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On mean values of some zeta-functions in the critical strip

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Dedicated to the memory of Robert Rankin

Résumé. Un entier $k \geq 3$ et un réel σ tel que $\frac{1}{2} < \sigma < 1$ étant fixés, on considère dans la formule asymptotique

$$\int_1^T |\zeta(\sigma + it)|^{2k}\,\mathrm{d}t = \sum_{n=1}^{\infty} d_k^2(n) n^{-2\sigma} T + R(k,\sigma;T),$$

le terme erreur $R(k,\sigma;T)$, pour lequel nous montrons de nouvelles bornes lorsque $\min(\beta_k, \sigma_k^*) < \sigma < 1$. Nous obtenons également des majorations nouvelles pour les termes erreur dans le développement des moments d'ordre deux des fonctions zeta de formes paraboliques holomorphes et des séries de Rankin-Selberg.

Abstract. For a fixed integer $k \geq 3$, and fixed $\frac{1}{2} < \sigma < 1$ we consider

$$\int_1^T |\zeta(\sigma + it)|^{2k}\,\mathrm{d}t = \sum_{n=1}^{\infty} d_k^2(n) n^{-2\sigma} T + R(k,\sigma;T),$$

where $R(k,\sigma;T) = o(T)$ $(T \to \infty)$ is the error term in the above asymptotic formula. Hitherto the sharpest bounds for $R(k,\sigma;T)$ are derived in the range $\min(\beta_k, \sigma_k^*) < \sigma < 1$. We also obtain new mean value results for the zeta-function of holomorphic cusp forms and the Rankin-Selberg series.

1. Introduction

The aim of this paper is to provide asymptotic formulas for the $2k$–th moment of the Riemann zeta-function $\zeta(s)$ and some related Dirichlet series in the so-called "critical strip" $\frac{1}{2} < \sigma = \Re\mathrm{e}\, s < 1$. For the zeta-function our results are relevant when $k \geq 3$ is a fixed integer, where henceforth $s = \sigma + it$ will denote a complex variable. Mean values of $\zeta(s)$ on the "critical line" $\sigma = \frac{1}{2}$ behave differently (see e.g., [4]), while the problem

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of mean values for $0 < \sigma < \frac{1}{2}$ can be reduced to the range $\frac{1}{2} < \sigma < 1$ by means of the functional equation for $\zeta(s)$, namely

$$\zeta(s) = \chi(s)\zeta(1-s), \quad \chi(s) = 2^s\pi^{s-1}\sin(\tfrac{1}{2}\pi s)\Gamma(1-s).$$

Mean values of $\zeta(s)$ for $\frac{1}{2} < \sigma < 1$ in the cases $k = 1$ and $k = 2$ have been extensively studied, and represent one of the central themes in zeta-function theory. One has (see [10, Theorem 2])

$$\begin{aligned}(1.1)\quad \int_1^T |\zeta(\sigma+it)|^2\,\mathrm{d}t = \zeta(2\sigma)T + \frac{\zeta(2\sigma-1)\Gamma(2\sigma-1)}{1-\sigma}\sin(\pi\sigma)T^{2-2\sigma} \\ + O(T^{2(1-\sigma)/3}\log^{2/9}T) \qquad (\tfrac{1}{2} < \sigma \le 1),\end{aligned}$$

and (see [7, Theorem 2])

$$(1.2)\quad \int_1^T |\zeta(\sigma+it)|^4\,\mathrm{d}t = \frac{\zeta^2(2\sigma)}{\zeta(4\sigma)}T + O(T^{2-2\sigma}\log^3 T) \qquad (\tfrac{1}{2} < \sigma \le 1),$$

which are the sharpest hitherto published asymptotic formulas valid in the whole range $\frac{1}{2} < \sigma \le 1$. These results have been obtained by special methods, and cannot be generalized to higher moments. The formula for the general $2k$–th moment of $\zeta(s)$ can be conveniently written (cf. [4, Chapter 8]) as

$$(1.3)\quad \int_1^T |\zeta(\sigma+it)|^{2k}\,\mathrm{d}t = \sum_{n=1}^\infty d_k^2(n)n^{-2\sigma}T + R(k,\sigma;T),\ R(k,\sigma;T) = o(T),$$

where $\frac{1}{2} < \sigma_0(k) \le \sigma \le 1, T \to \infty$, and the arithmetic function $d_k(n)$ denotes, as usual, the number of ways n may be written as a product of k factors (so that $d_k(n)$ is generated by $\zeta^k(s)$, and $d_2(n) = d(n)$ is the number of divisors of n). In [4, Chapter 8] it was proved that

$$(1.4)\qquad R(k,\sigma;T) \ll_\varepsilon T^{\frac{2-\sigma-\sigma_k^*}{2-2\sigma_k^*}+\varepsilon} \qquad (\sigma_k^* < \sigma < 1),$$

where henceforth ε denotes arbitrarily small constants, not necessarily the same ones at each occurrence, and σ_k^* is the infimum of σ^* $(\ge \frac{1}{2})$ for which

$$\int_1^T |\zeta(\sigma^*+it)|^{2k}\,\mathrm{d}t \ll_\varepsilon T^{1+\varepsilon}$$

holds for any given ε. Writing further the bounds for $R(k,\sigma;T)$ as

$$R(k,\sigma;T) \ll_\varepsilon T^{c_k(\sigma)+\varepsilon}$$

and using the known bounds for σ_k^* when $3 \le k \le 6$, it follows from (1.4) that we have

$$
\begin{aligned}
&c_3(\sigma) = \frac{17-12\sigma}{10} \quad (\tfrac{7}{12} < \sigma < 1), \quad c_4(\sigma) = \frac{11-8\sigma}{6} \quad (\tfrac{5}{8} < \sigma < 1),\\
&c_5(\sigma) = \frac{79-60\sigma}{38} \quad (\tfrac{41}{60} < \sigma < 1), \quad c_6(\sigma) = \frac{9-7\sigma}{4} \quad (\tfrac{5}{7} < \sigma < 1).
\end{aligned} \tag{1.5}
$$

As indicated in [4], explicit values for $c_k(\sigma)$ could be given for any fixed $k > 1$, but the expressions in general would be cumbersome, so only explicit values were given for $2 \le k \le 6$. The point of (1.3)-(1.5) lies in the fact that each value of $c_k(\sigma)$ satisfies $c_k(\sigma) < 1$ (i.e., when (1.3) becomes a true asymptotic formula), precisely for the range given in (1.5). However, as σ approaches 1, the values of $c_k(\sigma)$ become rather poor and they do not tend to zero, as one expects.

The problem of mean values of a Dirichlet series $F(s)$ (in this context $2k$–th moments of $F(s)$ can be regarded simply as the mean square of $F^k(s)$ $(k \in \mathbb{N})$) can be treated in various degrees of generality. Here we shall mention only the classes of Dirichlet series treated by Chandrasekharan–Narasimhan (see [2], [3]), Perelli [14], Richert [17] and Selberg [18]. Recently S. Kanemitsu et al. obtained in [12] a mean value theorem for a general class of Dirichlet series possessing a functional equation with multiple gamma-factors. The merit of their result, which is in part based on ideas of Matsumoto [13], is a relatively good value of the exponent in the error term as σ approaches the abscissa of absolute convergence of the Dirichlet series in question. In particular, the result of [12] can be applied to higher power moments of $\zeta(s)$. In this case in the notation of [12] one has

$$\alpha = 0,\ \mu = \nu = 1,\ \alpha_1 = 0,\ \gamma_1 = \tfrac12,\ \beta_1 = \tfrac12,\ H = 1,\ \eta = \tfrac12.$$

Their Theorem 4 gives then, in the notation of (1.3),

$$R(k,\sigma;T) \ll_\varepsilon T^{\frac{3k(1-\sigma)}{k+2-k\sigma}+\varepsilon} \tag{1.6}$$

for $k \ge 2$ and

$$1 - \frac{1}{k} + \varepsilon \le \sigma \le 1. \tag{1.7}$$

When (1.6) is compared with (1.3)–(1.4) it transpires that it holds for a poorer range, but the exponent in the error term is much sharper as σ grows, and it tends to 0 as $\sigma \to 1-0$, as one expects.

In what follows we may assume $\sigma < 1$, since we have the asymptotic formula

$$\int_1^T |\zeta(1+it)|^{2k}\,\mathrm{d}t = \sum_{n=1}^\infty |d_k(n)|^2 n^{-2} T + O((\log T)^{|k|^2}), \tag{1.8}$$

which was proved in [1]. In (1.8) one can take $k \in \mathbb{C}$ arbitrary, but fixed. Thus (1.8), obtained by a special method that cannot be adapted to the range $\sigma < 1$, yields a better error term than the one obtainable from any of the previous bounds (1.1)–(1.7).

The plan of the paper is as follows. In Section 2 we shall formulate the results (Theorem 1 and Theorem 2) concerning the higher moments of $\zeta(s)$, the proofs of which will be given in Section 3. In Section 4 we shall deal with the mean value of the Rankin-Selberg series, and in Section 5 with the mean values of the zeta-function of holomorphic modular forms and its square.

2. Higher moments of the zeta-function

The aim of this section is to furnish new bounds for $R(k,\sigma;T)$, which will improve both (1.4) and (1.6). We shall formulate now our results, with the remark that Theorem 2 is based on the use of the defining property of σ_k^* and it gives good bound for $R(k,\sigma;T)$ when σ is close to σ_k^*. Theorem 1, on the other hand, is derived by using the values of the constant β_k in the mean square estimates for the divisor problem. Namely we let, as usual,

$$\beta_k = \inf\left\{ b_k\ (\geq 0)\ :\ \int_1^x \Delta_k^2(y)\,\mathrm{d}y \ \ll\ x^{1+2b_k}\right\}, \tag{2.1}$$

where $\Delta_k(x)$ is the error term in the asymptotic formula for the summatory function of $d_k(n)$ (cf. (3.1)). Theorem 1 will provide good results for values of σ close to 1. Results of similar type for the general case and the case of the Rankin-Selberg series can be found in [12] and [13]. However in the proof of Theorem 1 we shall avoid using the Cauchy-Schwarz inequality and therefore obtain a sharper value of the exponent than we would obtain by following the ideas of [12] and [13].

Theorem 1. *For fixed σ satisfying $\max(\beta_k, \frac12) < \sigma < 1$ and every fixed integer $k \geq 3$, we have*

$$R(k,\sigma;T) \ \ll_\varepsilon\ T^{\frac{2(1-\sigma)}{1-\beta_k}+\varepsilon}. \tag{2.2}$$

Theorem 2. *For fixed σ satisfying $\sigma_k^* < \sigma < 1$ and every fixed integer $k \geq 3$, we have*

$$R(k,\sigma;T) \ \ll_\varepsilon\ T^{\frac{2(1-\sigma)}{2-\sigma_k^*-\sigma}+\varepsilon}. \tag{2.3}$$

Remark 1. Note that (2.2) improves (1.6). Namely we have

$$\frac{2}{1-\beta_k} \ \leq\ \frac{3k}{k+2-k\sigma}$$

for

$$2k\sigma \ \geq\ 4 - k + 3k\beta_k.$$

But from (1.7) it follows that

$$2k\sigma > 2k - 2 \geq 4 - k + 2k\beta_k$$

for

$$\beta_k \leq 1 - \frac{2}{k} \qquad (k \geq 3). \tag{2.4}$$

Equality in (2.4) holds only for $k = 3$, since $\beta_3 = \frac{1}{3}$ (see [4]). But we have $\beta_4 = \frac{3}{8}$ and $\beta_k \leq (k-1)/(k+2)$ for $k \geq 4$ (see [17]), hence in (2.4) we have strict inequality for $k > 3$. This means that (2.2) improves both the exponent of the error term in (1.6), and at the same time it holds in a wider interval than the one given by (1.7).

We also note that

$$\frac{2 - 2\sigma}{2 - \sigma_k^* - \sigma} \leq \frac{2 - \sigma - \sigma_k^*}{2 - 2\sigma_k^*}$$

is equivalent to

$$2\sigma\sigma_k^* \leq \sigma^2 + (\sigma_k^*)^2,$$

which is obvious. This means that (2.3) of Theorem 2 improves (1.4) in the whole range $\sigma_k^* < \sigma < 1$.

3. Proof of Theorem 1 and Theorem 2

We write as usual, for $k \in \mathbb{N}$,

$$D_k(x) := \sum_{n \leq x} d_k(n) = xP_{k-1}(\log x) + \Delta_k(x), \tag{3.1}$$

where $P_{k-1}(y)$ is a polynomial of degree $k-1$, whose coefficients (which depend on k) may be explicitly evaluated. Using the Stieltjes integral representation and (3.1) we have, for $1 \ll X \ll T^C$ $(C > 0)$, $\sigma > 1$ and $k \geq 2$ a fixed integer,

$$\begin{aligned} \zeta^k(s) &= \sum_{n \leq X} d_k(n)n^{-s} + \int_X^\infty x^{-s}\,\mathrm{d}D_k(x) \\ &= \sum_{n \leq X} d_k(n)n^{-s} + \int_X^\infty x^{-s}(Q_{k-1}(\log x)\,\mathrm{d}x + \mathrm{d}\Delta_k(x)), \end{aligned} \tag{3.2}$$

where $Q_{k-1} = P_{k-1} + P'_{k-1}$. From the definition (2.1) of β_k it follows that, for any given $Y \gg 1$, there exists $X \in [Y, 2Y]$ such that

$$\Delta_k(X) \ll_\varepsilon X^{\beta_k + \varepsilon}. \tag{3.3}$$

Henceforth we assume that X is chosen in such a way that it satisfies, besides $1 \ll X \ll T^C$, also the bound in (3.3). Repeated integration by parts yields

$$\begin{aligned}(3.4)\quad \int_X^\infty x^{-s}Q_{k-1}(\log x)\,\mathrm{d}x &= \frac{X^{1-s}}{s-1}Q_{k-1}(\log X)\\ &\qquad + \frac{1}{s-1}\int_X^\infty x^{-s}Q'_{k-1}(\log x)\,\mathrm{d}x\\ = \ldots = X^{1-s}&\left(\frac{Q_{k-1}(\log X)}{s-1} + \frac{Q'_{k-1}(\log X)}{(s-1)^2} + \ldots + \frac{Q^{(k-1)}_{k-1}(\log X)}{(s-1)^k}\right),\end{aligned}$$

which provides analytic continuation of the left-hand side of (3.4) to $\mathbb{C}$. We also have

$$(3.5)\qquad \int_X^\infty x^{-s}\,\mathrm{d}\Delta_k(x) = -X^{-s}\Delta_k(X) + s\int_X^\infty x^{-s-1}\Delta_k(x)\,\mathrm{d}x.$$

Note that the last integral converges absolutely for $\sigma > \beta_k$, in view of the Cauchy-Schwarz inequality for integrals and the definition (2.1) of β_k. Therefore from (3.1)–(3.5) we obtain, for $\max(\frac12,\beta_k) < \sigma \le 1$ and $T \le t \le 2T$,

$$\begin{aligned}(3.6)\quad \zeta^k(s) = \sum_{n\le X} d_k(n)n^{-s} &+ s\int_X^\infty x^{-s-1}\Delta_k(x)\,\mathrm{d}x\\ &+ O_\varepsilon\left(X^{\beta_k-\sigma+\varepsilon} + T^{-1}X^{1-\sigma}\log^{k-1}X\right).\end{aligned}$$

Observe now that (2.2) follows from

$$(3.7)\qquad \int_T^{2T}|\zeta(\sigma+it)|^{2k}\,\mathrm{d}t = \sum_{n=1}^\infty d_k^2(n)n^{-2\sigma}T + O_\varepsilon\left(T^{\frac{2(1-\sigma)}{1-\beta_k}+\varepsilon}\right)$$

on replacing T by $T2^{-j}$ ($j \in \mathbb{N}$) and summing all the results. To evaluate the integral in (3.7), we suppose that $\max(\frac12,\beta_k) < \sigma < 1$, we use (3.6) and

$$\begin{aligned}|a+b|^2 &= |a|^2 + |b|^2 + 2\Re\mathrm{e}\, a\bar b,\\ a &:= \sum_{n\le\frac12 X} d_k(n)n^{-s},\\ b &:= \sum_{\frac12 X<m\le X} d_k(m)m^{-s} + s\int_X^\infty x^{-s-1}\Delta_k(x)\,\mathrm{d}x\\ &\qquad + O_\varepsilon\left(X^{\beta_k-\sigma+\varepsilon} + T^{-1}X^{1-\sigma}\log^{k-1}X\right).\end{aligned}$$

The reason of this splitting of the sum in two sums is to have m and n differ by unity at least, which is expedient to have in the integration that will follow. Now note that we have, by the mean value theorem for Dirichlet polynomials (see [4, Chapter 4]) and $d_k(n) \ll n^{\varepsilon}$,

$$\begin{aligned}(3.8)\qquad \int_T^{2T} |a|^2\,\mathrm{d}t &= T\sum_{n\le\frac12 X} d_k^2(n)n^{-2\sigma} + O\Big(\sum_{n\le\frac12 X} d_k^2(n)n^{1-2\sigma}\Big)\\ &= T\sum_{n=1}^{\infty} d_k^2(n)n^{-2\sigma} + O_\varepsilon\left(TX^{1-2\sigma+\varepsilon} + X^{2-2\sigma+\varepsilon}\right).\end{aligned}$$

To evaluate the mean square of $|b|$ we may proceed directly by squaring out the modulus, or we may use Lemma 4 of [8], which says that

$$\int_{T_1}^{T_2}\Big|\int_\alpha^\beta g(x)x^{-s}\,\mathrm{d}x\Big|^2\mathrm{d}t \le 2\pi\int_\alpha^\beta g^2(x)x^{1-2\sigma}\,\mathrm{d}x \quad (s=\sigma+it\,, T_1<T_2,\ \alpha<\beta)$$

holds if $g(x)$ is a real-valued, integrable function on $[\alpha,\beta]$, a subinterval of $[2,\infty)$, which is not necessarily finite. We shall obtain

$$\begin{aligned}(3.9)\qquad \int_T^{2T}|b|^2\,\mathrm{d}t &\ll_\varepsilon T^2\int_T^{2T}\left|\int_X^\infty x^{-\sigma-it-1}\Delta_k(x)\,\mathrm{d}x\right|^2\mathrm{d}t\\ &\qquad + TX^{2\beta_k-2\sigma+\varepsilon} + TX^{1-2\sigma+\varepsilon} + X^{2-2\sigma+\varepsilon}\\ &\ll_\varepsilon T^2\int_X^\infty x^{-2\sigma-1}\Delta_k^2(x)\,\mathrm{d}x + TX^{2\beta_k-2\sigma+\varepsilon} + TX^{1-2\sigma+\varepsilon} + X^{2-2\sigma+\varepsilon}\\ &\ll_\varepsilon T^2X^{2\beta_k-2\sigma+\varepsilon} + TX^{1-2\sigma+\varepsilon} + X^{2-2\sigma+\varepsilon}.\end{aligned}$$

We have

$$\begin{aligned}(3.10)\qquad \int_T^{2T} a\bar b\,\mathrm{d}t &= \sum_{n\le\frac12 X} d_k(n)n^{-\sigma}\int_T^{2T} n^{-it}\Big\{\sum_{\frac12 X<m\le X} d_k(m)m^{-\sigma+it}\\ &\quad + (\sigma-it)\int_X^\infty x^{-\sigma-1+it}\Delta_k(x)\,\mathrm{d}x + O_\varepsilon\left(X^{\beta_k-\sigma+\varepsilon} + T^{-1}X^{1-\sigma+\varepsilon}\right)\Big\}\,\mathrm{d}t.\end{aligned}$$

By direct integration it is found that

$$\begin{aligned}(3.11)\qquad &\sum_{n\le\frac12 X} d_k(n)n^{-\sigma}\int_T^{2T} n^{-it}\sum_{\frac12 X<m\le X} d_k(m)m^{-\sigma+it}\,\mathrm{d}t\\ &\ll \sum_{n\le\frac12 X} d_k(n)n^{-\sigma}\sum_{\frac12 X<m\le X} d_k(m)m^{-\sigma}\left|\log\frac{m}{n}\right|^{-1}\\ &\ll_\varepsilon X^{\varepsilon-\sigma}\sum_{n\le\frac12 X} n^{-\sigma}\sum_{\frac12 X<m\le X}\left(\frac{X}{m-n}+1\right) \ll_\varepsilon X^{2-2\sigma+\varepsilon},\end{aligned}$$

on using the elementary inequality

$$\frac{1}{\log(1+x)} \;\le\; 1+\frac{1}{x} \qquad (x>0).$$

Similarly, by using the first derivative test (see [4, Lemma 2.1]), we obtain

$$\begin{aligned}(3.12)\qquad & \sum_{n\le\frac12 X} d_k(n)n^{-\sigma}\int_T^{2T} n^{-it}(\sigma-it)\int_X^\infty x^{-\sigma-1+it}\Delta_k(x)\,\mathrm{d}x\,\mathrm{d}t\\ &\ll T\sum_{n\le\frac12 X} d_k(n)n^{-\sigma}\int_X^\infty x^{-\sigma-1}|\Delta_k(x)|\left|\log\frac{x}{n}\right|^{-1}\,\mathrm{d}x\\ &\ll_\varepsilon TX^{1+\beta_k-2\sigma+\varepsilon},\end{aligned}$$

where the interchange of the order of integration is justified by absolute convergence. Therefore from (3.10)-(3.12) it follows that

$$(3.13)\qquad \int_T^{2T} a\bar{b}\,\mathrm{d}t \ll_\varepsilon T^\varepsilon\left(TX^{1+\beta_k-2\sigma+\varepsilon}+X^{2-2\sigma}\right),$$

so that finally from (3.8)–(3.10) and (3.13) we obtain

$$\begin{aligned}(3.14)\qquad \int_T^{2T}|\zeta(\sigma+it)|^{2k}\,\mathrm{d}t &= \sum_{n=1}^\infty d_k^2(n)n^{-2\sigma}T\\ &+O_\varepsilon\left\{\left(T^\varepsilon(TX^{1+\beta_k-2\sigma}+X^{2-2\sigma}+TX^{1-2\sigma}+T^2X^{2\beta_k-2\sigma}\right)\right\}.\end{aligned}$$

Now in (3.14) we set $X^{2-2\sigma}\asymp TX^{1+\beta_k-2\sigma}$, namely

$$(3.15)\qquad X \;=\; cT^{\frac{1}{1-\beta_k}},$$

where the constant $c>0$ is chosen in such a way that (3.3) is satisfied. With the choice (3.15) it is seen that (3.14) becomes (3.7), and the proof of Theorem 1 is completed. □

Corollary 1.

$$\int_1^T|\zeta(\sigma+it)|^6\,\mathrm{d}t = T\sum_{n=1}^\infty d_3^2(n)n^{-2\sigma}+O_\varepsilon\left(T^{3(1-\sigma)+\varepsilon}\right)\quad(\tfrac12<\sigma<1),$$

$$\int_1^T|\zeta(\sigma+it)|^8\,\mathrm{d}t = T\sum_{n=1}^\infty d_4^2(n)n^{-2\sigma}+O_\varepsilon\left(T^{\frac{16}{5}(1-\sigma)+\varepsilon}\right)\quad(\tfrac12<\sigma<1),$$

$$\int_1^T|\zeta(\sigma+it)|^{10}\,\mathrm{d}t = T\sum_{n=1}^\infty d_5^2(n)n^{-2\sigma}+O_\varepsilon\left(T^{\frac{40}{11}(1-\sigma)+\varepsilon}\right)\quad(\tfrac12<\sigma<1),$$

$$\int_1^T|\zeta(\sigma+it)|^{12}\,\mathrm{d}t = T\sum_{n=1}^\infty d_6^2(n)n^{-2\sigma}+O_\varepsilon\left(T^{4(1-\sigma)+\varepsilon}\right)\quad(\tfrac12<\sigma<1).$$

The formulas follow from Theorem 1 with the values $\beta_3 = \frac{1}{3}$, $\beta_4 = \frac{3}{8}$, $\beta_6 \le \frac{1}{2}$ (see [4]) and $\beta_5 \le \frac{9}{20}$ (see [20]).

Remark 2. It transpires that the Lindelöf hypothesis ($\zeta(\sigma+it) \ll |t|^\varepsilon$ for $\sigma > \frac{1}{2}$) is equivalent to

$$\int_1^T |\zeta(\sigma+it)|^{2k}\,\mathrm{d}t = \sum_{n=1}^\infty d_k^2(n)n^{-2\sigma}T + O_\varepsilon\left(T^{\frac{4k(1-\sigma)}{k+1}+\varepsilon}\right) \quad (\tfrac{1}{2} < \sigma \le 1,\ k \ge 2). \tag{3.16}$$

Namely the Lindelöf hypothesis implies that $\beta_k = (k-1)/(2k)$ for $k \ge 2$ (see [4, Chapter 13]), in which case (3.16) follows from (2.2) and Theorem 1. Conversely, if (3.16) holds, then by [4, Lemma 7.1] we have, for $T^\varepsilon \le H \le \frac{1}{2}T$ and $\frac{1}{2} < \sigma \le 1$,

$$\begin{aligned}|\zeta(\sigma+iT)|^{2k} &\ll 1 + \log T\int_{T-H}^{T+H} |\zeta(\sigma - \frac{1}{\log T} + it)|^{2k}\,\mathrm{d}t\\ &\ll_\varepsilon H\log T + T^{\frac{4k(1-\sigma)}{k+1}+\varepsilon},\end{aligned}$$

which yields the Lindelöf hypothesis on taking $H = \frac{1}{2}T$ and letting $k \to \infty$.

Remark 3. Other explicit results can be obtained from Theorem 1 with the bounds for β_k furnished by [9], some of which are hitherto the sharpest ones. Our method of proof can be used to obtain a sharpening of the general result proved in [12], since we did not use the Cauchy-Schwarz inequality in estimating $\int_T^{2T} a\bar{b}\,\mathrm{d}t$, which was done in [12] and [13]. Namely we integrated directly the expressions in question, which led to a sharper estimate than the one that would have resulted from the application of the Cauchy-Schwarz inequality.

Proof of Theorem 2. From the well-known Mellin inversion integral

$$e^{-x} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \Gamma(w)x^{-w}\,\mathrm{d}w \qquad (c>0,\ x>0)$$

we obtain

$$\sum_{n=1}^\infty d_k(n)e^{-n/Y}n^{-s} = \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} Y^w\Gamma(w)\zeta^k(s+w)\,\mathrm{d}w \tag{3.17}$$

for $1 \ll Y \ll T^C\,(C>0)\,, T \le t \le 2T$, $\sigma_k^* < \sigma < 1$. We move the line of integration in (3.17) to $\Re\mathrm{e}\, w = \sigma_k^* - \sigma$. In doing this we encounter the pole $w = 1-s$ with residue $O(T^{-A})$ for any fixed $A > 0$ in view of Stirling's formula for the gamma-function. There is also the simple pole at $w = 0$ with residue $\zeta^k(s)$. Therefore from (3.17) it follows that

$$\int_T^{2T} |\zeta(\sigma+it)|^{2k}\,\mathrm{d}t = \int_T^{2T} |F|^2\,\mathrm{d}t + \int_T^{2T} |G|^2\,\mathrm{d}t + 2\Re\mathrm{e}\int_T^{2T} F\bar{G}\,\mathrm{d}t,$$

where

$$F := \sum_{n\le Y\log^2 Y} d_k(n)e^{-n/Y}n^{-s},$$

$$G := O\left(Y^{\sigma_k^*-\sigma}\int_{-\log^2 T}^{\log^2 T} |\zeta(\sigma_k^*+it+iv)|^k e^{-|v|}\,\mathrm{d}v + T^{-A}\right).$$

Consequently we have, by the mean value theorem for Dirichlet polynomials,

$$\int_T^{2T} |F|^2\,\mathrm{d}t = T\sum_{n\le Y\log^2 Y} d_k^2(n)e^{-2n/Y}n^{-2\sigma} + O_\varepsilon(Y^{2-2\sigma+\varepsilon})$$
$$= T\sum_{n\le Y\log^2 Y} d_k^2(n)n^{-2\sigma} + O_\varepsilon(TY^{-1}\sum_{n\le Y\log^2 Y} d_k^2(n)n^{1-2\sigma}) + O(Y^{2-2\sigma+\varepsilon})$$
$$= T\sum_{n=1}^\infty d_k^2(n)n^{-2\sigma} + O_\varepsilon(T^{1+\varepsilon}Y^{1-2\sigma} + Y^{2-2\sigma+\varepsilon}).$$

We also have, by the definition of σ_k^*,

$$\int_T^{2T} |G|^2\,\mathrm{d}t \ll Y^{2\sigma_k^*-2\sigma}\int_{-\log^2 T}^{\log^2 T}\left(\int_T^{2T} |\zeta(\sigma_k^*+it+iv)|^{2k}\,\mathrm{d}t\right)\mathrm{d}v + T^{1-2A}$$
$$\ll_\varepsilon T^{1+\varepsilon}Y^{2\sigma_k^*-2\sigma},$$

on taking A sufficiently large. Finally by using the Cauchy-Schwarz inequality we obtain

$$\int_T^{2T} F\bar{G}\,\mathrm{d}t \ll_\varepsilon T^\varepsilon(T+Y^{2-2\sigma})^{1/2}T^{1/2}Y^{\sigma_k^*-\sigma}$$
$$\ll_\varepsilon T^{1+\varepsilon}Y^{\sigma_k^*-\sigma} + T^{1/2+\varepsilon}Y^{1+\sigma_k^*-2\sigma}.$$

Putting together all the estimates, replacing T by $T2^{-j}$ $(j \ge 1)$ and summing over j we obtain

$$R(k,\sigma;T) \ll_\varepsilon T^\varepsilon\left(TY^{1-2\sigma} + Y^{2-2\sigma} + TY^{\sigma_k^*-\sigma} + T^{1/2}Y^{1+\sigma_k^*-2\sigma}\right).$$

Now we take

$$Y = T^{\frac{1}{2-\sigma_k^*-\sigma}}$$

to obtain

$$R(k,\sigma;T) \ll_\varepsilon T^\varepsilon \left(T^{\frac{2-2\sigma}{2-\sigma_k^*-\sigma}} + T^{\frac{4+\sigma_k^*-5\sigma}{2(2-\sigma_k^*-\sigma)}} \right).$$

On noting that the condition

$$\frac{4+\sigma_k^*-5\sigma}{2(2-\sigma_k^*-\sigma)} \le \frac{2-2\sigma}{2-\sigma_k^*-\sigma}$$

reduces to $\sigma_k^* \le \sigma$, which is certainly true, we obtain then (2.3).

Corollary 2.

$$\int_1^T |\zeta(\sigma+it)|^6 \,\mathrm{d}t = T\sum_{n=1}^\infty d_3^2(n)n^{-2\sigma} + O_\varepsilon\left(T^{\frac{24(1-\sigma)}{17-12\sigma}+\varepsilon}\right) \quad (\tfrac{7}{12} < \sigma < 1),$$

$$\int_1^T |\zeta(\sigma+it)|^8 \,\mathrm{d}t = T\sum_{n=1}^\infty d_4^2(n)n^{-2\sigma} + O_\varepsilon\left(T^{\frac{16(1-\sigma)}{11-8\sigma}+\varepsilon}\right) \quad (\tfrac{5}{8} < \sigma < 1),$$

$$\int_1^T |\zeta(\sigma+it)|^{10} \,\mathrm{d}t = T\sum_{n=1}^\infty d_5^2(n)n^{-2\sigma} + O_\varepsilon\left(T^{\frac{120(1-\sigma)}{79-60\sigma}+\varepsilon}\right) \quad (\tfrac{41}{60} < \sigma < 1),$$

$$\int_1^T |\zeta(\sigma+it)|^{12} \,\mathrm{d}t = T\sum_{n=1}^\infty d_6^2(n)n^{-2\sigma} + O_\varepsilon\left(T^{\frac{14(1-\sigma)}{9-7\sigma}+\varepsilon}\right) \quad (\tfrac{5}{7} < \sigma < 1).$$

The above formulas follow from Theorem 2 with the values (see [4, Chapter 8]) $\sigma_3^* \le \frac{7}{12}$, $\sigma_4^* \le \frac{5}{8}$, $\sigma_5^* \le \frac{41}{60}$, $\sigma_6^* \le \frac{5}{7}$. They improve (1.5) and complement those furnished by Corollary 1.

4. The mean value of the Rankin–Selberg series

The arguments used in the proof of Theorem 1 and Theorem 2 are of a general nature and can be adapted to obtain mean value results for a wide class of Dirichlet series. Instead of working out the details in the general case, which would entail various technicalities, we prefer to conclude by considering two specific examples. In this section we shall deal with the mean value of the so-called Rankin–Selberg series (see R.A. Rankin [15], [16])

$$Z(s) := \zeta(2s)\sum_{n=1}^\infty |a(n)|^2 n^{1-\kappa-s} = \sum_{n=1}^\infty c_n n^{-s} \quad (\sigma > 1),$$

and in Section 5 we shall consider the zeta-function attached to holomorphic cusp forms. Here as usual $a(n)$ denotes the n-th Fourier coefficient of a holomorphic cusp form $\varphi(z)$ of weight κ with respect to the full modular group $SL(2,\mathbb{Z})$. We also suppose that $\varphi(z)$ is a normalized eigenfunction for the Hecke operators $T(n)$, so that $a(1) = 1$ and $a(n) \in \mathbb{R}$. We have (see [5], [11] and [13]) $c_n \ll_\varepsilon n^\varepsilon$,

$$\sum_{n\le x} c_n^2 \ll_\varepsilon x(\log x)^{1+\varepsilon}, \quad \sum_{n\le x} c_n = Ax + \Delta(x,\varphi) \qquad (A>0)$$

with Rankin's classical estimate (see [15]) $\Delta(x,\varphi) \ll x^{3/5}$, and

$$\int_1^X \Delta^2(x,\varphi)\,\mathrm{d}x \ll_\varepsilon X^{2+\varepsilon}.$$

This means that analogously to (3.6) we have

$$Z(s) = \sum_{n\le X} c_n + s\int_X^\infty \Delta(x,\varphi)x^{-s-1}\,\mathrm{d}x + O_\varepsilon(T^{-1}X^{1-\sigma} + X^{\frac12-\sigma+\varepsilon}) \tag{4.1}$$

for $\frac12 < \sigma \le 1$, $T \le t \le 2T$, where $1 \ll X \ll T^C$ and X ($\in [Y, 2Y]$) satisfies (this is the analogue of (3.3))

$$\Delta(X,\varphi) \ll_\varepsilon X^{\frac12+\varepsilon}.$$

Then we write

$$Z(s) := D + E$$

with

$$D := \sum_{n\le\frac12 X} c_n n^{-s},$$

$$E := \sum_{\frac12 X<m\le X} c_m m^{-s} + s\int_X^\infty \Delta(x,\varphi)x^{-s-1}\,\mathrm{d}x + O_\varepsilon(T^{-1}X^{1-\sigma} + X^{\frac12-\sigma+\varepsilon}),$$

and consider

$$\int_T^{2T} |Z(s)|^2\,\mathrm{d}t = \int_T^{2T} |D|^2\,\mathrm{d}t + \int_T^{2T} |E|^2\,\mathrm{d}t + 2\Re\mathrm{e}\int_T^{2T} D\bar{E}\,\mathrm{d}t.$$

Similarly as in the proof of Theorem 1 we find that

$$\int_T^{2T} |D|^2\,\mathrm{d}t = T\sum_{n=1}^\infty c_n^2 n^{-2\sigma} + O_\varepsilon\left\{(TX^{1-2\sigma} + X^{2-2\sigma})\log^{1+\varepsilon} X\right\},$$

$$\int_T^{2T} |E|^2\,\mathrm{d}t \ll_\varepsilon X^{2-2\sigma+\varepsilon} + T^2X^{1-2\sigma+\varepsilon},$$

$$\int_T^{2T} D\bar{E}\,\mathrm{d}t \ll_\varepsilon X^{2-2\sigma+\varepsilon} + TX^{\frac32-\sigma+\varepsilon}.$$

With the choice $X = bT^2$, where $b > 0$ is a suitable constant, we obtain

$$\int_T^{2T} |Z(\sigma+it)|^2\,\mathrm{d}t = T\sum_{n=1}^\infty c_n^2 n^{-2\sigma} + O_\varepsilon(T^{4-4\sigma+\varepsilon}),$$

which easily gives then

Theorem 3. *For fixed σ satisfying $\frac{1}{2} < \sigma \le 1$ we have*

$$\int_1^T |Z(\sigma+it)|^2\,\mathrm{d}t = T\sum_{n=1}^{\infty} c_n^2 n^{-2\sigma} + O_\varepsilon(T^{4-4\sigma+\varepsilon}). \tag{4.2}$$

Remark 4. The asymptotic formula (4.2) improves, for $\frac{3}{4} < \sigma \le 1$, the result of K. Matsumoto [13] who proved

$$\int_1^T |Z(\sigma+it)|^2\,\mathrm{d}t = T\sum_{n=1}^{\infty} c_n^2 n^{-2\sigma} + R(\sigma,T)$$

with

$$R(\sigma,T) \ll_\varepsilon \begin{cases} T^{\frac{5}{2}-2\sigma+\varepsilon} & \left(\frac{3}{4} < \sigma < \frac{12+\sqrt{19}}{20} = 0.81666\ldots\right), \\ T^{\frac{60(1-\sigma)}{29-20\sigma}+\varepsilon} & \left(\frac{12+\sqrt{19}}{20} < \sigma < 1\right). \end{cases}$$

For $\frac{1}{2} < \sigma \le \frac{3}{4}$ our result is slightly weaker than the corresponding result of [13], namely

$$R(\sigma,T) \ll_\varepsilon T^{4-4\sigma}(\log T)^{1+\varepsilon},$$

but it should be remarked that (4.2) is a true asymptotic formula only in the range $\frac{3}{4} < \sigma \le 1$.

5. The mean value of the zeta-function of cusp forms

We retain the notation of Section 4 and consider (see [6]) the Dirichlet series

$$F(s) := \sum_{n=1}^{\infty} \tilde{a}(n) n^{-s} \qquad (\sigma > 1), \tag{5.1}$$

which may be continued analytically to an entire function over $\mathbb{C}$. In (5.1) the arithmetic function

$$\tilde{a}(n) := a(n) n^{\frac{1}{2}(1-\kappa)} \tag{5.2}$$

is the "normalized" function of cusp form coefficients. This function is "small", since it satisfies $\tilde{a}(n) \ll d(n)$ by Deligne's classical estimate. We shall also consider

$$F^2(s) = \sum_{n=1}^{\infty} \tilde{a}*\tilde{a}(n) n^{-s} \qquad (\sigma > 1), \tag{5.3}$$

where

$$\tilde{a}*\tilde{a}(n) := \sum_{d|n} \tilde{a}(d)\tilde{a}(\frac{n}{d})$$

is the convolution of $\tilde{a}(n)$ with itself. The mean values of $F(s)$ and $F^2(s)$ were considered in [6]. It was proved there that, for σ fixed,

$$\int_1^T |F(\sigma+it)|^2 \,\mathrm{d}t = T\sum_{n=1}^\infty |\tilde{a}(n)|^2 n^{-2\sigma} + H(\sigma;T) \tag{5.4}$$

and

$$\int_1^T |F(\sigma+it)|^4 \,\mathrm{d}t = T\sum_{n=1}^\infty |\tilde{a}*\tilde{a}(n)|^2 n^{-2\sigma} + K(\sigma;T) \tag{5.5}$$

with

$$H(\sigma;T) \ll_\varepsilon T^{\frac{3}{2}-\sigma+\varepsilon} \qquad (\tfrac{1}{2}<\sigma<1) \tag{5.6}$$

and

$$K(\sigma;T) \ll_\varepsilon T^{\frac{11-8\sigma}{6}+\varepsilon} \qquad (\tfrac{1}{2}<\sigma<1). \tag{5.7}$$

Note that from (5.5) and (5.7) it transpires that we can obtain a true asymptotic formula for the fourth moment of $F(\sigma+it)$ for $\frac{5}{8}<\sigma<1$. This reflects the fact that we have (see [6])

$$\int_1^T |F(\sigma+it)|^4 \,\mathrm{d}t \ll_\varepsilon T^{1+\varepsilon} \tag{5.8}$$

only for $\sigma \ge \frac{5}{8}$, and any improvement of the range for which (5.8) holds would result in the improvement of the bound for $K(\sigma;T)$. We shall improve on (5.6) and (5.7) by proving

Theorem 4. *If $H(\sigma;T)$ is defined by* (5.4), *then for σ fixed we have*

$$H(\sigma;T) \ll_\varepsilon \begin{cases} T^{\frac{4(1-\sigma)}{3-2\sigma}+\varepsilon} & (\frac{1}{2}<\sigma\le\frac{3}{4}),\\ T^{\frac{8}{3}(1-\sigma)+\varepsilon} & (\frac{3}{4}\le\sigma\le 1).\end{cases} \tag{5.9}$$

Theorem 5. *If $K(\sigma;T)$ is defined by* (5.5), *then for σ fixed we have*

$$K(\sigma;T) \ll_\varepsilon \begin{cases} T^{\frac{16(1-\sigma)}{12-8\sigma}+\varepsilon} & (\frac{1}{2}<\sigma\le\frac{3}{4}),\\ T^{\frac{16}{5}(1-\sigma)+\varepsilon} & (\frac{3}{4}\le\sigma\le 1).\end{cases} \tag{5.10}$$

Proof of Theorem 4 and Theorem 5. The first bounds in (5.9) and (5.10) are the analogues of (2.3) of Theorem 2 corresponding to the values $\sigma_1^* = \frac{1}{2}$ and $\sigma_2^* = \frac{5}{8}$, which follow from (5.4) and (5.8), respectively. The method of proof of Theorem 2 may be used, since

$$\tilde{a}(n) \ll d(n) \ll_\varepsilon n^\varepsilon, \qquad \tilde{a}*\tilde{a}(n) \ll \sum_{\delta|n} d(\delta)d(\frac{n}{\delta}) \ll_\varepsilon n^\varepsilon. \tag{5.11}$$

Similarly the second bounds in (5.9) and (5.10) are the analogues of (2.2) of Theorem 1 corresponding to the values $\beta_1 = \frac{1}{4}$ and $\beta_2 = \frac{3}{8}$, respectively. Namely if we define

$$\rho = \inf\left\{ c \geq 0 \ : \ \int_1^X \Big(\sum_{n\leq x} \tilde{a}(n)\Big)^2 \,\mathrm{d}x \ll X^{1+2c} \right\}$$

and

$$\theta = \inf\left\{ c \geq 0 \ : \ \int_1^X \Big(\sum_{n\leq x} \tilde{a}*\tilde{a}(n)\Big)^2 \,\mathrm{d}x \ll X^{1+2c} \right\},$$

then $\rho = \frac{1}{4}$ (this corresponds to $\beta_2 = \frac{1}{4}$ in the classical Dirichlet divisor problem) and $\theta \leq \frac{3}{8}$ (see [6]; this corresponds to $\beta_4 = \frac{3}{8}$ in the Dirichlet divisor problem for $\Delta_4(x)$). Thus proceeding as in the proof of Theorem 1 and keeping in mind again that (5.11) holds, we shall obtain the second bounds in (5.9) and (5.10). Clearly the bounds in (5.9) and (5.10) improve (5.6) and (5.7), respectively. Although the first bound in (5.10) holds for $\frac{1}{2} < \sigma \leq \frac{3}{4}$, it is relevant only in the range $\sigma > \frac{5}{8}$, when $16(1-\sigma)/(11-8\sigma) < 1$, when (5.5) becomes a true asymptotic formula.

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