mathbf{N}_0^k$ is an ideal. By Lemma 2, there is a finite set W^* that generates I_r in the sense that $x \in \mathbf{N}_0^k$ is r-useless if and only if $x \geq w$ for some $w \in W^*$.

Let $(h_1, \ldots, h_r) \in \mathbf{N}_0^r$ and

$$\sigma(h_1, \ldots, h_r) \quad = \quad \{(x_1, \ldots, x_r) \in \mathbf{N}_0^k : \mathrm{ht}(x_j) = h_j \text{ for } j = 1, \ldots, r\}.$$

Then $f(\sigma(h_1,\ldots,h_r)) = h_1A_1 + \cdots + h_rA_r$, and $|h_1A_1 + \cdots + h_rA_r|$ is the number of lattice points in $\sigma(h_1,\ldots,h_r)$ that are not useless. For every subset $W \subseteq W^*$, we define the set

$$B(h_1,\ldots,h_r,W) = \{x \in \sigma(h_1,\ldots,h_r) : x \geq w \text{ for all } w \in W.\}$$

By the principle of inclusion-exclusion,

$$|h_1A_1 + \cdots + h_rA_r| = \sum_{W \subseteq W^*} (-1)^{|W|} |B(h_1,\ldots,h_r,W)|.$$

By Lemma 3, for all sufficiently large integers $h_1,\ldots,h_r$, the function $|B(h_1,\ldots,h_r,W)|$ is a polynomial in $h_1,\ldots,h_r$, and so $|h_1A_1+\cdots+h_rA_r|$ is a polynomial in $h_1,\ldots,h_r$. This completes the proof. $\square$

Remark. It would be interesting to describe the set of polynomials $f(t)$ such that $f(h) = |hA|$ for some finite set A and sufficiently large h. Similarly, one can ask for a description of the set of polynomials $f(t_1,\ldots,t_r)$ such that $f(h_1,\ldots,h_r) = |h_1A_1+\cdots+h_rA_r|$, where $A_1,\ldots,A_r$ are finite subsets of a semigroup S.

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Melvyn B. NATHANSON
Department of Mathematics
Lehman College (CUNY)
Bronx, New York 10468
USA
E-mail : `nathansn@alpha.lehman.cuny.edu`

Imre Z. RUZSA
Alfréd Rényi Mathematical Institute
Hungarian Academy of Sciences
Budapest, Pf. 127
H-1364 Hungary
E-mail : `ruzsa@renyi.hu`