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A Gauss-Kuzmin theorem for the Rosen fractions

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Résumé. En utilisant les extensions naturelles des transformations de Rosen, nous obtenons une représentation de la chaîne d'ordre infini associée à la suite des quotients incomplets des fractions de Rosen. Associé au comportement ergodique d'un certain système aléatoire homogène à liaisons complètes, ce fait nous permet de résoudre une version du problème de Gauss-Kuzmin pour le développement en fraction de Rosen.

Abstract. Using the natural extensions for the Rosen maps, we give an infinite-order-chain representation of the sequence of the incomplete quotients of the Rosen fractions. Together with the ergodic behaviour of a certain homogeneous random system with complete connections, this allows us to solve a variant of Gauss-Kuzmin problem for the above fraction expansion.

1. Introduction

The Rosen fractions, introduced in [11], form an infinite family which generalizes the nearest integer continued fractions. They are related to the so-called Hecke groups. There is an extended literature on these (see the papers by Rosen and Schmidt [12], Gröchenig and Haas [2], Schmidt [13], Haas and Series [3], and Lehner [7], [8]). It is only recently (see the papers by Burton, Kraaikamp and Schmidt [1] and Nakada [10]) that the ergodic properties of these expansions have been studied. It should be stressed that the ergodic theorem does not yield rates of convergence for mixing properties; for this a Gauss-Kuzmin theorem is needed.

The technique developped by Iosifescu in [5] for the case of the regular continued fraction expansion could be used for different types of continued fractions. The aim of this paper is to take up the Gauss-Kuzmin problem for the Rosen fractions in this manner.

This paper is organized as follows. Using the natural extensions for the Rosen maps, in Section 3 we give an infinite order-chain representation of the sequence of the incomplete quotients of the Rosen fractions. In Section 5 we show that the random systems with complete connections

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(RSCC_s) associated with the Rosen continued fraction expansion are with contraction and their transition operators are regular w.r.t. the Banach space of Lipschitz functions. This leads further, in Section 6, to a solution of a version of Gauss-Kuzmin problem for the Rosen fractions.

Let $\lambda = \lambda_q$ equal $2\cos\pi/q$ for $q \in \{3, 4, \dots\}$. Fix some such $q \geq 4$ and let $I_q = [-\lambda/2, \lambda/2)$. Then the map $\tau_q : I_q \to I_q$ defined by

$$\tau_q(x) = |x^{-1}| - \left\lfloor |x^{-1}|\lambda^{-1} + 1/2 \right\rfloor \lambda,\ x \neq 0; \tau_q(0) = 0 \tag{1}$$

is called a *Rosen map*. Putting

$$\varepsilon_n(x) = \operatorname{sgn}\left(\tau_q^{n-1}(x)\right),\ a_n(x) = \left\lfloor |(\tau_q^{n-1}(x))^{-1}|\lambda^{-1} + 1/2 \right\rfloor,\ n \in \mathbf{N}^*, \tag{2}$$

in case $\tau_q^{n-1}(x) \neq 0$, and $\varepsilon_n(x) = 0$, $a_n(x) = \infty$ in case $\tau_q^{n-1}(x) = 0$, and using the Rosen map τ_q, one easily sees that every $x \in I_q$ has a unique expansion

$$x = \cfrac{\varepsilon_1}{a_1\lambda + \cfrac{\varepsilon_2}{a_2\lambda + \ddots}} = [\varepsilon_1 a_1, \varepsilon_2 a_2, \dots], \tag{3}$$

which is called the *Rosen* or *λ-expansion* (λCF) of x.

Now, related to *Hecke* (*triangle*) *group* G_q, $q \geq 4$, we call x a *G_q-irrational* if x has a Rosen expansion of infinite length. For G_q-irrationals, there are restrictions on the set of admissible sequences of ε_i and a_i. These restrictions are determined by the orbit of $\lambda/2$ [cf. [11] and [1]).

2. Natural extensions for the Rosen maps

Consider (see [1]) the so-called *natural extension* $\mathcal{T}$ for any Rosen interval maps, which is defined as follows: for any fixed $q \geq 4$, let $\lambda = \lambda_q$, $\tau(x)$ be $\tau_q(x)$ and

$$\mathcal{T}(x, y) = \left(\tau(x), \frac{1}{a\lambda + \varepsilon y}\right),$$

where we have suppressed the dependence of $a = a_1$ and $\varepsilon = \varepsilon_1$ on x. Also notice that $\mathcal{T}$ is a transformation which on the first coordinate is simply the interval map while on the second coordinate is directly related to the "past" of the first coordinate.

It has been shown in [1], that $\mathcal{T}$ is a bijective transformation of a domain Ω in $\mathbf{R}^2$ except for a set of Lebesgue measure zero. We consider two cases.

2.1. **Even indices**. Fix $q = 2p$, with $p \geq 2$, and let I be the interval I_q. Putting $\phi_j = -\tau^j\lambda/2$, with $\phi_0 = -\lambda/2$, we construct a partition of I by considering the intervals

$$J_j = [\phi_{j-1}, \phi_j) \quad \text{for} \quad j \in \{1, \ldots, p-1\},$$
$$J_p = [0, \tfrac{\lambda}{2}).$$

Furthermore, let $K_j = [0, L_j]$, $j \in \{1, \ldots, p-1\}$ and $K_p = [0, R]$, where L_j, $1 \leq j \leq p-1$, and R satisfy the system

$$\begin{cases} R = \lambda - L_{p-1} \\ L_1 = \frac{1}{\lambda + R} \\ L_j = \frac{1}{\lambda - L_{j-1}} \quad \text{for} \quad j \in \{2, \ldots, p-1\} \\ R = \frac{1}{\lambda - L_{p-1}}. \end{cases}$$

Let $\Omega = \bigcup_{k=1}^{p} J_k \times K_k$. One has that $R = 1$, and that $\mathcal{T}$ is bijective on Ω except for a set of Lebesgue measure zero. In [1] it is shown that $\mathcal{T}$ preserves the probability measure ν (abolutely continuous with respect to the Lebesgue measure on Ω) with density $\frac{C}{(1+xy)^2}$, where C is a normalizing constant. Actually, for q even, the constant C is given by

$$C = \frac{1}{\ln \left[\frac{1 + \cos \pi/q}{\sin \pi/q} \right]}.$$

2.2. **Odd indices**. Fix an odd q and recycle notation as above. Let I be the interval I_q and $\phi_j = -\tau^j \lambda/2$, with $\phi_0 = -\lambda/2$. Putting $h = \frac{q-3}{2}$, we construct a partition of I by considering the intervals J_j, $j \in \{1, \ldots, 2h+2\}$, where

$$J_{2k} = [\phi_{h+k}, \phi_k) \quad \text{for} \quad k \in \{1, \ldots, h\},$$
$$J_{2k+1} = [\phi_k, \phi_{h+k+1}) \quad \text{for} \quad k \in \{0, 1, \ldots, h\},$$
$$J_{2h+2} = \left[0, \tfrac{\lambda}{2}\right).$$

Let $K_j = [0, L_j]$, $j \in \{1, \ldots, 2h+1\}$, and $K_{2h+2} = [0, R]$, where L_j, $1 \leq j \leq 2h+1$, and R satisfy the system

$$\begin{cases} R = \lambda - L_{2h+1} \\ L_1 = \frac{1}{2\lambda - L_{2h}} \\ L_2 = \frac{1}{2\lambda - L_{2h+1}} \\ L_j = \frac{1}{\lambda - L_{j-2}} \quad \text{for} \quad 2 < j < 2h+2 \\ R = \frac{1}{\lambda - L_{2h}}. \end{cases}$$

Let $q = 2h+3$, with $h \geq 1$ and $\Omega = \bigcup_{j=1}^{2h+2} J_j \times K_j$. One has that R satisfies the equation

$$R^2 + (2 - \lambda)R - 1 = 0,$$

and that $\mathcal{T}$ is bijective on Ω except for a set of Lebesgue measure zero. In particular, $\frac{\lambda}{2} < R < 1$. The transformation $\mathcal{T}$ preserves the probability measure ν with density $\frac{C}{(1+xy)^2}$, where

$$C = \frac{1}{\ln(1+R)}.$$

3. An infinite-order-chain representation

In this section we obtain an infinite-order-chain representation of the sequence of the incomplete quotients of the Rosen fractions. We treat simultaneously the two subfamilies of Rosen maps, those of odd indices and those of even one.

Let us consider the sets

$$\Delta = (0, R] \setminus \left\{\tfrac{1}{n\lambda} | n \in \mathbf{N}^*\right\},$$
$$A = \Delta \setminus \left[\left\{\tfrac{1}{(n+u)\lambda} | n \in \mathbf{N}^*, \quad u \in \left(\tfrac{1}{2}, 1\right)\right\} \cup \left(\tfrac{1}{\lambda}, R\right)\right],$$

and define $\sigma_q : \Delta \to \left(0, \frac{\lambda}{2}\right]$ by

$$\sigma_q(y) = \sigma(y) = \begin{cases} \tau(y), & \text{if } \; y \in A, \\ \tau(-y), & \text{if } \; y \in \Delta \setminus A. \end{cases}$$

Similarly to relations (2), for all $n \in \mathbf{N}^*$, let us define

$$\delta_n(y) = \begin{cases} 1, & \text{if } \; \sigma^{n-1}(y) \in A, \\ -1, & \text{if } \; \sigma^{n-1}(y) \in \Delta \setminus A, \end{cases}$$

$$b_n(y) = \left[\delta_n(y) \cdot (\sigma^{n-1}(y))^{-1} \lambda^{-1} + \frac{1}{2}\right],$$

which implies

$$\mathcal{T}^{-1}(x, y) = \left(\frac{1}{b_1(y)\lambda + \delta_1(y)x}, \sigma(y)\right), \quad (x, y) \in \Omega.$$

Writing $[\varepsilon_1 x_1, \varepsilon_2 x_2, \dots, \varepsilon_n x_n]$ for the finite continued fraction

$$\cfrac{\varepsilon_1}{x_1\lambda + \cfrac{\varepsilon_2}{x_2\lambda + \ddots + \cfrac{\varepsilon_n}{x_n\lambda}}},$$

for all $n \in \mathbf{N}^*$ we have

$$\mathcal{T}^n(x, y) = \left(\tau^n(x), [a_n(x), \varepsilon_n(x) a_{n-1}(x), \dots, \varepsilon_2(x)(a_1(x) + \varepsilon_1(x) y \lambda^{-1})]\right)$$

with $(x,y) \in \Omega$, $x \neq 0$, and

$$\mathcal{T}^{-n}(x,y) = \big([b_n(y), \delta_n(y)b_{n-1}(y), \dots, \delta_2(y)(b_1(y) + \delta_1(y)x\lambda^{-1})], \sigma^n(y)\big)$$

with $(x,y) \in \Omega$, $y \neq 0$.

Now, define the random variables $\bar{a}_n$, $n \in \mathbf{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$ on Ω by

$$\bar{a}_{n+1}(x,y) = \bar{a}_1(\mathcal{T}^n(x,y)),$$

where $\mathcal{T}^0$ denotes the identity map and $\bar{a}_1(x,y) = a_1(x)$. Clearly, on Ω,

$$\begin{aligned}
&\bar{a}_n(x,y) = a_n(x),\ n \in \mathbf{N}^*,\ x \neq 0,\\
&\bar{a}_0(x,y) = b_1(y),\ y \neq 0,\\
&\bar{a}_{-n}(x,y) = b_{n+1}(y),\ n \in \mathbf{N}^*,\ y \neq 0.
\end{aligned}$$

Since $\mathcal{T}$ preserves ν, the doubly infinite sequence $(\bar{a}_n)_{n\in\mathbf{Z}}$ is strictly stationary under ν.

Let us put $\bar{\varepsilon}_{n+1}(x,y) = \bar{\varepsilon}_1(\mathcal{T}^n(x,y))$, $\bar{\varepsilon}_1(x,y) = \varepsilon_1(x)$ for all $n \in \mathbf{Z}$ and $(x,y) \in \Omega$. Clearly,

$$\begin{aligned}
&\bar{\varepsilon}_n(x,y) = \varepsilon_n(x),\ n \in \mathbf{N}^*,\ x \neq 0,\\
&\bar{\varepsilon}_0(x,y) = \delta_1(y) \quad \text{and},\\
&\bar{\varepsilon}_{-n}(x,y) = \delta_{n+1}(y),\ n \in \mathbf{N}^*,\ y \neq 0.
\end{aligned}$$

Projecting ν, we obtain the invariant measure ν_x on $\left[-\frac{\lambda}{2}, \frac{\lambda}{2}\right]$ and ν_y on $[0,R]$. Since the invariant density $h(x,y) = \frac{C}{(1+xy)^2}$ on Ω has projections $h_x(t) = \int h(t,y)dy$ and $h_y(t) = \int h(x,t)dx$ on $\left[-\frac{\lambda}{2}, \frac{\lambda}{2}\right]$ and $[0,R]$, respectively, we have

$$\begin{aligned}
\nu_x(A) &= \int_A h_x(t)dt, \quad A \in \mathcal{B}_{[-\frac{\lambda}{2}, \frac{\lambda}{2}]},\\
\nu_y(B) &= \int_B h_y(t)dt, \quad B \in \mathcal{B}_{[0,R]},
\end{aligned}$$

where $\mathcal{B}_{[-\frac{\lambda}{2}, \frac{\lambda}{2}]}$ and $\mathcal{B}_{[0,R]}$ denote the collection of Borel sets on $\left[-\frac{\lambda}{2}, \frac{\lambda}{2}\right]$ and $[0,R]$, respectively.

Now, we are able to prove the following theorem which is very important in the sequel.

Theorem 1. *For any $x \in I$, we have*

$$\nu(\Omega_x | \bar{\varepsilon}_0\bar{a}_0, \bar{\varepsilon}_{-1}, \bar{a}_{-1}, \dots) = \frac{(2x+\lambda)(a\lambda+2)}{4\lambda(ax+1)} \quad \nu\text{-}a.s.,$$

where $\Omega_x = \left(\left[-\frac{\lambda}{2}, x\right) \times [0,R]\right) \cap \Omega$ and $a = [\bar{\varepsilon}_0\bar{a}_0, \bar{\varepsilon}_{-1}\bar{a}_{-1}, \dots]$ (the λ-continued fraction with incomplete quotients $\bar{a}_0, \bar{a}_{-1}, \dots$).

Proof. As is well known

$$\nu(\Omega_x | \bar{\varepsilon}_0\bar{a}_0, \bar{\varepsilon}_{-1}\bar{a}_{-1}, \dots) = \lim_{n\to\infty} \nu(\Omega_x | \bar{\varepsilon}_0\bar{a}_0, \bar{\varepsilon}_{-1}\bar{a}_{-1}, \dots, \bar{\varepsilon}_{-n}\bar{a}_{-n}) \quad \nu\text{-a.s.}$$

Let us denote by I_n the fundamental interval

$$I(\bar{\varepsilon}_0\bar{a}_0, \bar{\varepsilon}_{-1}\bar{a}_{-1}, \dots, \bar{\varepsilon}_{-n}\bar{a}_{-n}) \subset [0, R]$$

for any arbitrarily fixed values of the $\bar{\varepsilon}_i$ and $\bar{a}_i$, $i = 0, -1, \dots, -n$. Then we have

$$\nu(\Omega_x | \bar{\varepsilon}_0\bar{a}_0, \dots, \bar{\varepsilon}_{-n}\bar{a}_{-n}) = \frac{\nu\left(\left([-\frac{\lambda}{2}, x) \times I_n\right) \cap \Omega\right)}{\nu_y(I_n)} =$$

$$= \frac{\frac{(2x+\lambda)}{4\lambda}\int_{I_n} \frac{v\lambda+2}{vx+1}\nu_y(dv)}{\nu_y(I_n)} = \frac{(2x+\lambda)(v_n\lambda+2)}{4\lambda(v_n x+1)}$$

for some $v_n \in I_n$. Since

$$\lim_{n\to\infty} v_n = [\bar{\varepsilon}_0\bar{a}_0, \bar{\varepsilon}_{-1}\bar{a}_{-1}, \dots] = a,$$

the proof is complete. $\square$

Corollary. *For any $i \in \mathbf{Z}^*$ we have*

$$\nu(\bar{\varepsilon}_1\bar{a}_1 = i | \bar{\varepsilon}_0\bar{a}_0, \bar{\varepsilon}_{-1}\bar{a}_{-1}, \dots) = p_i(a) \quad \nu\text{-a.s.},$$

where $a = [\bar{\varepsilon}_0\bar{a}_0, \bar{\varepsilon}_{-1}\bar{a}_{-1}, \dots]$ and

$$p_i(a) = \begin{cases} \frac{4-a^2\lambda^2}{[-2a+(2i-1)\lambda][-2a+(2i+1)\lambda]}, & \text{if } i \le -2, \\ \frac{(-4+3\lambda^2)(a\lambda+2)}{4\lambda(-2a+3\lambda)}, & \text{if } i = -1, \\ \frac{(-4+3\lambda^2)(-a\lambda+2)}{4\lambda(2a+3\lambda)}, & \text{if } i = 1, \\ \frac{4-a^2\lambda^2}{[2a+(2i-1)\lambda][2a+(2i+1)\lambda]}, & \text{if } i \ge 2. \end{cases}$$

Proof. For $i = -1$ we have

$$(\bar{\varepsilon}_1\bar{a}_1 = -1) = (\bar{\varepsilon}_1 = -1) \cap (\bar{a}_1 = 1) = \Omega_{-2/3\lambda}.$$

Hence the conditional probability above equals ν-a.s. to

$$\frac{\left[2\left(-\frac{2}{3\lambda}\right) + \lambda\right](a\lambda+2)}{4\lambda\left[a\left(-\frac{2}{3\lambda}\right)+1\right]} = \frac{(-4+3\lambda^2)(a\lambda+2)}{4\lambda(-2a+3\lambda)} = p_{-1}(a).$$

For $i = 1$ we have

$$(\bar{\varepsilon}_1\bar{a}_1 = 1) = (\bar{\varepsilon}_1 = 1) \cap (\bar{a}_1 = 1) = \left(\frac{2}{3\lambda}, \frac{\lambda}{2}\right) \times [0, R].$$

Hence the conditional probability above equals ν-a.s. to

$$\frac{\left(2\cdot\frac{\lambda}{2}+\lambda\right)(a\lambda+2)}{4\lambda\left(a\cdot\frac{\lambda}{2}+1\right)} - \frac{\left(2\cdot\frac{2}{3\lambda}+\lambda\right)(a\lambda+2)}{4\lambda\left(a\cdot\frac{2}{3\lambda}+1\right)} =$$

$$= 1 - \frac{(4+3\lambda^2)(a\lambda+2)}{4\lambda(2a+3\lambda)} = \frac{(-4+3\lambda^2)(-a\lambda+2)}{4\lambda(2a+3\lambda)} = p_1(a).$$

Finally, the proof for $i \in \mathbf{Z}^* \setminus \{-1, 1\}$ is analogous. $\square$

Remarks. (i) The strict stationarity of $(\bar{a}_n)_{n\in\mathbf{Z}}$ under ν implies that the conditional probability

$$\nu(\bar{\varepsilon}_{n+1}\bar{a}_{n+1} = i \mid \bar{\varepsilon}_n\bar{a}_n, \bar{\varepsilon}_{n-1}\bar{a}_{n-1}, \dots), \quad i \in \mathbf{Z}^*,$$

does not depend on n and is ν-a.s. equal to $p_i(a)$, with

$$a = [\bar{\varepsilon}_n\bar{a}_n, \bar{\varepsilon}_{n-1}\bar{a}_{n-1}, \dots].$$

(ii) The process $(\bar{\varepsilon}_n\bar{a}_n)_{n\in\mathbf{Z}}$ is called an *infinite-order-chain* in the theory of dependence with complete connections (see [6], Section 5.5).
(iii) Many standard formulas for continued fractions (see [9]), hold for the Rosen fractions. Letting

$$\frac{p_n(x)}{q_n(x)} = [\varepsilon_1 a_1, \varepsilon_2 a_2, \dots, \varepsilon_n a_n],$$

where ε_i and a_i depend on $x \in I$, we find that

$$p_n(x) = a_n(x)\lambda p_{n-1}(x) + \varepsilon_n(x)p_{n-2}(x),$$

$$q_n(x) = a_n(x)\lambda q_{n-1}(x) + \varepsilon_n(x)q_{n-2}(x).$$

Therefore

$$x = \frac{p_n(x) + \tau^n(x)p_{n-1}(x)}{q_n(x) + \tau^n(x)q_{n-1}(x)}, \quad x \in I.$$

It follows that for any G_q-irrational number $x \in I$ and any positive integer n we have

$$q_n(x) > 0, \; q_{n+1}(x) > q_n(x).$$

Let us put $s_n = \frac{q_{n-1}}{q_n}$, $n \in \mathbf{N}^*$. Note that the equation $q_n = a_n\lambda q_{n-1} + \varepsilon_n q_{n-2}$ implies

$$s_n = \frac{1}{a_n\lambda + \varepsilon_n s_{n-1}}, \; n \in \mathbf{N}^*,$$

with $s_0 = 0$, i.e. $s_n = [a_n, \varepsilon_n a_{n-1}, \dots, \varepsilon_2 a_1]$; clearly, $s_n \in [0, R]$.

Motivated by Theorem 1, we shall consider the family of probability measures $(\nu^a)_{a\in[0,R]}$ on $\mathcal{B}_I$ defined by their distribution functions

$$\nu^a([-\lambda/2, x)) = \frac{(2x+\lambda)(a\lambda+2)}{4\lambda(ax+1)}, \; x \in I, \; a \in [0, R].$$

In particular ν^0 is the Lebesgue measure on I, if $\lambda = 1$. For any $a \in [0, R]$ put $s_0^a = a$ and

$$s_n^a = \frac{1}{a_n\lambda + \varepsilon_n s_{n-1}^a}.$$

Let us consider the quadruple $\{(W,\mathcal{W}),(X,\mathcal{X}),u,P\}$ where $W=[0,R]$, $\mathcal{W}=\mathcal{B}_W$, $X=\{-1,1\}\times\mathbf{N}^*$, $\mathcal{X}=\mathcal{P}(X)$, $u:W\times X\to W$

$$u(w,(l,i))=\begin{cases}\frac{1}{i\lambda+lw}, \text{ if } w\in W,\ (l,i)\in X\setminus\{(-1,1)\},\\ \frac{1}{\lambda-w}, \text{ if } w\in\left[0,\frac{\lambda R-1}{R}\right),\ (l,i)=(-1,1),\\ \frac{1}{\lambda-\frac{\lambda R-1}{R}}=R, \text{ if } w\in\left[\frac{\lambda R-1}{R},R\right],\ (l,i)=(-1,1),\end{cases}$$

and $P:W\times X\to[0,1]$ is given by

$$P(w,(l,i))=p_{li}(w)=\begin{cases}\frac{(-4+3\lambda^2)(lw\lambda+2)}{4\lambda(l2w+3\lambda)}, \text{ if } l=\pm1,\ i=1,\\ \frac{4-w^2\lambda^2}{[l2w+(2i-1)\lambda][l2w+(2i+1)\lambda]}, \text{ otherwise.}\end{cases}$$

By the corollary to Theorem 1, the sequence $(s_n^a)_{n\geq0}$ is an W-valued Markov chain on $(I,\mathcal{B}_I,\nu^a)$ which starts at $s_0^a=a$ and has the following transition mechanism: from state $s\in W$ the only possible transitions are those to states $\frac{1}{i\lambda+ls}$, $(l,i)\in X\setminus\{(-1,1)\}$, and to $\frac{1}{\lambda-s}$ or to R if $(l,i)=(-1,1)$ and $s\in\left[0,\frac{\lambda R-1}{R}\right)$ or $s\in\left[\frac{\lambda R-1}{R},R\right]$, respectively, the transition probability being $p_{li}(s)$.

Let $B(W)$ be the Banach space of bounded measurable complex-valued functions f on W under the supremum norm $|f|=\sup_{w\in W}|f(w)|$. Clearly, the transition operator of $(s_n^a)_{n\geq0}$ transforms $f\in B(W)$ into the function defined by

$$E(f(s_{n+1}^a)|s_n^a=s)=\sum_{(l,i)\in X}p_{li}(s)f(u(s,(l,i)))=Uf(s),\ s\in W,$$

whatever $a\in W$.

Note that for $a\in W$, $A=\sigma(a_{n+1},\dots)$ and $n\in\mathbf{N}^*$,

$$\nu^a(A\,|\,a_n,\varepsilon_na_{n-1},\dots,\varepsilon_2a_1,\varepsilon_1a)=\nu^{s_n^a}(\tau^n(A)).$$

This follows from Theorem 1 for all irrational $a\in W$ and by continuity for all rational $a\in W$. In particular, it follows that the Brodén-Borel-Lévy formula holds under ν^a for any $a\in W$, that is

$$\nu^a\left(\tau^n\in[-\lambda/2,x)\,|a_n,\varepsilon_na_{n-1},\dots,\varepsilon_2a_1,\varepsilon_1a\right)=\frac{(2x+\lambda)(s_n^a\lambda+2)}{4\lambda(s_n^ax+1)}, \tag{4}$$

for $x\in I$, $n\in\mathbf{N}^*$.

4. The Gauss-Kuzmin type equation

Let μ be an arbitrary non-atomic probability on $\mathcal{B}_I$ and define

$$F_n(x)=F_n(x,\mu)=\mu\left(\tau^n\in[-\lambda/2,x)\right),\quad n\in\mathbf{N},\ x\in I.$$

Clearly, $F_0(x) = \mu\left(\left[-\frac{\lambda}{2}, x\right)\right)$ because τ^0 is the identity map. Since $-\frac{\lambda}{2} \le \tau^{n+1} < x$ if and only if

$$(x + a_{n+1}(x)\lambda)^{-1} < \varepsilon_{n+1}(x)\tau^n(x) \le \left(-\lambda/2 + a_{n+1}(x)\lambda\right)^{-1},$$

we can write the Gauss-Kuzmin type equation as

$$F_{n+1}(x) = \sum_{(l,i)\in X} l\left[F_n\left(\frac{l}{-\frac{\lambda}{2}+i\lambda}\right) - F_n\left(\frac{l}{x+i\lambda}\right)\right],$$

$n \in \mathbf{N}$, $x \in I$, $X = \{-1, 1\} \times \mathbf{N}^*$.

Assuming that for some $m \in \mathbf{N}$ the derivative F'_m exists everywhere in I and is bounded, it is easy to see by induction that F'_{m+n} exists and is bounded for all $n \in \mathbf{N}^*$. Differentiating the Gauss-Kuzmin type equation we arrive at

$$F'_{n+1}(x) = \sum_{(l,i)\in X} \frac{1}{(x+i\lambda)^2} \cdot F'_n\left(\frac{l}{x+i\lambda}\right), \quad n \ge m, x \in I.$$

We consider two cases.

4.1. **Even indices**. Let $h(x,y) = \frac{C}{(1+xy)^2}$ be the invariant density on $\Omega = \bigcup_{k=1}^{p} J_k \times K_k$, where $q = 2p$, with $p \ge 2$ (see Subsection 2.1). For $t \in J_j = [\phi_{j-1}, \phi_j)$, $j \in \{1, \dots, p-1\}$ and $K_j = [0, L_j]$ we obtain

$$h_x(t) = \int_0^{L_j} h(t,y)dy = C\int_0^{L_j} \frac{1}{(1+ty)^2}dy = \frac{CL_j}{1+tL_j},$$

and, for $t \in J_p = \left[0, \frac{\lambda}{2}\right)$ and $K_p = [0, 1]$,

$$h_x(t) = C\int_0^1 \frac{1}{(1+ty)^2}dy = \frac{C}{1+t}.$$

Thus we have

$$h(t) = h_x(t) = \frac{CH_t}{1+tH_t}, \quad t \in I,$$

where

$$H_t = \begin{cases} L_j, & \text{if} \quad t \in J_j,\ j \in \{1, \dots, p-1\}, \\ 1, & \text{if} \quad t \in J_p. \end{cases}$$

Further, write

$$f_n(x) = \frac{C}{h(x)} \cdot F'_n(x) = \frac{1+xH_x}{H_x} \cdot F'_n(x), \quad x \in I$$

to get

$$f_{n+1}(x) = \frac{1+xH_x}{H_x} \sum_{(l,i)\in X} \frac{H_{\frac{l}{x+i\lambda}}}{(x+i\lambda)\left(x+i\lambda+lH_{\frac{l}{x+i\lambda}}\right)} \cdot f_n\left(\frac{l}{x+i\lambda}\right),$$

for $n \geq m$. Clearly, the above equation reduces to

$$f_{n+1} = V f_n, \quad n \geq m,$$

with V being the linear operator defined on $\mathcal{B}(I)$ by

$$V f(x) = \sum_{(l,i)\in X} q_{li}(x) f(v_{li}(x)),$$

where

$$q_{li}(x) = \begin{cases} 0, \quad \text{if} \quad x \in \left[-\frac{\lambda}{2}, \frac{2}{\lambda} - \lambda\right], \quad l = \pm 1,\ i = 1, \\ \frac{H_{\frac{l}{x+i\lambda}}}{H_x} \cdot \frac{1+xH_x}{(x+i\lambda)\left(x+i\lambda+lH_{\frac{l}{x+i\lambda}}\right)}, \quad \text{otherwise and,} \end{cases}$$

$$v_{li}(x) = \begin{cases} \frac{l\lambda}{2}, \quad \text{if} \quad x \in \left[-\frac{\lambda}{2}, \frac{2}{\lambda} - \lambda\right], \quad l = \pm 1,\ i = 1, \\ \frac{l}{x+i\lambda}, \quad \text{otherwise.} \end{cases}$$

Note that V is the transition operator of the homogeneous Markov chain $(y_n)_{n\geq 0}$ defined as

$$y_n = [\varepsilon_n a_n, \varepsilon_{n-1} a_{n-1}, \dots, \varepsilon_1 a_1], \quad n \in \mathbf{N}^*,$$

and $y_0 = 0$. Clearly, $(y_n)_{n\geq 0}$ satisfies the recursion equation

$$y_n = \frac{\varepsilon_n}{a_n \lambda + y_{n-1}}, \quad n \in \mathbf{N}^*,\ y_0 = 0,$$

and $y_n \in I$ for all indices n. Let us consider the quadruple

$$\{(I, \mathcal{B}_I), (X, \mathcal{X}), v, Q\},$$

where $v : I \times X \to I$ is given by $v(x,(l,i)) = v_{li}(x)$ and $Q : I \times X \to [0,1]$ is given by $Q(x,(l,i)) = q_{li}(x)$.

Let us prove that $\sum_{(l,i)\in X} q_{li}(x) = 1$, for all $x \in I$. First, we show that $\sum_{(l,i)\in X} q_{li}(x) = 1$, on $\left[0, \frac{\lambda}{2}\right)$. From

$$q_{li}(x) = \frac{1+xH_x}{H_x} \cdot \frac{H_{\frac{l}{x+i\lambda}}}{(x+i\lambda)\left(x+i\lambda+lH_{\frac{l}{x+i\lambda}}\right)}$$

for all $(l,i)\in X$ and, $H_x=1$ on $\left[0,\frac{\lambda}{2}\right)$, we obtain

$$\begin{aligned}
\sum_{(l,i)\in X} q_{li}(x) &= (1+x)\sum_{(l,i)\in X} l\left(\frac{1}{x+i\lambda}-\frac{1}{x+i\lambda+lH_{\frac{l}{x+i\lambda}}}\right)\\
&= (1+x)\left[\sum_{i\geq 1}\left(\frac{1}{x+i\lambda}-\frac{1}{x+i\lambda+H_{\frac{1}{x+i\lambda}}}\right)\right.\\
&\qquad\left.+\sum_{i\geq 1}-\left(\frac{1}{x+i\lambda}-\frac{1}{x+i\lambda-H_{\frac{-1}{x+i\lambda}}}\right)\right]\\
&= (1+x)\sum_{i\geq 1}\left(\frac{1}{x+i\lambda-H_{\frac{-1}{x+i\lambda}}}-\frac{1}{x+i\lambda+H_{\frac{1}{x+i\lambda}}}\right).
\end{aligned}$$

Since $-\frac{1}{x+i\lambda}\in J_{p-1}$ $\left(x\in\left[0,\frac{\lambda}{2}\right),\ i\geq 1\right)$ it follows that $H_{\frac{-1}{x+i\lambda}}=L_{p-1}=\lambda-1$. Also $H_{\frac{1}{x+i\lambda}}=1$, $\left(x\in\left[0,\frac{\lambda}{2}\right),\ i\geq 1\right)$. Hence the right-hand side of the above equation equals

$$\begin{aligned}
&(1+x)\sum_{i\geq 1}\left(\frac{1}{x+i\lambda-(\lambda-1)}-\frac{1}{x+i\lambda+1}\right)\\
&=(1+x)\sum_{i\geq 1}\left(\frac{1}{x+(i-1)\lambda+1)}-\frac{1}{x+i\lambda+1}\right)=1.
\end{aligned}$$

Next, to prove that $\sum_{(l,i)\in X} q_{li}=1$, on $J_{p-1}=\left[-\frac{1}{\lambda},0\right)$, we note that

$$\begin{aligned}
&H_x=L_{p-1}=\lambda-1,\\
&H_{\frac{1}{x+i\lambda}}=1\quad(i\geq 1),\\
&H_{\frac{-1}{x+\lambda}}=L_{p-2}=\lambda-\frac{1}{\lambda-1},\\
&H_{\frac{-1}{x+i\lambda}}=L_{p-1}=\lambda-1\quad(i\geq 2).
\end{aligned}$$

Hence

$$\begin{aligned}
\sum_{(l,i)\in X} q_{li}(x) &= \frac{1+x(\lambda-1)}{\lambda-1}\cdot\sum_{(l,i)\in X} l\left(\frac{1}{x+i\lambda}-\frac{1}{x+i\lambda+lH_{\frac{l}{x+i\lambda}}}\right)\\
&= \frac{1+x(\lambda-1)}{\lambda-1}\left[\sum_{i\geq 1}\left(\frac{1}{x+i\lambda}-\frac{1}{x+i\lambda+H_{\frac{1}{x+i\lambda}}}\right)-\right.\\
&\qquad\left.-\left(\frac{1}{x+i\lambda}-\frac{1}{x+i\lambda-H_{\frac{-1}{x+i\lambda}}}\right)\right]\\
&= \frac{1+x(\lambda-1)}{\lambda-1}\sum_{i\geq 1}\left(\frac{1}{x+i\lambda-H_{\frac{-1}{x+i\lambda}}}-\frac{1}{x+i\lambda+H_{\frac{1}{x+i\lambda}}}\right)\\
&= \frac{1+x(\lambda-1)}{\lambda-1}\cdot\frac{\lambda-1}{1+x(\lambda-1)}=1.
\end{aligned}$$

For the other cases, similar proofs hold, and we leave them to the reader.

Thus we have shown that the sequence $(y_n)_{n\geq 0}$ is an I-valued Markov chain on $(I,\mathcal{B}_I,\nu_x)$ which start at $y_0=0$ and has the following transition mechanism: from state $s \in I$, the only possible transitions are those to states $\frac{l}{s+i\lambda}$, $(l,i) \in X \setminus \{(-1,1),(1,1)\}$, and to $\frac{l\lambda}{2}$ or to $\frac{l}{s+\lambda}$ if $l = \pm 1$, $i = 1$ and $s \in \left[-\frac{\lambda}{2}, \frac{2}{\lambda} - \lambda\right]$ or $s \in \left(\frac{2}{\lambda} - \lambda, \frac{\lambda}{2}\right)$, respectively, the transition probability being $q_{li}(s)$.

4.2. **Odd indices**. We have the same goal as in the previous subsection. Let $q = 2h+3$, with $h \geq 1$, and recycle notation as above. Since $h(x,y) = \frac{C}{(1+xy)^2}$ is the invariant density on $\Omega = \bigcup_{j=1}^{2h+2} J_j \times K_j$ (see Subsection 2.2), it follows that for $t \in J_j$, $j \in \{1, \dots, 2h+1\}$ and $K_j = [0, L_j]$, $j \in \{1, \dots, 2h+1\}$ we have

$$h_x(t) = \int_0^{L_j} h(t,y)dy = \frac{CL_j}{1+tL_j},$$

while for $t \in J_{2h+2} = \left[0, \frac{\lambda}{2}\right)$ and $K_{2h+2} = [0, R]$, where $\frac{\lambda}{2} < R < 1$, we get

$$h_x(t) = \frac{CR}{1+tR}.$$

Thus

$$h(t) = h_x(t) = \frac{CH_t}{1+tH_t}, \quad t \in I,$$

where

$$H_t = \begin{cases} L_j, & \text{if} \quad t \in J_j, \quad j \in \{1, \dots, 2h+1\}, \\ R, & \text{if} \quad t \in J_{2h+2}. \end{cases}$$

Further, write $f_n(x) = \frac{C}{h(x)} F_n'(x)$, $x \in I$, to get $f_{n+1} = Vf_n$, $n \geq m$, where V is the linear operator defined on $\mathcal{B}(I)$ by

$$Vf(x) = \sum_{(l,i) \in X} q_{li}(x) f(v_{li}(x)),$$

with q_{li} and v_{li}, $(l,i) \in X$ defined as in the previous subsection. As in the even case, we may prove that $\sum_{(l,i) \in X} q_{li}(x) = 1$ on I. Also, it appears that V is the transition operator of the homogeneous Markov chain $(y_n)_{n \geq 0}$ on $(I, \mathcal{B}_I, \nu_x)$.

5. Characteristic properties of the transition operators U and V

Here we restrict our attention to the transition operators U and V introduced in Sections 3 and 4. In this section we deal with the properties of U and V on function spaces different from $B(W)$ and $B(I)$.

In connection with the operators U and V, we note the following properties. First, if we define

$$U^\infty f = \int_W f(w) \nu_y(dw), \quad f \in B(W),$$

and

$$V^\infty f = \int_I f(w)\nu_x(dw), \quad f \in B(I),$$

then we have $U^\infty U^n f = U^\infty f$ for all $f \in B(W)$ and $V^\infty V^n f = V^\infty f$ for all $f \in B(I)$ and $n \in \mathbf{N}^*$. Second, let $L(W)$ (respectively $L(I)$) be the Banach space of all bounded complex-valued Lipschitz functions on W (respectively I) under the usual norm $||f||_L = |f| + s(f)$, where

$$s(f) = \sup_{w \neq w'} \frac{|f(w) - f(w')|}{|w - w'|}.$$

Then U (respectively V) sends boundedly $L(W)$ (respectively $L(I)$) into itself. Moreover, we have the following results.

Theorem 2. *The random system with complete connections (RSCC)*

$$\{(W, \mathcal{W}), (X, \mathcal{X}), u, P\}$$

associated with the λCF-expansion is with contraction and its transition operator U is regular w.r.t $L(W)$.

Proof. We have for all $(l,i) \in X$

$$\sup_{w\in W} \left|\tfrac{d}{dw} u(w,(l,i))\right| \leq \sup_{w\in\left[0, \frac{\lambda R-1}{R}\right)} \tfrac{1}{(\lambda-w)^2} < 1,$$
$$\sup_{w\in W} \left|\tfrac{d}{dw} P(w,(l,i))\right| \leq \infty.$$

Hence the requirements of the definition of an RSCC with contraction are met with $k = 1$ (see Definition 3.1.15 in [6]). By Theorem 3.1.16 in [6], it follows that the Markov chain $(s_n^a)_{n\geq 0}$ associated with the RSCC $\{(W, \mathcal{W}), (X, \mathcal{X}), u, P\}$ is a Doeblin-Fortet chain. Hence by Definition 3.2.1 in [6], the Markov chain $(s_n^a)_{n\geq 0}$ is compact, because its state space is a compact metric space $(W,d) = ([0,R],d)$, with $d(x,y) = |x-y|$, $\forall x,y \in W$ and its transition operator is a Doeblin-Fortet operator.

To prove the regularity of U w.r.t. $L(W)$, let us define recursively $w_{n+1} = (w_n + 2)^{-1}$, $n \in \mathbf{N}$, with $w_0 = w$. A criterion of regularity is expressed in Theorem 3.2.13 in [6], in terms of the supports $\sum_n(w)$ of the n-step transition probability functions $P^n(w,\cdot)$, $n \in \mathbf{N}^*$ where, with the usual notation,

$$P(w,B) = \sum_{\{(l,i)\in X \mid u(w,(l,i))\in B\}} P(w,(l,i)), \quad w \in W,\ B \in \mathcal{W}.$$

Clearly $w_{n+1} \in \sum_1(w_n)$. Therefore, Lemma 3.2.14 in [6] and an induction argument lead to the conclusion that $w_n \in \sum_n(w)$, $n \in \mathbf{N}^*$. But $\lim_{n\to\infty} w_n = \sqrt{2} - 1$ for any $w \in W$. Consequently

$$\lim_n d\left(\textstyle\sum_n(w), \sqrt{2}-1\right) = 0.$$

Now, the regularity of U w.r.t. $L(W)$ follows from Theorem 3.2.13 in [6] and the proof is complete. $\square$

Theorem 3. *The random system with complete connections (RSCC)*

$$\{(I, \mathcal{B}_I), (X, \mathcal{X}), v, Q\}$$

is with contraction and its transition operator V is regular w.r.t. $L(I)$.

Proof. The proof parallels that one of Theorem 2. We have for all $(l,i) \in X$

$$\sup_{x\in I} \left|\tfrac{d}{dx} v(x,(l,i))\right| \le \sup_{x\in\left(\frac{2}{\lambda}-\lambda, \frac{\lambda}{2}\right)} \tfrac{1}{(x+\lambda)^2} < 1,$$
$$\sup_{x\in I} \left|\tfrac{d}{dx} Q(x,(l,i))\right| \le \infty.$$

We deduce that the Markov chain $(y_n)_{n\ge 0}$ associated with the RSCC $\{(I, \mathcal{B}_I),\ (X, \mathcal{X}), v, Q\}$ is a Doeblin-Fortet chain. Moreover, the Markov chain $(y_n)_{n\ge 0}$ is compact and its transition operator is a Doeblin-Fortet operator.

To prove the regularity of V w.r.t. $L(I)$ we proceed as in the preceding proof. Here the transition probability function is

$$Q(x,A) = \sum_{\{(l,i)\in X\,|\,v(x,(l,i))\in A\}} Q(x,(l,i)), \quad x \in I,\ A \in \mathcal{B}_I.$$

A similar argument leads to the regularity of V. $\square$

Remark. Theorem $1'$ in [4] shows that there exist positive constants K_1, K_2, $\beta < 1$ and $\theta < 1$ such that

$$||U^n f - U^\infty f||_L \le K_1 \beta^n ||f||_L \quad (n \ge 1, \quad f \in L(W)),$$
$$||V^n f - V^\infty f||_L \le K_2 \theta^n ||f||_L \quad (n \ge 1, \quad f \in L(I)).$$

6. A Gauss-Kuzmin type problem

The results obtained allow to a solution of a Gauss-Kuzmin type problem. The solution presented here is based on the ergodic behaviour of the RSCC associated with the transition operator V.

It should be mentioned that it is only very recently that there has been any investigation of the metrical properties of the Rosen fractions (Nakada (in [10]) started investigations of metrical properties of the Rosen fractions, but only with even q's). Thus, we may emphasize that our solution obtained in an elementary manner is surprisingly simple and could be used in similar contexts.

For any arbitrarily given $n \in \mathbf{N}^*$, take

$$\mu^a = \nu^a(\cdot \mid a_n, \varepsilon_n a_{n-1}, \ldots, \varepsilon_2 a_1, \varepsilon_1 a), \quad a \in W$$

(see Section 3). By equation (4), we have

$$\begin{aligned} F_n^a(x) &= \nu^a\left(\tau^n \in \left[-\tfrac{\lambda}{2}, x\right) | a_n, \varepsilon_n a_{n-1}, \dots, \varepsilon_1 a\right) \\ &= \tfrac{(2x+\lambda)(s_n^a\lambda+2)}{4\lambda(s_n^a x+1)}, \quad x \in I, \quad n \in \mathbf{N}^*,\ a \in W. \end{aligned}$$

By elementary computations we obtain

$$f_n^a(x) = \frac{1+xH_x}{H_x}(F_n^a)'(x) = \frac{4-\lambda^2(s_n^a)^2}{4\lambda} \cdot \frac{1+xH_x}{H_x(s_n^a x+1)^2},\ x \in I,$$

$$V^\infty f_n^a = \frac{C(4-\lambda^2(s_n^a)^2)}{4\lambda}\int_{\frac{-\lambda}{2}}^{\frac{\lambda}{2}} \frac{1}{(s_n^a x+1)^2}dx = C,$$

and $|f_n^a| < 2$, $s(f_n^a) < 1$, $||f_n^a||_L < 3$ for all possible values of s_n^a.

Now, we note that on account of the last remark in Section 5, we actually proved the following Gauss-Kuzmin type theorem.

Theorem 4 (Solution of the Gauss-Kuzmin problem). *For all $a \in W$ there exist two positive constants K and $\theta < 1$ such that*

$$\left|\mu^a\left(\tau^{-n}\left([-\lambda/2, t)\right)\right) - \rho\left([-\lambda/2, t)\right)\right| \le 3K\theta^n\rho\left([-\lambda/2, t)\right),\ (t \in I, n \in \mathbf{N}),$$

where ρ denotes the invariant measure ν_x for τ on $\mathcal{B}_I$.

Remark. Theorem 4 reduces for $a = 0$ and $\lambda = 1$ to a version of Gauss-Kuzmin type theorem for the nearest integer continued fraction, where μ^0 represents the Lebesgue measure on $\left[-\frac{1}{2}, \frac{1}{2}\right]$.

It should be emphasized that, to our knowledge, Theorem 4 is the first Gauss-Kuzmin result proved for the Rosen fractions.

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