Journal de Théorie des Nombres
de Bordeaux **15** (2003), 33–44

On sums of three squares

par James W. COGDELL

Résumé. Nous nous intéressons à la question de savoir quand un entier d'un corps de nombres totalement réel est somme de trois carrés d'entiers du corps, et plus généralement, s'il peut être représenté par une forme quadratique ternaire définie positive et entière sur le corps. Dans un travail récent avec Piatetski-Shapiro et Sarnak, nous avons montré que tout entier totalement positif et sans facteur carré assez grand possède une représentation intégrale globale si et seulement s'il en est de même localement partout, résolvant ainsi pour l'essentiel le dernier cas ouvert du onzième problème de Hilbert. Dans cet article, nous exposons les idées de la démonstration de ce résultat.

Abstract. We address the question of when an integer in a totally real number field can be written as the sum of three squared integers from the field and more generally whether it can be represented by a positive definite integral ternary quadratic form over the field. In recent work with Piatetski-Shapiro and Sarnak we have shown that every sufficiently large totally positive square free integer is globally integrally represented if and only if it is so locally at all places, thus essentially resolving the remaining open case of Hilbert's eleventh problem. In this paper we give an exposition of the ideas in the proof of this result.

The question of when an integer is representable as a sum of squares has a long venerable history. More generally, Hilbert's eleventh problem asks (among other things) which integers are integrally represented by a given quadratic form over a number field. The case of binary quadratic forms is equivalent to the theory of relative quadratic extensions and their class groups and class fields as developed by Hilbert. For forms in four or more variables the situation is quite different and has been understood for some time. The case of three variables has remained open.

Manuscrit reçu le 19 novembre 2001.

The author was supported in part by the NSA. While at the Institute for Advanced Study, where the original stages of the work was performed, the author was supported in part by a grant from the Ellentuck Fund and in part the research was conducted for the Clay Mathematics Institute.

The problem of which integers in a number field k are represented by the genus of a quadratic form $q(x)$ is completely answered by Siegel's mass formula, which gives the number of solutions in terms of products of local densities [23]. If there is only one class in the genus of $q(x)$, then this answers the question of representability for q. If $n \geq 4$ and q is indefinite at some archimedean place of k then Siegel showed by analytic methods that a number is represented by one form in a genus iff it is represented by all forms in the genus, thus reducing the global representability question to local ones [24]. These results were recovered and extended to indefinite ternary forms via algebraic techniques by Kneser [12] and Hsia [7] utilizing spinor genera. So we will restrict ourselves to the case of positive definite integral forms over a totally real field k. If the number of variables is at least five, one can proceed either by analytic methods, using bounds towards Ramanujan for Hilbert modular forms, or algebraic methods [8] to prove that there is an effective constant C_q such that if $\alpha \in \mathcal{O}$, the ring of integers of k, is totally positive and its norm $N(\alpha) > C_q$ then α is represented by q iff it is represented by q for every completion k_v of k. When the number of variables is four one must add a primitivity condition on the representations, both locally and globally [8]. Recently, in collaboration with I. I. Piatetski-Shapiro and P. Sarnak, we have established an analogue of this result in the case of positive definite integral ternary quadratic forms via analytic methods [3].

Theorem. *Let k be a totally real number field and let $q(x)$ be a positive definite integral ternary quadratic form over k. Then there is an ineffective constant C_q such that if α is a totally positive square free integer of k with $N(\alpha) > C_q$ then α is represented integrally by q iff it is locally integrally represented over every completion k_v of k.*

The result was previously known for $k = \mathbb{Q}$ by Duke and Schulze–Pillot [6, 18].

Of special interest is the case of the ternary form $q(x) = x_1^2 + x_2^2 + x_3^2$ giving the result on sums of squares in a number field alluded to in the title.

Corollary. *Let k be a totally real number field. Then there is an ineffective constant C_k such that every totally positive square free integer α with $N(\alpha) > C_k$ is the sum of three integral squares iff it is the sum of three integral squares locally for each place v of k.*

Of course over $\mathbb{Q}$ Legendre has given us the precise answer: $a \in \mathbb{Z}$ is the sum of three squares iff a is not of the form $4^n(8m+7)$. Over a number field, partial results have been obtained by algebraic methods by Donkar [4]. His methods, when applicable, give formulas for the number of ways such α can be represented but do not give the local to global result we present here.

In this note I would like to describe our proof of this theorem in the simplified case of k totally real of class number one.

This paper is an expanded version of the lecture I presented at the *XXII Journées Arithmétiques 2001* in Lille. I would like to thank the organizers of *JA 2001* for the opportunity to speak on this topic. I would also like to thank my collaborators, I. I. Piatetski-Shapiro and P. Sarnak, for allowing me to present this summary of our work here. In particular, I thank P. Sarnak for reading and providing critical comments on an earlier version of this note.

1. Theta series

We take k to be a totally real number field having class number one. Let $d = (k : \mathbb{Q})$ be the degree of k over $\mathbb{Q}$. We let $\mathcal{O}$ be the ring of integers of k. Let V be a vector space of dimension three over k equipped with a positive definite integral quadratic form $q(x)$. We will let $L = \mathcal{O}^3$ denote the integral lattice in V. So $q(x) \in \mathcal{O}$ for every $x \in L$.

The proof we will give is analytic in nature. Hence we begin with the theta series associated to $q(x)$ and L, Siegel's analytic class invariant [23],

$$\vartheta_q(\tau, L) = \sum_{x \in L} e^{2\pi i\, tr(q(x)\tau)}$$

where $\tau \in \mathfrak{H}^d$. This is a Hilbert modular form of weight $3/2$ for an appropriate congruence subgroup $\Gamma \subset SL_2(\mathcal{O})$. Its Fourier expansion is given by

$$\vartheta_q(\tau, L) = 1 + \sum_{\alpha >> 0} r_q(\alpha, L) e^{2\pi i\, tr(\alpha\tau)}$$

where $r_q(\alpha, L) = |\{x \in L | q(x) = \alpha\}|$ is the representation number of α by L.

There are two related theta series. Let $Spn(L)$ denote the spinor genus of L and $Gen(L)$ the genus of L [18, 19]. (Whether we fix the lattice and work with the genus of the form or fix the form and work with the genus of the lattice is all the same. Here we fix the form q and vary the lattice in its genus to conform to the work of Schulze–Pillot.) We set

$$\begin{aligned}\vartheta_q(\tau, Gen(L)) &= \frac{\sum_{M \in Gen(L)} o(M)^{-1} \vartheta_q(\tau, M)}{\sum_M o(M)^{-1}} \\ &= 1 + \sum_{\alpha >> 0} r_q(\alpha, Gen(L)) e^{2\pi i tr(\alpha\tau)}\end{aligned}$$

to be the weighted average over the genus of L, Siegel's analytic genus invariant [23], where $o(M)$ is the order of the group of units of M, and similarly for $\vartheta_q(\tau, Spn(L))$.

By the work of Siegel [23] we know that the coefficients $r_q(\tau, Gen(L))$ are given by a product of local densities and is non-zero iff α is locally represented integrally by (q, L_v) for all completions. So we need to be able to relate $r_q(\alpha, L)$ and $r_q(\alpha, Gen(L))$.

By classical results of Siegel, an algebraic proof of which can be found in Walling [27], we know that both

$$\vartheta_q(\tau, L) - \vartheta_q(\tau, Spn(L))$$

and

$$\vartheta_q(\tau, Spn(L)) - \vartheta_q(\tau, Gen(L))$$

are cusp forms of weight 3/2. Now there are two types of cusp forms of weight 3/2. Recall that we have the Shimura correspondence between cusp forms of weight 3/2 for Γ and modular forms of weight 2 for an appropriate subgroup $\Gamma' \subset PGL_2(\mathcal{O})$. If we denote by $S^1_{3/2}(\Gamma)$ the cusp forms associated to theta series attached to one dimensional quadratic forms and $S^0_{3/2}(\Gamma)$ its orthogonal complement with respect to the Petersson inner product then $S^0_{3/2}(\Gamma)$ consists of precisely those cusp forms that lift to cusp forms in $S_2(\Gamma')$ [25].

Following the work of Schulze–Pillot (see [19, 20] for related results) we can conclude that

$$\vartheta_q(\tau, L) - \vartheta_q(\tau, Spn(L)) \in S^0_{3/2}(\Gamma)$$

and

$$\vartheta_q(\tau, Spn(L)) - \vartheta_q(\tau, Gen(L)) \in S^1_{3/2}(\Gamma).$$

Fortunately the Fourier coefficients of the forms in $S^1_{3/2}(\Gamma)$ are quite sparse. In fact, it is known (see [7, 12]) that outside of an explicitly computable finite number of "genus exceptional" square classes we have

$$r_q(\alpha, Spn(L)) = r_q(\alpha, Gen(L)).$$

So for square free α there are only a finite number of such genus exceptions which we can avoid by taking α sufficiently large. Hence for all but finitely many square free α we have

$$r_q(\alpha, L) - r_q(\alpha, Gen(L)) = \tilde{a}(\alpha)$$

where $\tilde{a}(\alpha)$ is the α–Fourier coefficient of a cusp form $\tilde{f} \in S^0_{3/2}(\Gamma)$.

From the computation of the local densities (see [19]) one can conclude that for a locally represented α one has an ineffective lower bound

$$r_q(\alpha, Gen(L)) >> N(\alpha)^{\frac{1}{2}-\epsilon}$$

with the implied constant independent of α. The ineffectivity comes from an application of the Brauer–Siegel theorem and is the source of the ineffectivity of our theorem. Thus our result will follow if we can produce an estimate for the Fourier coefficients $\tilde{a}(\alpha)$ of half-integral weight forms in $S^0_{3/2}(\Gamma)$ of the form

$$|\tilde{a}(\alpha)| << N(\alpha)^{\frac{1}{2}-\delta}$$

for some fixed $\delta > 0$. This program was carried out in the case of $k = \mathbb{Q}$ by Duke and Schulze–Pillot [6] with a key ingredient being estimates on the Fourier coefficients of half-integral weight forms due to Iwaniec [9] and Duke [5].

2. Fourier coefficients and L-functions

By now, a common way to estimate Fourier coefficients of modular forms of half integral weight is to appeal to Waldspurger's formula [26] relating these coefficients to central values of the L-functions of the Shimura lift. Waldspurger established this relation only for $k = \mathbb{Q}$ but recently his result has been generalized to totally real fields by Shimura [22] using the Shimura correspondence and by Baruch and Mao using Jacquet's relative trace formula [1]. We will use Baruch and Mao's version of this relation.

Note that it suffices to prove our estimate for Hecke eigenforms since there is always a basis coming from such. Let $\tilde{\pi}$ denote the cuspidal representation of $\widetilde{SL}_2(\mathbb{A})$ generated by our $\tilde{f} \in S^0_{3/2}(\Gamma)$. Fix an additive character ψ of $k\backslash\mathbb{A}$ such that $\pi = \Theta(\tilde{\pi}, \psi)$ is the Shimura lift of $\tilde{\pi}$ to a cuspidal representation of $PGL_2(\mathbb{A})$. If α is a square free integer of $\mathcal{O}$ and we let $\psi^\alpha(x) = \psi(\alpha x)$ then we know [25]

$$\Theta(\tilde{\pi}, \psi^\alpha) = \pi \otimes \chi_\alpha(\det)$$

where χ_α is the quadratic character associated to the extension $k(\sqrt{\alpha})/k$. Classically χ_α is a ray class character mod (α). Let φ be the new form on $\mathfrak{H}^d$ associated to the new vector in the space of π and let $a(\alpha)$ denote its α-Fourier coefficient.

Let S denote the finite set of places of k where π, $\tilde{\pi}$, or ψ is ramified. Let S_∞ denote the archimedean places of k and let S_α denote the set of finite places v where $ord_v(\alpha) \neq 0$. Let

$$L(s, \pi) = \prod_{v \notin S_\infty} L(s, \pi_v)$$

be the finite (or classical) L-function of π. Then the formula of Baruch and Mao [1] can be stated as:

$$|\tilde{a}(\alpha)|^2 = |a(1)|^2 \frac{||\tilde{f}||^2}{||\varphi||^2} L(\tfrac{1}{2}, \pi \otimes \chi_\alpha) \prod_{v \in S \cup S_\alpha \cup S_\infty} c_v(\alpha)$$

where the $c_v(\alpha)$ are certain local constants given as ratios involving local norms, local Fourier coefficients, and local L-values. Similar formulas had been given earlier by Shimura [22]. For $v \notin S$ Baruch and Mao can explicitly compute the $c_v(\alpha)$ and for $v \in S$, a finite set of places, they can estimate them as α varies. As a consequence one gets an estimate

$$|\tilde{a}(\alpha)|^2 << N(\alpha)^{1/2} L(\tfrac{1}{2}, \pi \otimes \chi_\alpha)$$

with the implied constant independent of α.

Note that the convexity bound on $L(\frac{1}{2}, \pi \otimes \chi_\alpha)$ as α varies (see [10]) is

$$L(\tfrac{1}{2}, \pi \otimes \chi_\alpha) << N(\alpha)^{\frac{1}{2}+\epsilon}$$

which would give

$$|\tilde{a}(\alpha)| << N(\alpha)^{\frac{1}{2}+\epsilon}.$$

This is better than the Hecke bound, but not sufficient for our purposes. We must beat the convexity bound to obtain a first non-trivial estimate towards Ramanujan for the Fourier coefficients of half-integral weight forms over a totally real number field.

3. Subconvexity

Let π be a cuspidal automorphic representation of $PGL_2(\mathbb{A})$ corresponding to a holomorphic Hilbert modular form of even weight $k = (k, \ldots, k)$. Let $\varphi(\tau)$ be the associated new form. Let χ_1 be any ray class character modulo an ideal $\mathfrak{a}$. We will let χ_1 also denote its associated idele class character. The key to our proof of the stated theorem, and a result of interest in its own right, is the following breaking of convexity for $L(\frac{1}{2}, \pi \otimes \chi_1)$ in the conductor aspect as χ_1 varies.

Theorem. *We have*

$$L(\tfrac{1}{2}, \pi \otimes \chi_1) << N(\mathfrak{a})^{\frac{1}{2}-\frac{7}{130}+\epsilon}$$

with the implied constant depending on ϵ but independent of $\mathfrak{a}$.

Here again by $L(s, \pi \otimes \chi_1)$ we mean the classical (or finite) L-function

$$L(s, \pi \otimes \chi_1) = \prod_{v < \infty} L(s, \pi_v \otimes \chi_{1,v}).$$

If we apply this result to our previous situation with $\chi_1 = \chi_\alpha$ then this will complete the proof of our first Theorem.

We would now like to describe the proof of this Theorem, still in the case of class number one. We will work with the L-function in its additive form. To this end, we write

$$L(s,\pi\otimes\chi_1)=L(s,\varphi,\chi_1)=\sum_{\substack{\mu>>0\\ \mathrm{mod}\ U_+}}\lambda(\mu)\chi_1(\mu)N(\mu)^{-s}$$

where U_+ is the group of totally positive units in $U=\mathcal{O}^\times$. This is related to the Fourier expansion of φ

$$\varphi(\tau)=\sum_{\mu>>0}a(\mu)e^{2\pi i\ tr(d_K^{-1}\mu\tau)},$$

where d_K is the different of K, by $\lambda(\mu)=a(\mu)N(\mu)^{-(k-1)/2}$.

We first use the approximate functional equation for $L(s,\pi\otimes\chi_1)$ (see for example [13, 17]). This gives an expression of the form

$$L(\tfrac{1}{2},\pi\otimes\chi_1)=\sum_{\substack{\mu>>0\\ \mathrm{mod}\ U_+}}\frac{\lambda(\mu)\chi_1(\mu)}{\sqrt{N(\mu)}}V_1\left(\frac{N(\mu)}{X}\right)$$
$$+\,\epsilon(\pi\otimes\chi_1)\sum_{\substack{\mu>>0\\ \mathrm{mod}\ U_+}}\frac{\lambda(\mu)\chi_1^{-1}(\mu)}{\sqrt{N(\mu)}}V_2\left(\frac{N(\mu)}{X}\right)$$

as two sums of length essentially X, where we have taken $X=N(\mathfrak{a})$. Here V_1 and V_2 are functions having $V_i(0)=1$, smooth, and rapidly decaying at infinity. By using a smooth dyadic subdivision it suffices to estimate sums of the shape

$$J(\chi_1)=\sum_{\substack{\mu>>0\\ \mathrm{mod}\ U_+}}\frac{\lambda(\mu)\chi_1(\mu)}{\sqrt{N(\mu)}}\widetilde{W}\left(\frac{N(\mu)}{X}\right)$$

where now $\widetilde{W}$ is smooth of compact support say in the interval $(\frac{1}{2},2)$, so concentrated near 1. There are approximately $\log(N(\mathfrak{a}))$ such sums up to $N(\mathfrak{a})$. The crucial contribution for us will be again when X is of size $N(\mathfrak{a})$. Note that the presence of the cutoff function $\widetilde{W}$ in $J(\chi_1)$ forces $\sqrt{N(\mu)}\sim\sqrt{X}$ so that if we set $\widetilde{W}(x)=\sqrt{x}\ W(x)$ we have W is still smooth with compact support in $(\frac{1}{2},2)$ and

$$J(\chi_1)=X^{-1/2}S(\chi_1)$$

where

$$S(\chi_1)=\sum_{\substack{\mu>>0\\ \mathrm{mod}\ U_+}}\lambda(\mu)\chi_1(\mu)W\left(\frac{N(\mu)}{X}\right).$$

The sum $S(\chi_1)$ is the crucial sum we will have to estimate.

Our estimate will proceed by placing $S(\chi_1)$ into a family and then using arithmetic amplification. For a general exposition of these techniques one can refer to the talk of P. Michel at the XXII Journées Arithmétiques 2001 [13]. To get an appropriate family we need to work with a sum over the full set of totally positive integers. To this end we let F be a smoothed characteristic function of a fundamental domain for the action of the totally positive units U_+ on the hyperboloid defined by $N(x) = 1$ in $k_\infty^+ = \mathbb{R}_+^d$ which satisfies

$$\sum_{\epsilon \in U_+} F(\epsilon x) = 1$$

for every $x \in k_\infty^+$ with $N(x) = 1$. We extend this to all totally positive u by setting $F(u) = F\left(\frac{u}{N(u)}\right)$. Then we can write

$$S(\chi_1) = \sum_{\mu >> 0} \lambda(\mu)\chi_1(\mu)F(\mu)W\left(\frac{N(\mu)}{X}\right).$$

Note that both W and F act as cutoff functions – W cutting off in $N(\mu)$ and F cutting off in the "argument" of μ.

We now place $S(\chi_1)$ in a family. We do not use the family of all $S(\chi)$ where χ runs over all ray class characters mod $\mathfrak{a}$, which might seem more natural but could be too sparse. Indeed, the image of U_+ in $(\mathcal{O}/\mathfrak{a})^\times$ can be large [15]. Instead we use the family of all $S(\chi)$ as χ runs over all characters of the group $(\mathcal{O}/\mathfrak{a})^\times$. Let $C(\mathfrak{a})$ denote this group of characters. Then we will consider the average value of $|S(\chi)|^2$ in this family $\sum_{C(\mathfrak{a})} |S(\chi)|^2$. Note that the length of the sum is $|C(\mathfrak{a})|$ which is given by the generalized Euler totient function $\Phi(\mathfrak{a})$ and is trivially bounded by $N(\mathfrak{a})$ and is at least of size $N(\mathfrak{a})^{1-\epsilon}$, unlike the group of ray class characters.

In addition to averaging over this family, we will utilize the technique of arithmetic amplification. To this end we will take an auxiliary parameter M which will be of size X^δ with δ small. Its precise value will be determined in the course of the argument. We take a set $\{\nu\}$ of totally positive integers which should be relatively prime to $\mathfrak{a}$ and all have norm bounded by M. There should be roughly M of them and they should be balanced, in that for the archimedean embeddings v each ν_v should be roughly of the same size. For each ν we take a coefficient $c(\nu)$ such that $|c(\nu)| = 1$. We then consider the amplified sum

$$A = \sum_{C(\mathfrak{a})} |S(\chi)|^2 |\sum_{\nu} c(\nu)\chi(\nu)|^2.$$

To obtain an estimate on our original $S(\chi_1)$ we will take arithmetically defined coefficients $c(\nu) = \chi_1^{-1}(\nu)$ thus amplifying the term $|S(\chi_1)|^2$ by a factor of M^2.

To utilize A, we expand the norm squares, interchange the order of summation, and perform the character sum over $C(\mathfrak{a})$. This yields

$$A = \Phi(\mathfrak{a}) \sum_{\nu_1,\nu_2} c(\nu_1)\overline{c(\nu_2)} \times \sum_{\substack{\mu_1,\mu_2 \\ \mu_1\nu_1 \equiv \mu_2\nu_2 \pmod{\mathfrak{a}}}} \lambda(\mu_1)\overline{\lambda(\mu_2)} F(\mu_1)F(\mu_2) W\left(\frac{N(\mu_1)}{X}\right) \overline{W\left(\frac{N(\mu_2)}{X}\right)}.$$

We split this sum into two terms, the diagonal D and the off diagonal OD, where

$$D = \Phi(\mathfrak{a}) \sum_{\nu_1,\nu_2} c(\nu_1)\overline{c(\nu_2)} \times \sum_{\mu_1\nu_1 = \mu_2\nu_2} \lambda(\mu_1)\overline{\lambda(\mu_2)} F(\mu_1)F(\mu_2) W\left(\frac{N(\mu_1)}{X}\right) \overline{W\left(\frac{N(\mu_2)}{X}\right)}$$

and

$$OD = \Phi(\mathfrak{a}) \sum_{\nu_1,\nu_2} c(\nu_1)\overline{c(\nu_2)} \times \sum_{\substack{\mu_1\nu_1 - \mu_2\nu_2 = h \in \mathfrak{a} \\ h \neq 0}} \lambda(\mu_1)\overline{\lambda(\mu_2)} F(\mu_1)F(\mu_2) W\left(\frac{N(\mu_1)}{X}\right) \overline{W\left(\frac{N(\mu_2)}{X}\right)}.$$

The diagonal term is estimated simply using the Ramanujan bounds for the $\lambda(\mu)$, known in this case by Brylinski and Labesse [2], namely $|\lambda(\mu)| << N(\mu)^\epsilon$, and then analysing the size of the sums determined by the cutoff functions. These yield

$$D << N(\mathfrak{a}) X^{1+\epsilon} M^{1+\epsilon}.$$

The off diagonal term is more interesting. Let us write it as

$$OD = \Phi(\mathfrak{a}) \sum_{\nu_1,\nu_2} c(\nu_1)\overline{c(\nu_2)} \sum_{\substack{0 \neq h \in \mathfrak{a} \\ h \mod U_+}} B(\nu_1,\nu_2,h)$$

where in $B(\nu_1,\nu_2,h)$ we have resummed over U_+ which only has an effect on the cutoff functions F. The terms $B(\nu_1,\nu_2,h)$ are estimated using several variable Mellin inversion. We can write

$$B(\nu_1,\nu_2,h) = \frac{1}{(2\pi i)^d} \int_{Re(s)=a} D(s,\nu_1,\nu_2,h) H(s)\ ds$$

for suitably large a where $s=(s_1,\ldots,s_d)\in\mathbb{C}^d$. $D(s,\nu_1,\nu_2,h)$ is a type of Dirichlet series

$$D(s,\nu_1,\nu_2,h)=\sum_{\ell>>0}\sum_{\substack{\nu_1\mu_1+\nu_2\mu_2=\ell\\ \nu_1\mu_1-\nu_2\mu_2=h}}\lambda(\mu_1)\overline{\lambda(\mu_2)}\left(\frac{\sqrt{N(\nu_1\mu_1\nu_2\mu_2)}}{N(\ell)}\right)^{k-1}\ell^{-s}$$

where we have used a multi-index notation, so $\ell^{-s}=\ell_1^{-s_1}\cdots\ell_d^{-s_d}$ where $\ell_1,\ldots,\ell_d$ are the images of ℓ under the d embeddings of k into $\mathbb{R}$. This Dirichlet series carries the arithmetic information in $B(\nu_1,\nu_2,h)$. The function $H(s)$ is essentially the Mellin transform of the cutoff functions.

$H(s)$ is relatively simple to handle. It is entire, rapidly decreasing in $Im(s)$, and can be estimated by

$$|H(s)|<<(MX)^{\frac{1}{d}\sum\sigma_j+\epsilon}\prod_{j=1}^{d}(1+|t_j|)^{-5}$$

where as is common we have written $s_j=\sigma_j+it_j$.

The interesting bit of the estimate is in the Dirichlet series $D(s,\nu_1,\nu_2,h)$. It is essentially a Dirichlet series formed with products of shifted Fourier coefficients. Selberg has shown how to approach such Dirichlet series via Poincaré series [21]. To this end, let us set $g(\tau)=y^k\varphi(\nu_1\tau)\overline{\varphi(\nu_2\tau)}$. Then there is a Poincaré series $P_h(\tau,s)$ such that when we compute the Petersson inner product of g with $P_h(s)$ we find

$$\langle g,P_h(s)\rangle=N(\nu_1\nu_2)^{(1-k)/2}\prod_{j=1}^{d}\Gamma(s_j+k-1)D(s,\nu_1,\nu_2,h).$$

One now expands this inner product spectrally via Parseval's formula. If we let $\{\phi_j\}$ be a suitable orthonormal basis of Maass cusp forms then

$$\langle g,P_h(s)\rangle=\sum_j\langle g,\phi_j\rangle\langle\phi_j,P_h(s)\rangle+c(s)$$

where $c(s)$ is a similar expression involving the continuous spectrum and is estimated in a similar manner. Sarnak has developed a general method for estimating $\langle g,\phi_j\rangle$ (see for example [16]) which in this case yields

$$\langle g,\phi_j\rangle<<N(\nu_1\nu_2)^{(1-k)/2}\prod_{i=1}^{d}(1+|r_{j,i}|)^{k+1}e^{-\frac{\pi}{2}|r_{j,i}|}$$

where $r_j=(r_{j,1},\ldots,r_{j,d})$ is the spectral parameter of ϕ_j. The term $\langle\phi_j,P_h(s)\rangle$ is expressed in terms of the h-Fourier coefficient $\rho_j(h)$ of ϕ_j and the associated archimedean Γ–factors involving the spectral parameters r_j. We then estimate these using the bounds towards Ramanujan in both the archimedean and non-archimedean aspects due to Kim and Shahidi [11].

The sum is then controlled using Weyl's law. A similar type of estimate can be found [14] and in the appendix of [17].

In the final analysis, these estimates give that the Dirichlet series $D(s,\nu_1,\nu_2,h)$ has an analytic continuation to the domain $Re(s_j) > \frac{1}{2}+\frac{1}{9}$ and in this region satisfies

$$|D(s,\nu_1,\nu_2,h)| << N(\nu_1\nu_2)^{\frac{1}{2}} N(h)^{\frac{1}{9}+\epsilon} \prod_{j=1}^{d} |h_j|^{\frac{1}{2}-\sigma_i}(1+|t_j|)^{3+\epsilon}$$

where the h_j are the images of h under the d embeddings of k into $\mathbb{R}$. Note that the $\frac{1}{9}$ in the boundary of the domain of continuation comes from the archimedean estimates towards Ramanujan while the $\frac{1}{9}$ in the exponent of $N(h)$ is from the non-archimedean bound towards Ramanujan of Kim and Shahidi.

Returning now to our expression of $B(\nu_1,\nu_2,h)$ in terms of the inverse Mellin transform, we can now shift the lines of integration to $Re(s_j) = \frac{1}{2}+\frac{1}{9}+\epsilon = \frac{11}{18}+\epsilon$ to obtain

$$|B(\nu_1,\nu_2,h)| << M^{1+\epsilon}(MX)^{\frac{11}{18}+\epsilon}$$

which in turn results in

$$OD << N(\mathfrak{a})M^{4+\frac{11}{18}+\epsilon}X^{\frac{11}{18}+\epsilon}.$$

When we combine D and OD and choose $M = X^{\delta}$ to give them the same order of growth in X we find that $M = X^{7/65}$. Now taking $X = N(\mathfrak{a})$ to get the dominant term from our partition we get an estimate of our amplified sum

$$A = \sum_{C(\mathfrak{a})} |S(\chi)|^2 |\sum_{\nu} c(\nu)\chi(\nu)|^2 << N(\mathfrak{a})^{2+\frac{7}{65}+\epsilon}.$$

We now take $c(\nu) = \chi_1^{-1}(\nu)$ to amplify the term we are interested in. Then estimating this one term by the entire sum we find

$$M^2|S(\chi_1)|^2 << N(\mathfrak{a})^{2+\frac{7}{65}+\epsilon}$$

or

$$|S(\chi_1)| << N(\mathfrak{a})^{1-\frac{7}{130}+\epsilon}$$

which finally gives

$$L(\tfrac{1}{2},\pi\otimes\chi_1) << N(\mathfrak{a})^{\frac{1}{2}-\frac{7}{130}+\epsilon}$$

as desired.

Note that to our knowledge this estimate is better than the current best bounds even in the case $k = \mathbb{Q}$.

References

[1] M. Baruch, Z. Mao, *Central value of automorphic L-functions*, preprint, 2000.

[2] J.-L. Brylinkski, J.-P. Labesse, *Cohomologie d'intersection et fonctions L de certaines variétés de Shimura.* Ann. Sci. École Norm. Sup.(4) **17** (1984), 361–412.

[3] J. W. Cogdell, I. I. Piatetski-Shapiro, P. Sarnak, *Estimates on the critical line for Hilbert modular L-functions and applications*, in preparation.

[4] E. N. Donkar, *On sums of three integral squares in algebraic number fields.* Amer. J. Math., **99** (1977), 1297–1328.

[5] W. Duke, *Hyperbolic distribution problems and half-integral weight Maass forms.* Invent. Math., **92** (1988), 73–90.

[6] W. Duke, R. Schulze–Pillot, *Representations of integers by positive ternary quadratic forms and equidistribution of lattice points on ellipsoids.* Invent. Math. **89** (1990), 49–57.

[7] J. S. Hsia, *Representations by spinor genera.* Pacific J. Math. **63** (1976), 147–152.

[8] J. S. Hsia, Y. Kitaoka, M. Kneser, *Representations of positive definite quadratic forms.* J. Reine Angew. Math. **301** (1978), 132–141.

[9] H. Iwaniec, *Fourier coefficients of modular forms of half integral weight.* Invent. Math. **87** (1987), 385–401.

[10] H. Iwaniec, P. Sarnak, *Perspectives on the analytic theory of L-functions.* Geom. Funct. Anal. (GAFA), Special Volume, Part II (2000), 705–741.

[11] H. Kim, F. Shahidi, *Cuspidality of symmetric powers of $GL(2)$ with applications.* Duke Math. J. **112** (2002), 177–197.

[12] M. Kneser, *Darstellungsmasse indefiniter quadratischer Formen.* Math. Zeit. **77** (1961), 188–194.

[13] P. Michel, *Familles de fonctions L de formes automorphes et applications.* J. Théor. Nombres Bordeaux **15** (2003), 275–307.

[14] Y. Petridis, P. Sarnak, *Quantum unique ergodicity for $SL_2(\mathcal{O})\backslash\mathbf{H}^3$ and estimates for L-functions.* J. Evol. Equ. **1** (2001), 277–290.

[15] D. Rohrlich, *Non-vanishing of L-functions for $GL(2)$.* Invent. Math. **97** (1989), 381–403.

[16] P. Sarnak, *Integrals of products of eigenfunctions.* Internat. Math. Res. Notices (1994) 251–260.

[17] P. Sarnak, *Estimates for Rankin–Selberg L-functions and quantum unique ergodicity.* J. Funct. Anal **184** (2001), 419–453.

[18] R. Schulze-Pillot, *Thetareihen positiv definiter quadratischer Formen.* Invent. Math. **75** (1984), 283–299.

[19] R. Schulze-Pillot, *Darstellungsmasse von Spinorgeschlechtern ternärer quadratischer Formen.* J. Reine Angew. Math. **352** (1984), 114–132.

[20] R. Schulze-Pillot, *Ternary quadratic forms and Brandt matrices.* Nagoya Math. J. **102** (1986), 117–126.

[21] A. Selberg, *On the estimation of Fourier coefficients of modular forms.* Proc. Sympos. Pure Math., Vol. VIII, pp. 1–15 Amer. Math. Soc., Providence, R.I., 1965.

[22] G. Shimura, *On the Fourier coefficients of Hilbert modular forms of half-integral weight.* Duke Math. J. **72** (1993), 501–557.

[23] C. L. Siegel, *Über die analytische Theorie der quadratischer Formen I.* Ann. of Math. **36** (1935), 527–606; II, **37** (1936), 230–263; III, **38** (1937) 212–291.

[24] C. L. Siegel, *Indefinite quadratische Formen und Funktionentheorie I.* Math. Ann. **124** (1951) 17–54; II, 366–387.

[25] J.-L. Waldspurger, *Correspondance de Shimura.* J. Math. Pures Appl. **59** (1980), 1–113.

[26] J.-L. Waldspurger, *Sur les coefficients de Fourier des formes modulaires de poids demi-entier.* J. Math. Pures Appl. **60** (1981), 375–484 .

[27] L. Walling, *A remark on differences of theta series.* J. Number Theory **48** (1994), 243–251.

James W. Cogdell
Department of Mathematics
Oklahoma State University
Stillwater OK 74078
USA
E-mail : cogdell@math.okstate.edu