Journal de Théorie des Nombres
de Bordeaux **14** (2002), 603–611

Symmetry and folding of continued fractions

par Alfred J. van der POORTEN

To Michel Mendès France on his 65th birthday

Résumé. Le 'lemme de pliage' de Michel Mendès France fournit une nouvelle justification de la symétrie du développement en fraction continue d'un irrationnel quadratique.

Abstract. Michel Mendès France's 'Folding Lemma' for continued fraction expansions provides an unusual explanation for the well known symmetry in the expansion of a quadratic irrational integer.

1. Continued fractions

A *continued fraction* expansion

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{a_4 + \cfrac{1}{a_5 + \ddots}}}}}$$

is conveniently represented by $[\,a_0\,,\,a_1\,,\,a_2\,,\,\ldots]$. Set $[\,a_0\,,\,a_1\,,\,\ldots\,,\,a_h\,] = x_h/y_h$ for $h = 0,\ 1,\ 2,\ \ldots$. Then the definition $[\,a_0\,,\,a_1\,,\,\ldots\,,\,a_h\,] = a_0 + 1/[\,a_1\,,\,\ldots\,,\,a_h\,]$ immediately implies by induction on h that there is a correspondence

$$\begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix}\cdots\begin{pmatrix} a_h & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} x_h & x_{h-1} \\ y_h & y_{h-1} \end{pmatrix}$$
$$\longleftrightarrow [\,a_0\,,\,a_1\,,\,\ldots\,,\,a_h\,] = x_h/y_h$$

between certain products of two by two matrices and continued fractions. It follows by transposition of matrices that

Manuscrit reçu le 5 février 2001.
The author was supported in part by a grant from the Australian Research Council.

Proposition 1. $[\,a_h\,, a_{h-1}\,, \ldots\,, a_1\,] = y_h/y_{h-1}$, *for* $h = 1,\, 2,\, \ldots\,,$

and by taking determinants that

Proposition 2. $x_h y_{h-1} - x_{h-1} y_h = (-1)^{h+1}$, *for* $h = 0,\, 1,\, 2\, \ldots\,.$

One then sees that $x_h/y_h = a_0 + 1/y_0y_1 - 1/y_1y_2 + \cdots + (-1)^{h-1}/y_{h-1}y_h\,.$ It follows that if $\alpha = [\,a_0\,, a_1\,, a_2\,, \ldots\ldots\,] = \lim_{h\to\infty} x_h/y_h$ then

$$\alpha - x_h/y_h = (-1)^h\bigl(1/y_hy_{h+1} - 1/y_{h+1}y_{h+2} + 1/y_{h+2}y_{h+3} - \cdots\bigr).$$

Specifically, if the *partial quotients* $a_1,\, a_2,\, \ldots$ are positive integers, as is the case for the simple continued fraction expansions we discuss below, and α is irrational, then

$$|y_h\alpha - x_h| < 1/y_{h+1} < 1/a_{h+1}y_h \qquad \text{for } h = 0,\, 1,\, 2,\, \ldots,$$

illustrating the quality of approximation of the *convergents* x_h/y_h of α.

Continued fractions are well studied objects, so new elementary results concerning their structure are a surprise.

Proposition 3 (Folding Lemma [2]). *We have*

$$x_h/y_h + (-1)^h/cy_h^2 = [\,a_0\,, w_h\,, c - y_{h-1}/y_h\,] = [\,a_0\,, w_h\,, c\,, -\overleftarrow{w_h}\,]\,.$$

Here we have set $w_h = a_1, a_2 \ldots, a_h$; accordingly the notation $-\overleftarrow{w_h}$ denotes $-a_h, -a_{h-1}, \ldots, -a_1$. It will save space to write $\overline{a}$ for $-a$.

Proof. Here $\longleftrightarrow$ denotes the 'correspondence' between two by two matrices and continued fractions. We have

$$[\,a_0\,, w_h\,, c - y_{h-1}/y_h\,] \longleftrightarrow \begin{pmatrix} x_h & x_{h-1} \\ y_h & y_{h-1}\end{pmatrix}\begin{pmatrix} c - y_{h-1}/y_h & 1 \\ 1 & 0\end{pmatrix} =$$

$$\begin{pmatrix} cx_h - (x_hy_{h-1} - x_{h-1}y_h)/y_h & x_h \\ cy_h & y_h\end{pmatrix} \longleftrightarrow x_h/y_h - (-1)^{h+1}/cy_h^2\,,$$

as alleged. Moreover, $[\,c - y_{h-1}/y_h\,] = [\,c\,, -\overleftarrow{w_h}\,]$ by Proposition 1. $\square$

The adjective 'folding' is appropriate because iteration of the perturbed symmetry $w \longrightarrow w, c, -\overleftarrow{w}$ yields a pattern of signs on the word w corresponding to the pattern of creases in a sheet of paper repeatedly folded in half; see [1].

Incidentally, it is often convenient to drop subscripts and to use the convention whereby if, say, y denotes y_h, then y' denotes the previous y; that is, $y' = y_{h-1}$.

The folding lemma makes it easy, given the expansion of a partial sum of a series, to adjust a continued fraction expansion for an appended term if the intervening 'gap' is wide enough. Our example is from the function field case, where 'degree' in X replaces the logarithm of the absolute value. Specifically if the partial sum has degree $-n$ one can readily append a term of degree $-2n$ or less. Of course if the degree of the appended term is much less, that introduces a partial quotient of correspondingly high degree.

As an application of the lemma: the continued fraction expansion of the sum

$$F = X\sum_{h\geq 0} X^{-2^h} = 1+X^{-1}+X^{-3}+X^{-7}+X^{-15}+X^{-31}+X^{-63}+X^{-127}+\cdots$$

is given sequentially by $1+X^{-1} = [\,1\,,\,X\,]$, $1+X^{-1}+X^{-3} = [\,1\,,\,X\,,\,\overline{X}\,,\,\overline{X}\,]$, $1 + X^{-1} + X^{-3} + X^{-7} = [\,1\,,\,X\,,\,\overline{X}\,,\,\overline{X}\,,\,\overline{X}\,,\,X\,,\,X\,,\,\overline{X}\,]$, ... , where the addition of each term is done by a 'fold' with $c = -X$; see [4]*.

2. Symmetry

The computation

$$\begin{aligned} -y &= 0 + \overline{y} \\ -1/y &= \overline{1} + (y-1)/y \\ y/(y-1) &= 1 + 1/(y-1) \\ y - 1 &= \overline{1} + y \\ 1/y &= 0 + 1/y \\ y &= \end{aligned}$$

shows how to make negative partial quotients positive. Specifically

$$\begin{aligned} [\,a_0\,,\,w'\,,\,a_h\,,\,c\,,\,-\overleftarrow{w}\,] &= [\,a_0\,,\,w'\,,\,a_h\,,\,c\,,\,0\,,\,\overline{1}\,,\,1\,,\,\overline{1}\,,\,0\,,\,a_h\,,\,\overleftarrow{w}'\,] \\ &= [\,a_0\,,\,w'\,,\,a_h\,,\,c-1\,,\,1\,,\,a_h - 1\,,\,\overleftarrow{w}'\,]. \end{aligned}$$

Here we use $[\,\dots\,,\,a\,,\,0\,,\,b\,,\,\dots\,] = [\,\dots\,,\,a+b\,,\,\dots\,]$, having recalled that

$$\begin{pmatrix} a & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} b & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} a+b & 1 \\ 1 & 0 \end{pmatrix}.$$

Thus folding yields a symmetric continued fraction expansion, perturbed (cf [3]) in the given example by the word $a_h\,, c-1, 1, a_h - 1$.

This raises the question as to whether every such 'symmetric' expansion arises from folding. Mind you, it should not raise any such thing. If the given symmetric expansion yields the convergent X/Y and its 'first half' is x/y then necessarily $X/Y - x/y$ is of the shape $\pm 1/cy^2$, for some appropriate c.

So, a better question is to ask what one might learn from viewing a symmetric expansion as a folded expansion. The best known case of a symmetric expansion is the period of a real quadratic irrational integer. Below

*Jeffrey Shallit recently noticed that the evaluation just given is foreshadowed in remarks of Walter Leighton and W. T. Scott, 'A general continued fraction expansion', *Bull. Amer. Math. Soc.* **45** (1939) 596-605.

we study just that case and rediscover the conditions for being halfway in such an expansion.

3. Periodic continued fractions

Suppose the real quadratic irrational integer δ has norm n and trace t. That is, $\delta^2 - t\delta + n = 0$ with integers n and t, where $t^2 - 4n > 0$ is not a square in $\mathbb{Z}$. To fix δ, we suppose that $\delta > \overline{\delta}$, where $\overline{\delta}$ denotes the conjugate of δ.

Let $x_h/y_h = [a_0, a_1, \dots, a_h] = [a_0, w]$ be an initial segment of the continued fraction expansion of δ. We will play with the matrices

$$N_h = \begin{pmatrix} x_h & -ny_h \\ y_h & x_h - ty_h \end{pmatrix} = \begin{pmatrix} x & -ny \\ y & x - ty \end{pmatrix},$$

noticing first that the decomposition

$$(1)\quad N_h = \begin{pmatrix} x_h & -ny_h \\ y_h & x_h - ty_h \end{pmatrix} = \begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix}\cdots\begin{pmatrix} a_h & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & P_{h+1} \\ 0 & Q_{h+1} \end{pmatrix}$$

defines sequences $(P_h)_{h\geq 0}$ and $(Q_h)_{h\geq 0}$ of integers. Here, $(-1)^{h+1}Q_{h+1}$ is the determinant $x^2 - txy + ny^2$ of N_h. Also

$$(2)\qquad xP + x'Q = -ny, \qquad yP + y'Q = x - ty.$$

Because x/y is a convergent of δ we have $|x - \delta y| < 1/y$, and so Q is at most $\delta - \overline{\delta}$. It follows from several applications of the box principle that there are positive integers X and Y providing a solution to 'Pell's equation' $X^2 - tXY + nY^2 = \pm 1$.

Proposition 4. *If $X^2 - tXY + nY^2 = \pm 1$ and $X/Y = [a_0, a_1, \dots, a_r]$, then*

$$\delta = [\overline{a_0, a_1, \dots, a_r, 0}] = [a_0, \overline{a_1, \dots, a_{r-1}, a_r + a_0}].$$

Proof. Set $[\overline{a_0, a_1, \dots, a_r, 0}] = \gamma$; that is, $\gamma = [a_0, a_1, \dots, a_r, 0, \gamma]$. By the matrix correspondence† this is

$$\gamma \longleftrightarrow \begin{pmatrix} X & -nY \\ Y & X - tY \end{pmatrix}\begin{pmatrix} \gamma & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \gamma X - nY & X \\ X - tY + \gamma Y & Y \end{pmatrix} \longleftrightarrow \frac{\gamma X - nY}{X - tY + \gamma Y},$$

and so $(\gamma^2 - t\gamma + n)Y = 0$. But $Y \neq 0$ and $\gamma > \overline{\gamma}$, so $\gamma = \delta$. $\square$

Notice moreover that

$$[a_0, a_1, \dots, a_r, 0, t] \longleftrightarrow \begin{pmatrix} X & -nY \\ Y & X - tY \end{pmatrix}\begin{pmatrix} t & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} tX - nY & X \\ X & Y \end{pmatrix}.$$

Thus the word $a_0, a_1, \dots, a_{r-1}, a_r + t$ must be a palindrome‡.

†There is a subtle point here at which I tacitly suppose that $X^2 - tXY + nY^2 = (-1)^r$. However, that loses no generality, because if $X/Y = [\dots\dots, a_r]$ also $X/Y = [\dots\dots, a_r - 1, 1]$.

‡Anubis: Dog as a devil deified lived as a god.

4. Halfway in the period

Suppose, first, that the period of δ is of even length $r = 2h + 2$. Then $N = N_h$ corresponds to half the period in the sense that $N^2 = \left(\begin{smallmatrix} X & -nY \\ Y & X-tY \end{smallmatrix}\right)$ provides X/Y yielding the period as in Proposition 4.

Thus we consider

$$\frac{X}{Y} - \frac{x}{y} = \frac{(x^2 - ny^2)y - (2xy - ty^2)x}{Yy}$$
$$= \frac{-(x^2 - txy + ny^2)y}{(2x - ty)y^2} = (-1)^h \frac{Qy}{(2x - ty)y^2}\,;$$

whence

$$\frac{x}{y} + \frac{(-1)^h}{\frac{2x-ty}{Qy}y^2} = [\, a_0 \,,\, w \,,\, \frac{2x - ty}{Qy} - \frac{y'}{y}\,]\,.$$

By (2) we have

$$\frac{2x - ty}{Qy} = \frac{(2P + t)y + 2Qy'}{Qy} = \frac{2P + t}{Q} + 2\frac{y'}{y}\,,$$

and so, indeed, the Folding Lemma entails that

$$\frac{X}{Y} = \frac{x}{y} + \frac{(-1)^h}{\frac{2x-ty}{Qy}y^2} = [\, a_0 \,,\, w \,,\, \frac{2P + t}{Q} \,,\, \frac{y}{y'}\,] = [\, a_0 \,,\, w \,,\, (2P + t)/Q \,,\, \overleftarrow{w}\,]\,.$$

Proposition 5. *Suppose that the quadratic irrational integer δ satisfies $\delta > \overline{\delta}$ and has a continued fraction expansion period of even length $2h + 2$. Then*

$$\delta = [\, a_0 \,,\, \overline{w \,,\, (2P + t)/Q \,,\, \overleftarrow{w} \,,\, 2a_0 - t}\,]$$

if and only if Q divides $2P + t$.

Proof. The necessity of the condition is clear since otherwise the given expansion is not admissible. For its sufficiency we need to confirm that X/Y *does* yield a solution to 'Pell's equation'. That is not obvious: plainly $X^2 - tXY + nY^2 = Q^2$.

However, it turns out that both $Q \big| X$ and $Q \big| Y$. To see that, consider the conditions (2) modulo Q, that is, $Px \equiv -ny$ and $Py \equiv x - ty$.

Recall we suppose $2P+t \equiv 0$. Then $X = x^2 - ny^2 \equiv (P+t)xy + Pxy \equiv 0$ and $Y = 2xy - ty^2 = (x - ty + x)y \equiv (Py + (P+t)y)y \equiv 0$, as suggested. $\square$

Suppose, second, that the period of δ is of odd length $r = 2h + 1$. Then N corresponds to half the period in the sense that $NN' = \left(\begin{smallmatrix} X & -nY \\ Y & X-tY \end{smallmatrix}\right)$ provides X/Y yielding the period; here N' denotes N_{h-1}.

We now consider

$$\frac{X}{Y} - \frac{x}{y} = \frac{(xx' - nyy')y - (xy' + x'y - tyy')x}{Yy}$$
$$= \frac{-(x^2 - txy + ny^2)y'}{(xy' + x'y - tyy')y} = (-1)^h \frac{Qy'}{(xy' + x'y - tyy')y};$$

whence

$$\frac{x}{y} + \frac{(-1)^h}{(\frac{x-ty}{Qy} + \frac{x'}{Qy'})y^2} = [\,a_0\,,\,w\,,\,\frac{x - ty}{Qy} + \frac{x'}{Qy'} - \frac{y'}{y}\,].$$

By (2)

$$\frac{x - ty}{Qy} = \frac{P}{Q} + \frac{y'}{y} \qquad \frac{x'}{Qy'} = \frac{P' + t}{Q} + \frac{Q'}{Q}\frac{y''}{y'},$$

and so

$$X/Y = x/y + \frac{(-1)^h}{(\frac{x-ty}{Qy} + \frac{x'}{Qy'})y^2}$$
$$= [\,a_0\,,\,w\,,\,\frac{P' + P + t}{Q}\,,\,\frac{Q}{Q'}\frac{y'}{y''}\,] = [\,a_0\,,\,w\,,\,(P' + P + t)/Q\,,\,\overleftarrow{w}'\,],$$

provided that $Q = Q'$; here w' denotes the word $a_1, a_2, \ldots, a_{h-1}$.

Proposition 6. *Suppose that the quadratic irrational integer δ satisfies $\delta > \overline{\delta}$ and has a continued fraction expansion period of odd length $2h+1$. Then*

$$\delta = [\,a_0\,,\,\overline{w\,,\,(P' + P + t)/Q\,,\,\overleftarrow{w}'\,,\,2a_0 - t}\,]$$

if and only if $Q = Q'$.

Remark. We will see below that $P_k + P_{k+1} + t = a_kQ_k$ for $k = 0, 1, \ldots$, so when $Q' = Q$ the requirement $Q \big| (P' + P + t)$ is automatically satisfied.

Proof. The necessity of the condition is clear since otherwise the given expansion is not symmetric; in particular we would not have $a_h = (P' + P + t)/Q$. For its sufficiency we need to confirm that X/Y *does* yield a solution to 'Pell's equation'. That's not obvious: plainly $X^2 - tXY + nY^2 = -Q'Q = -Q^2$.

However, both $Q \big| X$ and $Q \big| Y$. Recall we have $Q' = Q$ and so $P' + P + t \equiv 0$ modulo Q. Then $xx' - nyy' \equiv (P + t)x'y + P'x'y \equiv 0$ and $Y = xy' + x'y - tyy' = (P + t)yy' + (P' + t)y'y - tyy' \equiv 0$. □

5. Remarks and explanations

5.1. Some identities. The equations $Px + Qx' = -ny$ and $Py + Qy' = x - ty$ together entail that

$$(xy' - x'y)P = -(xx' + nyy' - tx'y), \quad \text{or} \quad P = (-1)^h(xx' + nyy' - tx'y).$$

Brute computation now shows that

$$\det NN' = -QQ' = n + tP + P^2\,.$$

We will avoid the need for such brutality by showing that our also taking the equations (2) in the form $P'x' + Q'x'' = -ny'$ and $P'y' + Q'y'' = x' - ty'$ provides a base from which one can prove the facts we want by induction. Here we note that $x = ax' + x''$ and $y = ay' + y''$, where $a = a_h$. Then

$$\begin{aligned} 0 &= P'y' + Q'y'' - x' + ty' \\ &= P'y' + Q'y - x' + ty' - aQ'y'\,, \end{aligned}$$

and

$$\begin{aligned} 0 &= P'x' + Q'x'' + ny' \\ &= P'x' + Q'x + ny' - aQ'x'\,. \end{aligned}$$

Multiplying these equations respectively by x and by y, and subtracting, yields

$$(xy' - x'y)aQ' = (xy' - x'y)P' - (xx' + nyy' - tx'y) + (xy' - x'y)t\,.$$

That is, as promised,

$$aQ' = P' + P + t\,. \tag{3}$$

As just noted, we therefore have

$$\begin{aligned} 0 &= (P' + t - aQ')y' + Q'y - x' \\ &= Q'y - Py' - x', \end{aligned}$$

and

$$\begin{aligned} 0 &= (P' - aQ')x' + Q'x + ny' \\ &= Q'x - (P + t)x' + ny'. \end{aligned}$$

Multiplying these equations respectively by n and by P, and adding, yields

$$\begin{aligned} 0 &= Q'(Px + ny) - (n + tP + P^2)x' \\ &= -Q'Qx' - (n + tP + P^2)x', \end{aligned}$$

and so, indeed

$$-QQ' = n + tP + P^2\,. \tag{4}$$

5.2. The continued fraction expansion. The significance of (3) is its announcement that

$$(\delta + P_h)/Q_h = a_h - (\overline{\delta} + P_{h+1})/Q_h \tag{5}$$

is step h in the continued fraction expansion of δ. Just so, (4) fixes the reciprocal of the *remainder* in (5) above and thus Q_{h+1} and the next *complete quotient* $(\delta + P_{h+1})/Q_{h+1}$. Moreover, because the Q's are positive it follows from (4) alone that $-\delta < P_{h+1} < -\overline{\delta}$ for $h = 0, 1, \ldots$. Mind you, we also know that the complete quotients satisfy $(\delta + P)/Q > 1$ and the remainders $-1 < (\overline{\delta} + P)/Q' < 0$.

Given the rules (3) and (4) defining the continued fraction expansion it is an interesting exercise to retrieve the conditions (2) and so the decompositions (1) of the matrices N_h. The trick is to use induction, showing that $P'x' + Q'x'' = -ny'$ and $P'y' + Q'y'' = x' - ty'$ entails $Px + Qx' = -ny$ and $Py + Qy' = x - ty$. The point of this all is to emphasise that the two approaches *are* equivalent, confirming that the matrices N_h detail the continued fraction expansion of δ.

5.3. Binary quadratic forms. The matrix N_h determines a quadratic form $(-1)^{h+1}Qu^2 - (2P + t)uv + (-1)^h Q'v^2$ of discriminant $4P^2 + 4Pt + t^2 + 4QQ' = t^2 - 4n = (\delta - \overline{\delta})^2$. That is, it provides P and Q, and the discriminant, and therefore Q'.

It might take us a little too far afield to provide details, so I leave confirmation of the following to the interested reader. It turns out that multiplication of a pair N_{i-1} and N_{j-1} *is* composition of the corresponding quadratic forms. Indeed, this is a natural way to define composition. There are provisos. Whereas the forms determined by the N_h are reduced, the composite, which a priori has leading coefficient Q_iQ_j, is in general not reduced. Moreover, it may not be primitive. Its coefficients will have the common factor G^2, where $G = \gcd(Q_i, Q_j, P_i + P_j + t)$. One must therefore always consider the 'primitive' matrix product $G^{-1}N_{i-1}N_{j-1}$.

5.4. The 'results', specifically Propositions 5 and 6, given here are of course well known. I doubt however whether they have heretofore been proved by the Folding Lemma. My remarks on the matrices N_h contain some previously unpublished observations. One motivation for viewing continued fractions of quadratic irrationals as a study of the N_h is that doing it suggests an immediate generalisation to approximation algorithms for higher degree algebraic irrationals.

References

[1] F. M. DEKKING, M. MENDÈS FRANCE, A. J. VAN DER POORTEN, *FOLDS!*. Math. Intelligencer **4** (1982), 130–138; II: *Symmetry disturbed*. ibid. 173–181; III: *More morphisms*. ibid. 190–195. Erratum **5** (1983), page 5.

[2] M. Mendès France, *Sur les fractions continues limitées.* Acta Arith. **23** (1973), 207–215.
[3] M. Mendès France, *Principe de la symétrie perturbée.* Seminar on Number Theory, Paris 1979–80, 77–98, Progr. Math. **12**, Birkhäuser, Boston, Mass., 1981. [MR 83a:10089]
[4] A. J. van der Poorten, J. Shallit, *Folded continued fractions.* J. Number Theory **40** (1992), 237–250.

Alf van der Poorten
ceNTRe for Number Theory Research
Macquarie University
Sydney, NSW 2109
Australia
E-mail : alf@math.mq.edu.au