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The metric simultaneous Diophantine approximations over formal power series

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Résumé. Nous étudions les propriétés métriques de l'approximation diophantienne simultanée dans le cas non archimédien. Nous prouvons d'abord une loi du $0-1$ de type Gallagher, que nous utilisons ensuite pour obtenir un résultat de type Duffin-Schaeffer.

Abstract. We discuss the metric theory of simultaneous Diophantine approximations in the non-archimedean case. First, we show a Gallagher type 0-1 law. Then by using this theorem, we prove a Duffin-Schaeffer type theorem.

Introduction

In the study of metric Diophantine approximations, one of the questions asks for conditions on a non-negative function ψ which guarantee that the inequality

$$(1) \qquad \left| x - \frac{p}{q} \right| < \frac{\psi(q)}{q}, \qquad (p,q) = 1$$

has infinitely many solutions $\frac{p}{q}$ for almost every x. In the one-dimensional case it is known that if the following two conditions hold then (1) has infinitely many solutions for almost every x, Duffin-Schaeffer [1]:

(i) $\sum \psi(q) = \infty$

(ii) $\displaystyle\sum_{q \leq Q} \psi(q) < c \sum_{q \leq Q} \psi(q) \frac{\phi(q)}{q}$,

where c is a positive constant and ϕ is the Euler function.
A similar result holds for s-dimensional Diophantine approximations.

In the previous paper [4], we discussed an analogue of Duffin-Schaeffer theorem for formal Laurent power series (one-dimensional case). As its continuation we consider the same question for higher dimensional case in this paper. To state our results we start with some fundamental definitions and notations.

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We use the following notations:

$\mathbb{F}$: a finite field with q elements,
$\mathbb{F}[X]$: the set of polynomials with $\mathbb{F}$-coefficients,
$\mathbb{F}(X)$: the fraction field of $\mathbb{F}[X]$,
$\mathbb{F}((X^{-1}))$: the set of formal Laurent power series.

We define the absolute value of $f \in \mathbb{F}((X^{-1}))$ by $|f| = q^{\deg f}$ and put

$$\mathbb{L} = \{f = a_{-1}X^{-1} + \cdots + a_{-i}X^{-i} + \cdots \mid a_i \in \mathbb{F} \text{ for } \quad i \leq -1\}$$

which is a compact abelian group with the metric $d(f,g) = |f-g|$. We denote by m the normalized Haar measure on $\mathbb{L}$. This implies that for any $b_i \in \mathbb{F}$ $(1 \leq i \leq l)$,

$$m\{f = a_{-1}X^{-1} + a_{-2}X^{-2} + \cdots \mid a_{-i} = b_i \quad \text{for } 1 \leq i \leq l\} = \frac{1}{q^l}. \tag{2}$$

In [4], we showed that

$$\left|f - \frac{P}{Q}\right| < \frac{\psi(Q)}{|Q|} \qquad P, Q \in \mathbb{F}[X]$$

has either infinitely many solutions for m-a.e. $f \in \mathbb{L}$ or only finitely many solutions for m-a.e. $f \in \mathbb{L}$. Here, ψ is a $\{q^{-n} \,|n \geq 0\} \cup \{0\}$- valued function and P and Q have no non-trivial common factor. We call this "Gallagher type theorem" since this property was proved by Gallagher in the classical case. Then, we showed "Duffin-Schaeffer type theorem" :

Let ψ be a $\{q^{-n} \mid n \geq 0\} \cup \{0\}$-valued function, which satisfies

$$\sum_{\substack{\deg Q=1 \\ Q:\text{monic}}}^{\infty} \psi(Q) = \infty.$$

Suppose for a positive constant C, there are infinitely many positive integers n such that

$$\sum_{\substack{\deg Q \leq n \\ Q:\text{monic}}} \psi(Q) < C \sum_{\substack{\deg Q \leq n \\ Q:\text{monic}}} \psi(Q)\frac{\Phi(Q)}{|Q|}$$

holds, then

$$\left|f - \frac{P}{Q}\right| < \frac{\psi(Q)}{|Q|}, \qquad (P,Q) = 1$$

has infinitely many solutions $\frac{P}{Q}$ for m-a.e. $f \in \mathbb{L}$. Here,

$$\Phi(Q) = \#\left\{Q' \,\middle|\, \begin{array}{l} \deg Q' < \deg Q,\ Q', Q \ : \text{ monic polynomials} \\ Q' \text{ and } Q \text{ have no non-trivial common factor} \end{array}\right\}.$$

The purpose of this paper is to show the s-dimensional version of these theorems for formal Laurent power series ($s \geq 2$). For a given ψ as above, we consider the following inequalities:

$$\left|f_1 - \tfrac{P_1}{Q}\right| < \tfrac{\psi(Q)}{|Q|}\ , \dots ,\ \left|f_s - \tfrac{P_s}{Q}\right| < \tfrac{\psi(Q)}{|Q|} \tag{3}$$
$$(P_1, Q) = (P_2, Q) = \cdots = (P_s, Q) = 1.$$

We shall give a sufficient condition on ψ so that (3) has infinitely many solutions $(\frac{P_1}{Q}, \dots , \frac{P_s}{Q})$ for almost every $(f_1, \dots , f_s) \in \mathbb{L}^s$ with respect to m^s, which is the s-fold product of m. For this purpose, let

$$E_Q = \left\{ (f_1, \dots , f_s) \in \mathbb{L}^s \ \middle|\ \begin{array}{l} \left|f_i - \frac{P_i}{Q}\right| < \frac{\psi(Q)}{|Q|}\ , \text{ for some } P_i \quad \text{s.t.} \\ \deg P_i < \deg Q, (P_i, Q) = 1\,, 1 \leq i \leq s \end{array} \right\}$$

for a monic polynomial Q and

$$E = \bigcap_{n=1}^{\infty} \bigcup_{\deg Q \geq n} E_Q,$$

we first prove the following theorem in §1.

Theorem 1. *For any ψ, either $m^s(E) = 0$ or 1 holds, equivalently, (3) has infinitely many solutions of $(Q, P_1, \dots , P_s)$ for m^s-a.e. $(f_1, \dots , f_s) \in \mathbb{L}^s$ or has only finitely many solutions for m^s-a.e. $(f_1, \dots , f_s) \in \mathbb{L}^s$.*

We call this the s-dimensional version of the Gallagher type theorem. Secondly, making use of this theorem, we can show the s-dimensional version of the Duffin-Schaeffer type theorem in §2.

Theorem 2. *Let ψ be a $\{q^{-n} \mid n \geq 0\} \cup \{0\}$-valued function which satisfies*

$$\sum_{\substack{\deg Q = 1 \\ Q:\text{monic}}}^{\infty} \psi^s(Q) = \infty.$$

Suppose for a positive constant C, there are infinitely many positive integers n such that

$$\sum_{\substack{\deg Q \leq n \\ Q:\text{monic}}} \psi^s(Q) < C \sum_{\substack{\deg Q \leq n \\ Q:\text{monic}}} \psi^s(Q) \frac{\Phi^s(Q)}{|Q|^s} \tag{4}$$

holds. Then (3) has infinitely many solutions $\left(\frac{P_1}{Q}, \dots , \frac{P_s}{Q}\right)$ for m^s-a.e. $(f_1, \dots , f_s) \in \mathbb{L}^s$.

Finally, we explain a Duffin-Schaeffer type conjecture in formal Laurent power series case in §3.

1. Proof of Theorem 1

In the sequel, we study the metric property of s-dimensional simultaneous approximations (3) for $s \geq 2$. For $f = a_l X^l + \cdots + a_1 X + a_0 + a_{-1} X^{-1} + \cdots \in \mathbb{F}((X^{-1}))$ $(a_l \neq 0)$, we put

$$[f] = a_l X^l + a_{l-1} X^{l-1} + \cdots + a_1 X + a_0.$$

For given $h_i \in \mathbb{F}[X]$ such that

$$h_i \;=\; a_{i\,l_i} X^{l_i} \;+\; a_{i\,l_i - 1} X^{l_i - 1} \;+\; \cdots \;+\; a_{i\,1} X \;+\; a_{i\,0},$$

$$a_{i\,j} \in \mathbb{F},\; 1 \leq i \leq s,\; 0 \leq j \leq l_i,\; a_{i\,l_i} \neq 0,$$

we define the cylinder set $\langle h_1\,, \ldots, h_s \rangle$ as follows:

$$\langle h_1\,, \ldots, h_s \rangle := \big\{ (f_1\,, \ldots, f_s) \in \mathbb{L}^s \mid [X^{l_1+1} \cdot f_1] = h_1\,, \ldots, [X^{l_s+1} \cdot f_s] = h_s \big\}.$$

Then we see the following,

$$m^s(\langle h_1\,, \ldots, h_s \rangle) = \frac{1}{q^{l_1+1}} \cdots \frac{1}{q^{l_s+1}}.$$

Lemma 1. *Let* $\{\langle h_{1\,k}\,, h_{2\,k}\,, \ldots, h_{s\,k}\rangle \mid k \geq 1\}$ *be a sequence of cylinder sets defined as above with*

$$\lim_{k\to\infty} \deg\, h_{i\,k} \;=\; \infty,$$

and $\{\, E_k \mid k \;\geq\; 1\}$ *be a sequence of measurable sets of* $\mathbb{L}^s$ *for which* $E_k \subset \langle\, h_{1\,k}\,, \ldots, h_{s\,k}\,\rangle$. *Suppose there exists* $\delta > 0$ *such that* $m^s(\,E_k\,) \;\geq\; \delta m^s(\langle h_{1\,k}\,, \ldots, h_{s\,k}\rangle)$ *for any* $k \geq 1$. *Then*

$$m^s\left(\bigcap_{l=1}^{\infty} \bigcup_{k=l}^{\infty}\; E_k\right) \;=\; m^s\left(\bigcap_{l=1}^{\infty} \bigcup_{k=l}^{\infty} \langle h_{1\,k}, \ldots, h_{s\,k}\rangle\right).$$

Proof. Let

$$H \;:=\; \bigcap_{l=1}^{\infty} \bigcup_{k=l}^{\infty} \langle h_{1\,k}, \ldots, h_{s\,k}\rangle, \quad E_l^* \;=\; \bigcup_{k=l}^{\infty} E_k, \quad H_l^* \;:=\; H \setminus E_l^*.$$

We show that $m^s(H_l^*) = 0$ for any $l \geq 1$, which implies the assertion of this lemma. Suppose there exists a $k_0 \in \mathbb{Z}_+$ such that $m^s(H_k^*) > 0$ for $k \geq k_0$. There is a natural correspondence between cylinder sets defined for $\mathbb{L}$ as in (2) and q-adic rational intervals, and so $(\mathbb{L}, m)$ is isomorphic to $[0,1]$ with the Lebesgue measure. Similarly, $(\mathbb{L}^s, m^s)$ is isomorphic to $[0,1]^s$ with the Lebesgue measure. So by using cylinder sets $\langle h_1, \ldots, h_s \rangle \subset \mathbb{L}^s$ instead of $I_1 \times \cdots \times I_s \subset [0,1]^s$, we can apply Lebesgue density theorem. Then we get, since $\{H_l^* \mid l \geq 1\}$ is an increasing sequence of sets,

$$\frac{m^s(H_k^* \cap \langle h_{1\,k}, \ldots, h_{s\,k}\rangle)}{m^s(\langle h_{1\,k}, \ldots, h_{s\,k}\rangle)} \;>\; 1 - \frac{\delta}{2} \tag{5}$$

for some k. On the other hand,

$$H_k^* \cap E_k^* = \emptyset.$$

From the assumption of this lemma,

$$\begin{aligned}
& m^s(\langle h_{1\,k}, \dots, h_{s\,k}\rangle) \\
& \quad \geq \; m^s(E_k) + m^s(H_k^* \cap \langle h_{1\,k}, \dots, h_{s\,k}\rangle) \\
& \quad \geq \; \delta m^s(\langle h_{1\,k}, \dots, h_{s\,k}\rangle) + m^s(H_k^* \cap \langle h_{1\,k}, \dots, h_{s\,k}\rangle).
\end{aligned}$$

That is

$$(1-\delta)\, m^s(\langle h_{1\,k}, \dots, h_{s\,k}\rangle) \geq m^s(H_k^* \cap \langle h_{1\,k}, \dots, h_{s\,k}\rangle),$$

which contradicts (4). $\square$

Lemma 2. *For any polynomial $h_i \in \mathbb{F}[X]\,(h_i \neq 0)$ and $g_i \in \mathbb{L}$, $1 \leq i \leq s$, the map T of $\mathbb{L}^s$ onto itself defined by*

$$T(f_1, \dots, f_s) = (h_1 f_1 + g_1 - [h_1 f_1 + g_1], \dots, h_s f_s + g_s - [h_s f_s + g_s])$$

for $(f_1, \dots, f_s) \in \mathbb{L}^s$ is ergodic.

Proof. It is easy to see that each map

$$T_i(f_i) = h_i f_i + g_i - [h_i f_i + g_i], \qquad 1 \leq i \leq s$$

is a Bernoulli transformation of $\mathbb{L}$. In other words, if we put

$$W_k(f_i) = [h_i \cdot T_i^{k-1} f_i + g_i], \qquad \text{for } f_i \in \mathbb{L},$$

then $\{W_k \mid k \geq 1\}$ gives a sequence of independent and identically distributed random variables. In particular, T_i is weak mixing. Since the s-fold product of weak mixing transformations is ergodic (see [5] Prop. 4.2.), this yields the assertion of the lemma. $\square$

Proof of Theorem 1. In the case of formal power series, we mean by $(P_i, Q) = 1$ that P_i and Q have no non-trivial common factor.

If

$$\lim_{\deg Q \to \infty} \frac{\psi(Q)}{q^{\deg Q}} > 0,$$

then we can find a sequence of polynomials $Q_1, Q_2, Q_3, \dots$ and a positive integer l such that $\frac{\psi(Q_k)}{q^{\deg Q_k}} > q^{-l}$ for any $k \geq 1$. In this case, for any $f_i \in \mathbb{L}$ and a sufficiently large k, we can find $P_i\,(\deg P_i < \deg Q_k)$, such that

$$\left| f_i - \frac{P_i}{Q_k} \right| < \frac{1}{q^l} \;\; \left(< \frac{\psi(Q_k)}{q^{\deg Q_k}} \right), \qquad (1 \leq i \leq s)$$

and P_i and Q_k have no non-trivial common factor. Indeed, the number of polynomials $\hat{P}$ such that

$$\left| f_i - \frac{\hat{P}}{Q_k} \right| < \frac{1}{q^l}$$

is $q^{\deg Q_k - l}$. If all such polynomials $\hat{P}$ are not relatively prime to Q_k, then Q_k has more than $q^{\deg Q_k - l}$ factors, which is impossible if $\deg Q_k$ is sufficiently large. This implies

$$E = \mathbb{L}^s.$$

Now we show the assertion of the theorem when

$$\lim_{\deg Q \to \infty} \frac{\psi(Q)}{q^{\deg Q}} = 0. \tag{6}$$

For fixed Q, $P_1, \dots, P_{s-1}$ and P_s, there exist polynomials $h_1, \dots, h_s$ such that $\left\{(f_1, \dots, f_s) \mid \left| f_i - \frac{P_i}{Q} \right| < \frac{\psi(Q)}{|Q|} \right\} = \langle h_1, \dots, h_s \rangle$. Then (6) implies $\deg h_i \to \infty$, $1 \le i \le s$ as $\deg Q$ tends to ∞. Thus we can apply Lemma 1 when (6) holds. Then we evaluate the measure of $\bigcap_{n=1}^{\infty} \bigcup_{\deg Q \ge n} E_Q$. Let R be an irreducible polynomial and consider

$$\left| f_i - \frac{P_i}{Q} \right| < \frac{\psi(Q)|R|^{n-1}}{|Q|}, \qquad (P_i, Q) = 1 \tag{7}$$

for $n \ge 1$ and $1 \le i \le s$.

We put

$$E_0(n:R) = \left\{ (f_1, \dots, f_s) \in \mathbb{L}^s \;\middle|\; \begin{array}{c} \text{(7) has infinitely many solutions} \\ P_i, Q \text{ with } R \nmid Q \quad \text{for } 1 \le i \le s \end{array} \right\}$$

and

$$E_1(n:R) = \left\{ (f_1, \dots, f_s) \in \mathbb{L}^s \;\middle|\; \begin{array}{c} \text{(7) has infinitely many solutions} \\ P_i, Q \text{ with } R \parallel Q \quad \text{for } 1 \le i \le s \end{array} \right\}$$

Then we see

$$E_j(1:R) \subset E_j(2:R) \subset E_j(3:R) \subset \cdots$$

and

$$E_j(1:R) \subset E$$

for $j = 0, 1$. From Lemma 1, we find that

$$m^s(E_j(n:R)) = m^s(E_j(1:R)) = m^s\Big(\bigcup_{n \ge 1} E_j(n:R)\Big).$$

Let

$$T_j(f_1, \dots, f_s) = \begin{cases} (R \cdot f_1 - [R \cdot f_1], \dots, R \cdot f_s - [R \cdot f_s]) & j = 0 \\ \big(R \cdot f_1 + \frac{1}{R} - [R \cdot f_1 + \frac{1}{R}], \dots, \\ \qquad R \cdot f_s + \frac{1}{R} - [R \cdot f_s + \frac{1}{R}]\big) & j = 1, \end{cases}$$

for $(f_1, \dots, f_s) \in \mathbb{L}^s$. Suppose (7), we have

$$\left| R \cdot f_i - \frac{R \cdot P_i}{Q} \right| < \frac{\psi(Q)|R|^n}{|Q|}$$

and see

$$\Big(R \cdot P_i \,,\, Q \Big) = 1.$$

Also, we have

$$\left| \Big(R \cdot f_i + \frac{1}{R} \Big) - \frac{R \cdot P_i + \frac{Q}{R}}{Q} \right| < \frac{\psi(Q)|R|^n}{|Q|}.$$

Here,

$$\Big(R \cdot P_i + \frac{Q}{R}, Q \Big) = 1.$$

These imply

$$T_j \Big(\bigcup_{n \geq 1} E_j(n : R) \Big) = \bigcup_{n \geq 2} E_j(n : R))$$

for $j = 0, 1$. Hence from Lemma 2, we have

$$m^s \Big(\bigcup_{n \geq 1} E_j(n : R) \Big) = 0 \text{ or } 1.$$

for $j = 0, 1$. Thus, if either $m^s(E_0(1 : R))$ or $m^s(E_1(1 : R)) > 0$ for some irreducible polynomial R, then we have $m^s(E) = 1$.

Now we assume that $m^s(E_0(1 : R)) = m^s(E_1(1 : R)) = 0$ for any irreducible polynomial R. We put

$$F(R) = \left\{ (f_1, \dots, f_s) \in \mathbb{L}^s \;\middle|\; \begin{array}{r} (8) \text{ has infinitely many solutions} \\ P_i,\, Q \text{ such that } R^2 \mid Q \end{array} \right\},$$

where (8) refers to:

$$\left| f_i - \frac{P_i}{Q} \right| < \frac{\psi(Q)}{|Q|}, \qquad (P_i, Q) = 1, \qquad 1 \leq i \leq s. \tag{8}$$

Suppose (8), we have

$$\left| \Big(f_i + \frac{U}{R} \Big) - \frac{P_i + \frac{QU}{R}}{Q} \right| < \frac{\psi(Q)}{|Q|},$$

for any polynomial U with $0 \leq \deg U < \deg R$. Here, we see

$$\Big(P_i + \frac{QU}{R}, Q \Big) = 1,$$

which implies that if $(f_1, \dots, f_s) \in F(R)$, then $(f_1 + \frac{U}{R}, \dots, f_s + \frac{U}{R}) \in F(R)$. Also we put

$$S(U; R) = \{ (f_1, \dots, f_s) \in \mathbb{L}^s \mid [Rf_1] = U, \dots, [Rf_s] = U \},$$

then its measure is $\frac{1}{q^{s\deg R}}$ and

$$\bigcup_{U:0\leq \deg U<\deg R} S(U;R) \bigcup \{(f_1,\dots,f_s)\in \mathbb{L}^s \mid \deg f_i \ < -\deg R\} = \mathbb{L}^s.$$

Since $F(R)$ is $(\cdot + \frac{U}{R})$-invariant, $S(U;R)$ $(0 \leq \deg U < \deg R)$ and $\{(f_1,\dots,f_s) \in \mathbb{L}^s \mid \deg f_i \ < -\deg R\}$ have the same measure. Hence we have

$$m^s(F(R)\cap S(U;R)) = \frac{m^s(F(R))}{q^{s\deg R}},$$

which implies

$$\frac{m^s(F(R)\cap S(U;R))}{m^s(S(U;R))} = m^s(F(R)).$$

Suppose $m^s(E) > 0$, since $E = F(R)\cup E_0(1,R)\cup E_1(1,R)$, we see that $m^s(F(R)) > 0$ for any irreducible R. By the density theorem, we have $m^s(E) = m^s(F(R)) = 1$ where R is so chosen that $\deg R$ is sufficiently large. Otherwise, $m^s(E) = 0$. $\square$

2. Proof of Theorem 2

In the sequel, we always assume that Q, Q_1, Q' and Q'_1 are monic. Recall

$$E_Q = \left\{ (f_1,\dots,f_s)\in \mathbb{L}^s \;\middle|\; \begin{array}{l} \left|f_i - \frac{P_i}{Q}\right| < \frac{\psi(Q)}{|Q|}\ ,\ \text{for some } P_i \quad \text{s.t.} \\ \deg\ P_i < \deg\ Q, (P_i,Q)=1\,, 1\leq i\leq s \end{array} \right\}.$$

From (6), if $\frac{\psi(Q)}{|Q|}$ is sufficiently small, then we have

$$m^s(E_Q) = \psi^s(Q)\,\frac{\Phi^s(Q)}{|Q|^s}. \tag{9}$$

Now consider the measure of $E_Q\cap E_{Q'}$ $(\deg Q' \leq \deg Q)$.We let $N(Q,Q')$ be the number of pairs of polynomials P and P' which satisfies

$$\left|\frac{P}{Q}-\frac{P'}{Q'}\right| < \frac{\psi(Q)}{|Q|}+\frac{\psi(Q')}{|Q'|}\ , \tag{10}$$

$$(P,Q) = (P',Q') = 1,\ \deg P < \deg Q,\ \deg P' < \deg Q'$$

for given Q and Q'. Then we show that the measure is bounded as follows,

$$m^s(E_Q\cap E_{Q'}) \leq \left\{\min\left(\frac{\psi(Q)}{|Q|},\frac{\psi(Q')}{|Q'|}\right)\cdot N(Q,Q')\right\}^s.$$

Suppose

$$PQ' - P'Q = R \tag{11}$$

holds for some polynomial R and $D = (Q, Q')$. If D divides R, we may write

$$Q = DQ^* \,,\; Q' = DQ'^* \,,\; R = DR^*,$$

and have

$$PQ'^* - P'Q^* = R^* \qquad (Q^*, Q'^*) = 1. \tag{12}$$

If P^* and P'^* also satisfy (12), then

$$P^*Q'^* - P'^*Q^* = R^*. \tag{13}$$

From (12),(13), we get

$$P = P^* + KQ^*, \qquad K \,:\, \text{a polynomial.} \tag{14}$$

From (14), we see

$$|P - P^*| = |K||Q^*| < |Q| = |D||Q^*|,$$

which implies that

$$|K| < |D|$$

must hold. Thus the possible number of polynomials P satisfying (11) for a given R is no more than $q^{\deg D}$. Since (10) implies

$$0 \neq |R| < |Q'|\psi(Q) + |Q|\psi(Q')$$

and R is divisible by D, we find that

$$N(Q, Q') \leq \frac{|Q'|\psi(Q) + |Q|\psi(Q')}{|D|} \cdot |D| = |Q'|\psi(Q) + |Q|\psi(Q').$$

Then

$$m^s(E_Q \cap E_{Q'}) \leq \left[\min\left(\frac{\psi(Q)}{|Q|}, \frac{\psi(Q')}{|Q'|}\right) \cdot \{|Q'|\psi(Q) + |Q|\psi(Q')\}\right]^s$$
$$= 2^s \psi^s(Q)\psi^s(Q').$$

Because we assume $\sum_{\deg Q \leq n} \psi^s(Q) = \infty$,

$$\sum_{\deg Q \leq n} \psi^s(Q) \leq \left(\sum_{\deg Q \leq n} \psi^s(Q)\right)^2$$

holds for sufficiently large n. Therefore we have

$$\sum_{\deg Q, \deg Q' \leq n} m^s(E_Q \cap E_{Q'}) \leq 2^s \sum_{\substack{\deg Q' \leq \deg Q \leq n \\ Q \neq Q'}} \psi^s(Q)\psi^s(Q') + \sum_{\deg Q \leq n} \psi^s(Q)$$
$$< 2^s \Big(\sum_{\deg Q \leq n} \psi^s(Q)\Big)^2$$

for sufficiently large $\deg Q$. From (4) and (9), we have

$$\sum_{\deg Q, \deg Q' \leq n} m^s(E_Q \cap E_{Q'}) < 2^s C^2 \left(\sum_{\deg Q \leq n} m^s(E_Q) \right)^2 \tag{15}$$

for infinitely many n. Then $m^s(E) > (2^s C^2)^{-1}$, by (15) and Lemma 5 of [8](pp. 17–18). Finally using to Theorem 1, we complete the proof of this theorem. □

Example. Put

$$\psi(Q) = \begin{cases} \frac{1}{|Q|^{\frac{1}{s}}} & \text{if Q is irreducible,} \\ 0 & \text{otherwise.} \end{cases}$$

Then, from Theorem 2.2 of [7], we have

$$\sum_{\deg Q \leq n}^{\infty} \psi^s(Q) \gg \sum_{k=1}^{\infty} \left(\frac{1}{q^{\frac{k}{s}}} \right)^s \cdot \frac{1}{k} \cdot q^k = \infty$$

and it is easy to see that

$$\sum_{\deg Q \leq n} \psi^s(Q) \leq C \sum_{\deg Q \leq n} \psi^s(Q) \frac{\Phi^s(Q)}{|Q|^s}$$

holds. Thus we see that there are infinitely many solutions $\left(\frac{P_1}{Q}, \ldots, \frac{P_s}{Q}\right)$ with irreducible Q's of

$$\left| f_i - \frac{P_i}{Q} \right| < \frac{1}{|Q|^{\frac{s+1}{s}}}, \quad \text{for } 1 \leq i \leq s$$

for a.e. $(f_1, \ldots, f_s) \in \mathbb{L}^s$.

Remark. It is natural to ask whether we can get a necessary and sufficient condition instead of (4) in Theorem 2. In this sense, we give the s-dimensional Duffin-Schaeffer type conjecture in the following.

Conjecture. (3) *has infinitely many solutions* $(\frac{P_1}{Q}, \ldots \frac{P_s}{Q})$ *for* m^s*-a.e.* $(f_1, \cdots, f_s) \in \mathbb{L}^s$ *if and only if*

$$\sum_{\substack{\deg Q = 1 \\ Q:\text{monic}}}^{\infty} \psi^s(Q) \frac{\Phi^s(Q)}{|Q|^s}$$

diverges.

In the classical case, the s-dimensional Duffin-Schaeffer conjecture was proved Pollington and Vaughan [6] for $s \geq 2$. We may also prove this conjecture for the s-dimensional formal power series, $s \geq 2$, if we estimate the lower bound of $\Phi(Q)$. We will discuss it in another occasion.

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