Journal de Théorie des Nombres
de Bordeaux **15** (2003), 1–10

Arithmetic Gevrey series and transcendence. A survey

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Résumé. Nous passons en revue les principaux résultats de la théorie des séries Gevrey de type arithmétique introduite dans [3] [4], leurs applications à la transcendance, ainsi que quelques conjectures diophantiennes sur la sommation de séries divergentes.

Abstract. We review the main results of the theory of arithmetic Gevrey series introduced in [3] [4], their applications to transcendence, and a few diophantine conjectures on the summation of divergent series.

1. Introduction: ubiquity of Gevrey series in complex analysis

1.1. Let us begin with a classical definition.

1.1.1. Definition. Let s be a rational number. A formal power series

$$f = \sum_0^\infty a_n x^n \in \mathbf{C}[[x]]$$

with complex coefficients is *Gevrey of order* s if and only if the associated series

$$f^{[s]} = \sum_0^\infty \frac{a_n}{n!^s}\, x^n$$

has a non-zero radius of convergence, *i.e.* if and only if

$$\exists C > 0, \forall n > 0, \;\; |a_n| < C^n.n!^s.$$

It is *Gevrey of precise order* s if and only if $f^{[s]}$ has a *finite* non-zero radius of convergence.

1.1.2. Remarks. i) for $s = 0$, a Gevrey series is just a convergent series.
ii) Gevrey series of order $s > 0$ are in general divergent.
iii) For $s < 0$, Gevrey series define entire functions. More precisely, it is not difficult to see that f is Gevrey of order s if and only if it is *entire of exponential order* $\leq 1/|s|$.

Manuscrit reçu le 11 septembre 2001.

1.2. At the begining of the 20th century, it was discovered that Gevrey series are remarkably ubiquitous in complex analysis. This was emphasized by M. Gevrey [8].

For instance, it turns out that any formal power series arising in the asympotic expansion of any solution of a linear or non-linear analytic differential equation is Gevrey[1] of precise order s for some rational number s.

Gevrey series also occur in the context of singular perturbations, difference equations ...

Besides their ubiquity, divergent Gevrey series have the remarkable property, in many natural analytic contexts, of being summable in an essentially canonical way, in suitable sectors. This is the starting point (*cf.* [14]) of the extensive theory of summability/ multisummability/ resurgence. We refer to [11] for an inspiring overview.

1.2.1. Examples. i) The Airy function is

$$Ai(x) = \frac{1}{\pi}\int_0^\infty \cos(xt + t^3/3)dt.$$

Its Taylor expansion at the origin is

$$= \frac{1}{9^{1/3}\Gamma(2/3)}\sum_0^\infty \frac{x^{3n}}{9^n(2/3)_n n!} - \frac{1}{3^{1/3}\Gamma(1/3)}\sum_0^\infty \frac{x^{3n+1}}{9^n(4/3)_n n!}$$

Here both formal power series in x are *Gevrey of precise order* $-2/3$.

Its asymptotic expansion at ∞, in a suitable sector bissected by the positive real axis, is given by

$$\exp(-2/3x^{3/2}).[\frac{1}{2\sqrt{\pi}}(1/2)^{1/4}\sum_0^\infty (3/4)^{2n}\frac{(1/6)_{2n}(5/6)_{2n}}{(2n)!}(1/x)^{3n}$$

$$-\frac{1}{2\sqrt{\pi}}(1/2)^{3/4}\sum_0^\infty (3/4)^{2n+1}\frac{(1/6)_{2n+1}(5/6)_{2n+1}}{(2n+1)!}(1/x)^{3n+1}].$$

Here both formal power series in $1/x$ are *Gevrey of precise order* $+2/3$.

(The Pochhammer symbol $(a)_n$ stands for $a(a+1)\dots(a+n-1)$).

ii) The Barnes generalized hypergeometric series ${}_pF_{q-1}(x^r)$ are Gevrey of precise order $s = \frac{p-q}{r}$.

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[1] the fact that any formal power solution of a non-linear analytic differential equation is always Gevrey was already known in 1903 [9]; and, for the precise order - in the algebraic linear case - by O. Perron in 1910 [10].

2. Arithmetic Gevrey series

2.1. The idea of arithmetic Gevrey series came out from an empirical observation: leafing through the classical treatises on special functions, one remarks that the Gevrey series occurring in Taylor or asymptotic expansions of classical special functions have algebraic or even rational coefficients, and that the growth of the *denominators* of these coefficients is inversely proportional to the archimedean growth.

This observation led to the following definition:

2.1.1. Definition. Let s be a rational number. A formal power series

$$f = \sum_0^\infty a_n x^n \in \mathbf{C}[[x]]$$

is an *arithmetic Gevrey series of order* s if and only if

$$\forall n, \ \ a_n \in \bar{\mathbf{Q}},$$

$$\exists C > 0, \forall n > 0, \ \ Max|\text{conjugates } (\frac{a_n}{n!^s})| < C^n, \ den(\frac{a_0}{0!^s}, \dots, \frac{a_n}{n!^s}) < C^n.$$

2.1.2. Remarks. i) Since $s \in \mathbf{Q}$, one has $n!^s \in \bar{\mathbf{Q}}$, so that the conjugates and common denominators of finitely many $\frac{a_n}{n!^s}$'s are well-defined.
ii) Infinite[2] arithmetic Gevrey series of order s give rise to Gevrey series of *precise* order s for any embedding $\bar{\mathbf{Q}} \hookrightarrow \mathbf{C}$.
iii) The arithmetic Gevrey series of order s form a subalgebra of $\bar{\mathbf{Q}}[[x]]$, stable under derivation and integration.

2.2. The above observation (about growth of denominators, 2.1) becomes less surprising if one remarks that the coefficients of the Gevrey series in question are generally expressed in terms of Pochhammer symbols with rational parameters. One can then apply the following lemma which goes back to Mayer and Siegel [13]:

2.2.1. Lemma. *Let $a_1, \dots, a_p, b_1 \dots, b_q$ be rational numbers (and not negative integers). Then*

$$\exists C > 0, \forall n > 0, \ \ den(\frac{\Pi_i (a_i)_0}{\Pi_j (b_j)_0}, \dots, \frac{\Pi_i (a_i)_n}{\Pi_j (b_j)_n}) < C^n.n!^{q-p}.$$

2.2.2. Examples. i) By the lemma, the series occurring in the expansion of $Ai(x)$ at 0, namely

$$\sum_0^\infty \frac{x^{3n}}{9^n (2/3)_n n!}, \ \ \sum_0^\infty \frac{x^{3n+1}}{9^n (4/3)_n n!}$$

[2]= non-polynomial

are *arithmetic Gevrey of order* $-2/3$. The series occurring in its asympotic expansion at $+\infty$, namely

$$\sum_0^\infty (3/4)^{2n} \frac{(1/6)_{2n}(5/6)_{2n}}{(2n)!}(1/x)^{3n},$$
$$\sum_0^\infty (3/4)^{2n+1} \frac{(1/6)_{2n+1}(5/6)_{2n+1}}{(2n+1)!}(1/x)^{3n+1}$$

are *arithmetic Gevrey of order* $+2/3$.

ii) The Barnes generalized hypergeometric series ${}_pF_{q-1}(x^r)$ with rational parametes are arithmetic Gevrey of order $s = \frac{p-q}{r}$.

iii) Any series which is algebraic over $\bar{\mathbf{Q}}(x)$ is arithmetic Gevrey of order 0 (Eisenstein). More generally, Siegel's G-functions [13] are nothing but *holonomic* arithmetic Gevrey series of order 0.

Here and in the sequel, *holonomic* means: solution of a linear differential equation with coefficients in $\mathbf{C}(x)$.

3. Structure of differential operators annihilating arithmetic Gevrey series

3.1. Let us consider a linear differential operator

$$\varphi = \frac{d^\mu}{dx^\mu} + \gamma_{\mu-1}\frac{d^{\mu-1}}{dx^{\mu-1}} + \cdots + \gamma_0, \ \gamma_i \in \mathbf{C}(z).$$

At every point ξ which is not a singularity *i.e.* not a pole of any γ_i, φ admits a basis of analytic solutions at ξ. The converse is not always true, but when it holds, the singularity is called trivial[3].
On the other hand, if $\xi \in \mathbf{C}$ is any singularity of φ, it is known that there exists a basis of "formal solutions" which are finite linear combinations

$$\sum u_i(x-\xi).(x-\xi)^{\alpha_i}.\log^{k_i}(x-\xi).\exp(Q_i((x-\xi)^{-1/\mu!})),$$

where $\alpha_i \in \mathbf{C}$, k_i is a non-negative integer, Q_i is a polynomial, and u_i is a Gevrey series of (some) precise order $s \in \mathbf{Q}$.
When no non-zero polynomial Q_i occurs[4], one says that ξ is a regular singularity. In that case, the u_i's are Gevrey series of order ≤ 0, *i.e.* convergent (Frobenius' theorem).

[3] traditionally, the expression "apparent singularity" is reserves for a point ξ where there is a basis of meromorphic, not necessarily homolomorphic, solutions.

[4] such finite linear combinations y are sometimes called Nilsson functions.

3.2. The presence of Gevrey series of certain orders among the solutions of φ gives some information about the "formal structure" of φ at the given point ξ, but nothing beyond.

We shall see, in strong contrast, that the presence of *arithmetic* Gevrey series in the solutions of φ at some point determines somehow the global behaviour of φ. In particular, together with the remark at the beginning of §2, this provides an "arithmetic explanation" for another elementary observation which can be made when leafing through the treatises on special functions: namely, that the classical linear differential operators occurring there are of two types: either fuchsian (the "non-confluent type"), or with two singularities, 0 and ∞, one of them being regular (the "confluent type" or "Hamburger type").

3.3. To express our main result in the most economic way, it is convenient to consider, besides arithmetic Gevrey series, finite ramifications, logarithms, and transcendental constants. This leads to the following

3.3.1. Definition. An *arithmetic Nilsson-Gevrey series* of order $s \in \mathbf{Q}$ is a finite linear combination

$$y = \sum \lambda_i . u_i(x) . x^{e_i} . \log^{k_i} x$$

where $\lambda_i \in \mathbf{C}$, $e_i \in \mathbf{Q}$, $k_i \in \mathbf{N}$, and u_i is an arithmetic Gevrey series of order s.

They form a $\mathbf{C}(x)$-algebra (stable by derivation and integration) denoted by $NGA\{x\}_s$ in [3].

3.3.2. Example. The Taylor expansion at 0 of the Airy function belongs to $NGA\{x\}_{-2/3}$. The factor of $\exp(-2/3x^{3/2})$ in the asymptotic expansion at $+\infty$ belongs to $NGA\{1/x\}_{+2/3}$.

3.4. Let y be a holonomic arithmetic Nilsson-Gevrey series of order s (we assume for simplicity that y is not a polynomial). Let D_y be a non-zero-element of $\mathbf{C}(x)[\frac{d}{dx}]$ of minimal degree in $\frac{d}{dx}$ such that

$$D_y\,(y) = 0$$

(such a differential operator is unique up to left multiplication by an element of $\mathbf{C}(x)^*$).

3.4.1. Theorem. 1) *Assume $s = 0$. Then D_y is fuchsian, i.e. has only regular singularities (even at ∞).*
Moreover, for any $\xi \in \bar{\mathbf{Q}}$, D_y has a basis *of solutions in $NGA\{x - \xi\}_0$ and a basis of solutions in $NGA\{1/x\}_0$.*

2) *Assume $s < 0$. Then D_y has only two non-trivial singularities: 0 and ∞ (in other words: D_y has a basis of analytic solutions at any point $\neq 0, \infty$). Moreover:*

i) 0 *is a regular singularity with rational exponents, and* D_y *has a* basis *of solutions in* $NGA\{x\}_s$,

ii) ∞ *is an irregular singularity, and* D_y *has a* basis *of solutions of the form* $f_i.\exp(\zeta_i x^{-1/s})$ *with* $f_i \in NGA\{1/x\}_{-s}$, $\zeta_i \in \bar{\mathbf{Q}}$.

3) *Assume* $s > 0$. *Then* D_y *has only two non-trivial singularities:* 0 *and* ∞. *Moreover:*

i) 0 *is an irregular singularity, and* D_y *has a* basis *of solutions of the form* $f_i.\exp(\zeta_i x^{-1/s})$ *with* $f_i \in NGA\{x\}_s$, $\zeta_i \in \bar{\mathbf{Q}}$,

ii) ∞ *is a regular singularity with rational exponents, and* D_y *has a* basis *of solutions in* $NGA\{1/x\}_{-s}$.

The theorem is proved in [3]. The proof involves an arithmetic study of the Fourier-Laplace transform (which reduces 2) and 3) to 1)).

The main feature of this theorem is that an arithmetic property of one solution at one point accounts for the global behaviour the operator. In particular, *whenever arithmetic Gevrey series of order* s *occur at* 0, *then arithmetic Gevrey series of order* $-s$ *occur at* ∞, *and conversely.* The Airy differential operator $D_{Ai} = d^2/dx^2 - x$ offers a typical illustration of this phenomenon, with $s = -2/3$.

4. On special values of arithmetic Gevrey series

4.1. Let K be a number field, and let $f \in K[[x]]$ be a *holonomic arithmetic Gevrey series of order* $s \in \mathbf{Q}$. We may assume that $D_f \in K[x, d/dx]$.

We shall study separately the cases $s < 0$, $s > 0$ and $s = 0$.

4.2. The case $s < 0$. Recall that for any embedding $v : K \hookrightarrow \mathbf{C}$, f_v defines an entire function of exponential order $-1/s$. In particular, it can be evaluated at any point $\xi \in K^*$.

4.2.1. Corollary. *Assume that for every complex embedding* v, $f_v(\xi) = 0$. *Then* ξ *is a singularity of* D_f.

Proof. Let us write $f = \sum a_n x^n$, $g := \frac{f}{x-\xi} = \sum b_n x^n$, and show that g is also a holonomic arithmetic Gevrey series of order s.

We have $\frac{b_n}{n!^s} = -\frac{1}{\xi}\sum_0^n \xi^{k-n}(\frac{n!}{k!})^{-s}\frac{a_k}{k!^s}$, which shows that g satisfies, like f, the denominator condition in the definition of arithmetic Gevrey series of order s. To prove that g is arithmetic Gevrey of order s, it remains to prove that for any complex embedding v, g_v is Gevrey of order s in the ordinary sense. Since $s < 0$, this amounts to being entire of exponential order $\leq -1/s$ (like f). Since $g_v = \frac{f_v}{x-\xi}$ has no pole at ξ by assumption, this is clear. The holonomicity of g is also clear: $D_g = D_f \circ (x - \xi)$.

We can therefore apply the theorem to g. We get that D_g has a basis of analytic solutions at any point $\neq 0, \infty$, in particular at ξ. But this implies that D_f has a basis of solutions in $(x-\xi)K[[x-\xi]]$, hence that ξ is a (trivial) singularity of D_f. □

4.3. This corollary is in fact a transcendence theorem in disguise.
Let us first indicate how it easily implies the *Lindemann-Weierstrass theorem.* We embed K in $\mathbf{C}$. Let $\alpha_1,\ldots,\alpha_m$ be distinct elements of K and $\beta_1,\ldots,\beta_m$ be non-zero elements of K. Let us show that

$$\beta_1\exp(\alpha_1)+\cdots+\beta_m\exp(\alpha_m)\neq 0.$$

Let L be the Galois closure of $K(\alpha_i,\beta_j)$ in $\mathbf{C}$. We set

$$f=\prod_{\sigma\in Aut(L)}\left(\sum\beta_i\exp(\alpha_i x)\right)\ \in\mathbf{Q}[[x]].$$

It is enough to show that $f(1)\neq 0$. Note that f is an exponential polynomial with constant coefficients, hence an arithmetic Gevrey series of order -1, with $D_f\in L[d/dx]$ (constant coefficients). In particular, 1 is not a singularity of D_f, and the corollary shows that $f(1)\neq 0$ as wanted. □

4.4. A modification of this argument - replacing the fact that D_f has constant coefficients by a zero lemma for solutions of linear differential operators - leads to a new and qualitative proof of the *Siegel-Shidlovsky theorem*, which can be expressed as follows:

4.4.1. Theorem. *Let f be a holonomic arithmetic Gevrey series of order $s<0$. Let μ be the order of D_f.*
Then for any $\xi\in\bar{\mathbf{Q}}^*$ distinct from the singularities of D_f, the transcendence degree of $f(\xi),f'(\xi),\ldots,f^{(\mu-1)}(\xi)$ over $\bar{\mathbf{Q}}$ equals the transcendence degree of $f,f',\ldots,f^{(\mu-1)}$ over $\bar{\mathbf{Q}}(x)$.

cf. [4] (and also [5] for details about the zero lemma).

The theorem applies to confluent hypergeometric series ${}_pF_{q-1}(x)$ with rational parameters and $p<q$.

4.5. Actually, although theorem 3.4.1 deals with arithmetic Nilsson-Gevrey series and not only with arithmetic Gevrey series, we have not been able to generalize the above corollary to that more general case, hence to derive transcendence results for arithmetic Nilsson-Gevrey series. We nevertheless propose the following

4.5.1. Conjecture. *Let y be a holonomic arithmetic Nilsson-Gevrey series of order $s<0$. Let μ be the order of D_y. Assume that $y,y',\ldots,y^{(\mu-1)}$ are algebraically independent over $\mathbf{C}(x)$.*
Then for any $\xi\in\bar{\mathbf{Q}}^*$ distinct from the singularities of D_y, with finitely many exceptions, $y(\xi),y'(\xi),\ldots,y^{(\mu-1)}(\xi)$ are algebraically independent over $\bar{\mathbf{Q}}$.

The necessity to allow some exceptions is already patent on the example $y = \exp x - e$. The conjecture would imply for instance that the Airy function takes transcendental values at almost every algebraic point.

4.6. The case $s > 0$**.** A prototype is the Euler series $\sum(-1)^n n! x^n$. Evaluations at points $\xi \neq 0$ diverge but one can turn the difficulty in two ways: either by evaluating such series p-adically for almost all p, or by using some canonical process of (archimedean) resummation in a sector bissected by $\vec{0\xi}$ [5].

We concentrate on the second viewpoint, which is actually much older than the first one: L. Euler already proposed four methods for summing the series $\sum(-1)^n n!$ and gave some evidence that they lead to the same result [7].

4.6.1. Conjecture. *The previous conjecture should also hold if* $s > 0$, *provided* $y(\xi), y'(\xi), \ldots, y^{(\mu-1)}(\xi)$ *are interpreted as canonical resummations.*

4.6.2. Proposition. *Conjecture 4.5.1 implies 4.6.1.*

This follows from theorem 3.4.1, part 3ii). □

By symmetry, the Siegel-Shidlovsky theorem suggests the following

4.6.3. Conjecture. *The statement of 4.4.1 should also hold if* $s > 0$, *provided* $f(\xi), f'(\xi), \ldots, f^{(\mu-1)}(\xi)$ *are interpreted as canonical resummations.*

A proof of this conjecture would be of considerable interest because special values of resummations of arithmetic Gevrey series of positive order (such as the Euler series) involve the *Euler constant, the values of Riemann* ζ *function at integers, and more generally the derivatives of Euler's* Γ *function at rational points.* In fact, if we had restricted the definition of arithmetic Nilsson-Gevrey series by allowing only constants λ_i in $\bar{\mathbf{Q}}(\Gamma^{(n)}(p/q))$, theorem 3.4.1 would still hold, *cf.* [3] (remark at the end of 6.2).

This might be a way of investigating the arithmetic nature of those mysterious constants, via arithmetic Gevrey series of order $\neq 0$, whereas the usual approach involves arithmetic Gevrey series of order 0 (variants of polylogarithms). A lot of work remains to be done in this direction.

4.7. The case $s = 0$**.** This case is more delicate than the case $s < 0$, because there do exist infinite "exceptional sets" of points $\xi \in \bar{\mathbf{Q}}$ in general where such transcendental series take algebraic values[6].

[5] if this is a singular (= anti-Stokes) direction, one has to use the middle summation of Écalle-Ramis-Martinet.

[6] nevertheless, under a strong assumption of simultaneous uniformization imposed on the series under consideration, a statement analogous to the Siegel-Shidlovsky theorem is proven in [2] in the case $s = 0$.

A striking example, due to F. Beukers and J. Wolfart involves ${}_2F_1(\frac{1}{12}, \frac{5}{12}, \frac{1}{2}; x)$, which takes algebraic values at infinitely many algebraic points, *e.g.*

$$ {}_2F_1(\frac{1}{12}, \frac{5}{12}, \frac{1}{2}; \frac{1323}{1331}) = \frac{3}{4}\, {}^4\sqrt{11}. $$

In fact, these "accidental relations" have nice geometric interpretations, even in a very general setting. Indeed, it is expected that for any G-function (*i.e.* holonomic arithmetic Gevrey series of order 0), D_f "comes from geometry", *i.e.* is a product of factors of Picard-Fuchs differential operators (conjecture of Bombieri-Dwork). This is certainly the case for any G-function of generalized hypergeometric type.

In such a situation, special values of G-fonctions can be interpreted as rational functions in periods of smooth projective algebraic varieties defined over $\bar{\mathbf{Q}}$. A transcendence conjecture of Grothendieck then predicts that any relation between such periods should come from algebraic cycles over suitable powers of these varieties - or, if one prefers, should have an interpretation in terms of isomorphisms of motives. We refer to [1] for a detailed discussion of this topic. This motivic interpretation also leads to the following prediction: whenever an "accidental relation" occurs between special values of G-functions, one should expect a *similar relation* between the p-adic evaluations of the same series at the same point for every prime p for which the evaluation makes sense (this can be justified, relying upon some "standard conjectures" in the theory of motives in characteristic zero and p, see [1] for details).

As an illustration, one gets in the above example

$$ {}_2F_1(\frac{1}{12}, \frac{5}{12}, \frac{1}{2}; \frac{1323}{1331}) = \frac{1}{4}\, {}^4\sqrt{11} $$

7-adically (note that $1323 = 3^3.7^2$), as was checked by Beukers [6].

References

[1] Y. André, *Théorie des motifs et interprétation géométrique des valeurs p-adiques de G-fonctions (une introduction).* Number theory (Paris, 1992–1993), 37–60, London Math. Soc. Lecture Note Ser., 215, Cambridge Univ. Press, Cambridge, 1995.

[2] Y. André, *G-fonctions et transcendance.* J. Reine Angew. Math. **476** (1996), 95–125.

[3] Y. André, *Séries Gevrey de type arithmétique I. Théorèmes de pureté et de dualité.* Ann. of Math. **151** (2000), 705–740.

[4] Y. André, *Séries Gevrey de type arithmétique II. Transcendance sans transcendance.* Ann. of Math. **151** (2000), 741–756.

[5] D. Bertrand, *On André's proof of the Siegel-Shidlovsky theorem.* Colloque Franco-Japonais: Théorie des Nombres Transcendants (Tokyo, 1998), 51–63, Sem. Math. Sci., 27, Keio Univ., Yokohama, 1999.

[6] F. Beukers, *Algebraic values of G-fonctions.* J. Reine Angew. Math. **434** (1993), 45–65.

[7] L. Euler, *De seriebus divergentibus.* In Opera omnia I. 14 Teubner (1925), 601–602.

[8] M. Gevrey, *La nature analytique des solutions des équations aux dérivées partielles.* Ann. Sci. École Norm. Sup. (3) **25** (1918), 129–190.

[9] E. MAILLET, *Sur les séries divergentes et les équations différentielles*. Ann. Sci. École Norm. Sup. (1903), 487–518.
[10] O. PERRON, *Über lineare Differentialgleichungen mit rationalen Koeffizienten*. Acta math. **34** (1910), 139–163.
[11] J.-P. RAMIS, *Séries divergentes et Théories asymptotiques*. Panoramas et Synthèses **21**, SMF Paris, 1993.
[12] A. SHIDLOVSKY, *Transcendental Numbers*. de Gruyter Studies in Math. **12**, 1989; translation of Transtsendentnye Chisla, Nauka, 1987.
[13] C. L. SIEGEL, *Über einige Anwendungen diophantischer Approximationen*. Abh. Preuss. Akad. Wiss. Phys.-Math. Kl. **1** (1929), 1–70.
[14] G. WATSON, *A theory of Asymptotic series*. Philos. Trans. Roy. Soc. London Ser. A **211** (1911), 279–313.

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