Journal de Théorie des Nombres
de Bordeaux **15** (2003), 223–229

A note on circular units in $\mathbb{Z}_p$-extensions

par Radan KUČERA

Résumé. Nous nous intéressons aux limites projectives des groupes de Sinnott et des groupes de Washington des unités circulaires dans la $\mathbb{Z}_p$-extension d'un corps abélien. Nous montrons par un exemple qu'en général ces deux limites ne coïncident pas.

Abstract. In this note we consider projective limits of Sinnott and Washington groups of circular units in the cyclotomic $\mathbb{Z}_p$-extension of an abelian field. A concrete example is given to show that these two limits do not coincide in general.

1. Introduction

For any positive integer m, let $\zeta_m = e^{2\pi i/m}$ and let $\mathbb{Q}^{(m)} = \mathbb{Q}(\zeta_m)$ be the mth cyclotomic field. Let K be an abelian field of conductor m, i.e., m is the smallest positive integer such that K is a subfield of $\mathbb{Q}^{(m)}$. Let $C(K)$ be the Sinnott group of circular units of K (see [S]). It is well known (see [L]) that this group can be defined as the intersection of the group $E(K)$ of all units in K and the subgroup of the multiplicative group of K generated by -1 and by all norms $N_{\mathbb{Q}^{(r)}/\mathbb{Q}^{(r)}\cap K}(1-\zeta_r^a)$, where $1 < r \mid m$ and $(a,r)=1$. Let $\overline{C}(K)$ be the group of cyclotomic units of K mentioned in [W, page 143], i.e., $\overline{C}(K) = E(K) \cap C(\mathbb{Q}^{(m)})$. We shall call the latter group the Washington group of cyclotomic units.

Now, let p be an odd prime which does not ramify in K. Let $B/\mathbb{Q}$ be the cyclotomic $\mathbb{Z}_p$-extension of the rational numbers, i.e., $\mathbb{Q} = B_0 \subset B_1 \subset B_2 \subset \dots$ are abelian fields ramified only at p, $[B_n : \mathbb{Q}] = p^n$, and $B = \bigcup_{n=0}^{\infty} B_n$. Hence $KB = \bigcup_{n=0}^{\infty} KB_n$ is the cyclotomic $\mathbb{Z}_p$-extension of K.

We shall consider the following projective limits (with respect to norms) $C = \varprojlim(C(KB_n)\otimes\mathbb{Z}_p)$, and $\overline{C} = \varprojlim(\overline{C}(KB_n)\otimes\mathbb{Z}_p)$. It has been proved in [KN] that C is of finite index in $\overline{C}$. But there has remained an open question whether $C = \overline{C}$ in general or not. This note is devoted to the negative answer to this question. More precisely, by an explicit construction we shall prove the following

Manuscrit reçu le 3 décembre 2001.
Supported by the grant 201/01/0471 of the Grant Agency of the Czech Republic.

Theorem. *For $p=3$, there is an abelian field K of degree 9 over $\mathbb{Q}$ such that there is $\eta=(\eta_n)_{n=0}^{\infty}\in\varprojlim\overline{C}(KB_n)$ satisfying $\eta^3\in\varprojlim C(KB_n)$ and $\eta\notin\varprojlim(C(KB_n)\otimes\mathbb{Z}_p)$.*

Let us mention that by an easy and straightforward modification of our construction one can get, for any given odd prime p, an example of an abelian field of degree p^2, ramified at $2p+1$ primes, whose cyclotomic $\mathbb{Z}_p$-extension satisfies $p\mid[\overline{C}:C]$.

A similar example has been given independently by J.-R. Belliard, who describes in [B] an abelian field K of degree p^2, ramified at $p+1$ primes, and shows by means of an explicit construction in $\mathbb{Z}_p$-extension KB/K that C is not a Λ-free Λ-module, where $\Lambda=\mathbb{Z}_p[[\mathrm{Gal}(KB/K)]]$. Moreover, he proves here that there exists $\varepsilon\in\overline{C}$ such that $\varepsilon^p\in C$ and $\varepsilon\notin C$ if and only if C is not a Λ-free Λ-module. To compare these two constructions: on the one hand our example needs more ramified primes, but on the other hand we do not need to tensor by $\mathbb{Z}_p$ to be able to define units $\eta_n\in\overline{C}(KB_n)$ giving the norm-coherent sequence $(\eta_n)\in\overline{C}$, so — in some sense — our example is more explicit.

Notation. Let $q_1,q_2,\dots,q_7$ be different primes, all congruent to 1 modulo 9, chosen in such a way that $q_1\not\equiv 1 \pmod{27}$ and that for different $i,j\in\{1,2,\dots,7\}$ the prime q_i is a cubic residue modulo q_j. For each $i\in\{1,2,\dots,7\}$, let χ_i be a cubic Dirichlet character modulo q_i. Let K be the abelian field corresponding to the group of Dirichlet characters generated by $\chi_1\chi_2\chi_3\chi_4\chi_5$ and $\chi_2\chi_3\chi_4^2\chi_5^2\chi_6\chi_7$. So $[K:\mathbb{Q}]=9$ and K has the following four subfields of degree 3:

K_1 corresponding to $\chi_2\chi_3\chi_4^2\chi_5^2\chi_6\chi_7$ of conductor $f_1=q_2q_3q_4q_5q_6q_7$,
K_2 corresponding to $\chi_1\chi_4^2\chi_5^2\chi_6^2\chi_7^2$ of conductor $f_2=q_1q_4q_5q_6q_7$,
K_3 corresponding to $\chi_1\chi_2^2\chi_3^2\chi_6\chi_7$ of conductor $f_3=q_1q_2q_3q_6q_7$,
K_4 corresponding to $\chi_1\chi_2\chi_3\chi_4\chi_5$ of conductor $f_4=q_1q_2q_3q_4q_5$.

For the sake of brevity, let us write $K_0=K$, $f_0=q_1q_2q_3q_4q_5q_6q_7$, $K_5=\mathbb{Q}$, $f_5=1$.

For any prime $q\mid 3f_0$, let $\mathrm{Frob}(q)$ mean a fixed Frobenius automorphism of q on $KB/\mathbb{Q}$, i.e., any chosen extension to KB of the Frobenius automorphism of q on the maximal subfield of KB unramified at q. Since q_i is a cube modulo f_0/q_i for any $i=1,2,\dots,7$, the previous choice can be done in such a way that $\mathrm{Frob}(q_i)\in\mathrm{Gal}(KB/K)$ for each $i=1,2,\dots,7$.

We shall introduce generators of $C(KB_n)$. For any $i=0,1,\dots,4$ and any integer $n\geq 0$ let

$$\varepsilon_{i,n}=\begin{cases} N_{\mathbb{Q}^{(f_i)}/K_i}(1-\zeta_{f_i}) & \text{for } n=0,\\ N_{\mathbb{Q}^{(3^{n+1}f_i)}/K_iB_n}(1-\zeta_{3^{n+1}f_i}) & \text{for } n>0.\end{cases}$$

Moreover, let $\varepsilon_{5,0} = -1$ and

$$\varepsilon_{5,n} = \frac{(1-\zeta_{3^{n+1}}^4)(1-\zeta_{3^{n+1}}^{-4})}{(1-\zeta_{3^{n+1}})(1-\zeta_{3^{n+1}}^{-1})}$$

for any positive integer n. The following lemma shows that units $\varepsilon_{i,j}$ with $0 < j < n$ are generated by all conjugates of the units $\varepsilon_{i,j}$, where $j = 0$ or $j = n$.

Lemma 1. *For any integer $n \geq 0$, the Sinnott group of circular units $C(KB_n)$ is the Galois module generated by*

$$\{\varepsilon_{i,0};\ 0 \leq i \leq 5\} \cup \{\varepsilon_{i,n};\ 0 \leq i \leq 5\}.$$

Proof. Using [L], $C(KB_n)$ is the intersection of $E(KB_n)$ and of the Galois module D generated by -1 and by all norms $N_{\mathbb{Q}^{(r)}/\mathbb{Q}^{(r)}\cap KB_n}(1-\zeta_r)$, where $1 < r \mid 3^{n+1} f_0$. We can write $r = 3^j q$, where $0 \leq j \leq n+1$, and $q \mid f_0$. Let

$$f_i = \max\{f_l;\ 0 \leq l \leq 5,\ f_l \mid q\}.$$

So, denoting $B_{-1} = \mathbb{Q}$, we have $\mathbb{Q}^{(r)} \cap KB_n = K_i B_{j-1}$. Therefore

$$N_{\mathbb{Q}^{(r)}/\mathbb{Q}^{(r)}\cap KB_n}(1-\zeta_r) = N_{\mathbb{Q}^{(3^j f_i)}/K_i B_{j-1}}\left(N_{\mathbb{Q}^{(r)}/\mathbb{Q}^{(3^j f_i)}}(1-\zeta_r)\right).$$

But

$$N_{\mathbb{Q}^{(r)}/\mathbb{Q}^{(3^j f_i)}}(1-\zeta_r) = (1-\zeta_{3^j f_i})^{\prod_q (1-\mathrm{Frob}(q)^{-1})},$$

where q in the product runs over all primes $q \mid r,\ q \nmid 3^j f_i$ (an empty product equals 1). Hence D is generated by -1, by $\varepsilon_{i,j}$ for all $i = 0, 1, \ldots, 4$ and $j = 0, 1, \ldots, n$, and by

$$N_{\mathbb{Q}^{(3^j)}/B_{j-1}}(1-\zeta_{3^j}) = N_{B_n/B_{j-1}}\left(N_{\mathbb{Q}^{(3^{n+1})}/B_n}(1-\zeta_{3^{n+1}})\right)$$

for $j = 1, 2, \ldots, n+1$. If $0 < j < n$ then

(1) $$\varepsilon_{i,j} = N_{K_i B_n / K_i B_j}(\varepsilon_{i,n}).$$

and $N_{\mathbb{Q}^{(3^{n+1})}/B_n}(1-\zeta_{3^{n+1}}) = (1-\zeta_{3^{n+1}})(1-\zeta_{3^{n+1}}^{-1})$. Therefore D is the Galois module generated by -1, by $\varepsilon_{i,j}$ for all $i = 0, 1, \ldots, 4$ and $j \in \{0, n\}$, and by $(1-\zeta_{3^{n+1}})(1-\zeta_{3^{n+1}}^{-1})$. It is well-known that all the numbers but the last one are units and that the quotient of any two conjugates of the last number is a unit. The lemma follows using the fact that the automorphism sending each root of unity to its fourth power generates the Galois group of $B_n/\mathbb{Q}$. □

The generators described by Lemma 1 are not independent. For any integers $j \geq 0$ and $i, l \in \{0, 1, \ldots, 5\}$ such that $l = 0$ or $i = 5$ we have the following relation

$$N_{K_lB_j/K_iB_j}(\varepsilon_{l,j}) = \varepsilon_{i,j}^{\prod_q(1-\mathrm{Frob}(q)^{-1})}, \tag{2}$$

where q in the product runs over all primes dividing f_l/f_i. Moreover, for any integer $j \geq 0$ and $i \in \{0, 1, \dots, 5\}$

$$N_{K_iB_j/K_i}(\varepsilon_{i,j}) = \varepsilon_{i,0}^{1-\mathrm{Frob}(3)^{-1}}. \tag{3}$$

Construction. As we have already mentioned, the Frobenius automorphism $\mathrm{Frob}(q_i) \in \mathrm{Gal}(KB/K)$ for each $i = 1, 2, \dots, 7$. Moreover, $q_i \equiv 1 \pmod 9$ implies that $\mathrm{Frob}(q_i)$ is a cube in $\mathrm{Gal}(KB/K) \simeq (\mathbb{Z}_3, +)$. Therefore there is unique $\psi \in \mathrm{Gal}(KB/K)$ such that $\psi^3 = \mathrm{Frob}(q_1)^{-1}$. Because $q_1 \not\equiv 1 \pmod{27}$, ψ is a topological generator of $\mathrm{Gal}(KB/K)$. Hence, for each $i = 2, 3, \dots, 7$ there is a uniquely determined 3-adic integer α_i satisfying $\mathrm{Frob}(q_i)^{-1} = \psi^{3\alpha_i}$.

It is easy to see that $\mathrm{Gal}(KB_n/B_n) \setminus \{\mathrm{id}\}$ is the disjoint union

$$\mathrm{Gal}(KB_n/B_n) \setminus \{\mathrm{id}\} = \bigcup_{i=1}^{4} \big(\mathrm{Gal}(KB_n/K_iB_n) \setminus \{\mathrm{id}\}\big).$$

Therefore

$$\varepsilon_{0,n}^3 = \big(N_{KB_n/B_n}(\varepsilon_{0,n})\big)^{-1} \prod_{i=1}^{4} N_{KB_n/K_iB_n}(\varepsilon_{0,n}). \tag{4}$$

Let us consider the group $C(KB_n)/(C(KB_n))^3$ written additively, where we shall identify any unit of $C(KB_n)$ with its image in $C(KB_n)/(C(KB_n))^3$. Therefore the identity (4) means

$$0 = -N_{KB_n/B_n}(\varepsilon_{0,n}) + \sum_{i=1}^{4} N_{KB_n/K_iB_n}(\varepsilon_{0,n}). \tag{5}$$

Let us fix $n \geq 2$. By abuse of notation, we shall denote the restriction of ψ to KB_n again by ψ. We shall apply

$$\tau = (1 - \psi^{2\cdot 3^{n-1}}) \sum_{r=0}^{3^{n-2}-1} \psi^{3r} \in \mathbb{Z}[\mathrm{Gal}(KB_n/K)]$$

on (5). Since $\psi^{3^n} = 1$, we have

$$\tau(1-\psi^3) = (1-\psi^{2\cdot 3^{n-1}})(1-\psi^{3^{n-1}}) = 3 - (1 + \psi^{3^{n-1}} + \psi^{2\cdot 3^{n-1}}). \tag{6}$$

It is easy to see that $1+\psi^{3^{n-1}}+\psi^{2\cdot 3^{n-1}}$ is the norm operator of KB_n/KB_{n-1}. Using (6) and the norm relations (2) and (1) we obtain

$$\begin{aligned} \tau N_{KB_n/K_1B_n}(\varepsilon_{0,n}) &= \tau(1-\mathrm{Frob}(q_1)^{-1})\varepsilon_{1,n} = \tau(1-\psi^3)\varepsilon_{1,n} \\ &= 3\varepsilon_{1,n} - N_{KB_n/KB_{n-1}}(\varepsilon_{1,n}) = -\varepsilon_{1,n-1}. \end{aligned} \tag{7}$$

Similarly, for any $i = 2, 3, 4$ using (2) we have

$$\begin{aligned}\tau N_{KB_n/K_iB_n}(\varepsilon_{0,n}) &= \tau(1-\mathrm{Frob}(q_{2i-1})^{-1})(1-\mathrm{Frob}(q_{2i-2})^{-1})\varepsilon_{i,n} \\ &= \tau(1-\psi^{3\alpha_{2i-1}})(1-\psi^{3\alpha_{2i-2}})\varepsilon_{i,n}.\end{aligned}$$

Let $a_j^{(n)}$ be a positive integer satisfying $a_j^{(n)} \equiv \alpha_j \pmod{3^n\mathbb{Z}_3}$. Then

$$\begin{aligned}\tau N_{KB_n/K_iB_n}(\varepsilon_{0,n}) &= \tau(1-\psi^{3a_{2i-1}^{(n)}})(1-\psi^{3a_{2i-2}^{(n)}})\varepsilon_{i,n} \\ &= \tau(1-\psi^3)\sigma_{2i-1}^{(n)}(1-\psi^{3a_{2i-2}^{(n)}})\varepsilon_{i,n},\end{aligned}$$

where

$$\sigma_i^{(n)} = \sum_{r=0}^{a_i^{(n)}-1} \psi^{3r},$$

and (6) and (1) give

$$\begin{aligned}\tau N_{KB_n/K_iB_n}(\varepsilon_{0,n}) &= -\sigma_{2i-1}^{(n)}(1-\psi^{3a_{2i-2}^{(n)}})N_{KB_n/KB_{n-1}}(\varepsilon_{i,n}) \\ &= -\sigma_{2i-1}^{(n)}(1-\psi^{3a_{2i-2}^{(n)}})\varepsilon_{i,n-1}.\end{aligned} \tag{8}$$

A similar computation gives

(9)

$$\tau N_{KB_n/B_n}(\varepsilon_{0,n}) = \tau\left(\prod_{i=1}^{7}(1-\psi^{3a_i^{(n)}})\right)\varepsilon_{5,n} = -\left(\prod_{i=2}^{7}(1-\psi^{3a_i^{(n)}})\right)\varepsilon_{5,n-1}.$$

Putting (5), (7), (8), and (9) together, we obtain

$$\varepsilon_{1,n-1} + \sum_{i=2}^{4}\sigma_{2i-1}^{(n)}(1-\psi^{3a_{2i-2}^{(n)}})\varepsilon_{i,n-1} - \left(\prod_{i=2}^{7}(1-\psi^{3a_i^{(n)}})\right)\varepsilon_{5,n-1} = 0.$$

This equality in $C(KB_n)/(C(KB_n))^3$ means that there is $\eta_{n-1} \in C(KB_n)$ such that

$$\eta_{n-1}^3 = \varepsilon_{1,n-1}\cdot\left(\prod_{i=2}^{4}\varepsilon_{i,n-1}^{\sigma_{2i-1}^{(n)}(1-\psi^{3a_{2i-2}^{(n)}})}\right)\cdot\varepsilon_{5,n-1}^{-\prod_{i=2}^{7}(1-\psi^{3a_i^{(n)}})}. \tag{10}$$

Since KB_n is a real abelian field, η_{n-1} is uniquely determined by the previous identity. It is easy to see that $\eta_{n-1}^3 \in C(KB_{n-1})$, so $\eta_{n-1}^3 \in KB_{n-1}$. Since $KB_n/\mathbb{Q}$ is abelian, we have $\eta_{n-1} \in KB_{n-1}$, hence $\eta_{n-1} \in \mathbb{Q}^{(3^n f_0)}$. On the other hand $\eta_{n-1} \in C(KB_n)$ implies $\eta_{n-1} \in C(\mathbb{Q}^{(3^{n+1} f_0)})$. But for cyclotomic fields, the Sinnott groups of circular units satisfy the Galois descent (see [GK]), so

$$\eta_{n-1} \in \mathbb{Q}^{(3^n f_0)} \cap C(\mathbb{Q}^{(3^{n+1} f_0)}) = C(\mathbb{Q}^{(3^n f_0)}).$$

This means $\eta_{n-1} \in \overline{C}(KB_{n-1})$.

Since $a_j^{(n)} \equiv a_j^{(n+1)} \pmod{3^n}$, we have

$$\varepsilon_{i,n-1}^{1-\psi^{3a_j^{(n)}}} = \varepsilon_{i,n-1}^{1-\psi^{3a_j^{(n+1)}}}.$$

It is easy to see that for any positive integers a and b

$$(1-\psi^{3a}) \sum_{r=0}^{b-1} \psi^{3r} = (1-\psi^3) \left(\sum_{s=0}^{a-1} \psi^{3s}\right) \sum_{r=0}^{b-1} \psi^{3r} = (1-\psi^{3b}) \sum_{r=0}^{a-1} \psi^{3r}.$$

Using the previous identities we can easily prove that the numbers $a_j^{(n)}$ can be replaced by $a_j^{(n+1)}$ in (10). Using (1) this implies that

$$N_{KB_n/KB_{n-1}}(\eta_n^3) = \eta_{n-1}^3,$$

so

$$N_{KB_n/KB_{n-1}}(\eta_n) = \eta_{n-1}.$$

Denoting $\eta_0 = N_{KB_1/K}(\eta_1)$, we have proved

$$\eta = (\eta_n)_{n=0}^{\infty} \in \varprojlim \overline{C}(KB_n). \tag{11}$$

But $\eta_n^3 \in C(KB_n)$, so $\eta^3 \in \varprojlim C(KB_n)$. To show that $\eta \notin \varprojlim C(KB_n)$, it is enough to prove that $\eta_1 \notin C(KB_1)$. For this purpose we shall describe a basis of $C(KB_1)$.

Lemma 2. *There is a basis of $C(KB_1)$ consisting of*
(i) 4 *conjugates of each of the units $\varepsilon_{1,1}$, $\varepsilon_{2,1}$, $\varepsilon_{3,1}$, and $\varepsilon_{4,1}$,*
and
(ii) *of* 2 *conjugates of each of the units $\varepsilon_{1,0}$, $\varepsilon_{2,0}$, $\varepsilon_{3,0}$, $\varepsilon_{4,0}$, and $\varepsilon_{5,1}$.*

Proof. Since

$$\operatorname{rank}_{\mathbb{Z}} C(KB_1) = \operatorname{rank}_{\mathbb{Z}} E(KB_1) = [KB_1 : \mathbb{Q}] - 1 = 26,$$

which is the number of units mentioned in the lemma, it is enough to show that they together with -1 generate $C(KB_1)$. We have chosen the primes $q_1, \ldots, q_7$ in such a way that all automorphisms $\mathrm{Frob}(q_1), \ldots, \mathrm{Frob}(q_7)$ act trivially on KB_1. Hence relations (4) and (2) for $n = 0, 1$ give $\varepsilon_{0,n}^3 = 1$, so $\varepsilon_{0,0} = \varepsilon_{0,1} = 1$. Because $\varepsilon_{5,0} = -1$, Lemma 1 gives that the multiplicative group, generated by -1 and by all conjugates of units mentioned in the lemma, equals $C(KB_1)$.

Let us fix any $i = \{1,2,3,4\}$ and consider $\varepsilon_{i,1} \in K_iB_1$. The Galois groups $\mathrm{Gal}(K_iB_1/K_i)$ and $\mathrm{Gal}(K_iB_1/B_1)$ are groups of order 3; let μ and ν be their generators, respectively. Then μ and ν generate $\mathrm{Gal}(K_iB_1/\mathbb{Q})$, so $\varepsilon_{i,1}^{\mu^r\nu^s}$, with $r, s = 0, 1, 2$, is the system of all conjugates of $\varepsilon_{i,1}$. Using the trivial action of Frobenius automorphisms, (2) implies

$$\varepsilon_{i,1}^{1+\nu+\nu^2} = N_{K_iB_1/B_1}(\varepsilon_{i,1}) = \varepsilon_{5,1}^{\prod_{q|f_i}(1-\mathrm{Frob}(q)^{-1})} = 1.$$

Similarly, (3) gives

$$\varepsilon_{i,1}^{1+\mu+\mu^2} = N_{K_iB_1/K_i}(\varepsilon_{i,1}) = \varepsilon_{i,0}^{1-\mathrm{Frob}(3)^{-1}}.$$

Therefore all conjugates of $\varepsilon_{i,1}$ are generated by the four conjugates $\varepsilon_{i,1}^{\mu^r\nu^s}$, with $r, s = 0, 1$, and by all three conjugates of $\varepsilon_{i,0}$.

Similarly, one can show that each of the units $\varepsilon_{1,0}$, $\varepsilon_{2,0}$, $\varepsilon_{3,0}$, $\varepsilon_{4,0}$, and $\varepsilon_{5,1}$ has precisely three conjugates and the product of them (i.e., its absolute norm) equals one. The lemma follows. □

Since ψ^3 acts trivially on KB_1, equality (10) gives $\eta_1^3 = \varepsilon_{1,1}$. But Lemma 2 shows that $\varepsilon_{1,1}$ is not a cube in $C(KB_1)$, hence $\eta_1 \notin C(KB_1)$, which gives $\eta \notin \varprojlim C(KB_n)$.

The theorem will be proved, if we show that there exist primes $q_1, \ldots, q_7$ satisfying all assumptions made in the first sentence of Notation. But this can be done either by a standard technique using Tchebotarev density theorem (e.g., see Theorem 3.1 in [R]) or just by checking that primes

$$p_1 = 19,\ p_2 = 487,\ p_3 = 4591,\ p_4 = 65557,$$
$$p_5 = 186391,\ p_6 = 675253,\ p_7 = 1414207$$

satisfy all these assumptions.

References

[B] J.-R. Belliard, *Sous-modules d'unités en théorie d'Iwasawa*, to appear in Publications mathématiques de l'Université de Franche-Comté.

[GK] R. Gold, J. Kim, *Bases for cyclotomic units.* Compositio Math. **71** (1989), 13–27.

[KN] R. Kučera, J. Nekovář, *Cyclotomic units in $\mathbb{Z}_p$-extensions.* J. Algebra **171** (1995), 457–472.

[L] G. Lettl, *A note on Thaine's circular units.* J. Number Theory **35** (1970), 224–226.

[R] K. Rubin, *The main conjecture*, appendix in S. Lang, Cyclotomic Fields I and II, Springer-Verlag, New York, 1990.

[S] W. Sinnott, *On the Stickelberger ideal and the circular units of an abelian field.* Invent. Math. **62** (1980), 181–234.

[W] L. C. Washington, *Introduction to cyclotomic fields.* Springer-Verlag, New York, 1996.

Radan Kučera
Department of Mathematics
Masaryk Univerzity
Janáčkovo nám. 2a
662 95 Brno
Czech Republic
E-mail : kucera@math.muni.cz