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An explicit formula for the Mahler measure of a family of 3-variable polynomials

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To Michel Mendès France on the occasion of his 65th birthday

Résumé. On montre une formule explicite pour la mesure de Mahler du polynôme $a + bx^{-1} + cy + (a + bx + cy)z$ en termes de dilogarithmes et trilogarithmes.

Abstract. An explicit formula for the Mahler measure of the 3-variable Laurent polynomial $a + bx^{-1} + cy + (a + bx + cy)z$ is given, in terms of dilogarithms and trilogarithms.

1. Introduction

Over recent years there has been some interest in calculating explicit formulae for the Mahler measure of polynomials. By an *explicit formula* I mean, roughly, one not involving integrals or infinite sums, but involving only standard functions (possibly defined by integrals!). For a polynomial $R(x_1, \ldots, x_n)$ its (logarithmic) Mahler measure, a kind of height function, is defined as

$$m(R) = \tfrac{1}{(2\pi)^n} \int_0^{2\pi} \cdots \int_0^{2\pi} \log \left| R(e^{i\theta_1}, \cdots, e^{i\theta_n}) \right| \, d\theta_1 \, \ldots \, d\theta_n.$$

For a 1-variable polynomial $R(x) = a \prod_j (x - \alpha_j)$, Jensen's Theorem

$$\tfrac{1}{2\pi} \int_0^{2\pi} \log |e^{i\theta} - \alpha| \, d\theta = \log_+ |\alpha|$$

shows that $m(R) = \log |a| + \sum_j \log_+ |\alpha_j|$. Here $\log_+ x = \max(0, \log x)$. For 2-variable polynomials the situation is much more interesting, with many examples of explicit formulae for their measures having been given in terms of L-functions of quadratic characters – see Boyd [B1], [B2], [B4], Ray [R], Smyth [Sm]. Such formulae can readily be re-cast in terms of dilogarithms evaluated at associated roots of unity. Also, Boyd [B2], [B4], Rodriguez Villegas [RV] have produced many formulae (many proved, but some still conjectural) for 2-variable Mahler measures which are rational multiples of the derivative of the L-function of the associated elliptic curve

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evaluated at 0. Further, Boyd and Rodriguez Villegas [BRV] have obtained explicit formulae for 2-variable Mahler measures of polynomials of the form $p(x)y - q(x)$, where p and q are cyclotomic polynomials; their formulae involve the Bloch-Wigner dilogarithm evaluated at certain algebraic points.

For polynomials in four or more variables, no non-trivial explicit formulae are known. For 3-variable polynomials, up till now there has been only one non-trivial example $m(1+x+y+z)$ of an explicit formula–see Corollary 2 below.

In this paper the Mahler measure of the Laurent polynomial

$$P_{a,b,c}(x,y,z) = a + bx^{-1} + cy + (a+bx+cy)z$$

is evaluated explicitly, for a, b and c any real numbers. To give the formulae for this measure, we need some definitions. We take $\mathrm{Li}_n(x)$ to be the classical n^{th} polylogarithm function (see also Lemma 1 below) and, following Zagier [Z] put

$$\mathrm{L}_n(x) = \sum_{r=0}^{n-1} \frac{(-\log|x|)^r}{r!}\,\mathrm{Li}_{n-r}(x), \quad (n = 1,2,3,\ldots) \tag{1.1}$$

a modified n^{th} polylogarithm function which for $n \geqslant 2$ is real for all real x (Lemma 4). In particular

$$\mathrm{L}_2(x) = \mathrm{Li}_2(x) + \log|x| \cdot \mathrm{Log}(1-x) \tag{1.2}$$

$$\mathrm{L}_3(x) = \mathrm{Li}_3(x) - \log|x| \cdot \mathrm{Li}_2(x) - \tfrac{1}{2}\log^2|x| \cdot \mathrm{Log}(1-x) \tag{1.3}$$

where Log denotes the principal value of the complex logarithm, having imaginary part in $(-\pi, \pi]$.

Next, for $x, y \neq 0$ put

$$g_2(x,y) = \mathrm{L}_2\left(\frac{1+x+y}{x}\right),$$
$$g_3(x,y) = \mathrm{L}_3\left(\frac{1+x+y}{x}\right) + \mathrm{L}_3\left(\frac{1+x+y}{y}\right) - \mathrm{L}_3\left(\frac{1+x+y}{-xy}\right),$$

and for $i = 2$ and 3

$$G_i(x,y) = g_i(x,y) - g_i(x,-y) - g_i(-x,y) + g_i(-x,-y).$$

We can now state our main result, giving $m(P_{0,1,c})$ and $m(P_{1,b,c})$, from which the formulae for general $m(P_{a,b,c})$ follow as an immediate corollary.

Theorem. (i) *Let* $0 \leqslant c \leqslant 1$. *Then*

$$m(P_{0,1,c}) = \tfrac{2}{\pi^2}\left(\mathrm{Li}_3(c) - \mathrm{Li}_3(-c)\right).$$

(ii) *Let* $b > 0$ *and* $c > 0$. *Then*

$$m(P_{1,b,c}) = \tfrac{1}{\pi^2}(G_3(b,c) + \log c \cdot G_2(b,c) + \log b \cdot G_2(c,b)).$$

From the fact that $m(\lambda P) = \log|\lambda| + m(P)$ and $m(P(\pm x, \pm y, \pm z)) = m(P)$ (see (3.13)) we obtain the following.

Corollary 1. (i) *For b and c real, not both 0, with $|b| \geqslant |c|$*

$$m(P_{0,b,c}) = m(P_{0,c,b}) = \log|b| + \tfrac{2}{\pi^2}\left(\mathrm{Li}_3\left(\left|\tfrac{c}{b}\right|\right) - \mathrm{Li}_3\left(-\left|\tfrac{c}{b}\right|\right)\right).$$

(ii) *For a, b, c real and all non-zero*

$$m(P_{a,b,c}) = \log|a| + \tfrac{1}{\pi^2}\Big(G_3\left(\left|\tfrac{b}{a}\right|, \left|\tfrac{c}{a}\right|\right) + \log\left|\tfrac{c}{a}\right| \cdot G_2\left(\left|\tfrac{b}{a}\right|, \left|\tfrac{c}{a}\right|\right) + \log\left|\tfrac{b}{a}\right| \cdot G_2\left(\left|\tfrac{c}{a}\right|, \left|\tfrac{b}{a}\right|\right)\Big).$$

The case b or $c = 0$ of (ii) is trivial: the 2-variable polynomials $P_{a,0,c}$ and $P_{a,c,0}$ both have measure $m(a + cy) = \log(\max(|a|, |c|))$.

Boyd [B1] has conjectured that the set of all $m(R)$, for R a Laurent polynomial in any number of variables and having integer coefficients, is a closed subset of $\mathbb{R}$. Our theorem gives explicit formulae for two three-variable Mahler measures in this set.

Corollary 2. *We have*

$$m(1 + x + y + z) = \tfrac{7}{2\pi^2}\zeta(3)$$

and

$$m(1 + x^{-1} + y + (1 + x + y)z) = \tfrac{14}{3\pi^2}\zeta(3),$$

where ζ is the Riemann zeta function.

The first of these results is not new - see [B1], Appendix 1. Two more examples (specialisations of the Theorem) are given at the end of the paper.

The results of the Theorem are proved using two main auxiliary results. The first (Proposition 1) enables us to replace certain integrals over a whole torus by the same integral over part of the torus. The resulting integrals can then be calculated with the aid of certain identities (Proposition 2). These identities were derived by simplifying the results of computing the indefinite integrals $\int\int \log(x + y)\frac{dx\,dy}{xy}$ and $\int\int \log(1 + x + y)\frac{dx\,dy}{xy}$ using Mathematica. In particular, the Mathematica result of the second integral needed a great deal of simplification, Mathematica originally producing screenfuls of output! The proofs of these identities are now independent of Mathematica, though I would not have found them without its help.

For the proof we will also need some results about $\mathrm{L}_n(x)$ and also about $\Lambda_n(x)$ (defined by (3.6) below), which is an analytic version of L_n, and related to Kummer's n^{th} polylogarithm [K]. (The function $\mathrm{L}_n(x)$ of the complex variable x is only real analytic in the two variables $\Re x, \Im x$–see [Z].)

In Section 2 we state and prove Proposition 1. In Section 3 we give some results concerning Li_n, L_n and a related function Λ_n and the connections between them. In Section 4 we state and prove Propositions 2 and 3, and then prove the Theorem in Section 5. In Section 6 we prove Corollary 2. Section 7 contains two further examples.

2. Restricting the integral to part of the torus

Suppose that we have a polynomial H of the special form

$$H(x_1,...,x_n,y_1,...,y_r,z) = P(x_1,...,x_n) + Q(y_1,...,y_r) + (P(x_1,...,x_n) + Q(y_1^{-1},...,y_r^{-1}))z,$$

a polynomial in $n+r+1$ variables. Here P and Q are polynomials with real coefficients. Then $m(H)$ can be expressed in the following form.

Proposition 1. *We have*

$$m(H) = 4\left(\tfrac{1}{2\pi i}\right)^{n+r} \int \log|P(\mathbf{x}) + Q(\mathbf{y})| \tfrac{dx_1}{x_1} \cdots \tfrac{dx_n}{x_n}\tfrac{dy_1}{y_1} \cdots \tfrac{dy_r}{y_r}$$

where the integral is taken over the region where $|\mathbf{x}| = |\mathbf{y}| = 1$ *and* $\Im P(\mathbf{x}) \geqslant 0$, $\Im Q(\mathbf{y}) \geqslant 0$.

Here for instance $|\mathbf{x}| = 1$ denotes the torus $|x_1| = \cdots = |x_n| = 1$.

Proof. Applying Jensen's Theorem to the z-variable we have

$$m(H) = m(P(\mathbf{x}) + Q(\mathbf{y}^{-1})) + \left(\tfrac{1}{2\pi}\right)^{n+r} \int \log_+ \left|\frac{P(\mathbf{x}) + Q(\mathbf{y})}{P(\mathbf{x}) + Q(\mathbf{y}^{-1})}\right| d\boldsymbol{\theta} d\boldsymbol{\phi}$$

where $\mathbf{y}^{-1} = (y_1^{-1},...,y_r^{-1})$, $d\boldsymbol{\theta} = d\theta_1...d\theta_n$, $d\boldsymbol{\phi} = d\phi_1...d\phi_r$, $x_j = e^{i\theta_j}$ $(j = 1,...,n)$, $y_k = e^{i\phi_k}(k = 1,...,r)$, and the integral is taken over the whole $n+r$ torus $0 \leqslant \theta_j < 2\pi$ $(j = 1,...,n)$, $0 \leqslant \phi_k \leqslant 2\pi (k = 1,...,r)$. Now $|P+Q| \geqslant |P + \overline{Q}|$ for $\Im P \geqslant 0$ and $\Im Q \geqslant 0$ or $\Im P \leqslant 0$ and $\Im Q \leqslant 0$, but otherwise $|P+Q| < |P+\overline{Q}|$. So, as $Q(\mathbf{y}^{-1}) = \overline{Q(\mathbf{y})}$ on the r-torus $|\mathbf{y}| = 1$, we have

$$(2\pi)^{n+r} m(H) = (2\pi)^{n+r} m(P+\overline{Q}) + \int_{R_{++}\cup R_{--}} \log\left|\frac{P+Q}{P+\overline{Q}}\right| d\boldsymbol{\theta} d\boldsymbol{\phi},$$

where

$$R_{++} = \{(\mathbf{x},\mathbf{y}) \mid \Im P(\mathbf{x}) \geqslant 0,\ \Im Q(\mathbf{y}) \geqslant 0\}$$
$$R_{--} = \{(\mathbf{x},\mathbf{y}) \mid \Im P(\mathbf{x}) < 0,\ \Im Q(\mathbf{y}) < 0\}.$$

Hence, defining R_{+-}, R_{-+} similarly,

$$\begin{aligned}(2\pi)^{n+r}m(H) &= \int_{R_{++}\cup R_{+-}\cup R_{-+}\cup R_{--}} \log\left|P+\overline{Q}\right| d\boldsymbol{\theta} d\boldsymbol{\phi} \\ &\qquad + \int_{R_{++}\cup R_{--}} (\log|P+Q| - \log\left|P+\overline{Q}\right|) d\boldsymbol{\theta} d\boldsymbol{\phi} \\ &= \int_{R_{++}\cup R_{--}} \log|P+Q|\, d\boldsymbol{\theta} d\boldsymbol{\phi} + \int_{R_{-+}\cup R_{+-}} \log\left|P+\overline{Q}\right| d\boldsymbol{\theta} d\boldsymbol{\phi} \\ &= 2\int_{R_{++}\cup R_{--}} \log|P+Q|\, d\boldsymbol{\theta} d\boldsymbol{\phi} = 4\int_{R_{++}} \log|P+Q|\, d\boldsymbol{\theta} d\boldsymbol{\phi}.\end{aligned}$$

Finally, as $d\theta_j = -\frac{idx_j}{x_j}$, $d\phi_k = -\frac{idy_k}{y_k}$, we have the result.

3. Li_n, $\boldsymbol{\Lambda}_n$ and L_n.

In this section we discuss Li, Λ_n and L, and the relations between them. Not all of these results are needed for the proof of the Theorem. Many are well–known, or minor variants of well–known results. We seem to need these three varieties of polylogarithm. We have already used Li_n and L_n to state the main theorem. Also, the function Λ_n, defined below, is analytic in the cut plane $\mathbb{C}\backslash(-\infty,0]$, and is therefore particularly convenient to work with. Furthermore, its values can be given in terms of the L_r (Lemma 5).

We first recall that for $n \geqslant 1$

$$\mathrm{Li}_n(x) = \sum_{k=1}^{\infty} \frac{x^k}{k^n} \qquad (|x|<1),$$

that the principal value of $\mathrm{Li}_1(x)$ is

$$\mathrm{Li}_1(x) = -\operatorname{Log}(1-x),$$

and that for $n \geqslant 2$

$$\mathrm{Li}_n'(x) = \frac{\mathrm{Li}_{n-1}(x)}{x}, \tag{3.1}$$

leading to the expression of Li_n as an n-fold integral. It is useful here to note that Li_n can also be written as a single integral, as follows.

Lemma 1. *The principal value of* $\mathrm{Li}_n(x)$ *is given for* $n \geqslant 1$ *and all* $x \in \mathbb{C}$ *by*

$$\mathrm{Li}_n(x) = \tfrac{1}{(n-1)!}\int_0^x (\operatorname{Log}(x/t))^{n-1}\frac{dt}{1-t}, \tag{3.2}$$

where the integral is say taken along a ray from 0 *to* x, *unless* x *is real and greater than* 1, *in which case the path* $0 \to x$ *should pass just below the pole at* $t=1$.

The function Li_n *is analytic in* $\mathbb{C}\backslash[1,\infty)$, *and for* x *on the cut* $(1,\infty)$, *the discontinuity* $\lim_{\delta\searrow 0}\mathrm{Li}_n(x+i\delta)-\mathrm{Li}_n(x)$ *is equal to* $\frac{2\pi i}{(n-1)!}\log^{n-1}(x)$.

Proof. We have $\mathrm{Li}_1(x) = -\operatorname{Log}(1-x)$ and it is easily verified that (3.1) holds for $n = 2, 3, \ldots$. This integral form of $\mathrm{Li}_n(x)$ shows that it is analytic in $\mathbb{C}\backslash[1,\infty)$. To evaluate the discontinuity at $x > 1$, one can, as noted in [Z], use (3.1) and induction. One can also obtain it directly from (3.2) by observing that, by Cauchy's residue theorem,

$$\mathrm{Li}_n(x+i\delta)-\mathrm{Li}_n(x) = \tfrac{1}{(n-1)!}\int_x^{x+i\delta}(\operatorname{Log}(x/t))^{n-1}\frac{dt}{1-t} + \tfrac{2\pi i}{(n-1)!}\log^{n-1}(x),$$

which gives the result on letting $\delta \searrow 0$.

We next state some very simple facts that we need. Firstly, for the principal value Log of log,

$$\operatorname{Log}(xy) = \operatorname{Log} x + \operatorname{Log} y \tag{3.3}$$

when $-\pi < \arg x + \arg y \leqslant \pi$. From this we have that

$$\operatorname{Log}(x/y) = \operatorname{Log} x - \operatorname{Log} y \tag{3.4}$$

when $-\pi < \arg(x/y) + \arg y \leqslant \pi$.

Secondly, we need the following identity coming from the binomial theorem:

$$\tfrac{1}{(n-1)!}\int_0^x (a+b(t))^{n-1}\,\frac{dt}{1-t} = \sum_{r=0}^{n-1}\frac{a^r}{r!}\int_0^x \frac{b(t)^{n-r}}{(n-r)!}\,\frac{dt}{1-t}, \tag{3.5}$$

where b is an integrable function of t, and a and b may also be functions of x.

Now define Λ_n for all $x \in \mathbb{C}$ by

$$\Lambda_n(x) = \tfrac{1}{(n-1)!}\int_0^x (-\operatorname{Log}(t))^{n-1}\frac{dt}{1-t}, \tag{3.6}$$

which is a slight variant of Kummer's n^{th} polylogarithm (See [L], p178, and [K]). Then $\Lambda_1(x) = \mathrm{Li}_1(x)$, analytic in $\mathbb{C}\backslash[1,\infty)$, while for $n \geqslant 2$, Λ_n is analytic in the same region that Log is, namely in $\mathbb{C}\backslash(-\infty,0]$, the integrand of (3.6) then having no pole at $t = 1$.

The next result relates Li_n and Λ_n.

Lemma 2. *For* $n \geqslant 1$ *we have*

$$\mathrm{Li}_n(x) = \sum_{r=0}^{n-1}\frac{(\operatorname{Log} x)^r}{r!}\Lambda_{n-r}(x)$$

and

$$\Lambda_n(x) = \sum_{r=0}^{n-1} \frac{(-\operatorname{Log} x)^r}{r!} \operatorname{Li}_{n-r}(x).$$

Proof. From Lemma 1 and (3.4),

$$\begin{aligned}\operatorname{Li}_n(x) &= \tfrac{1}{(n-1)!} \int_0^x (\operatorname{Log} x - \operatorname{Log} t)^{n-1} \frac{dt}{1-t} \\ &= \sum_{r=0}^{n-1} \frac{(\operatorname{Log} x)^r}{r!} \Lambda_{n-r}(x),\end{aligned}$$

using (3.5) with $a = \log x, b = -\operatorname{Log} t$. The second formula also follows from (3.5), using (3.6) and (3.4) to write $\Lambda_n(x)$ as

$$\Lambda_n(x) = \tfrac{1}{(n-1)!} \int_0^x (-\operatorname{Log} x + \operatorname{Log}(x/t))^{n-1} \frac{dt}{1-t},$$

and then using (3.2).

We also need some properties of Λ_n.

Lemma 3. *Let $n \geqslant 2$. Then*

(a) *For all $x \in \mathbb{C} \backslash (-\infty, 0]$ we have the functional equation*

$$\Lambda_n\left(\tfrac{1}{x}\right) = (-1)^{n-1} \Lambda_n(x) + \frac{(\operatorname{Log} x)^n}{n!} + \begin{cases} 0 & n \text{ odd} \\ 2\Lambda_n(1) & n \text{ even} \end{cases}.$$

(b) *For x on the cut $(-\infty, 0)$*

$$\Delta_n(x) := \lim_{\delta \searrow 0} \Lambda_n(x - i\delta) - \Lambda_n(x) = \sum_{r=1}^{n-1} \frac{(2i\pi)^r}{r!} \Lambda_{n-r}(x).$$

Further, $\Delta_n(x)$ is imaginary $(x < 0)$.

Clearly $\Delta_1(x) = 2\pi i$ for x on the cut $(1, \infty)$ of Λ_1. It amusing to note that for $n = 1$ (a) becomes

$$\Lambda_1\left(\tfrac{1}{x}\right) = \Lambda_1(x) + \operatorname{Log} x \pm 2\pi i$$

where the $+$ sign is taken if $\Im x < 0$ or $x > 1$, and the $-$ sign otherwise.

Proof. (a) Now one readily checks that

$$\begin{aligned}\left(\Lambda_n\left(\tfrac{1}{x}\right)\right)' &= \frac{(\operatorname{Log} x)^{n-1}}{(n-1)!} \left(\frac{1}{1-x} + \frac{1}{x}\right) \\ &= (-1)^{n-1} \Lambda_n'(x) + \left(\frac{(\operatorname{Log} x)^n}{n!}\right)'.\end{aligned}$$

Then the result follows on integration, evaluating the constant by putting $x = 1$.

(b) Take $x < 0$, and put $u = -\operatorname{Log} t$. Then from the definition of Λ_n, Δ_n is given by

$$\begin{aligned}\Delta_n &= \tfrac{1}{(n-1)!}\int_0^x ((2i\pi + u)^{n-1} - u^{n-1})\frac{dt}{1-t}\\ &= \sum_{r=1}^{n-1} \tfrac{(2i\pi)^r}{r!}\Lambda_{n-r}(x),\end{aligned}$$

using (3.5) with $a = 2i\pi, b = -\operatorname{Log} t$. Also, as $\bar{u} = u + 2i\pi$ for $t < 0$, we see that for $x < 0$

$$\Delta_n = \tfrac{1}{(n-1)!}\int_0^x \left(\bar{u}^{\,n-1} - u^{n-1}\right)\frac{dt}{1-t}$$

is imaginary. In particular

(3.7) $\qquad \Delta_2 = 2i\pi\Lambda_1(x) = -2i\pi \log(1-x) \qquad (x < 0)$

and

(3.8) $\qquad \Delta_3 = 2i\pi\Lambda_2(x) - 2\pi^2\log(1-x) = 2i\pi\Re\Lambda_2(x) \qquad (x < 0)$

since Δ_3 has zero real part.

We now express L_n as an integral.

Lemma 4. *We have for $n \geqslant 2$ and all x*

$$\mathrm{L}_n(x) = \tfrac{1}{(n-1)!}\int_0^x (-\log|t|)^{n-1}\frac{dt}{1-t}.$$

This is real for all real x.

Proof. Using Lemma 1 and the definition of L_n we have

$$\begin{aligned}\mathrm{L}_n(x) &= \sum_{r=0}^{n-1} \frac{(-\log|x|)^r}{r!}\int_0^x \frac{(\operatorname{Log}(x/t))^{n-r-1}}{(n-r-1)!}\frac{dt}{1-t}\\ &= \tfrac{1}{(n-1)!}\int_0^x (\operatorname{Log}(x/t) - \log|x|)^{n-1}\frac{dt}{1-t}\end{aligned}$$

using (3.5). Then, as x/t is real and positive when we integrate along the ray from 0 to x, (3.4) gives the result. For $n \geqslant 2$ the integral has no pole at $t = 1$, so the integral is then real for all x.

We can now compare Λ_n and L_n. Clearly $\mathrm{L}_n(x) = \Lambda_n(x)$ for x real and positive.

Lemma 5. *For $\theta = \arg x \in (-\pi, \pi]$ we have*

$$\Lambda_n(x) - \mathrm{L}_n(x) = \sum_{r=1}^{n-1} \tfrac{(-i\theta)^r}{r!}\,\mathrm{L}_{n-r}(x) = -\sum_{r=1}^{n-1} \tfrac{(i\theta)^r}{r!}\Lambda_{n-r}(x).$$

Proof. Put $v = -\log|t|$. Then for $\arg t = \arg x = \theta$, $-\operatorname{Log} t = v - i\theta$, so that

$$\begin{aligned}\Lambda_n(x) - \mathrm{L}_n(x) &= \tfrac{1}{(n-1)!}\int_0^x \left((v-i\theta)^{n-1} - v^{n-1}\right)\tfrac{dt}{1-t}\\ &= \sum_{r=1}^{n-1} \tfrac{(-i\theta)^r}{r!}\,\mathrm{L}_{n-r}(x),\end{aligned}$$

using (3.6). Similarly, for $w = -\operatorname{Log} t = -\log|t| - i\theta$

$$\begin{aligned}\mathrm{L}_n(x) - \Lambda_n(x) &= \tfrac{1}{(n-1)!}\int_0^x \left((w+i\theta)^{n-1} - w^{n-1}\right)\tfrac{dt}{1-t}\\ &= \sum_{r=1}^{n-1} \tfrac{(i\theta)^r}{r!}\Lambda_{n-r}(x),\end{aligned}$$

giving the second result.

In particular, for $x < 0$ we have

$$\Lambda_2(x) - \mathrm{L}_2(x) = i\pi\log(1-x) \tag{3.9}$$

and

$$\Lambda_3(x) - \mathrm{L}_3(x) = -i\pi\,\mathrm{L}_2(x) + \tfrac{\pi^2}{2}\log(1-x) \tag{3.10}$$

Remark. One can readily write down the generating function $\mathrm{Li}(x,T) := \sum_{n=1}^{\infty}\mathrm{Li}_n(x)T^n$ for the Li_n, with similar generating functions $\mathrm{L}(x,T)$ and $\Lambda(x,T)$ for the L_n and the Λ_n, respectively. Then, using the integral representations (3.2), (3.6) and Lemma 4 for these functions we easily obtain

$$\mathrm{Li}(x,T) = T\int_0^x \left(\frac{x}{t}\right)^T \frac{dt}{1-t} = x^T\Lambda(x,T) = |x|^T\,\mathrm{L}(x,T).$$

By expanding these identities one gets alternative proofs of Lemmas 2 and 5, with Lemma 3(b) following similarly.

We next need the following identity for special values of L_3.

Lemma 6. *We have* $2\,\mathrm{L}_3(3) - \mathrm{L}_3(-3) = \frac{13}{6}\zeta(3)$.

Proof. For this somewhat *ad hoc* proof we use the following sequence of results from Lewin [L], with his equation numbers and values of his variables given:

For Li_2 :

(L1.10, $y = 3$) $\qquad \mathrm{Li}_2(3) + \mathrm{Li}_2\left(\tfrac{1}{3}\right) = \tfrac{\pi^2}{3} - \tfrac{1}{2}\log^2 3 - i\pi\log 3$

(L1.31, $x = -1$) $\qquad \mathrm{Li}_2(-3) + 2\,\mathrm{Li}_2\left(\tfrac{1}{3}\right) = -\log^2 3$

from which

$$2\,\mathrm{Li}_2(3)-\mathrm{Li}_2(-3)=\frac{2\pi^2}{3}-2i\pi\log 3. \tag{3.11}$$

For Li_3 :
From its definition we know that

(a) $\mathrm{Li}_3(1)=\zeta(3)$

and also

(b) $\mathrm{Li}_3(-1)=-\frac{3}{4}\zeta(3)$ (L6.5)

which from (1.3) also give $\mathrm{L}_3(1)=\zeta(3)$, $\mathrm{L}_3(-1)=-\frac{3}{4}\zeta(3)$. Further

(c) $\mathrm{Li}_3\left(\frac{1}{4}\right)-4\,\mathrm{Li}_3\left(-\frac{1}{2}\right)-4\,\mathrm{Li}_3\left(\frac{1}{2}\right)=0$ $\left(\text{L6.4, } x=\frac{1}{2}\right)$

(d) $\mathrm{Li}_3(-3)-\mathrm{Li}_3\left(-\frac{1}{3}\right)=-\frac{\pi^2}{6}\log 3-\frac{1}{6}\log^3 3$ (L6.6, $x=3$)

(e) $\mathrm{Li}_3(3)-\mathrm{Li}_3\left(\frac{1}{3}\right)=\frac{\pi^2}{3}\log 3-\frac{1}{6}\log^3 3-\frac{i\pi}{2}\log^2 3$ (L6.7, $y=3$)

(f) $\mathrm{Li}_3\left(-\frac{1}{2}\right)+\mathrm{Li}_3\left(\frac{2}{3}\right)+\mathrm{Li}_3\left(\frac{1}{3}\right)=\zeta(3)+\frac{\pi^2}{6}\log\left(\frac{2}{3}\right)+\frac{1}{2}\log 3\log^2\left(\frac{2}{3}\right)+\frac{1}{6}\log^3\left(\frac{2}{3}\right)$ (L6.10, $x=\frac{1}{3}$)

(g) $\mathrm{Li}_3\left(\frac{3}{4}\right)+\mathrm{Li}_3\left(\frac{1}{4}\right)+\mathrm{Li}_3(-3)=\zeta(3)-\frac{\pi^2}{6}\log 4-\frac{1}{2}\log 3\log^2 4+\frac{1}{3}\log^3 4$ (L6.11, $x=-3$)

(h) $\mathrm{Li}_3\left(\frac{1}{3}\right)-\mathrm{Li}_3\left(-\frac{1}{3}\right)-2\,\mathrm{Li}_3\left(\frac{2}{3}\right)+\frac{1}{2}\,\mathrm{Li}_3\left(\frac{3}{4}\right)=2\,\mathrm{Li}_3\left(\frac{1}{2}\right)-\frac{7}{4}\zeta(3)+\frac{\pi^2}{6}\log\left(\frac{3}{2}\right)-\frac{1}{3}\log^3\left(\frac{3}{2}\right)$ (L6.33, $x=\frac{1}{2}$)

where

$$\mathrm{Li}_3\left(\frac{1}{2}\right)=\frac{7}{8}\zeta(3)-\frac{\pi^2}{12}\log 2+\frac{1}{6}\log^3(2). \tag{L6.12}$$

Then $(c)-2\times(d)+6\times(e)+4\times(f)-(g)+2\times(h)$ gives

$$2\,\mathrm{Li}_3(3)-\mathrm{Li}_3(-3)=\frac{13}{6}\zeta(3)+\frac{2}{3}\pi^2\log 3-i\pi\log^2 3. \tag{3.12}$$

Using (1.3) again, and (3.11), (3.12)

$$\begin{aligned}2\,\mathrm{L}_3(3)-\mathrm{L}_3(-3)&=(2\,\mathrm{Li}_3(3)-\mathrm{Li}_3(-3))-\log 3\cdot(2\,\mathrm{Li}_2(3)-\mathrm{Li}_2(-3))\\&\quad-\frac{1}{2}\log^2 3\cdot(2\,\mathrm{Log}(-2)-\log 4)\\&=\frac{13}{6}\zeta(3)\end{aligned}$$

as claimed.

Lemma 7. (Schinzel [Sc, Cor. 8, p. 226-7]) *For an n-variable polynomial $P(\mathbf{x})$, and a non-singular $n\times n$ integer matrix V, we have*

$$m(P(\mathbf{x}))=m(P(\mathbf{x}^V)).$$

Here as usual $\mathbf{x}^V$ denotes $(\prod_j x_j^{v_{1j}},\dots,\prod_j x_j^{v_{nj}})$, where $\mathbf{x}=(x_1,\dots,x_n)^T$, $V=(v_{ij})$.

Proof. For $\mathbf{x}=(e^{2\pi i\theta_1},...,e^{2\pi i\theta_n})^T$, and $\boldsymbol{\theta}=(\theta_1,...,\theta_n)^T$, $\boldsymbol{\psi}=V\boldsymbol{\theta}$ we have

$$\mathbf{x}^V=(e^{2\pi i\psi_1},...,e^{2\pi i\psi_n})^T.$$

Then the map $\boldsymbol{\theta} \mapsto \boldsymbol{\psi}$ from $\mathbb{R}^n/\mathbb{Z}^n$ to itself is a $|\det V|$-fold linear covering of $\mathbb{R}^n/\mathbb{Z}^n$. On the other hand this map has Jacobian $|\det V|$, so that

$$\begin{aligned} m(P(\mathbf{x}^V)) &= \int_{\mathbb{R}^n/\mathbb{Z}^n} \log|P(e^{2\pi i(V\boldsymbol{\theta})_1}, \ldots, e^{2\pi i(V\boldsymbol{\theta})_n})|\, d\theta_1 \ldots d\theta_n \\ &= |\det V| \int_{\mathbb{R}^n/\mathbb{Z}^n} \log\left|P(e^{2\pi i\psi_1}, ..., e^{2\pi i\psi_n})\right| \frac{d\psi_1 \ldots d\psi_n}{|\det V|} \\ &= m(P(\mathbf{x})). \end{aligned}$$

In particular, the lemma immediately gives

$$m(P(\pm x, \pm y, \pm z)) = m(P) \tag{3.13}$$

4. Two partial derivative identities

We now study functions $\hat{g}_3$ and $\hat{f}$, which are bi-analytic versions of the functions g_3 and f (defined in (5.6) below). The hat denotes 'lambdafication'.

Proposition 2. (a) *The functions*

$$\hat{g}_3(x,y) := \Lambda_3\left(\tfrac{1+x+y}{x}\right) + \Lambda_3\left(\tfrac{1+x+y}{y}\right) - \Lambda_3\left(\tfrac{1+x+y}{-xy}\right)$$

$$\begin{aligned} \hat{f}(x,y) := \hat{g}_3(x,y) &+ (\operatorname{Log} y - i\pi)\Lambda_2\left(\tfrac{1+x+y}{x}\right) \\ &+ (\operatorname{Log} x - i\pi)\Lambda_2\left(\tfrac{1+x+y}{y}\right) - i\pi \operatorname{Log} x \cdot \operatorname{Log} y. \end{aligned}$$

are both analytic for x and y in the upper half-plane $\mathcal{H}$.

Furthermore we have the following identities.

(b) *For $x, y \in \mathcal{H}$*

$$\frac{\partial^2}{\partial x \partial y}\hat{f}(x,y) = \frac{-\operatorname{Log}(1+x+y)}{xy}.$$

(c) *For $x, y \in \mathcal{H}$ with $|x| < |y|$ we have*

$$\frac{\partial^2}{\partial x \partial y}\left(-\operatorname{Li}_3(-x/y) - \tfrac{1}{2}\operatorname{Log} x \cdot \operatorname{Log}^2 y\right) = -\frac{\operatorname{Log}(x+y)}{xy}.$$

Proof. (a) First note that if $\Im x > 0, \Im y > 0$ then none of $\frac{1+x+y}{x}$, $\frac{1+x+y}{y}$ or $\frac{1+x+y}{-xy}$ can be real and negative. For if $\frac{1+x+y}{x} = -\lambda < 0$, then $1 + (1+\lambda)x + y = 0$ while $\Im(1 + (1+\lambda)x + y) > 0$. If $\frac{1+x+y}{-xy} = -\lambda < 0$, then $\left(1 + \frac{1}{x}\right)\left(1 + \frac{1}{y}\right) = 1 + \lambda$. On the other hand $\arg\left(1 + \frac{1}{x}\right)$ and $\arg\left(1 + \frac{1}{y}\right) \in (-\pi, 0)$, so $\left(1 + \frac{1}{x}\right)\left(1 + \frac{1}{y}\right)$ cannot be real and positive, and hence is not $1 + \lambda$. This shows that the image of $\mathcal{H} \times \mathcal{H}$ under each of the maps $(x,y) \mapsto \frac{1+x+y}{x}$, $(x,y) \mapsto \frac{1+x+y}{y}$, and $(x,y) \mapsto \frac{1+x+y}{-xy}$ lies in the cut plane $\mathbb{C}\backslash(-\infty,0]$ where Λ_2, Λ_3 and Log are all analytic. Hence $\hat{g}_3$ and $\hat{f}$ are both bi-analytic in $\mathcal{H}\times\mathcal{H}$.

(b) We now assume for the moment that x and y are both near enough to the positive imaginary axis so that $\arg(1+x+y) \in \left(0,\frac{\pi}{2}\right)$, $\arg x, \arg y \in \left(\frac{\pi}{2}-\delta,\frac{\pi}{2}+\delta\right)$ say, and so

$$\arg\left(\tfrac{1+x+y}{x}\right),\ \arg\left(\tfrac{1+x+y}{y}\right) \in \left(\tfrac{-\pi}{2}-\delta,\delta\right)$$

and $\arg\left(\frac{1+x+y}{-xy}\right) \in \left(-\delta,\frac{\pi}{2}+\delta\right)$. Then, writing $\ell = \operatorname{Log}(1+x+y)$ and using (3.4) we have

$$\begin{aligned}
\operatorname{Log}\left(\tfrac{1+x+y}{x}\right) &= \ell - \operatorname{Log} x\\
\operatorname{Log}\left(\tfrac{1+x+y}{y}\right) &= \ell - \operatorname{Log} y\\
\operatorname{Log}\left(\tfrac{1+x+y}{-xy}\right) &= \ell - \operatorname{Log}(-xy) = \ell - \operatorname{Log} x - \operatorname{Log} y + i\pi.
\end{aligned}$$

Using these identities we readily calculate, using (3.3), that

$$\frac{\partial^2 \hat{g}_3}{\partial x\partial y} = \frac{-(1+y)\operatorname{Log} x - (1+x)\operatorname{Log} y + (1+x+y)\ell + i\pi}{(1+x+y)xy}$$

$$\frac{\partial^2}{\partial x\partial y}\left((\operatorname{Log} y - i\pi)\Lambda_2\left(\frac{1+x+y}{x}\right)\right) = \frac{\operatorname{Log} x}{xy} - \frac{\operatorname{Log} y}{x(1+x+y)} - \frac{\ell}{xy}.$$

Swapping x and y in the second identity and then adding all three identities, we see that (b) is valid for x and y both near the positive imaginary axis, as assumed above. In fact, as $\arg(1+x+y) \in (0,\pi]$ for $x, y \in \mathcal{H}$ we see that, by continuity, that (b) actually holds for all such x and y.

(c) Here, direct calculation using $\mathrm{L}_n'(x) = \mathrm{L}_{n-1}(x)/x$ gives

$$\frac{\partial^2}{\partial y\partial x}\left(\operatorname{Li}_3(-x/y) + \tfrac{1}{2}\operatorname{Log} x\cdot \operatorname{Log}^2 y\right) = \tfrac{1}{xy}\left(\operatorname{Log}\left(1+\tfrac{x}{y}\right) + \operatorname{Log} y\right).$$

Now if $|x| < |y|$ then $\arg\left(1+\frac{x}{y}\right) \in \left(-\frac{\pi}{2},\frac{\pi}{2}\right)$, $\arg\left(1+\frac{x}{y}\right) + \arg y \in \left(-\frac{\pi}{2},\frac{3\pi}{2}\right)$. But as

$$\arg\left(1+\tfrac{x}{y}\right) + \arg(y) \equiv \arg(x+y) \mod 2\pi$$

and $\arg(x+y) \in (0,\pi)$, $\arg\left(1+\frac{x}{y}\right) + \arg y = \arg(x+y)$, and so

$$\operatorname{Log}\left(1+\tfrac{x}{y}\right) + \operatorname{Log} y = \operatorname{Log}(x+y),$$

giving the result.

Next, we need to study $\Re\hat{f}(x,y)$ for x, y in the upper half plane, as they each tend to points on the real axis. To do this, we need to define $r(x,y)$ for x and y real by

$$r(x,y) = \begin{cases} \Re\hat{f}(x,y) - 2\pi^2\log\left(\frac{1+y}{-x}\right) & \text{if } x<0,\ y>0,\ 1+x+y>0 \\ \Re\hat{f}(x,y) - 2\pi^2\log\left(\frac{1+x}{-y}\right) & \text{if } x>0,\ y<0,\ 1+x+y>0 \\ \Re\hat{f}(x,y) & \text{otherwise.} \end{cases}$$

Proposition 3. *Let $x_0,\ y_0 \in \mathbb{R}\backslash\{0\}$. Then for $x, y \in \mathcal{H}$,*

$$\lim_{\substack{x\to x_0\\ y\to y_0}} \Re\hat{f}(x,y) = r(x_0,y_0)$$

Proof. We separate the proof into three cases corresponding to the cases in the definition of r above.

We first consider the case $x_0 < 0,\ y_0 > 0,\ 1+x_0+y_0 > 0$, and note that for $x = x_0 + i\delta,\ y = y_0 + i\delta'$,

$$\Im\left(\frac{1+x+y}{x}\right) = \frac{x_0\delta' - (1+y_0)\delta}{x\bar{x}}.$$

Then, as x_0 and $-(1+y_0)$ are both negative, $\frac{1+x+y}{x} \to \frac{1+x_0+y_0}{x_0}$ from the lower half plane. Hence

$$\begin{aligned} &\lim_{\substack{x\to x_0\\ y\to y_0}} \Re\left((\operatorname{Log} y - i\pi)\Lambda_2\left(\tfrac{1+x+y}{x}\right)\right) = \Re((\log y_0 - i\pi)\lim_{\substack{x\to x_0\\ y\to y_0}}\Lambda_2\left(\tfrac{1+x+y}{x}\right)) \\ &= \log y_0\cdot\Lambda_2\left(\tfrac{1+x_0+y_0}{x_0}\right) + (-i\pi)(-2i\pi)\log\left(1-\tfrac{1+x_0+y_0}{x_0}\right) \quad \text{using (3.7)} \\ &= \log y_0\cdot\Lambda_2\left(\tfrac{1+x_0+y_0}{x_0}\right) - 2\pi^2\log\left(\tfrac{1+y_0}{-x_0}\right). \end{aligned}$$

This proves the first case of the Proposition.

The second case comes from interchanging x and y. For the third case, first note that since by Lemma 3 the jump Δ_3 for Λ_3 having argument on the negative real axis is imaginary,

$$\lim_{\substack{x\to x_0\\ y\to y_0}} \Re\hat{g}_3(x,y) = \Re\hat{g}_3(x_0,y_0).$$

Also, as $\operatorname{Log} y - i\pi$ is real for $y<0$, and by Lemma 3 the jump Δ_2 for Λ_2 on the negative real axis is also imaginary, we have

$$\lim_{\substack{x\to x_0\\ y\to y_0}} \Re\left((\operatorname{Log} y - i\pi)\Lambda_2\left(\tfrac{1+x+y}{x}\right)\right) = \Re\left((\operatorname{Log} y_0 - i\pi)\Lambda_2\left(\tfrac{1+x_0+y_0}{x_0}\right)\right)$$

when $y_0 < 0$ or $\frac{1+x_0+y_0}{x_0} > 0$. A similar result holds for $x_0 < 0$ or $\frac{1+x_0+y_0}{y_0} > 0$. This proves the third case of the Proposition.

5. Proof of the Theorem

We can now evaluate the Mahler measures of the Theorem and Corollary 1. First of all, by Proposition 1 we have

$$\pi^2 m(P_{0,c,1}) = -\int_c^{-c} \tfrac{dx}{x} \int_1^{-1} \tfrac{dy}{y} \log|x+y| \tag{5.1}$$

where the integrals are taken on the semicircles $x = ce^{i\theta} \quad (0 \leqslant \theta \leqslant \pi)$ and $y = e^{i\phi} \quad (0 \leqslant \phi \leqslant \pi)$,

$$= \Re\left(-\int_c^{-c} \tfrac{dx}{x} \int_1^{-1} \tfrac{dy}{y} \operatorname{Log}(x+y)\right) \tag{5.2}$$

as $\frac{dx}{x}.\frac{dy}{y} = -d\theta d\phi$ is real. Here cx has been replaced by x in the integrand.

We now apply Proposition 2(ii) for $c < 1$, which gives

$$\begin{aligned} \pi^2 m(P_{0,c,1}) &= \Re\left(\left(\left(-\operatorname{Li}_3(-x/y) - \tfrac{1}{2}\operatorname{Log} x \,.\, \operatorname{Log}^2 y\right|_{x=c}^{-c}\right|_{y=1}^{-1}\right) \\ &= \Re\left(2\operatorname{Li}_3(c) - 2\operatorname{Li}_3(-c) - \tfrac{1}{2}(\operatorname{Log}(-c) - \log c)\operatorname{Log}^2(-1)\right) \\ &= 2\operatorname{Li}_3(c) - 2\operatorname{Li}_3(-c). \end{aligned} \tag{5.3}$$

Since the Mahler measure of a polynomial is a continuous function of its coefficients [B3], this formula is also valid for $c = 1$. Then we get the same formula for $m(P_{0,1,c})$, as $m(P_{0,1,c}) = m(P_{0,c,1})$ from (5.1).

We now evaluate $m(P_{1,b,c})$. From Proposition 1 we obtain the formula

$$m(P_{1,b,c}) = -\Re\int\int \tfrac{dx}{x}\tfrac{dy}{y} \operatorname{Log}(1+x+y) \tag{5.4}$$

in a similar manner to (5.2). Here we have replaced bx and cy by x and y respectively, so that the integrals are taken over the semicircles $x = be^{i\theta}(0 \leqslant \theta \leqslant \pi)$, $y = ce^{i\phi}(0 \leqslant \theta \leqslant \pi)$. Since the right-hand side of (5.4) is symmetrical in x and y, we can assume that $b \geqslant c > 0$. Then from Propositions 2(ii) and 3 we have that

$$m(P_{1,b,c}) = r(b,c) - r(b,-c) - r(-b,c) + r(-b,-c). \tag{5.5}$$

Next, define for real x and y

$$f(x,y) = G_3(x,y) + \log|y| \cdot G_2(x,y) + \log|x| \cdot G_2(y,x). \tag{5.6}$$

Our aim is to derive a more computable form of (5.6), namely (5.11), which is (5.6) with r replaced by f. To do this, we next compute the four terms $r(\pm b, \pm c)$ of (5.5) in terms of f. Firstly, using (3.7) and (3.8)

$$
\begin{aligned}
(5.7)\quad r(b,c) &= \Re\left(\Lambda_3\left(\tfrac{1+b+c}{b}\right) + \Lambda_3\left(\tfrac{1+b+c}{c}\right) - \Lambda_3\left(\tfrac{1+b+c}{-bc}\right)\right)\\
&\qquad + \Re\left((\log c - i\pi)\Lambda_2\left(\tfrac{1+b+c}{b}\right) + (\log b - i\pi)\Lambda_2\left(\tfrac{1+b+c}{c}\right)\right)\\
&= \mathrm{L}_3\left(\tfrac{1+b+c}{b}\right) + \mathrm{L}_3\left(\tfrac{1+b+c}{c}\right) - \left(\mathrm{L}_3\left(\tfrac{1+b+c}{-bc}\right) + \tfrac{\pi^2}{2}\log\left(1 - \tfrac{1+b+c}{-bc}\right)\right)\\
&\qquad + \log c \cdot \mathrm{L}_2\left(\tfrac{1+b+c}{b}\right) + \log b \cdot \mathrm{L}_2\left(\tfrac{1+b+c}{c}\right)\\
&= f(b,c) - \tfrac{\pi^2}{2}\log\left(\tfrac{(1+b)(1+c)}{bc}\right).
\end{aligned}
$$

Next, for $r(b,-c)$, we have $1+b-c>0$, $\frac{1+b-c}{b}>0$, $\frac{1+b-c}{-c}<0$, so that, using (3.8) again, and the definitions of $\hat{f}$ and r,

$$
\begin{aligned}
r(b,-c) = &\,\mathrm{L}_3\left(\tfrac{1+b-c}{b}\right) + \left(\mathrm{L}_3\left(\tfrac{1+b-c}{-c}\right) + \tfrac{\pi^2}{2}\log\left(1 - \tfrac{1+b-c}{-c}\right)\right)\\
&- \mathrm{L}_3\left(\tfrac{1+b-c}{-b(-c)}\right) + \Re\left((\mathrm{Log}(-c) - i\pi)\Lambda_2\left(\tfrac{1+b-c}{b}\right)\right)\\
&+ \Re\left((\log b - i\pi)\,\Lambda_2\left(\tfrac{1+b-c}{-c}\right) - i\pi\log b\cdot\mathrm{Log}(-c)\right) - 2\pi^2\log\left(\tfrac{1+b}{c}\right).
\end{aligned}
$$

Now as $\mathrm{Log}(-c) - i\pi = \log c$ is real, (3.7) gives

$$
\begin{aligned}
(5.8)\qquad r(b,-c) &= f(b,-c) + \tfrac{\pi^2}{2}\log\left(\tfrac{1+b}{c}\right) - i\pi\left(i\pi\log\left(1 - \tfrac{1+b-c}{-c}\right)\right)\\
&\qquad\qquad + \pi^2\log b - 2\pi^2\log\left(\tfrac{1+b}{c}\right)\\
&= f(b,-c) - \tfrac{\pi^2}{2}\log\left(\tfrac{1+b}{c}\right) + \pi^2\log b.
\end{aligned}
$$

For $r(-b,c)$, we distinguish two cases (i) $b > c+1$ and (ii) $b \leqslant c+1$.

In case (i), $1-b+c<0$, $\frac{1-b+c}{-b}>0$, $\frac{1-b+c}{c}<0$, so that similarly

$$
\begin{aligned}
(5.9)\quad r(-b,c) &= \mathrm{L}_3\left(\tfrac{1-b+c}{-b}\right) + \left(\mathrm{L}_3\left(\tfrac{1-b+c}{c}\right) + \tfrac{\pi^2}{2}\log\left(1 - \tfrac{1-b+c}{c}\right)\right)\\
&- \left(\mathrm{L}_3\left(\tfrac{1-b+c}{-(-b)c}\right) + \tfrac{\pi^2}{2}\log\left(1 - \tfrac{1-b+c}{-(-b)c}\right)\right) + \Re(\log c - i\pi)\,\mathrm{L}_2\left(\tfrac{1-b+c}{-b}\right)\\
&+ (\mathrm{Log}(-b) - i\pi)\,\mathrm{L}_2\left(\tfrac{1-b+c}{b}\right) - \Re(i\pi\,\mathrm{Log}(-b)\cdot\log c)\\
&= f(-b,c) - \tfrac{\pi^2}{2}\log\left(\tfrac{1+c}{b}\right) + \pi^2\log c.
\end{aligned}
$$

In case (ii), $1-b+c>0$, $\frac{1-b+c}{-b}<0$, $\frac{1-b+c}{c}>0$ so that

$$
\begin{aligned}
r(-b,c) = &\left(\mathrm{L}_3\left(\tfrac{1-b+c}{-b}\right) + \tfrac{\pi^2}{2}\log\left(1 - \tfrac{1-b+c}{-b}\right)\right) + \mathrm{L}_3\left(\tfrac{1-b+c}{c}\right)\\
&- \mathrm{L}_3\left(\tfrac{1-b+c}{-(-b)c}\right) + \Re\left((\log b - i\pi)\left(\mathrm{L}_2\left(\tfrac{1-b+c}{-b}\right) + i\pi\log\left(1 - \tfrac{1-b+c}{-b}\right)\right)\right)\\
&+ (\mathrm{Log}(-b) - i\pi)\,\mathrm{L}_2\left(\tfrac{1-b+c}{c}\right) - \Re(i\pi\,\mathrm{Log}(-b)\cdot\log c) - 2\pi^2\log\left(\tfrac{1+c}{b}\right)
\end{aligned}
$$

which again gives (5.9) on simplification.

For $r(-b,-c)$ we distinguish two cases (i) $b+c \geqslant 1$ and (ii) $b+c<1$.

For (i), $1-b-c<0$, $\frac{1-b-c}{-b}>0$, $\frac{1-b-c}{-c}>0$ so that

$$\begin{aligned}(5.10)\qquad r(-b,-c) &= \mathrm{L}_3\left(\tfrac{1-b-c}{-b}\right)+\mathrm{L}_3\left(\tfrac{1-b-c}{-c}\right)-\mathrm{L}_3\left(\tfrac{1-b-c}{-(-b)(-c)}\right)\\ &\quad-(\mathrm{Log}(-c)-i\pi)\,\mathrm{L}_2\left(\tfrac{1-b-c}{-b}\right)-(\mathrm{Log}(-b)-i\pi)\,\mathrm{L}_2\left(\tfrac{1-b-c}{-c}\right)\\ &\quad-\Re(i\pi\,\mathrm{Log}(-b)\cdot\mathrm{Log}(-c))\\ &= f(-b,-c)+\pi^2\log(bc).\end{aligned}$$

For (ii), $1-b-c>0$, $\frac{1-b-c}{-b}<0$, $\frac{1-b-c}{-c}<0$ so that

$$\begin{aligned}r(-b,-c) &= \left(\mathrm{L}_3\left(\tfrac{1-b-c}{-b}\right)+\tfrac{\pi^2}{2}\log\left(1-\tfrac{1-b-c}{-b}\right)\right)+\left(\mathrm{L}_3\left(\tfrac{1-b-c}{-c}\right)\right.\\ &\quad\left.+\tfrac{\pi^2}{2}\log\left(1-\tfrac{1-b-c}{-c}\right)\right)-\left(\mathrm{L}_3\left(\tfrac{1-b-c}{-(-b)(-c)}\right)+\tfrac{\pi^2}{2}\log\left(1-\tfrac{1-b-c}{-(-b)(-c)}\right)\right)\\ &\quad-\log c\cdot\,\mathrm{L}_2\left(\tfrac{1-b-c}{-b}\right)-\log b\cdot\,\mathrm{L}_2\left(\tfrac{1-b-c}{-c}\right)-\Re(i\pi\,\mathrm{Log}(-b)\cdot\mathrm{Log}(-c))\end{aligned}$$

giving (5.10) again in this case.

Using (5.5), (5.7), (5.8), (5.9) and (5.10) we now get

$$(5.11)\qquad m(P_{1,b,c})=f(b,c)-f(b,-c)-f(-b,c)+f(-b,-c)$$

which, from (5.6) gives the formula stated in the Theorem.

6. Proof of Corollary 2

Now by Lemma 7 with $V=\begin{pmatrix}1&1&0\\2&0&1\\1&1&1\end{pmatrix}$ we have

$$\begin{aligned}m(1+x+y+z) &= m(1+xy+x^2z+xyz)\\ &= m(x^{-1}+y+(x+y)z)\\ &= \tfrac{2}{\pi^2}(\mathrm{Li}_3(1)-\mathrm{Li}_3(-1))\\ &= \tfrac{7}{2\pi^2}\zeta(3)\end{aligned}$$

using the Theorem for $P_{0,1,1}$, and (a),(b) from the proof of Lemma 6.

For the second result

$$\begin{aligned}\pi^2 m(1+x^{-1}+y+(1+x+y)z) &= G_3(1,1)\\ &= g_3(1,1)-g_3(1,-1)-g_3(-1,1)+g_3(-1,-1)\\ &= 2\,\mathrm{L}_3(3)-\mathrm{L}_3(-3)-2\,\mathrm{L}_3(-1)+\mathrm{L}_3(1)\\ &= \tfrac{14}{3}\zeta(3),\end{aligned}$$

using Lemma 6.

7. Further examples

We have from the Theorem that

$$m(P_{112}) = \frac{1}{\pi^2}(G_3(1,2) + \log 2 \cdot G_2(1,2))$$
$$= \frac{1}{\pi^2}\left(\mathrm{L}_3(4) + 2\ \mathrm{L}_3(2) - 2\ \mathrm{L}_3(-2) + \log 2(\mathrm{L}_2(4) + \mathrm{L}_2(2) - \mathrm{L}_2(-2))\right).$$

This can be re-written using polylogarithms with argument $-\frac{1}{2}$ only by

$$m(P_{112}) = \frac{1}{\pi^2}\left(\frac{21}{4}\zeta(3) + 2\ \mathrm{Li}_3\left(-\frac{1}{2}\right) + \log 2 \cdot \mathrm{Li}_2\left(-\frac{1}{2}\right) + \frac{1}{6}\log^3 2\right) + \frac{7}{12}\log 2$$
$$= 0.9221164988$$

using (L6.4, $x = 2$), (L6.6, $x = 2$), (L6.7, $y = 2$) and (L6.12).

Similarly,

$$m(P_{123}) = \frac{1}{\pi^2}\left(G_3(2,3) + \log 3 \cdot G_2(2,3) + \log 2 \cdot G_2(3,2)\right)$$
$$= \frac{1}{\pi^2}\Big(\mathrm{L}_3(3) + 2\ \mathrm{L}_3(2) - 2\ \mathrm{L}_3(-1) - 2\ \mathrm{L}_3\left(\tfrac{2}{3}\right) + \mathrm{L}_3\left(\tfrac{1}{3}\right) + \mathrm{L}_3\left(\tfrac{4}{3}\right)$$
$$+ \log 3\left(\mathrm{L}_2(3) - \mathrm{L}_2(-1) + \mathrm{L}_2(2)\right) + \log 2\left(\mathrm{L}_2(2) - \mathrm{L}_2\left(\tfrac{2}{3}\right) + \mathrm{L}_2\left(\tfrac{4}{3}\right)\right)\Big)$$

which can, again using standard formulae, be shown to be given in terms of classical di- and trilogarithms by

$$m(P_{123}) = \frac{1}{\pi^2}\Big(\frac{11}{12}\zeta(3) - 2\,\mathrm{Li}_3\left(-\tfrac{1}{2}\right) + 2\,\mathrm{Li}_3\left(\tfrac{1}{3}\right) + \log 6 \cdot \mathrm{Li}_2\left(\tfrac{1}{3}\right) + \tfrac{1}{3}\log^3 2$$
$$- \log^2 2 \cdot \log 3 + \tfrac{1}{2}\log 2 \cdot \log^2 3 + \tfrac{1}{6}\log^3 3\Big) + \frac{5}{12}\log 2 + \frac{2}{3}\log 3$$
$$= 1.388343758.$$

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