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The Bloch-Kato conjecture on special values of L-functions. A survey of known results

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Résumé. Cet article présente un survol des cas connus de la conjecture de Bloch-Kato. Nous ne cherchons pas à passer en revue tous les cas connus de la conjecture de Beilinson, et nous laissons de côté la conjecture de Birch et Swinnerton-Dyer. L'article commence par une description de la conjecture générale. À la fin, nous indiquons brièvement les démonstrations des cas connus.

Abstract. This paper contains an overview of the known cases of the Bloch-Kato conjecture. It does not attempt to overview the known cases of the Beilinson conjecture and also excludes the Birch and Swinnerton-Dyer point. The paper starts with a brief review of the formulation of the general conjecture. The final part gives a brief sketch of the proofs in the known cases.

Introduction

This is an extended version of a talk at the "Journées arithmétiques" at Lille in July 2001. Our purpose is to survey the known results of the Bloch-Kato conjecture on special values of L-functions. We restrict ourselves here only to the cases where actually something is known about the L-value itself and not only up to rational numbers (Beilinson conjectures). For the Beilinson conjectures there are other surveys (see [34], although there is some progress since then [28]).

The known cases concern only certain number fields (or more generally Artin motives), certain elliptic curves with complex multiplication and adjoint motives of modular forms. Moreover all cases where the exact formula of the Birch-Swinnerton-Dyer conjecture is known give examples of the Bloch-Kato conjecture. Since we do not attempt an overview of the Birch-Swinnerton-Dyer conjecture and there are many special cases checked individually, we will say nothing about this.

The structure of the paper is as follows: In the first section we review briefly the formulation of the conjecture for Chow motives. As our K-theory

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knowledge is limited, we also formulate a weak version of the conjecture. The second section reviews the known cases and has naturally two parts: the first treats the case of Artin motives (mainly abelian) and the second treats elliptic curves with complex multiplication (mainly over an imaginary quadratic field). The third section explains finally the main ingredients of the proofs. Here two things are decisive: on the one hand techniques from Iwasawa theory and the main conjecture and on the other hand motivic polylogarithms and their realizations.

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1. Review of the Bloch-Kato conjecture

In this section we recall briefly the formulation of the Bloch-Kato conjecture, referring for more details to the articles of Fontaine [13], Fontaine and Perrin-Riou [14] and Kato [23]. We follow mainly the excellent survey [13]. To understand the connection with the Iwasawa main conjecture, [23] is indispensable.

1.1. Motives and their realizations. Fix two number fields K and E and consider the category of Chow motives $\mathcal{M}_K(E)$ over K with coefficients in E as defined in [21] § 4. To each object $M=(X,q,r)\in\mathcal{M}_K(E)$, where X/K is a smooth, projective variety, q an idempotent, and $r\in\mathbb{Z}$, are associated several realizations:

- $M_{DR}:=\bigoplus_{w\in\mathbb{Z}} q^*H^{w+2r}_{DR}(X/K)(r)\otimes_{\mathbb{Q}}E$ the *de Rham realization* equipped with its Hodge filtration $D^iM_{DR}:=\bigoplus_{w\in\mathbb{Z}} q^*\operatorname{Fil}^{i+r}H^{w+2r}_{DR}(X/K)\otimes_{\mathbb{Q}}E$.
- $M_B:=\bigoplus_{w\in\mathbb{Z}} q^*H^{w+2r}_{B}(X\times_K\mathbb{C},(2\pi i)^rE)$ the *Betti realization* each summand is equipped with a pure $E\otimes\mathbb{R}$-Hodge structure over $\mathbb{R}$ on $M_B\otimes_{\mathbb{Q}}\mathbb{R}$.
- $M_p:=\bigoplus_{w\in\mathbb{Z}} q^*H^{w+2r}_{\text{et}}(X\times_K\overline{K},\mathbb{Q}_p(r))\otimes_{\mathbb{Q}}E$ the *p-adic étale realization* with its $\operatorname{Gal}(\overline{K}/K)$ action, where p is a prime number.

Remark. Note that one expects that there is a decomposition

$$M\cong\bigoplus_{w\in\mathbb{Z}}h^{w+2r}(M)(r)$$

in $\mathcal{M}_K(E)$, such that the realization of $h^{w+2r}(M)(r)$ is $H^{w+2r}_{DR}(X/K)(r)$ etc. In particular it should be possible to restrict attention to *pure* motives $h^{w+2r}(M)(r)$. This is known for curves [32] and for abelian varieties [31]. As we do not know it in general, we have to use the above realizations.

For a motive $M=(X,q,r)$ we define its n-th Tate twist to be $M(n):=(X,q,r+n)$ and let $M^\vee:=(X,q^t,\dim X-r)$ be the dual of M if X has pure dimension. Here q^t is the transpose of the projector q. Denote by

$CH^r(X)^0 \otimes E$ the subgroup of the Chow group of X consisting of cycles homologically equivalent to 0. We define the following objects for $M = (X, q, r)$:

- $\tan_M := M_{DR}/D^0 M_{DR}$
- $H^0(K, M) := q^* CH^r(X) \otimes E / q^* CH^r(X)^0 \otimes E$
- $H^1(K, M) := q^* CH^r(X)^0 \otimes E \oplus \bigoplus_{w \in \mathbb{Z}, w \neq -1} q^*(K_{-w-1}(X) \otimes E)^{(r)}$ this is motivic cohomology

Recall that one has Chern class maps defined by Soulé

$$(K_{-w-1}(X) \otimes E)^{(r)} \to H^1_{\text{cont}}(K, H^{w+2r}_{\text{et}}(X \times_K \overline{K}, \mathbb{Q}_p(r)))$$

where $H^1_{\text{cont}}(K, \dots)$ denotes the continuous Galois cohomology. Together with the Abel Jacobi map $CH^r(X)^0 \to H^1_{\text{cont}}(K, H^{-1+2r}_{\text{et}}(X \times_K \overline{K}, \mathbb{Q}_p(r)))$ we get

$$r_p : H^1(K, M) \to H^1_{\text{cont}}(K, M_p)$$

for all p. Consider for all finite primes $\mathfrak{p}$ of K the group $H^1_f(K_\mathfrak{p}, M_p)$ as in [13] 3.2. This is a subgroup of $H^1(K_\mathfrak{p}, M_p)$ and we denote the quotient by $H^1_{/f}(K_\mathfrak{p}, M_p)$. Then let

$$H^1_f(K, M_p) := \operatorname{Ker}\left(H^1_{\text{cont}}(K, M_p) \to \bigoplus_{\mathfrak{p}} H^1_{/f}(K_\mathfrak{p}, M_p)\right)$$

and define

$$H^1_f(K, M) := \{x \in H^1(K, M) | r_p(x) \in H^1_f(K, M_p) \text{ for all } p\}.$$

Conjecture 1.1.1. *a) The E vector spaces $H^0(K, M)$ and $H^1_f(K, M)$ are finite dimensional*
b) The map r_p induces an isomorphism $H^1_f(K, M) \otimes_\mathbb{Q} \mathbb{Q}_p \cong H^1_f(K, M_p)$
c) $H^0(K, M) \otimes_\mathbb{Q} \mathbb{Q}_p \cong H^0(K, M_p)$ (Tate conjecture)

On the other hand there should be the following relation of $H^1_f(K, M)$ and $H^0(K, M)$ to the Betti and de Rham realization: Consider the comparison isomorphism

$$I_\infty : M_B \otimes_\mathbb{Q} \mathbb{C} \cong M_{DR} \otimes_\mathbb{Q} \mathbb{C}$$

this induces the *period map*

$$\alpha_M : M_B^+ \otimes_\mathbb{Q} \mathbb{R} \to \tan_M \otimes_\mathbb{Q} \mathbb{R},$$

where $M_B^+ := \bigoplus_{\mathfrak{p}|\infty} H^0(K_\mathfrak{p}, M_B)$ are the invariants under complex conjugation on M_B. One has the following conjecture (see [13] 6.10):

Conjecture 1.1.2. *There exists a long exact sequence*

$$0 \to H^0(K,M)\otimes_{\mathbb{Q}}\mathbb{R} \to \operatorname{Ker}\alpha_M \xrightarrow{r_\infty^\vee} H^1_f(K,M^\vee(1))^\vee \otimes_{\mathbb{Q}}\mathbb{R}$$
$$\to H^1_f(K,M)\otimes_{\mathbb{Q}}\mathbb{R} \xrightarrow{r_\infty} \operatorname{Coker}\alpha_M \to H^0(K,M^\vee(1))^\vee \otimes_{\mathbb{Q}}\mathbb{R} \to 0.$$

The maps are given by the Beilinson regulators r_∞ as in [13] *6.9.*

1.2. The conjecture of Bloch and Kato. Consider the motive $M = (X,q,r)$. Denote by $\det_E$ the highest exterior power of an E vector space and by $\det_E^{-1}$ its dual.

Definition 1.2.1. *Assume that the conjecture 1.1.1 a) is true and define the fundamental line following Fontaine and Perrin-Riou: Let*

$$L_f(M) := \det_E H^0(K,M)\otimes \det_E^{-1} H^1_f(K,M)$$
$$L_f(M^\vee(1)) := \det_E H^0(K,M^\vee(1))\otimes \det_E^{-1} H^1_f(K,M^\vee(1)).$$

The fundamental line *is then*

$$\Delta_f(M) := L_f(M)\otimes L_f(M^\vee(1))\otimes \det_E^{-1}(M_B^+)\otimes \det_E \tan_M,$$

where M_B^+ are the invariants under complex conjugation.

Now we want to investigate the special values of L-functions. Let us define an L-function for M: first consider a prime $\mathfrak{p}$ of K not lying over p. Put $D_{\mathfrak{p}}(M_p) := M_p^{I_{\mathfrak{p}}}$, where $I_{\mathfrak{p}}$ is the inertia subgroup at $\mathfrak{p}$. For $\mathfrak{p}|p$ we let

$$D_{\mathfrak{p}}(M_p) := (B_{\mathrm{cris},\mathfrak{p}}\otimes_{\mathbb{Q}_p} M_p)^{\mathrm{Gal}(\overline{K}_{\mathfrak{p}}/K_{\mathfrak{p}})}.$$

Define an Euler product over all finite places $\mathfrak{p}$ of K by

$$L(M,s) := \prod_{\mathfrak{p}} \det(1-\mathrm{Fr}_{\mathfrak{p}}\,\mathrm{N}\,\mathfrak{p}^{-s}|D_{\mathfrak{p}}(M_p))^{-1},$$

where $\mathrm{Fr}_{\mathfrak{p}}$ is the geometric Frobenius at $\mathfrak{p}$.

Definition 1.2.2. *We say that M has an L-function, if*

$$\det(1-\mathrm{Fr}_{\mathfrak{p}}\,\mathrm{N}\,\mathfrak{p}^{-s}|D_{\mathfrak{p}}(M_p)) \in E[\mathrm{N}\,\mathfrak{p}^{-s}]$$

for all $\mathfrak{p}$ and $L(M,s)$ converges for $\Re s \gg 0$.

Note that $L(M,s)$ is an $E\otimes_{\mathbb{Q}}\mathbb{C}$-valued function. The order of vanishing of $L(M,s)$ at $s=0$ is conjecturally given as follows:

Conjecture 1.2.3. *The L-function of M exists and it has a meromorphic continuation to a neighborhood of* 0*, moreover*

$$d_M := \operatorname{ord}_{s=0} L(M,s) = \dim H^1_f(K,M^\vee(1)) - \dim H^0(K,M^\vee(1)).$$

Let us define the leading coefficient of the Taylor series expansion of $L(M,s)$ at 0:

$$L(M,0)^* := \lim_{s\to 0} \frac{L(M,s)}{s^{d_M}} \in E \otimes_{\mathbb{Q}} \mathbb{C}.$$

Note that $L(M(n),0)^* = L(M,n)^*$, if $M(n)$ is the Tate twist of M. The Bloch-Kato conjecture determines this value as follows: Assume that conjecture 1.1.2 holds, then one can define an isomorphism recalled below

$$\iota_\infty : \Delta_f \otimes_{\mathbb{Q}} \mathbb{R} \cong \mathbb{R} \otimes_{\mathbb{Q}} E. \tag{1}$$

If conjecture 1.1.1 b) and c) holds, then one can define an isomorphism recalled below

$$\iota_p : \Delta_f \otimes_{\mathbb{Q}} \mathbb{Q}_p \cong \mathbb{Q}_p \otimes_{\mathbb{Q}} E. \tag{2}$$

Conjecture 1.2.4 (Bloch-Kato). *The assumptions are as above then:*
a) *There is an element $\delta_f(M) \in \Delta_f$ such that*

$$\iota_\infty(\delta_f(M) \otimes 1) L(M,0)^* = 1$$

(Beilinson conjecture).
b) *For all prime numbers p, the element*

$$\iota_p(\delta_f(M) \otimes 1)$$

is a unit in $\mathbb{Z}_p \otimes \mathcal{O}_E$.

Remark. There is also an equivariant version of the conjecture, which is very important if one wants to understand the relation to the Iwasawa main conjecture. This is due to Kato [23] and [24]. A generalization to non abelian coefficients can be found in [7].

Let us recall very briefly how one defines ι_∞ and ι_p: The isomorphism ι_∞ is defined by taking the determinants in the exact sequence in conjecture 1.1.2 and the determinants in

$$0 \to \operatorname{Ker} \alpha_M \to M_B^+ \otimes_{\mathbb{Q}} \mathbb{R} \to \tan_M \otimes_{\mathbb{Q}} \mathbb{R} \to \operatorname{Coker} \alpha_M \to 0.$$

The definition of ι_p is slightly more complicated. Fix a finite set of places of K containing the infinite places and the ones over p, such that M_p is an étale sheaf over $\mathcal{O}_S := \mathcal{O}_K[1/S]$. As in [13] 4.5. one defines a distinguished triangle

$$R\Gamma_c(\mathcal{O}_S, M_p) \to R\Gamma_f(\mathcal{O}_S, M_p) \to \Big(\bigoplus_{\mathfrak{p}\in S_{\mathrm{fin}}} R\Gamma_f(K_{\mathfrak{p}}, M_p) \oplus \bigoplus_{\mathfrak{p}\in S_\infty} R\Gamma(K_{\mathfrak{p}}, M_p)\Big).$$

One gets an isomorphism of rank 1 $E_p := \mathbb{Q}_p \otimes_{\mathbb{Q}} E$-modules

$$\begin{aligned}&\det\nolimits_{E_p} R\Gamma_f(\mathcal{O}_S, M_p) \cong \\ &\qquad \det\nolimits_{E_p} R\Gamma_c(\mathcal{O}_S, M_p) \otimes \det\nolimits_{E_p}^{-1}(\tan_M \otimes_{\mathbb{Q}} \mathbb{Q}_p) \otimes \det\nolimits_{E_p}(M_B^+ \otimes_{\mathbb{Q}} \mathbb{Q}_p).\end{aligned}$$

Here one uses $M_B^+ \otimes \mathbb{Q}_p \cong \bigoplus_{\mathfrak{p}|\infty} H^0(K_\mathfrak{p}, M_p)$ for primes $\mathfrak{p}$ dividing ∞ and an identification of $\det_{E_p} R\Gamma_f(K_\mathfrak{p}, M_p)$ with $\det_{E_p}^{-1}(\tan_M \otimes_\mathbb{Q} \mathbb{Q}_p)$ as follows: for $\mathfrak{p} \in S_{\text{fin}}$ and $\mathfrak{p} \nmid p$ one has $R\Gamma(K_\mathfrak{p}, M_p) \cong [M_p^{I_\mathfrak{p}} \xrightarrow{1-\mathrm{Fr}_\mathfrak{p}} M_P^{I_\mathfrak{p}}]$, where $I_\mathfrak{p}$ is the inertia group at $\mathfrak{p}$ and the complex is in degree 0 and 1. Similarly for $\mathfrak{p} \mid p$ one has $R\Gamma(K_\mathfrak{p}, M_p) \cong [D_{\text{cris}}(M_p) \xrightarrow{(1-\varphi,\pi)} D_{\text{cris}}(M_p) \oplus \tan_M]$ where $D_{\text{cris}}(M_p) := (B_{\text{cris},\mathfrak{p}} \otimes M_p)^{\mathrm{Gal}(\overline{K}_\mathfrak{p}/K_\mathfrak{p})}$, φ is the Frobenius on $D_{\text{cris}}(M_p)$ (normalized as in [13]) and π is the canonical projection of $D_{\text{cris}}(M_p)$ onto $\tan_M$.

Note that the identification of $\det_{E_p} R\Gamma_c(\mathcal{O}_S, M_p)$ with $\det_{E_p}^{-1}(\tan_M \otimes_\mathbb{Q} \mathbb{Q}_p)$ introduces Euler factors at the primes in S_{fin} (see [13] 4.4. and [19] the discussion before definition 1.2.3). A calculation, which can be found in [13] 4.5., shows that this gives an isomorphism

$$\det_E \Delta_f \otimes_\mathbb{Q} \mathbb{Q}_p \cong \det_{E_p} R\Gamma_c(\mathcal{O}_S, M_p).$$

Now take any $\mathrm{Gal}(\overline{K}/K)$ stable $\mathcal{O}_E \otimes_\mathbb{Z} \mathbb{Z}_p$ lattice T_p in M_p. Then the lattice $\det_{\mathcal{O}_E \otimes_\mathbb{Z} \mathbb{Z}_p} R\Gamma_c(\mathcal{O}_S, T_p)$ in $\det_{E_p} R\Gamma_c(\mathcal{O}_S, M_p)$ is independent of T_p. Take any generator of this rank 1 lattice and use this to define

$$\iota_p : \Delta_f \otimes_\mathbb{Q} \mathbb{Q}_p \cong \mathbb{Q}_p \otimes_\mathbb{Q} E,$$

i.e. mapping this generator to $1 \otimes 1$.

1.3. The case of pure motives. Let us discuss several simplifications which occur if we restrict to certain types of motives, in particular if we restrict the weight w of the motive M.

Definition 1.3.1. *Let $M = (X, q, r)$ be a motive. The integers w such that $q^* H_B^{w+2r}(X/K)(r) \neq 0$ are called the* weights *of M. The motive M is* pure of weight w *if w is the only integer such that $q^* H_B^{w+2r}(X/K)(r) \neq 0$.*

Remark. It is of course equivalent to define the weights with the de Rham or the étale realization.

Lemma 1.3.2.
a) *Assume that conjecture 1.1.1 c) is true, then $H^0(K, M) = 0$ if 0 is not a weight of M.*
b) *Assume that conjecture 1.1.1 b) is true, then $H_f^1(K, M) = 0$ if all weights w of M satisfy $w > -1$.*

Proof. Clear from conjecture 1.1.1 and the direct sum decomposition of the Soulé regulator r_p. ☐

Consider a motive $M = (X, q, r)$ pure of weight w. We will discuss now several special cases of the Bloch-Kato conjecture (cf. [13] § 7). For this write $V_\mathbb{R} := V \otimes_\mathbb{Q} \mathbb{R}$ for any $\mathbb{Q}$ vector space V.

1. ($w \le -3$): then $M^\vee(1)$ has weight $-w-2 \ge 1$ and the lemma implies that $H^0(K, M^\vee(1)) = 0 = H^1_f(K, M^\vee(1))$ and the order of vanishing of $L(M,s)$ at $s=0$ is $d_M = 0$. Also by assumption and the lemma $H^0(K,M) = 0$. In particular the exact sequence of conjecture 1.1.2 reduces to

$$r_\infty : H^1_f(K,M)_{\mathbb{R}} \cong \operatorname{Coker} \alpha_M,$$

where r_∞ is the Beilinson regulator. Note that $(\Delta_f)_{\mathbb{R}}$ reduces to

$$\det{}^{-1}_{E_{\mathbb{R}}} H^1_f(K,M)_{\mathbb{R}} \otimes \det{}^{-1}_{E_{\mathbb{R}}} \operatorname{Coker} \alpha_M.$$

2. ($w \ge 1$): here $d_M = \dim \ker \alpha_M$ and the exact sequence of conjecture 1.1.2 reduces to

$$r_\infty^\vee : \ker \alpha_M \cong H^1_f(K, M^\vee(1))^\vee_{\mathbb{R}}.$$

(This is the Beilinson conjecture after identifying $\operatorname{Coker} \alpha_M$ with Deligne cohomology.)

3. ($w = -2$): The dual motive $M^\vee(1)$ has weight 0 and consequently the lemma implies that $H^0(K,M) = 0$ and by definition of K-theory $H^1_f(K, M^\vee(1)) = 0$. In particular the order of vanishing of $L(M,s)$ at $s=0$ is $-\dim H^0(K, M^\vee(1))$. One gets

$$0 \to H^1_f(K,M)_{\mathbb{R}} \xrightarrow{r_\infty} \operatorname{Coker} \alpha_M \to H^0(K, M^\vee(1))^\vee \to 0 \tag{3}$$

and the fundamental line $(\Delta_f)_{\mathbb{R}}$ is accordingly:

$$\det{}_{E_{\mathbb{R}}} H^0(K, M^\vee(1)) \otimes \det{}^{-1}_{E_{\mathbb{R}}} H^1_f(K,M) \otimes \det{}_{E_{\mathbb{R}}} \operatorname{Coker} \alpha_M.$$

4. ($w = 0$): One gets

$$0 \to H^0(K,M)_{\mathbb{R}} \to \ker \alpha_M \xrightarrow{r_\infty^\vee} H^1_f(K, M^\vee(1))^\vee \to 0 \tag{4}$$

and the fundamental line $(\Delta_f)_{\mathbb{R}}$ is

$$\det{}_{E_{\mathbb{R}}} H^0(K,M) \otimes \det{}^{-1}_{E_{\mathbb{R}}} H^1_f(K, M^\vee(1)) \otimes \det{}^{-1}_{E_{\mathbb{R}}} \ker \alpha_M.$$

5. ($w = -1$): Here one conjectures that $\ker \alpha_M = 0 = \operatorname{Coker} \alpha_M$, i.e. that the map $H^1_f(K, M^\vee(1))^\vee_{\mathbb{R}} \to H^1_f(K,M)_{\mathbb{R}}$ is an isomorphism. This is equivalent to the statement that it induces a perfect *height* pairing

$$H^1_f(K,M)_{\mathbb{R}} \times H^1_f(K, M^\vee(1))_{\mathbb{R}} \to \mathbb{R}.$$

The formula for the L-value is the generalization of the Birch-Swinnerton-Dyer conjecture.

Finally we want to formulate a weak version of the Bloch-Kato conjecture, which is useful, if we can only construct a certain subspace of the motivic cohomology $H^1_f(K,M)$ (which conjecturally is the *same* as $H^1_f(K,M)$).

Conjecture 1.3.3 (Weak Bloch-Kato conjecture). *Let M be a pure motive of weight w.*
1) ($w \le -2$): *Suppose there is an E sub-vector space $\mathcal{R}(M) \subset H^1_f(K,M)$ such that the sequence*

$$0 \to \mathcal{R}(M) \xrightarrow{r_\infty} \operatorname{Coker} \alpha_M \to H^0(K, M^\vee(1))^\vee \to 0$$

is exact. Then the Bloch-Kato conjecture 1.2.4 is true if one replaces everywhere $H^1_f(K,M)$ by $\mathcal{R}(M)$.
2) ($w \ge 0$): *Suppose there is an E sub-vector space $\mathcal{R}(M^\vee(1))$ of $H^1_f(K, M^\vee(1))$ such that the sequence*

$$0 \to H^0(K,M)_\mathbb{R} \to \ker \alpha_M \xrightarrow{r_\infty^\vee} \mathcal{R}(M^\vee(1)) \to 0$$

is exact. Then the Bloch-Kato conjecture 1.2.4 is true if one replaces everywhere $H^1_f(K, M^\vee(1))$ by $\mathcal{R}(M^\vee(1))$.

Let us finally make a remark about the compatibility with the expected functional equation of the L-function. In fact one conjectures that the L-function $L(M,s)$ has a meromorphic continuation to $\mathbb{C}$ and satisfies

$$L_\infty(M,s)L(M,s) = \varepsilon(M,s)L_\infty(M^\vee(1),-s)L(M^\vee(1),-s),$$

for certain Γ factors $L_\infty(M,s), L_\infty(M^\vee(1),-s)$ and an ε factor $\varepsilon(M,s)$.

Conjecture 1.3.4. *The Bloch-Kato conjecture for M implies the one for $M^\vee(1)$.*

Remark. Perrin-Riou developed a theory of p-adic L-functions of motives starting from the Bloch-Kato conjecture. Decisive is her exponential map, which interpolates certain exponential maps of Bloch and Kato. For this she conjectured an "explicit reciprocity law" proven by Benois, Colmez, Kato-Kurihara-Tsuji (alphabetical order). This "explicit reciprocity law" also allows to deduce Kato's explicit reciprocity law, which can be used to prove the above conjecture in certain cases (see e.g. [19] appendix B).

2. A survey of results

Here we collect the known results of the Bloch-Kato conjecture to the best of the knowledge of the author. As we do not attempt an overview of the Beilinson conjecture (i.e. part a) of conjecture 1.2.4) we concentrate only on the cases where part b) is known (at least for some primes p). Moreover we do not say anything about the weight -1 case, which is essentially the (generalized) Birch-Swinnerton-Dyer conjecture.

The organization of this section is as follows: we first treat Dirichlet characters and abelian number fields, where the conjecture is known in the most complete form. Then we consider arbitrary number fields, where only very special cases are known, which follow from the class number formula. The second group of examples concerns elliptic curves with complex multiplication and Hecke characters of imaginary quadratic fields.

2.1. Artin motives. Consider the category $\mathcal{M}_{\mathbb{Q}}(E)$ of Chow motives over $\mathbb{Q}$ with coefficients in E. An object $M := (\operatorname{Spec} F, q, 0)$ of $\mathcal{M}_{\mathbb{Q}}(E)$, where $F/\mathbb{Q}$ is a number field is called an *Artin motive*. We also consider Tate twists $M(r) := (\operatorname{Spec} F, q, r)$. An Artin motive is called *abelian*, if $F/\mathbb{Q}$ is Galois with $\mathrm{Gal}(F/\mathbb{Q})$ abelian.

Definition 2.1.1. *Let $\chi : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$ be a Dirichlet character and $G := \mathrm{Gal}(\mathbb{Q}(\mu_N)/\mathbb{Q})$. Suppose that E contains all values of Dirichlet characters of $(\mathbb{Z}/N\mathbb{Z})^\times$. Let*

$$q_{\chi^{-1}} := \frac{1}{\#G} \sum_{\sigma \in G} \chi(\sigma)\sigma^*$$

be the projector onto the χ^{-1}-eigenspace in $End(h^0(\mathbb{Q}(\mu_N)))$. We call

$$h(\chi) := (\mathbb{Q}(\mu_N), q_{\chi^{-1}}, 0)$$

the associated Dirichlet motive. *Note that the element $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts via $\chi(\sigma)$ on $h(\chi)_{\mathrm{et}}$. We normalize the reciprocity map so that the* geometric *Frobenius $\chi(\mathrm{Fr}_p) = \chi(p)$. We also denote by $h(\chi)(r)$ the r-th Tate twist of $h(\chi)$.*

Note that we get a decomposition $h^0(\mathbb{Q}(\mu_N)) \cong \bigoplus_\chi h(\chi)$, where χ runs through all Dirichlet characters of $(\mathbb{Z}/N\mathbb{Z})^\times$. In particular, every abelian Artin motive is a direct sum of Dirichlet motives. The $E \otimes_{\mathbb{Q}} \mathbb{C}$-valued L-function of $h(\chi)$ exists and is given by

$$L(h(\chi), s) = \sum_{(n,f)=1} \frac{\chi(n)}{n^s}$$

for $\Re s > 1$, where f is the conductor of χ. This has a meromorphic continuation to $\mathbb{C}$. Note that $h(\chi)^\vee \cong h(\chi^{-1})$. We collect what is known about the conjectures in § 1, necessary to formulate the Bloch-Kato conjecture:

Theorem 2.1.2. *Consider the motive $h(\chi)(r)$.*
1) (Borel): *Conjecture 1.1.1 a) and c) is true.*
2) (Soulé): *Conjecture 1.1.1 b) is true.*
3) *Conjectures 1.1.2 and 1.2.3 are true.*

Concerning the Bloch-Kato conjecture we have:

Theorem 2.1.3. *Let $h(\chi)$ be a Dirichlet motive and $r \in \mathbb{Z}$. Then there is an element $\delta_f(h(\chi)(r))$ in $\Delta_f(h(\chi)(r))$, such that*
a) (Deligne [8], Beilinson [2]): $\iota_\infty(\delta_f(h(\chi)(r)) \otimes 1) = 1/L(h(\chi)(r), 0)^*$

and
b) (Burns-Greither, [6], Huber-Kings, [19]): *for all prime numbers* $p \neq 2$

$$\iota_p(\delta_f(h(\chi)(r)) \otimes 1) \in (\mathcal{O}_E \otimes_{\mathbb{Z}} \mathbb{Z}_p)^\times,$$

i.e. the Bloch-Kato conjecture is true up to powers of 2.

Remark.

1. The case $h^0(\mathbb{Q})$ is due to Bloch and Kato [5] Theorem 6.1 up to a conjecture (6.2 loc. cit.), which has been proved in [20] and later with a different method in [18]
2. A proof of the equivariant conjecture for Tate motives $\mathbb{Q}(r)$ and $r \leq 0$ over abelian number fields and $p \neq 2$ is given by Burns and Greither in [6].
3. In [23] § 6 Kato proves the p-part of the equivariant conjecture for $h^0(\mathbb{Q}(\mu_{p^n}))^+$ for $r \geq 2$ even.
4. The proof of Beilinson of part a) of conjecture 1.2.4, had some gaps, which were filled by [35] and [12].
5. Fontaine [13] § 10 mentions the case of Dirichlet motives $h(\chi)$ but he assumes that p^2 resp. p does not divide N or $\varphi(N)$ (where N is the conductor of χ) depending on parity conditions and no proof is given.
6. The Bloch-Kato conjecture for $h^0(F)(r)$ with coefficients in $\mathbb{Q}$ and $r \leq 0$ is easily seen to be equivalent to the cohomological Lichtenbaum conjecture (see [18] corollary 1.4.2.). The case $h^0(F)$ where F is totally real and $r \leq 0$ is odd is is a direct consequence of the main conjecture due to Wiles (see 2.1.5 below). In [30] this conjecture is proven for abelian number fields for all primes p, which do not divide the degree of the number field over $\mathbb{Q}$. However, the formula given there has erroneous Euler factors. (The Euler factors are fixed in [3] and [29]).
7. In recent work [4], Nguyen and Benois show how to reduce the Bloch-Kato conjecture for $h^0(F)$ and $r \geq 1$ to the case of $r \leq 0$ by showing compatibility under the functional equation.
8. Burns and Greither [6] proved even an equivariant version of the above theorem.

For general Artin motives we have the following results: The first is a reformulation of the class number formula.

Proposition 2.1.4. *The Bloch-Kato conjecture is true for* $h^0(F)$ *and* $h^0(F)(1)$.

This is mentioned in [13] with some indications of the proof. A full proof can be found in [19] 2.3. (at least for $p \neq 2$).
Remark. The equivariant conjecture for $h^0(F)$, where $F/\mathbb{Q}$ is a very special non abelian extension and $r = 0$ is considered in [7] § 7. We refer to this for the exact description of their result.

The next result is a consequence of the main conjecture for totally real fields by Wiles:

Theorem 2.1.5 (Wiles, [38] 1.6.). *Let F be a totally real field and $r \geq 1$ odd. Then the Bloch-Kato conjecture is true for $h^0(F)(-r)$ and all prime numbers $p \neq 2$.*

2.2. Elliptic curves with complex multiplication and Hecke characters of imaginary quadratic fields. There are two types of results in the case of Hecke characters of imaginary quadratic fields: there is the critical case for weights $w \leq -2$ and the case of all L-values at negative integers (recall that we do not treat the Birch-Swinnerton-Dyer case).

Let us start with the motives in question. Let K be an imaginary quadratic field and X/K be an elliptic curve with complex multiplication by $\mathcal{O}_K$, the ring of integers in K. Note that this restricts the class number of K to 1. The motive $h(X) := (X, \mathrm{id}, 0)$ over K has a decomposition with respect to the zero section

$$h(X) \cong h^0(X) \oplus h^1(X) \oplus h^2(X),$$

where $h^0(X) \cong (\operatorname{Spec} K, \mathrm{id}, 0)$ and $h^2(X) \cong (\operatorname{Spec} K, \mathrm{id}, -1)$. Consider $h^1(X)$ as object in the category of Chow motives $\mathcal{M}_K(K)$ over K with coefficients in K. This is pure of weight 1. We fix an embedding $\lambda : K \subset \mathbb{C}$ such that $j(X) = j(\mathcal{O}_K)$. For any $w \geq 1$, the motive $h^1(X)^{\otimes w}$ has multiplication by $K^{\otimes w}$, where the tensor product is taken over $\mathbb{Q}$. The idempotents in $K^{\otimes w}$ are parameterized by the $\mathrm{Aut}(\mathbb{C})$ orbits of the embeddings $\mathrm{Hom}_{\mathbb{Q}}(K^{\otimes w}, \mathbb{C})$. These in turn are given by $\lambda^{\otimes a} \otimes \overline{\lambda}^{\otimes b}$, for all $a, b \geq 0$ with $a + b = w$. Denote the motive associated to this projector by $h^1(X)^{a,b}$. This is a motive in $\mathcal{M}_K(K)$ which is pure of weight w. Fix an isomorphism $\theta : \mathcal{O}_K \cong \mathrm{End}_K(X)$ such that the composition $\theta : \mathcal{O}_K \cong \mathrm{End}_K(X) \to \mathrm{End}_K(\Omega^1_{X/K}) \cong K$ is the natural inclusion. Let $\varphi : \mathbb{A}_K^\times \to K^\times$ be the Serre-Tate character associated to X and $\mathfrak{f}$ be its conductor. We will write

$$h(\varphi^a \overline{\varphi}^b) := h^1(X)^{a,b}$$

and $h(\varphi^a \overline{\varphi}^b)(r)$ for its Tate twists. According to Deninger [10] 1.3.2. we have

$$L(h(\varphi^a \overline{\varphi}^b), s) = \prod_{\mathfrak{p} \nmid \mathfrak{f}} \left(1 - \frac{\varphi^a \overline{\varphi}^b(\mathfrak{p})}{N\mathfrak{p}^s}\right)^{-1},$$

where the product runs through all primes $\mathfrak{p}$ of K and $\Re s > \frac{w}{2} + 1$. This function has a meromorphic continuation to $\mathbb{C}$. Note that $L(h(\varphi^a \overline{\varphi}^b), s)$

has a first order zero at $s=-l$ and $l\in\mathbb{Z}$ for those l such that

$$-l \le \min(a,b) \text{ if } a\neq b \text{ and} \tag{5}$$

$$-l < a = b = \frac{w}{2} \text{ otherwise.} \tag{6}$$

The L-function $L(h(\varphi^a\overline{\varphi}^b),s)$ is critical at $s=-l$ and $l\in\mathbb{Z}$ if

$$\min(a,b) < -l \le \max(a,b).$$

Remark. The above construction can be carried out mutatis mutandis if the curve X is already defined over $\mathbb{Q}$. The resulting motives $h(\varphi^a\overline{\varphi}^b)$ are then in $\mathcal{M}_{\mathbb{Q}}(K)$ (see [10] § 4). To get the right L-function one has also to descend the coefficients from K to $\mathbb{Q}$. This is carried out in *loc. cit.*

The result in the critical case is:

Theorem 2.2.1. *Let $X/\mathbb{Q}$ be an elliptic curve with complex multiplication by $\mathcal{O}_K$ and consider $h(\overline{\varphi}^{j+k})(k)\in\mathcal{M}_{\mathbb{Q}}(\mathbb{Q})$, with $k>j>0$. Assume that $L(h(\overline{\varphi}^{j+k})(k),0)\neq 0$. Then:*
a) (Goldstein-Schappacher, [15] thm. 1.1)*: There is an element*

$$\delta_f(h(\overline{\varphi}^{j+k})(k))\in\Delta_f(h(\overline{\varphi}^{j+k})(k))$$

such that

$$\iota_\infty(\delta_f(h(\overline{\varphi}^{j+k})(k))\otimes 1)=\frac{1}{L(h(\overline{\varphi}^{j+k})(k),0)}.$$

b) (Guo, [16] thm. 1, Harrison, [17])*: For all prime numbers $p>k+1$ where X has good ordinary reduction,*

$$\iota_p(\delta_f(h(\overline{\varphi}^{j+k})(k))\otimes 1)\in\mathbb{Z}_p^\times.$$

Remark.

1. Note that $L(h(\overline{\varphi}^{j+k})(k),0)\neq 0$ if $k>j+1$.
2. Harrison [17] considered the case $j=0$ and $k>1$ over $K=\mathbb{Q}(i)$.
3. The result of Goldstein-Schappacher concerns all critical values and treats imaginary quadratic fields.

We turn to the non critical, K-theory case: Let for a subspace $\mathcal{R}(h(\varphi)^\vee(r+2))\subset H^1(K,h(\varphi)^\vee(r+2))$ be

$$\widetilde{\Delta}_f(h(\overline{\varphi})(-r)):=\det{}^{-1}\mathcal{R}(h(\varphi)^\vee(r+2))\otimes\det H^1(X(\mathbb{C}),\mathbb{Q}(r+1)).$$

Theorem 2.2.2. *Let X/K be an elliptic curve with complex multiplication by $\mathcal{O}_K$. Consider the motive $h(\overline{\varphi})(-r)$ with $r\ge 0$. Then $L(h(\overline{\varphi})(-r),s)$ has a zero of order 1 at $s=0$ and:*
a) (Bloch $r=0$, Deninger, [9] 11.3.2): *There is a subspace $\mathcal{R}(h(\overline{\varphi})^\vee(r))$ of $H^1(K,h(\overline{\varphi})^\vee(r))$ and an element $\widetilde{\delta}_f(h(\overline{\varphi})(-r))\in\widetilde{\Delta}_f(h(\overline{\varphi})(-r))$ such that*

$$\iota_\infty(\widetilde{\delta}_f(h(\overline{\varphi})(-r))\otimes 1)=\frac{1}{L(h(\overline{\varphi})(-r),0)},$$

i.e. the weak Beilinson conjecture is true.
b) (Bloch-Kato $r = 0$, Kings, [27] theorem 1.1.5.): *Assume that for* $p \nmid 6\mathfrak{f}\bar{\mathfrak{f}}$ *the group* $H^2_{et}(\mathcal{O}_K[1/p\mathfrak{f}\bar{\mathfrak{f}}], h(\varphi)(2+r))$ *is finite. Then, for all prime numbers* p *not dividing* $6\mathfrak{f}\bar{\mathfrak{f}}$,

$$\iota_p(\widetilde{\delta}_f(h(\overline{\varphi})(-r)) \otimes 1) = 1,$$

i.e. the weak Bloch-Kato conjecture is true.

Remark.

1. In the case where X is already defined over $\mathbb{Q}$ and $r = 0$ part b) is due to Bloch and Kato [5] at least for all primes p, which are regular.
2. The case X already defined over $\mathbb{Q}$ and $r \geq 0$ was deduced from part b) in [1].
3. Deninger's result is much more general. The methods for proving b) generalize to this case.
4. If we fix p, the finiteness of $H^2_{et}(\mathcal{O}_K[1/p\mathfrak{f}\bar{\mathfrak{f}}], h(\varphi)(2+r))$ is known for almost all r (see the discussion in [27] 1.1.4).

2.3. Adjoint motives of modular forms. In this section we describe briefly the results by Diamond, Flach and Guo [11] about the Bloch-Kato conjecture for the (critical) values $L(A, 0)$ and $L(A, 1)$, where A is the adjoint motive of a newform f of weight $k \geq 2$. We will omit a lot of details because they are very technical and give only a rough sketch.

Let f be a newform of weight $k \geq 2$ and level N with coefficients in the number field K. Scholl [37] has constructed a motive M_f with coefficients in K, such that for all places λ of K, the Galois representation $M_{f,\lambda}$ coincides with the one associated to f. The adjoint motive A^0_f is then essentially defined to be the kernel of the canonical map $M_f \otimes M_f^\vee \to K$. Define a set of places T_f of K to consist of the $\lambda | Nk!$ and the ones where the Galois representation $\mathcal{M}_{f,\lambda}/\lambda\mathcal{M}_{f,\lambda}$ ($\mathcal{M}_{f,\lambda}$ a lattice in $M_{f,\lambda}$) is not absolutely irreducible when restricted to G_F, where $F = \mathbb{Q}(\sqrt{(-1)^{(l-1)/2}l})$ and $\lambda | l$.

Theorem 2.3.1 (Diamond, Flach, Guo). a) *For* $i = 0, 1$, *there is an element*

$$\delta_f(A^0_f(i)) \in \Delta_f(A^0_f(i))$$

such that

$$\iota_\infty(\delta_f(A^0_f(i)) \otimes 1) = \frac{1}{L(A^0_f, i)}.$$

(This is essentially due to Rankin and Shimura)
b) *For all places* $\lambda \notin T_f$

$$\iota_\lambda(\delta_f(A^0_f(i)) \otimes 1) \in \mathcal{O}^*_{K,\lambda},$$

where ι_λ *is the* λ*-part of* ι_p.

The proof is mainly a computation of a Selmer group, which can be computed because it arises as the tangent space of a deformation problem as in the theory of Wiles, Taylor-Wiles.

3. Ingredients in the proofs

Here we want to explain the most important ingredients in the proofs of the above results. We concentrate on the more difficult non critical case. Then not only Iwasawa theory enters but also the motivic polylogarithm classes and their twisting property. The idea of proof is the same in all cases and follows the line of ideas already used in [5]: One describes the Galois cohomology groups $H^1_f(K, M_p)$ with the help of certain Iwasawa modules. These modules are independent of the Tate twist. The image of the element $\delta_f(M)$ in $H^1_f(K, M_p)$ extends to an Euler system and the conjecture can be reduced to the main conjecture in Iwasawa theory. To make this reduction possible one needs a twist compatibility of $\delta_f(M)$. This is the most difficult part of the proof (besides the use of the main conjecture). The twist compatibility can be shown in the known cases because the elements $\delta_f(M)$ are given by motivic polylogarithm classes.

We divide this section into two parts, the first treats Dirichlet motives and the second CM elliptic curves.

3.1. Dirichlet motives. In the case of Artin motives one can not only reduce the statement of the Bloch-Kato conjecture to the Iwasawa main conjecture but also use the Bloch-Kato conjecture to prove the main conjecture. This uses ideas of Kato [23] and follows Rubin's arguments (see [36]) to prove the main conjecture. Let us cite from [19] how the argument works:

1. One proves the Bloch-Kato conjecture directly in the case of the full motive $h^0(F)$ where F is a number field and $r = 0$. This is the class number formula. By the compatibility with the functional equation (which can be checked directly in this case) the conjecture also holds for $h^0(F)$ and $r = 1$.
2. Then one uses the Euler system methods to establish a divisibility statement for Iwasawa modules in the case $\chi(-1) = (-1)^{r-1}$. By the class number trick — with the class number formula replaced by the Bloch-Kato conjecture for $F = \mathbb{Q}(\mu_N)$ and $r = 1$ — one proves the main conjecture of Iwasawa theory from it.
3. Using Kato's explicit reciprocity law, one deduces the Bloch-Kato conjecture for $r \geq 1$ and $\chi(-1) = (-1)^r$ from the main conjecture. The necessary computation was already used in the "class number trick" in the previous step.
4. Using the precise understanding of the regulators of cyclotomic elements one shows the Bloch-Kato conjecture for $r \leq 0$ and $\chi(-1) =$

$(-1)^{r-1}$ from the main conjecture. However, this argument does not work for $r = 0$ and $\chi(p) = 1$, the case of "trivial zeroes".

5. Using the compatibility of the Bloch-Kato conjecture under the functional equation, one deduces the remaining cases for $r > 1$ and $r < 0$.
6. From the Bloch-Kato conjecture at positive and negative integers, we can deduce different versions of the main conjecture. Conversely, these new versions allow to prove the Bloch-Kato conjecture for $r = 0$ and $r = 1$ unless there are trivial zeroes.
7. The last exceptional case $r = 0$ and $\chi(p) = 1$ follows again by the functional equation.

We explain the most important steps in the next section in more detail. There we only consider the case of $h(\chi)(r)$ with $\chi(-1) = (-1)^r$ and $r \in \mathbb{Z}$. The functional equation 1.3.4 implies then (except for $r = 0$ and $r = 1$) that the result is also true for $h(\chi^{-1})(1-r)$ for all $r \in \mathbb{Z}$. This takes care of the characters of the other parity.

3.2. The cyclotomic polylogarithm. Let us consider the motive $h(\chi)(r)$ with $r < 0$ and $\chi(-1) = (-1)^r$. The fundamental line is

$$\Delta_f(h(\chi)(r)) = \det_E^{-1} H^1_f(\mathbb{Q}, h(\chi^{-1})(1-r)) \otimes \det_E^{-1} h(\chi^{-1})(r)^+_B.$$

Choose a lattice $t_B(\chi)\mathcal{O}_E$ of $h(\chi)_B$ and let $T_p(\chi) := t_B(\chi)\mathcal{O}_E \otimes_{\mathbb{Z}} \mathbb{Z}_p \subset h(\chi)_p$. We let $t_B(\chi^{-1})$ the dual basis element in $h(\chi^{-1})_B$. Recall the following theorem of Beilinson (see theorem 5.2.3. of [19]):

Theorem 3.2.1 (Beilinson [2], Neukirch [35], Esnault [12]). *There is an element*

$$b_{1-r}(\chi^{-1}) \in H^1_f(\mathbb{Q}, h(\chi^{-1})(1-r))$$

such that the element

$$\delta_f(h(\chi)(r)) := \left(-\frac{N^{-r}(-r)!}{2} b_{1-r}(\chi^{-1})\right)^{-1} \otimes \left((2\pi i)^r t_B(\chi^{-1})\right)^{-1}$$

maps to $(L(\chi, r)^*)^{-1}$ under the isomorphism ι_∞.

Define an element in $H^1_f(\mathbb{Q}, T_p(\chi^{-1})(1-r))$ by

$$c_{1-r}(\chi^{-1}) := (1 - \chi^{-1}(p)p^{-r}) N^{-r}(-r)! r_p(b_{1-r}(\chi^{-1})).$$

One can now reformulate part b) of conjecture 1.2.4 as follows:

Proposition 3.2.2. *The element $\iota_p(\delta_f(h(\chi)(r)))$ is a unit in $\mathcal{O}_{E_p} := \mathcal{O}_E \otimes_{\mathbb{Z}} \mathbb{Z}_p$, i.e. part b) of the Bloch-Kato conjecture 1.2.4 is true, if and only if $c_{1-r}(\chi^{-1})$ is a generator of $\det_{\mathcal{O}_{E_p}} R\Gamma_f(\mathbb{Q}, T_p(\chi^{-1})(1-r))$ via the isomorphism*

$$H^1_f(\mathbb{Q}, T_p(\chi^{-1})(1-r) \otimes_{\mathbb{Q}} \mathbb{Q}_p \cong R\Gamma_f(\mathbb{Q}, T_p(\chi^{-1})(1-r)) \otimes_{\mathbb{Q}} \mathbb{Q}_p.$$

To study $\det_{\mathcal{O}_{E_p}} R\Gamma_f(\mathbb{Q}, T_p(\chi^{-1})(1-r))$ we use Iwasawa theory. Let $\mathbb{Q}_\infty$ be the maximal $\mathbb{Z}_p$-extension inside $\mathbb{Q}(\zeta_{p^\infty})$ and $\mathbb{Q}_n \subset \mathbb{Q}_\infty$ with $\mathrm{Gal}(\mathbb{Q}_n/\mathbb{Q}) = \mathbb{Z}/p^n\mathbb{Z}$. We let $\Lambda := \varprojlim_n \mathcal{O}_{E_p}[\mathrm{Gal}(\mathbb{Q}_n/\mathbb{Q})]$ be the Iwasawa algebra. Let $G := \mathrm{Gal}(\mathbb{Q}(\zeta_N)/\mathbb{Q})$ and $\chi : G \to \mathbb{C}^\times$ be a Dirichlet character of conductor N and $T_p(\chi)$ be a Galois stable lattice in $h(\chi)_p$. Let $\mathbb{Z}_n$ be the ring of integers in the $\mathbb{Z}/p^n\mathbb{Z}$-extension $\mathbb{Q}_n$ of $\mathbb{Q}$ and define

$$\mathbf{H}^q(T_p(\chi^{-1})(1-r)) := \varprojlim_n H^q_{et}(\mathbb{Z}_n[1/p], T_p(\chi^{-1})(1-r))$$

where the limit is taken with respect to the corestriction maps. Let

$$\varepsilon_{\mathrm{cycl}} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathbb{Z}_p^\times$$

be the cyclotomic character and write $\varepsilon_{\mathrm{cycl}} = \varepsilon \times \varepsilon_\infty$ according to the decomposition $\mathbb{Z}_p^\times \cong (\mathbb{Z}/p\mathbb{Z})^\times \times \mathbb{Z}_p$. We write $\mathcal{O}_p(\varepsilon_\infty)$ for the $\mathcal{O}_p$ module of rank 1 with Galois action given by ε_∞. Then

$$\mathbf{H}^q(T_p(\chi^{-1})(1-r)) \otimes \mathcal{O}_p(\varepsilon_\infty) \cong \mathbf{H}^q(T_p(\chi^{-1}\varepsilon^{-1})(2-r)).$$

The elements $c_{1-r}(\chi^{-1})$ extend to Euler systems and in particular we get elements (also denoted by $c_{1-r}(\chi^{-1})$ by abuse of notation)

$$c_{1-r}(\chi^{-1}) \in \mathbf{H}^1(T_p(\chi^{-1})(1-r)).$$

The miracle is:

Theorem 3.2.3 (Huber-Wildeshaus, [20]). *Under the above isomorphism the element $c_{1-r}(\chi^{-1})$ maps to $c_{2-r}(\chi^{-1}\varepsilon^{-1})$.*

Remark. The theorem consists in showing that the $c_{1-r}(\chi^{-1})$ are specializations of the motivic cyclotomic polylogarithm. A different approach via the elliptic polylogarithm is developed in [18].

3.2.1. *Reduction to the Iwasawa main conjecture.* Write

$$\det_\Lambda \mathbf{R}\Gamma(T_p(\chi^{-1})(1-r)) := \bigotimes_{i=0}^{2} \det_\Lambda^{(-1)^i} \mathbf{H}^i(T_p(\chi^{-1})(1-r))$$

then we get by taking coinvariants under $\mathrm{Gal}(\mathbb{Q}_\infty/\mathbb{Q})$

$$\det_{\mathcal{O}_{E_p}} R\Gamma_{et}(\mathbb{Z}[1/p], T_p(\chi^{-1})(1-r)) \cong \det_\Lambda \mathbf{R}\Gamma(T_p(\chi^{-1})(1-r)) \otimes^{\mathbb{L}}_\Lambda \mathcal{O}_{E_p}$$

and the image of $c_{1-r}(\chi^{-1})$ is the old $c_{1-r}(\chi^{-1})$. Now one can use the property of the elements $c_{1-r}(\chi^{-1}) \in \mathbf{H}^1(T_p(\chi^{-1})(1-r))$ that they are compatible under twisting to define them for all $r \in \mathbb{Z}$ by taking simply the twist for positive r. It turns out:

Proposition 3.2.4 (theorem 3.3.2. in [18]). *The Bloch-Kato conjecture for $h(\chi)(r)$ for $r \geq 1$ and $\chi(-1) = (-1)^r$ is true, if and only if $c_{1-r}(\chi^{-1})$ is a generator of* $\det_\Lambda^{-1} \mathbf{R}\Gamma(T_p(\chi^{-1})(1-r))$.

Thus the propositions 3.2.2 and 3.2.4 give:

Proposition 3.2.5. *The Bloch-Kato conjecture for $h(\chi)(r)$ for all $r \in \mathbb{Z}$ and $\chi(-1) = (-1)^r$ is equivalent to*

$$\det_\Lambda c_{1-r}(\chi^{-1})\Lambda \cong \det_\Lambda^{-1} \mathbf{R}\Gamma(T_p(\chi^{-1})(1-r))$$

Remark.

1. Note that the statement is independent of r. For $r = 0$ it is a restatement of the main conjecture of Iwasawa theory in the form
$$\operatorname{char} \overline{\mathcal{E}}_\infty^\chi / \overline{\mathcal{C}}_\infty^\chi = \operatorname{char} \mathcal{X}_\infty^\chi$$
at least if the conductor of χ is not divisible by p and $\chi(p) \neq 1$. Here $\overline{\mathcal{E}}_\infty^\chi$ is the χ component of the inverse limit of the global units and $\mathcal{X}_\infty^\chi$ is the inverse limit of the ideal class groups (see [36] 3.2.8).
2. The above proposition turns out to be the right formulation of the Iwasawa main conjecture if $p \mid$ conductor χ. The main point of the proof of this more general main conjecture is that the formulation is *independent* of the choice of lattices $T_B(\chi)$. In particular the role of the cyclotomic elements is explained by the fact that they span $\delta_f(h(\chi)(r))$, i.e. map to the complex L-value under ι_∞.

3.3. Elliptic curves with complex multiplication. The proof follows the same lines as in the case of Artin motives.

We consider the motive $h(\overline{\varphi})(-r)$. Note that if we knew that the motivic cohomology $H^1_f(K, h(\overline{\varphi})^\vee(r+2))$ had the right dimension, we would have

$$\Delta_f(h(\overline{\varphi})(-r)) = \det_E^{-1} H^1_f(K, h(\overline{\varphi})^\vee(r+2)) \otimes \det_E^{-1} \operatorname{Coker} \alpha_{h(\overline{\varphi})(-r)}.$$

The cokernel can be identified with $H^1(X(\mathbb{C}), (2\pi i)^{r+1}\mathbb{Q})$. Fix as in [27] theorem 1.2.2 a generator $(2\pi i)^{r+1}\eta$ of this group.

Theorem 3.3.1 (Beilinson, Deninger [9]). *There is an element*

$$\xi_{-r} \in H^1_f(K, h(\overline{\varphi})^\vee(r+2))$$

such that

$$\widetilde{\delta}_f(h(\overline{\varphi})(-r)) := (\xi_{-r})^{-1} \otimes ((2\pi i)^{r+1}\eta)^{-1}$$

maps under ι_∞ to $1/L(\overline{\varphi}, -r)^*$.

Remark. Note that the ξ here is not the one from [27] 1.2.2. because there some Euler factors are excluded from $L(\overline{\varphi}, -r)$.

Let $L_p(\overline{\varphi}, -r)$ be the Euler factor (or the product of Euler factors) at the primes dividing p in K. Define

$$e_{-r} := L_p(\overline{\varphi}, -r)^{-1}\xi_{-r}.$$

Let $X[p^n]$ be the n torsion points of X and let $T_p(X) := \varprojlim_n X[p^n]$. Define $K_n := K(X[p^{n+1}])$ and denote by $\mathcal{O}_n$ its ring of integers. Let

$S := \{$ primes in Kdividing $p\mathfrak{f}\bar{\mathfrak{f}}\}$. Let $\Lambda := \varprojlim \mathcal{O}_K[[\mathrm{Gal}(K_\infty/K]]$. As in the cyclotomic case we can define

$$\mathbf{H}^i(T_p(X)(r+1)) := \varprojlim{}_n H^i_{et}(\mathcal{O}_n[1/S], T_p(X)(r+1))$$

and

$$\det{}_\Lambda \mathbf{R}\Gamma(T_p(X)(r+1)) := \bigotimes_{i=0}^{2} \det{}_\Lambda^{(-1)^i} \mathbf{H}^i(T_p(X)(r+1)).$$

The element e_{-r} extends to an Euler system and gives in particular elements

$$e_{-r} \in \mathbf{H}^1(T_p(X)(r+1)).$$

Assume that $H^2_{et}(\mathcal{O}_n[1/S], T_p(X)(r+1))$ is finite. One has:

Proposition 3.3.2. *The Bloch-Kato conjecture for $h(\overline{\varphi})(-r)$ is true for p, if*

$$\det{}_\Lambda e_{-r}\Lambda \cong \det{}_\Lambda^{-1} \mathbf{R}\Gamma(T_p(X)(r+1)).$$

Note that the right hand side is independent of r. Again a miracle happens:

Theorem 3.3.3 (Kings [27] 5.2.1)**.** *The isomorphism of Λ modules*

$$\mathbf{H}^1(T_p(X)(r+1)) \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(1) \cong \mathbf{H}^1(T_p(X)(r+2))$$

maps e_{-r} to e_{1-r}.

This theorem allows to reduce to the case $r = 0$, which is just a reformulation of Rubin's main conjecture for imaginary quadratic fields.

Remark.

1. This last theorem is the most difficult step in the proof of the Bloch-Kato conjecture for $h(\overline{\varphi})(-r)$ and the main contents of [27].
2. The idea of proof is roughly as follows: first one uses the result from [18] theorem 2.2.4 that the element e_{-r} is essentially the specialization of the elliptic polylogarithm. Then one has to compute the p-adic realization of the elliptic polylogarithm. For this a "geometric" approach to the polylog sheaf is developed in § 3 of [27]. The polylog extension can then be written as a projective limit of one-motives (section 4.1. in loc. cit.). The specialization can then be expressed in terms of elliptic units. This gives the twist compatibility and the relation to Rubin's main conjecture.

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