Journal de Théorie des Nombres de Bordeaux **14** (2002), 585–601

From Planck to Ramanujan: a quantum $1/f$ noise in equilibrium

par Michel PLANAT

Résumé. J'introduis un nouveau modèle de bosons thermiques sans masse ; il prédit, pour les fluctuations, un spectre hyperbolique aux basses fréquences. On trouve que la fonction de partition par mode est la fonction génératrice d' Euler pour le nombre de partitions $p(n)$. Les quantités thermodynamiques ont une structure arithmétique profonde : elles sont données par des séries, dont les coefficients de Fourier sont les fonctions sommatoires $\sigma_k(n)$ des puissances des diviseurs de n, avec $k=-1$ pour l'énergie libre, $k=0$ pour le nombre de particules, et $k=1$ pour l'énergie interne. Les contributions de basse fréquence sont calculées par l'usage de transformées de Mellin. En particulier, l'énergie interne par mode diverge comme $\frac{\tilde{E}}{kT}=\frac{\pi^2}{6x}$, avec $x=\frac{h\nu}{kT}$, au contraire de l'énergie de Planck $\tilde{E}=kT$. La théorie est appliquée à la correction de la loi de rayonnement du corps noir et au solide de Debye. Les fluctuations fractionnaires de l'énergie présentent un spectre en $1/\nu$ aux basse fréquences. On en déduit un modèle satisfaisant pour les fluctuations d'un résonateur à quartz. On rappelle aussi les résultats essentiels de la théorie mathématique des partitions de Ramanujan-Rademacher.

Abstract. We describe a new model of massless thermal bosons which predicts an hyperbolic fluctuation spectrum at low frequencies. It is found that the partition function per mode is the Euler generating function for unrestricted partitions $p(n)$. Thermodynamical quantities carry a strong arithmetical structure: they are given by series with Fourier coefficients equal to summatory functions $\sigma_k(n)$ of the power of divisors, with $k=-1$ for the free energy, $k=0$ for the number of particles and $k=1$ for the internal energy. Low frequency contributions are calculated using Mellin transform methods. In particular the internal energy per mode diverges as $\frac{\tilde{E}}{kT}=\frac{\pi^2}{6x}$ with $x=\frac{h\nu}{kT}$ in contrast to the Planck energy $\tilde{E}=kT$. The theory is applied to calculate corrections in black body radiation and in the Debye solid. Fractional energy fluctuations are found to show a $1/\nu$ power spectrum in the

Manuscrit reçu le 21 juin 2001.

low frequency range. A satisfactory model of frequency fluctuations in a quartz crystal resonator follows. A sketch of the whole Ramanujan–Rademacher theory of partitions is reminded as well.

1. Introduction

According to the equipartition law of statistical mechanics, the available noise power $\tilde{P}$ in the frequency interval $d\nu$ is equal to $kTd\nu$ [15]: this result is essentially Nyquist's theorem for the voltage noise $\langle v^2 \rangle$ at a resistor R, i.e., $\tilde{P} = \frac{\langle v^2 \rangle}{4R} = kTd\nu$, where $\langle\rangle$ means the average value. Since $\tilde{E} = kT$ is the mean energy per mode, Nyquist proposed to add quantum corrections as $\frac{\tilde{E}}{kT} = p(x)$, with the Planck factor $p(x) = \frac{x}{\exp(x)-1}$ in which $x = h\nu/kT$. This result was generalized as $\frac{\tilde{E}}{kT} = p(x) + \frac{x}{2}$ to account for the zero point energy. There are still controversies concerning the physical relevance of these relations: the Planck factor removed the ultraviolet divergence but this was reincorporated in the frame of quantum electrodynamics [1],[4].

At the present stage, quantum statistical mechanics does not include infrared corrections of the $1/\nu$ type. The infrared catastrophe was studied previously in non-stationary processes such as the scattering of electrons in an atomic field [3],[9].

Here we derive an alternative approach in which $1/\nu$ noise is a property of non-degenerate bosons in equilibrium. We first observe that the quantum mechanical partition function of a boson gas, with equally spaced energy levels, is the Euler generating function of $p(n)$: the number of indiscernible collections of the integer n. It relates to elliptic modular functions which are very exactly known. As a result the main contribution in the mean energy per mode is the infrared term $\frac{\tilde{E}}{kT} = \frac{\pi^2}{6x}$ instead of unity. The new law also leads to $1/\nu$ fractional energy fluctuations of the whole gas.

Using the new approach and the density of states of the conventional approach we calculate the corrections to black-body radiation laws, including the density of photons, the emissivity and infrared fluctuations. We also apply the calculations to the phonon gas in a quartz resonator.

The partition function Z of a non degenerate boson gas is given from

$$\ln Z = -\sum_s \ln[1 - \exp(-\beta\epsilon_s)], \tag{1}$$

where the summation is performed over all the states s of the assembly. In the conventional approach it is thus considered that the partition function $\tilde{Z}$ per mode of frequency $\epsilon_s = h\nu_s$ is such that $\ln \tilde{Z} = -\ln[1 - \exp(-\beta\epsilon_s)]$.

In black-body radiation one accounts for the wave character of the quantum states by counting the number l of wavelengths in a cubic box of size L

$$l^2 = \frac{\nu_s^2 L^2}{c^2} = l_1^2 + l_2^2 + l_3^2, \tag{2}$$

where the summation (1) should be performed over all integers $l_1,\ l_2,\ l_3$ obeying (2).

This can be achieved by removing the discreteness of energy levels and replacing the sum (1) by an integral

$$\ln Z = -\int_0^{+\infty} D(\nu) \ln[1 - \exp(-\beta h\nu)] d\nu, \tag{3}$$

with $D(\nu) = 2 \times \frac{4\pi V \nu^2}{c^3}$ the density of states [11]: the factor 2 occurs due to the two degrees of freedom of polarization, c is the light velocity and $V = L^3$ the volume of the cavity.

From now we consider that with each mode is associated a set of equally spaced energy levels $nh\nu$, n integer, so that the partition per mode becomes

$$\ln \tilde{Z} = -\sum_{n\geq 1} \ln[1 - \exp(-n\beta h\nu]. \tag{4}$$

As shown in Section (2.1) this accounts for new multiparticle microstates not considered so far.

The summation above is well known in number theory and can be very accurately described using elliptic modular functions. At is will be shown, there are drastic consequences in the low frequency part of the spectrum, while the high frequency part is left unchanged.

In the following the thermodynamical quantities will be defined as usual

$$N = -\frac{\partial \ln Z}{\partial(\beta\epsilon_s)}, \text{ the occupation number,} \tag{5}$$

$$E = -\frac{\partial \ln Z}{\partial \beta}, \text{ the internal energy,} \tag{6}$$

$$S = \frac{\partial F}{\partial T}, \text{ the entropy,} \tag{7}$$

$$u = kT^2 \frac{\partial \ln Z}{\partial T}, \text{ the spectral energy density,} \tag{8}$$

$$F = -kT \ln Z, \text{ the free energy,} \tag{9}$$

$$\epsilon^2 = kT^2 \frac{\partial E}{\partial T}, \text{ the fluctuations of the internal energy.} \tag{10}$$

In all the paper the subscript $\sim$ will indicate that we restrict the calculation to one single mode.

2. Thermodynamics of the Euler gas

2.1. Euler generating function. The partition function per mode $\tilde{Z}$ in (4) can be written in the Euler form [14]

$$\tilde{Z}(y) = \prod_{n\geq 1} \frac{1}{1-y^n} = \sum_{n\geq 1} p(n) y^n, \tag{11}$$

with $y = \exp(-x)$ and $x = \frac{h\nu}{kT}$. This is equivalent to the Boltzmann summation

$$\tilde{Z}(y) = \sum_{n\geq 1} p(n) \exp(-nx), \tag{12}$$

where $p(n)$ is the degeneracy parameter of the energy level $nh\nu$. It is known in number theory as the number of unrestricted partitions of the integer n, that is the number of different ways of calculating n as a sum of integers.

For example with $n = 4$ we have $p(4) = 5$ and the corresponding indiscernible collections are

$$\begin{aligned} &4 = 4+0+0+0 \ (a), \\ &4 = 1+1+1+1 \ (b), \ 4 = 2+2+0+0 \ (c), \\ &4 = 3+1+0+0 \ (d), \ 4 = 2+1+1+0 \ (e). \end{aligned} \tag{13}$$

This can be pictured in terms of the energy levels. The collection (a) means one particle on the level of index 4 and the three remaining particles on the ground state of index 0, i.e. $4h\nu = 1 \times 4h\nu + 3 \times 0h\nu$. This collection is the only one considered in the conventional approach. The other microstates from (b) to (d) correspond to different possibilities of bunching of the particles. Collection (b) means the four particles on the level of index 1, i.e. $4h\nu = 4 \times h\nu$, collection (c) means two particles on the level 2 and two particles on the ground state, i.e. $4h\nu = 2 \times 2h\nu + 2 \times 0h\nu$, and so on.

Properties of Euler generating function were studied in full details by Ramanujan [10] in 1918 and completed by Rademacher [14] in 1973. An important result is the asymptotic formula

$$p(n) \sim \frac{1}{4n\sqrt{3}} \exp(\pi\sqrt{2n/3}) \text{ when } n \to \infty. \tag{14}$$

2.2. Riemann zeta function and the free energy of the Euler gas. The partition function $\tilde{Z}(x)$ defined in (4) is related to the Riemann zeta function $\zeta(s)$ through the Mellin transform as follows (Ref. [12], Eq. (6.3) p. 464)

$$\Gamma(s)\zeta(s)\zeta(s+1) = \int_0^\infty (-\ln \tilde{Z}(x)) x^{s-1} dx. \tag{15}$$

Introducing $\sigma_k(n)$ as the sum of k^{th} powers of the divisors of n and the Dirichlet series

$$\zeta(s)\zeta(s-k) = \sum_{n\geq 1} \frac{\sigma_k(n)}{n^s}, \tag{16}$$

and computing the inverse Mellin transform one obtains ([12], p. 467) the free energy $\tilde{F}$ as follows

$$\frac{\tilde{F}}{kT} = -\ln \tilde{Z}(x) = \sum_{n\geq 1} \ln(1-\exp(-nx)) = -\sum_{n\geq 1} \sigma_{-1}(n)\exp(-nx). \tag{17}$$

with $\sigma_{-1}(n) = \sigma_1(n)/n$. There is thus a close relationship between the arithmetic of $p(n)$ and that of divisors. This will be confirmed in the derivation of the other thermodynamical quantities.

Since the main contributions to $\tilde{Z}(x)$ are given by the poles at $s=0$ of $\Gamma(s)$ and $\zeta(s+1)$ and at $s=1$ of $\zeta(s)$, the free energy may be approximated in the low frequency part of the spectrum ([5], p. 58)

$$\frac{\tilde{F}}{kT} \simeq -\frac{\pi^2}{6x} - \frac{1}{2}\ln(\frac{x}{2\pi}) + \frac{x}{24}, \tag{18}$$

with the error term $\sum_{l\geq 1} \ln(1-\exp(-4\pi^2 l/x))$.

2.3. Dedekind eta function and the internal energy of the Euler gas. One easily shows that the Mellin transform of the occupation number $\tilde{N}(x)$ is $\Gamma(s)\zeta(s)^2$. A similar derivation to the one performed in Section (2.2) leads to

$$\tilde{N}(x) = -\frac{\partial(\ln \tilde{Z}(x))}{\partial(nx)} = \sum_{n\geq 1} \frac{1}{\exp(nx)-1} = \sum_{n\geq 1} \sigma_0(n)\exp(-nx). \tag{19}$$

In the low frequency region one gets (see [6], p. 27)

$$\tilde{N}(x) \simeq \frac{-\ln x + \gamma}{x}. \tag{20}$$

where $\gamma(x) \simeq 0.577$ is Euler constant.

The Mellin transform of the internal energy $\tilde{E}(x)$ is $\Gamma(s)\zeta(s)\zeta(s-1)$ and from the same method than above

(21)

$$\tilde{E}(x) = -\frac{\partial(\ln \tilde{Z}(x))}{\partial(\beta)} = h\nu \sum_{n\geq 1} \frac{n}{\exp(nx)-1} = h\nu \sum_{n\geq 1} \sigma_1(n)\exp(-nx).$$

An alternative derivation involves elliptic modular aspects. According to Ninham [12]: *All mathematics is a tautology, and all physics uses mathematics to look in different ways at a fundamental problem of philosophy - how to bridge the discrete and continuous.*

The link between the modular group and the Euler generating function is from the equality [14]

$$\tilde{Z}(y) = \prod_{n\geq 1} \frac{1}{1-y^n} = \sum_{n\geq 1} p(n) y^n = \frac{\exp(i\pi\tau/12)}{\eta(\tau)}, \tag{22}$$

where the domain of integration of $\tilde{Z}(y)$ is taken to be the upper half complex plane of the new variable τ

$$y = \exp(2i\pi\tau),\ \Im(\tau) > 0. \tag{23}$$

Here we have

$$\Im(\tau) = \frac{x}{2\pi},\ x = \frac{h\nu}{kT}. \tag{24}$$

As shown in Section (5) Dedekind eta function acts on the full modular group $SL(2, Z)$. At this stage we do not enter into the full ramifications of the theory and only emphasize the connexion to the modular Eisenstein function

$$G_2(\tau) = \sum_{m,n}{}' \frac{1}{(m\tau + n)^2}, \tag{25}$$

where the summation is performed over all non-zero relative integers m and n and $\Im(\tau) > 0$. It can be shown ([16], p. 29) that $G_2(\tau)$ connects to the logarithmic derivative of $\eta(\tau)$

$$G_2(\tau) = -4i\pi \frac{d(\ln(\eta(\tau)))}{d\tau}, \tag{26}$$

with the Fourier expansion

$$G_2(\tau) = 2\zeta(2) + 2(2i\pi)^2 \sum_{n\geq 1} \sigma_1(n) \exp(2i\pi n\tau). \tag{27}$$

Using (22), (23) and (25)-(27) the relation (21) is easily recovered.

The low frequency expansion of internal energy is as follows

$$\frac{\tilde{E}}{kT} \simeq \frac{\pi^2}{6x} - \frac{1}{2} + \frac{x}{24}, \tag{28}$$

instead of the Planck result $\tilde{E} \simeq kT$.

One can also compute the entropy $\tilde{S}$ from

$$\frac{\tilde{S}}{k} = \frac{\tilde{E}}{kT} + \ln(\tilde{Z}) = \frac{\tilde{E} - \tilde{F}}{kT} = \sum_{n\geq 1} \sigma_1(n)\,(x + 1/n) \exp(-nx)$$

$$\simeq \frac{\pi^2}{3x} + \frac{1}{2} \ln \frac{x}{2\pi} - \frac{1}{2} \text{ when } x \to\ 0. \tag{29}$$

At very low frequency it results that the internal energy equals the opposite of free energy and half the entropy.

3. Application to black-body radiation

3.1. Stefan-Boltzmann constant revisited. The Stefan-Boltzmann constant is an integrated measure of the emissivity of a black body [11]. In the conventional approach the partition function is calculated from the integral (3) using the density of states $D(\nu) = \frac{8\pi V \nu^2}{c^3}$ that is

$$\ln Z = 8\pi V \left(\frac{kT}{ch}\right)^3 \times 2\zeta(4), \tag{30}$$

with $\zeta(4) = \pi^4/90$ and we used the Mellin integral formula

$$\zeta(s+1) = -\frac{1}{\Gamma(s)} \int_0^\infty x^{s-1} \ln(1-\exp(-x))dx. \tag{31}$$

The Stefan-Boltzmann constant σ_{SB} is defined from the free energy

$$F = -kT \ln Z = -\frac{4\sigma_{\mathrm{SB}}}{3c} V T^4 \text{ with } \sigma_{\mathrm{SB}} = \tfrac{2\pi^5 k^4}{15 c^2 h^3}. \tag{32}$$

If instead of (30) one uses the general formula

$$\ln Z = -8\pi V \left(\frac{kT}{ch}\right)^3 \int_0^\infty x^2 \sum_{n\geq 1} \ln(1-\exp(-nx))dx, \tag{33}$$

the interchange of the integral and the sum leads to

$$\ln Z = -8\pi V \left(\frac{kT}{ch}\right)^3 \times 2\zeta(4) \times \left(\sum_{n\geq 1} \frac{1}{n^3}\right). \tag{34}$$

As a result we find a free energy (and a modified Stefan-Boltzmann constant) in excess with a factor $\sum_{n\geq 1} \frac{1}{n^3} = \zeta(3) \simeq 1.20$. If one uses the alternative derivation in terms of the divisors one recovers the mathematical formula (16) with $s = 3$ and $k = -1$.

3.2. The density of photons. The number of photons in the bandwidth $d\nu$ is

$$dN(\nu, T) = D(\nu)\tilde{N}(\nu, T)d\nu, \tag{35}$$

with the occupation number $\tilde{N}(\nu, T) = (\exp(\beta h\nu) - 1)^{-1}$ in the conventional approach. Integrating one gets per unit volume

$$\frac{N(T)}{V} = 8\pi \left(\frac{kT}{ch}\right)^3 \int_0^{+\infty} \frac{x^2 dx}{\exp(x) - 1} = 8\pi \left(\frac{kT}{ch}\right)^3 \times 2\zeta(3). \tag{36}$$

In the general approach the occupation number is defined from the summation (19) and we need to evaluate

$$\int_0^\infty x^2 \sum_{n\geq 1} \frac{1}{\exp(nx)-1} = 2\zeta(3)\left(\sum_{n\geq 1} n^{-3}\right) = 2\zeta(3)^2. \tag{37}$$

This is in excess of a factor $\zeta(3) \simeq 1.20$ as for the free energy. If one compares the calculation in terms of divisors one recovers the mathematical formula (16) with $s = 3$ and $k = 0$.

3.3. Planck radiation formula revisited. The energy within the bandwidth $d\nu$ is defined as

$$dE(\nu, T) = D(\nu)\tilde{E}(\nu, T)d\nu = u(\nu, T)d\nu, \tag{38}$$

with $\tilde{E}(\nu, T) = h\nu(\exp(\beta h\nu) - 1)^{-1}$ in the conventional approach and with $u(\nu, T)$ the energy spectral density. We get the Planck radiation formula

$$u(\nu, T) = \frac{8\pi hV}{c^3}\frac{\nu^3}{\exp(\beta h\nu) - 1}. \tag{39}$$

The black-body emissivity is defined as

$$e_b(\nu, T) = \frac{c}{4V}u(\nu, T). \tag{40}$$

At very low frequency the conventional result leads to the Rayleigh-Jeans formula

$$[e_b(\nu, T)]_{\mathrm{RJ}} = 2\pi\frac{k}{c^2}\nu^2 T, \tag{41}$$

which is independent of Planck constant and is proportional to the inverse of the square of wavelength $\lambda = c/\nu$.

In the new approach the emissivity is

$$e_b(\nu, T) = \frac{2\pi h}{c^2}\sum_{n\geq 1}\frac{n\nu^3}{\exp(n\beta h\nu) - 1}. \tag{42}$$

At very low frequency one uses (28) with the result

$$[e_b(\nu, T)]_{\mathrm{LF}} = \frac{\pi^3}{3}\frac{k^2}{c^2 h}\nu T^2. \tag{43}$$

Thus the $\nu^2 T$ dependence is replaced by the νT^2 dependence, the low frequency emissivity now depends on the Planck constant and on the inverse wavelength; there is a ratio $\frac{\pi^2}{6x}$ between the new result and the one predicted by the Rayleigh-Jeans formula.

3.4. Radiative atomic transitions. Let us now consider the equilibrium between atoms and a radiation field, allowing the emission or absorption of photons of frequency

$$\nu = \nu_1 - \nu_2, \tag{44}$$

where $\epsilon_2 = h\nu_2$ is the energy in the upper state and $\epsilon_1 = h\nu_1$ in the lower state.

The conventional theory, as derived for the first time by Einstein[11], states that the rate at which atoms make a transition $1 \to 2$ in which one photon is absorbed is equal to the rate at which atoms emit photons, so that

$$B_{12}N_1u(\nu) = A_{21}N_2 + B_{21}N_2u(\nu), \tag{45}$$

where N_1 and N_2 are the occupation numbers of atoms in levels 1 and 2, A_{21} is the spontaneous absorption rate, B_{12} is the induced emission rate and $u(\nu)$ is the energy density in the radiation field as given in (39).

In thermal equilibrium the occupation numbers in states 1 and 2 obey the Boltzmann law

$$\frac{N_2}{N_1} = \exp(-\frac{h\nu}{kT}). \tag{46}$$

Using (46) and (39) one gets the well known formulas

$$A_{21} = A,\ B_{12} = B_{21} = B \text{ and } \tfrac{A}{B} = \tfrac{8\pi h}{\lambda^3}, \tag{47}$$

where $\lambda = c/\nu$ is the wavelength of the radiation field.

If one uses the general formula one gets low frequency corrections in the spontaneous to stimulated emission ratio. Using the low frequency expression (28) for the internal energy this yields

$$\left[\frac{A}{B}\right]_{\mathrm{LF}} = \frac{A}{B} \times \frac{\pi^2}{6x} = \frac{4\pi^3 kT}{3c\lambda^2}. \tag{48}$$

The A/B low frequency ratio now depends on the inverse of the square of wavelength λ and is independent of the Planck constant h, in contrast to the h/λ^3 dependence of the standard ratio. The spontaneous to stimulated absorption rate is enhanced a factor $\frac{\pi^2}{6x}$ over the conventional one.

3.5. Einstein's fluctuation law revisited. According to the conventional Einstein's approach [13] the energy fluctuations of a system in equilibrium within a larger system of temperature T are

$$\epsilon^2 = \left\langle (E - \langle E\rangle)^2 \right\rangle = -\frac{\partial \langle E\rangle}{\partial \beta} = kT^2 \frac{\partial \langle E\rangle}{\partial T}. \tag{49}$$

One can reformulate this relation for the fluctuations of the energy $dE = u(\nu, T)d\nu$ in the bandwidth $d\nu$

$$d\epsilon^2 = kT^2 \frac{\partial u}{\partial T} d\nu = S_u(\nu) d\nu, \tag{50}$$

with u the energy spectral density and $S_u(\nu)$ the power spectral density of the fluctuations of u. One gets([13], p. 429)

$$d\epsilon^2 = (h\nu u + \frac{c^3}{8\pi\nu^2 V} u^2) V d\nu. \tag{51}$$

The first term on the right hand side is the one corresponding to the high frequency part of the spectrum (Wien's law): it is of a pure quantum nature and is corpuscular-like; the second one corresponds to the low frequency (Rayleigh-Jeans) region: it is purely classical and wavelike. In the low frequency part of the spectrum they are fractional energy fluctuations of the random walk type

$$\left[\frac{S_u(\nu)}{u^2}\right]_{\text{RJ}} = \frac{c^3}{8\pi V} \frac{1}{\nu^2}. \tag{52}$$

In the new approach one uses the low frequency energy density $u(\nu, T) \simeq \frac{4\pi^3 V}{3c^3 h} \frac{\nu d\nu}{\beta^2}$ so that instead of (52) one gets

$$\left[\frac{S_u(\nu)}{u^2}\right]_{\text{LF}} \frac{3}{2} \frac{hc^3}{\pi^3 V} \frac{1}{kT\nu}. \tag{53}$$

This is the announced quantum $1/\nu$ fluctuation spectrum. There is a reduced low frequency noise and the ratio between the new result and the Einstein-Rayleigh-Jeans one is $\frac{12x}{\pi^2}$.

4. Application to a phonon gas and to the $1/f$ frequency noise of a quartz resonator

4.1. The specific heat of a phonon gas revisited. The properties of the phonon gas are quite similar to those of the photon gas except for the new form of the density of modes as $g(\nu) = \frac{12\pi V}{c_{\text{ph}}^3}\nu^2$ with $\frac{3}{c_{\text{ph}}^3} = \frac{2}{c_t^3} + \frac{1}{c_l^3}$ where c_{ph} represents the average wave velocity and c_t and c_l are the transverse and longitudinal velocities for an isotropic solid [11]. The maximal vibrational frequency ν_m (Debye frequency) is defined from the total number of allowed quantum states

$$3N_0 = \frac{12\pi V}{c_{\text{ph}}^3} \int_0^{\nu_{\text{m}}} \nu^2 d\nu \text{ that is } \nu_m = \left(\frac{3N_0 c_{\text{ph}}^3}{4\pi V}\right)^{1/3}, \tag{54}$$

where N_0 is the number of atoms in the volume V.

In the conventional theory [11] we get

$$\ln Z_{\mathrm{ph}} = -\frac{9N_0}{\nu_{\mathrm{m}}^3}\int_0^{v_{\mathrm{m}}} \nu^2 \ln\left[1-\exp(-\beta h\nu)\right] d\nu. \tag{55}$$

The internal energy follows from the formula

$$E = \frac{9RT}{x_{\mathrm{m}}^3} D(x_{\mathrm{m}}), \tag{56}$$

and the constant volume specific heat $C_{\mathrm{v}} = \partial E/\partial T$ equals

$$C_{\mathrm{v}} = 3R\left[D(x_{\mathrm{m}}) - x_{\mathrm{m}} D'(x_{\mathrm{m}})\right], \tag{57}$$

with $D(x_{\mathrm{m}}) = \frac{3}{x_{\mathrm{m}}^3}\int_0^{x_{\mathrm{m}}} \frac{x^3 dx}{\exp(x)-1}$ the Debye function, and $x_{\mathrm{m}} = \frac{\theta_{\mathrm{D}}}{T}$, with $\theta_{\mathrm{D}} = h\nu_{\mathrm{m}}/k$ the Debye characteristic temperature.

The case $T\theta_{\mathrm{D}}$, $D(\theta_{\mathrm{D}}) \to 1$ corresponds to the Dulong-Petit value $C_{\mathrm{v}} \sim 3R$. At very low temperatures one gets the cubic temperature dependence $C_{\mathrm{v}} \sim \frac{4\pi^4}{5} \times 3R\left(\frac{T}{\theta_{\mathrm{D}}}\right)^3$.

In the new approach

$$\ln Z_{\mathrm{ph}} = -\frac{9N_0}{\nu_{\mathrm{m}}^3}\int_0^{v_{\mathrm{m}}} \sum_{n\geq 1} \nu^2 \ln\left[1-\exp(-n\beta h\nu)\right] d\nu. \tag{58}$$

Debye results are found unchanged except for an extra multiplicative factor in the specific heat as was the case for the integrated emissivity in Section (3.1) that is

$$\frac{[C_{\mathrm{v}}]_{\mathrm{new}}}{C_{\mathrm{v}}} = \zeta(3) \sim 1.20. \tag{59}$$

At very low temperatures the electronic contribution to the specific heat which decreases as T, dominates the lattice contribution, which decreases as T^3. This is accounted for in the conventional way.

4.2. 1/f noise in a quartz resonator. Specific heat is involved in the energy fluctuations of a canonical ensemble from the relation

$$\epsilon^2 = kT^2 C_{\mathrm{v}}. \tag{60}$$

The relative energy fluctuations follows as $\frac{\epsilon^2}{E^2} = \left(\frac{2}{3N_0}\right)^{1/2}$ which is of the order 10^{-11}. The Avogadro number equals $N_0 = 6.02 \times 10^{23}$.

For energy fluctuations in the bandwidth $d\nu$, the main difference with the conventional theory lies in the low frequency region, as was the case of the photon gas. We find the quantum $1/\nu$ formula

$$\left[\frac{S_u(\nu)}{u^2}\right]_{\mathrm{LF}} = \frac{9hc_{\mathrm{ph}}^3}{4\pi^3 V}\frac{1}{kT\nu} = \frac{A_{\mathrm{ph}}}{V\nu}. \tag{61}$$

The method can be used to predict fractional frequency fluctuations in a quartz crystal resonator from the formula [1][7]-[8]

$$\frac{S_\omega(\nu)}{\omega^2} = \frac{1}{4Q^4}\frac{A_{\text{ph}}}{V\nu} = \frac{h_{-1}}{\nu}. \tag{62}$$

where ω and Q are the frequency and quality factor of the resonator. Using $c_{\text{ph}} \sim 3.5 \times 10^3\ m/s$, we find $A_{\text{ph}} \sim 5 \times 10^{-4}$. For a 5 MHz P5 quartz crystal resonator with $Q \sim 2 \times 10^6$, the active region under the electrodes has thickness $t = 5\lambda/2 \sim 3\ mm$, and section $S \sim 3\ cm^2$, that is $V \sim 1\ cm^3$. The resulting $1/\nu$ factor is $h_{-1} = \frac{A_{\text{ph}}}{4Q^4V} \sim \frac{10^2}{Q^4} \sim 6 \times 10^{-24}$. This is the order of magnitude found in experiments [7].

5. Ramanujan–Rademacher theory of partitions: a short reminder

Besides the low frequency approximations encountered in Section (2) there is a an exact method to calculate the number of partitions $p(n)$ first discovered by Ramanujan [10] and improved by Rademacher [14] thanks to an integration along Ford circles in the complex half plane. For completeness we remind here the main points of the theory from which the results in Section (2) may also be derived .

From well-known mathematical arguments [10] (p. 113) one can get the leading term for the case $0 < y < 1$ and $y \to 1$ from the expression

$$\ln \tilde{Z}(y) \sim \frac{\pi^2}{6(1-y)}. \tag{63}$$

The use of $y = \exp(-h\nu/kT)$ corresponds to the low frequency approximation at $\nu \to 0$, that is $1 - y = 1 - \exp(-h\nu/kT) \sim h\nu/kT$. This leads to the leading low frequency term in the free energy (18) and internal energy (28).

They are similar formulas associated with rational points which are located at

$$y_{pq} = \exp(2i\pi\frac{p}{q}), \tag{64}$$

on the unit circle $|y| = 1$. The leading term in the expansion of $\tilde{Z}(y)$ corresponds to the fundamental mode $\frac{p}{q} = \frac{1}{1}$.

The general method to compute rational contributions is a master piece of the mathematics of the 20th century([10]), ([14]). It uses the connection of $\tilde{Z}(y)$ to the elliptic modular functions.

[1] To establish the formula one writes the equation for a lossy harmonic oscillator and one postulates that the $1/\nu$ fluctuations are present in the loss coefficient.

5.1. The fundamental contribution. To compute the contribution of the fundamental point 1/1 of the unit circle $|y| = 1$ one uses the property ([2], p. 96, [5], p. 58)

$$\tilde{Z}(y) = \frac{y^{1/24}}{\sqrt{2\pi}}(\ln\frac{1}{y})^{1/2}\exp\left[\frac{\pi^2}{6\ln\frac{1}{y}}\right]\tilde{Z}(y') \text{ with } y' = \exp\left[\frac{4\pi^2}{\ln y}\right]. \tag{65}$$

In the low frequency region $y = \exp(-x) \sim 1$ so that $y' = \exp(-4\pi^2/x) \sim 0$ and $\tilde{Z}(y') \sim 1$. Low frequency approximations of the free energy (18) and of the internal energy (28) follow. There are similar formulas associated with the other rational points of the circle as shown below.

To get the leading term in $p(n)$ one uses the Cauchy formula

$$p(n) = \frac{1}{2i\pi}\oint\frac{\tilde{Z}(y)}{y^{n+1}}dy, \tag{66}$$

where $\oint$ means an arbitrary closed loop encircling the origin.

Substituting (65) in (66) with $\tilde{Z}(y') = 1$ one can obtain

$$p(n) = \frac{1}{2\pi\sqrt{2}}\frac{d}{dn}(\frac{\exp(K\lambda_n)}{\lambda_n}) \text{ with } \lambda_n = \sqrt{n-\tfrac{1}{24}} \text{ and } K = \pi\sqrt{\tfrac{2}{3}}. \tag{67}$$

This includes (14) in the limit $n \to \infty$ but is much more accurate.

5.2. Farey contributions and Ford circles. From now on we extend the domain of definition of $\tilde{Z}(y)$ to the complex plane and we introduce the new variable τ and the Dedekind eta function $\eta(\tau)$ as defined in (22)-(24).

It can be shown that $\eta(\tau)$ is a modular form of degree $-1/2$ on the full modular group. It acts on the generators of such a group through the relations [2] [2]

$$\eta(\tau+1) = \exp(i\pi/12)\eta(\tau);\ \eta(-1/\tau) = (\eta/i)^{1/2}\eta(\tau). \tag{68}$$

To express the partition function one uses the Cauchy formula

$$p(n) = \frac{1}{2i\pi}\oint\frac{\tilde{Z}(y)}{y^{n+1}}dy = \int_{\tau_0}^{\tau_0+1}\tilde{Z}\left[\exp(2i\pi\tau)\right]\exp(-2i\pi\tau n)d\tau. \tag{69}$$

In the third term above this corresponds to a path of unit length starting at an arbitrary point in $\mathcal{H}$.

[2]For more general modular transformations, we have

$$\eta(\frac{p\tau+p'}{q\tau+q'}) = \epsilon(p,p',q,q')\sqrt{\frac{q\tau+q'}{i}}\eta(\tau),$$

with ϵ a $24th$ root of unity related to Dedekind sums as defined in (79). See [14], p. 160.

The choice of the integration path comes along in a natural way by using the connection of $\tilde{Z}(y)$ to the modular group. Let us observe that the set of images of the line $\tau = X + i$, X real, under all modular transformations

$$\tau' = \frac{p\tau + p'}{q\tau + q'}, \text{ with } p, p', q, q' \text{ integers and } |pq' - q'p| = 1, \tag{70}$$

can be written as

$$\left|\tau - (\frac{p}{q} + \frac{i}{2q^2})\right| = \frac{1}{2q^2}. \tag{71}$$

Equation (71) defines circles $C(p,q)$ centered at points $\tau = \frac{p}{q} + \frac{i}{2q^2}$ with radius $1/2q^2$. They are named after L. R. Ford who first studied their properties in 1938 [14]. Ford circles are easily generated by using the ordered Farey sequence

$$\frac{0}{1} < \cdots < \frac{p_1}{q_1} < \frac{p_1 + p_2}{q_1 + q_2} < \frac{p_2}{q_2} < \cdots < \frac{1}{1}. \tag{72}$$

To each $\frac{p}{q}$ belongs a Ford circle in the upper half plane, which is tangent to the real axis at $\tau = \frac{p}{q}$. It can be observed that Ford circles never intersect. They are tangent to each other if and only if they belong to fractions which are adjacent in some Farey sequence.

If $\frac{p_1}{q_1} < \frac{p}{q} < \frac{p_2}{q_2}$ are three adjacent fractions in a Farey sequence then $C(p,q)$ touches $C(p_1,q_1)$ and $C(p_2,q_2)$ respectively at the points

$$\tau_{pq}^L = \frac{p}{q} + \zeta_{pq}^L \text{ and } \tau_{pq}^R = \tfrac{p}{q} + \zeta_{pq}^R, \tag{73}$$

where

$$\zeta_{pq}^L = -\frac{q_1}{q(q^2 + q_1^2)} + \frac{i}{q^2 + q_1^2} \text{ and } \zeta_{pq}^R = \tfrac{q_2}{q(q^2+q_2^2)} + \tfrac{i}{q^2+q_2^2}. \tag{74}$$

In Rademacher's approach (which improves Ramanujan's one) the unit length path on $\mathcal{H}$ is chosen so as to go along Ford circles

$$p(n) = \sum_{0 \le p \le q \le N}^{(p,q)=1} \int_{\gamma_{pq}} \tilde{Z}\left[\exp(2i\pi\tau)\right] \exp(-2i\pi\tau n) d\tau, \tag{75}$$

where γ_{pq} is the upper arc on a Ford circle which connects points of tangency at τ_{pq}^L and τ_{pq}^R.

Each Ford circle $C(p,q)$ is labelled by the expression $\tau = \frac{p}{q} + \zeta$, where the variable ζ runs on an arc of the circle $|\zeta_- \frac{i}{2q^2}| = \frac{1}{2q^2}$. If one uses z such that $\zeta = \frac{iz}{q^2}$, a Ford circle is mapped onto the circle $|z - \frac{1}{2}| = \frac{1}{2}$ and (75)

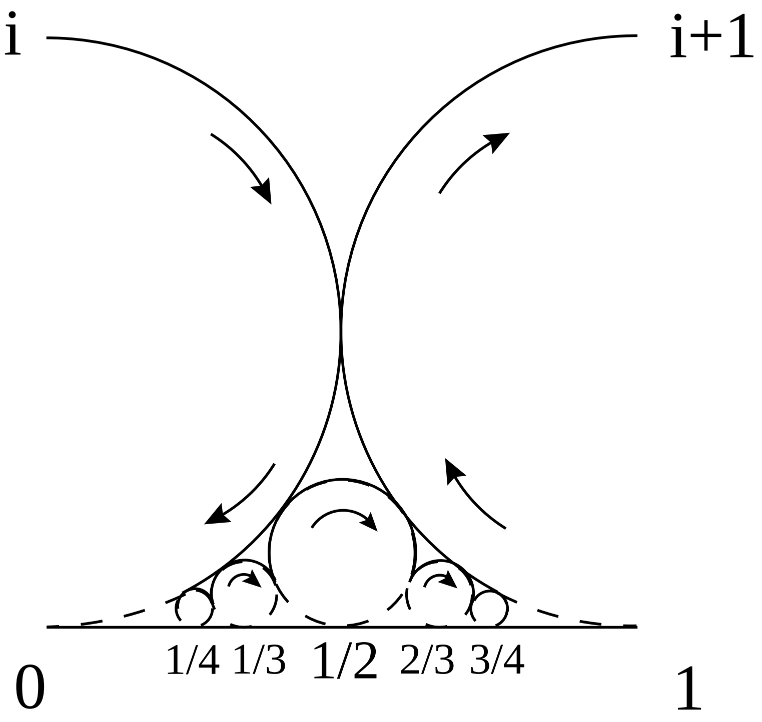

FIGURE 1. Rademacher's path of integration.

transforms as

$$p(n) = \sum_{0\leq p\leq q\leq N}^{(p,q)=1} \{\frac{i}{q^2}\exp(-2i\pi n\frac{p}{q})$$

$$\times \int_{z^{\mathrm{L}}_{pq}}^{z^{\mathrm{R}}_{pq}} \tilde{Z}\left[\exp(2\pi i\frac{p}{q} - \frac{2\pi z}{q^2})\right]\exp(\frac{2\pi n z}{q^2})dz\}, \tag{76}$$

where z^L_{pq} and z^R_{pq} follow from (73).

5.3. Farey contributions and Dedekind sums. To compute (76) one uses the transformation formula (22). After some manipulations and using z/q instead of z (see Ref. [14], p. 269), one gets the formula which generalizes (65)

$$\tilde{Z}(y) = \omega_{pq}\left(\frac{z}{q}\right)^{1/2}\exp(\frac{\pi}{12z} - \frac{\pi z}{12q^2})\tilde{Z}(y'), \tag{77}$$

with

$$y = \exp(\frac{2i\pi p}{q} - \frac{2\pi z}{q^2}),\ y' = \exp(\frac{2i\pi p'}{q} - \frac{2\pi}{z})\text{and } pp' = -1\ mod\ q. \tag{78}$$

The so-called Dedekind sums $s(p,q)$ are introduced by

$$\omega_{pq} = \exp\left(i\pi s(p,q)\right) \text{ with } s(p,q) = \sum_{l=1}^{q} \left(\frac{l}{q}\right)\left(\frac{pl}{q} - \left[\frac{pl}{q}\right]\right), \tag{79}$$

where [ ] in (79) denotes the integer part.

For the calculation of $p(n)$ one uses an approximation similar to the one used in (65). If z is a small positive real number, then y is near $\exp(2i\pi\frac{p}{q})$, the modulus at that point $|y'| = \exp(-\frac{2\pi}{z}) \sim 0$ and $\tilde{Z}(y') \sim 1$.

As a result (76) can be readily integrated and the final result is

$$p(n) = \frac{1}{\pi\sqrt{2}} \sum_{q\geq 1} \sqrt{q} A_q(n) \frac{d}{dn}\left(\frac{\sinh(K_q\lambda_n)}{\lambda_n}\right), \tag{80}$$

$$\text{with } K_q = \frac{\pi}{q}\sqrt{\frac{2}{3}},\ \lambda_n = \sqrt{n - \frac{1}{24}},$$
$$\text{and } A_q(n) = \sum_{p \bmod(\mathrm{q})} \omega_{pq} \exp(-2i\pi n \tfrac{p}{q}). \tag{81}$$

The physical meaning of these higher modes has still to be understood.

Acknowledgements

A first version of this manuscript was presented at the 8th Van der Ziel Quantum $1/f$ Noise Symposium in St-Louis (Missouri), June 5-6 (2000). I thanks P. Handel and C. Van Vliet for their useful comments.

References

[1] D. A. ABBOTT, B. R. DAVIS, N. J. PHILLIPS, K. ESHRAGHIAN, *Quantum vacuum fluctuations, zero point energy and the question of observable noise.* In: Unsolved Problems of Noise in Physics, Biology, Electronic Technology and Information Technology, eds Ch. R. Doering, L. B. Kiss and M. F. Shlesinger, World Scientific, 1997, 131–138.

[2] T. M. APOSTOL, *Modular Functions and Dirichlet Series in Number Theory* (Second Edition). Springer Verlag, New York, 1990.

[3] F. BLOCH, A. NORDSIECK, *Note on the Radiation Field of the Electron.* Phys. Rev. **52** (1937), 54–59.

[4] H. B. CALLEN, T. A. WELTON, *Irreversibility and generalized noise.* Phys. Rev. **83** (1951), 34–40.

[5] E. ELIZALDE, *Ten Physical Applications of Spectral Zeta Functions.* Springer Verlag Lecture Notes in Physics Vol. m35, Berlin, 1995.

[6] P. FLAJOLET, X. GOURDON, P. DUMAS, *Mellin transforms and asymptotics: harmonic sums.* Theoretical Computer Science **144** (1995), 3–58.

[7] J.-J. GAGNEPAIN, J. UEBERSFELD, G. GOUJON, P. H. HANDEL, *Relation between $1/f$ noise and Q-factor in quartz resonators at room and low temperature, first theoretical interpretation.* In: Proc. 35th Annual Symposium on Frequency Control, Philadelphia, 1981, 476–483.

[8] P. H. HANDEL, *Nature of $1/f$ frequency fluctuations in quartz crystal resonators.* Solid State Electronics **22** (1979), 875–876.

[9] P. H. HANDEL, *Quantum $1/f$ noise in the presence of a thermal radiation background.* In: Proc. II Int. Symp. on $1/f$ Noise, eds C. M. Van Vliet and E. R. Chenette, Orlando, 1980, 96–110. See also P. H. HANDEL, *Infrared divergences, radiative corrections, and Bremsstrahlung in the presence of a thermal-equilibrium background.* Phys. Rev. **A 38** (1988), 3082–3085.

[10] G. H. HARDY, *Ramanujan: Twelve Lectures on Subjects Suggested by his Life and Work.* Cambridge Univ. Press, London, 1940 (reprinted by Chelsea, New York, 1962).

[11] J. Kestin, J. R. Dorfman, *A Course in Statistical Thermodynamics*. Academic Press, New York, 1971.

[12] B. W. Ninham, B. D. Hughes, N. E. Frankel, M. L. Glasser, *Möbius, Mellin and mathematical physics*. Physica **A 186** (1992), 441–481.

[13] A. Pais, *"Subtle is the Lord..." The Science and the Life of Albert Einstein*, Oxford Univ. Press, Cambridge, 1982.

[14] H. Rademacher, *Topics in Analytic Number Theory*. Springer Verlag, New York, 1973.

[15] A. van der Ziel, *Noise in Measurements*. John Wiley and Sons, New York, 1976.

[16] A. Weil, *Elliptic functions according to Eisenstein and Kronecker*. Springer Verlag, Berlin, 1976.

Michel Planat
Laboratoire de Physique
et Métrologie des Oscillateurs du CNRS
32, Avenue de l'Observatoire
25044 Besançon Cedex
France
E-mail : `planat@lpmo.edu`