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Logarithmic density of a sequence of integers and density of its ratio set

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Résumé. Nous donnons des conditions suffisantes pour que l'ensemble $R(A)$ des fractions d'un ensemble d'entiers A soit dense dans $\mathbb{R}^+$, en termes des densités logarithmiques de A. Ces conditions diffèrent sensiblement de celles précédemment obtenues en termes des densités asymptotiques.

Abstract. In the paper sufficient conditions for the (R)-density of a set of positive integers in terms of logarithmic densities are given. They differ substantially from those derived previously in terms of asymptotic densities.

1. Preliminaries

Denote by $\mathbb{N}$ and $\mathbb{R}^+$ the set of all positive integers and positive real numbers, respectively. For $A \subset \mathbb{N}$ and $x \in \mathbb{R}^+$ let $A(x) = \{a \in A;\ a \leq x\}$. Denote by $R(A) = \{\frac{a}{b};\ a \in A,\ b \in A\}$ the *ratio set of* A and say that a set A is (R)-dense if $R(A)$ is (topologically) dense in the set $\mathbb{R}^+$ (see [3]). Let us notice that the (R)-density of a set A is equivalent to the density of $R(A)$ in the set $(1, \infty)$.
Define

$$\underline{d}(A) = \liminf_{x\to\infty} \frac{\#A(x)}{x}, \quad \overline{d}(A) = \limsup_{x\to\infty} \frac{\#A(x)}{x}, \quad d(A) = \lim_{x\to\infty} \frac{\#A(x)}{x}.$$

the *lower asymptotic density*, *upper asymptotic density*, and *asymptotic density* (if defined), respectively.
Similarly, define

$$\underline{\delta}(A) = \liminf_{x\to\infty} \frac{\sum\limits_{a\in A(x)} \frac{1}{a}}{\ln x}, \quad \overline{\delta}(A) = \limsup_{x\to\infty} \frac{\sum\limits_{a\in A(x)} \frac{1}{a}}{\ln x}, \quad \delta(A) = \lim_{x\to\infty} \frac{\sum\limits_{a\in A(x)} \frac{1}{a}}{\ln x},$$

the *lower logarithmic density*, *upper logarithmic density*, and *logarithmic density* (if defined), respectively.

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The following relations between asymptotic density and (R)-density are known

(S1) If $d(A) > 0$ then A is (R)-dense (see [3], [4]).

(S2) If $\underline{d}(A) \geq \frac{1}{2}$ then A is (R)-dense and for all $b \in \langle 0, \frac{1}{2})$ there is a set B such that $\underline{d}(B) = b$ and B is not (R)-dense (see [2], [5]).

(S3) If $\overline{d}(A) = 1$ then A is (R)-dense and for all $b \in \langle 0, 1)$ there is a set B such that $\overline{d}(B) = b$ and B is not (R)-dense (see [3], [4]).

Notice that the results (S1), (S2) and (S3) can be formulated in a common way as results about maximal sets (with respect to the corresponding density) which are not (R)-dense as follows. Denote by $\mathbb{D} = \{A \subset \mathbb{N};\ A \text{ is not } (R) - \text{dense}\}$. Then we have

(S1) $\{d(A);\ A \in \mathbb{D}\} = \{0\}$.

(S2) $\{\underline{d}(A);\ A \in \mathbb{D}\} = \langle 0, \frac{1}{2})$.

(S3) $\{\overline{d}(A);\ A \in \mathbb{D}\} = \langle 0, 1)$.

The aim of this paper is to prove corresponding relations for logarithmic density. It appears (see Theorem 2 and Corollary 1) that they differ substantially from the above ones for asymptotic density.

2. Logarithmic density and (R)-density

First, let us introduce a useful technique for calculation densities. It can be easily seen that in practical calculation of densities of a set A, the following method can be used.

Write the set A as $A = \bigcup\limits_{n=1}^{\infty} \langle p_n + 1, q_n \rangle \cap \mathbb{N}$, where $0 \leq p_1 < q_1 \leq p_2 < q_2 \leq \cdots$ are integers. Then

$$\underline{d}(A) = \liminf_{n\to\infty} \frac{\sum\limits_{i=1}^{n} (q_i - p_i)}{p_{n+1}}, \qquad \overline{d}(A) = \limsup_{n\to\infty} \frac{\sum\limits_{i=1}^{n} (q_i - p_i)}{q_n},$$

and

$$\underline{\delta}(A) = \liminf_{n\to\infty} \frac{\sum\limits_{i=1}^{n} \ln \frac{q_i}{p_i}}{\ln p_{n+1}}, \qquad \overline{\delta}(A) = \limsup_{n\to\infty} \frac{\sum\limits_{i=1}^{n} \ln \frac{q_i}{p_i}}{\ln q_n}.$$

In practice the bounds $p_n,\ q_n$ of intervals determining the set A are often real numbers instead of integers. Then it may be convenient to use the following lemma. In fact, we will use it in later calculations.

Lemma 1. *Let $0 \leq p_1 < q_1 \leq p_2 < q_2 \leq \cdots$ be given real numbers such that $\sum_{n=1}^{\infty} \frac{1}{p_n} < \infty$ and let $|a_n| \leq d,\ |b_n| \leq d$ for each $n \in \mathbb{N}$ and some* *fixed* $d \in \mathbb{N}$*. Let* $A = \bigcup_{n=1}^{\infty} \langle p_n + a_n + 1, q_n + b_n \rangle \cap \mathbb{N}$*. Then*

$$\underline{d}(A) = \liminf_{n\to\infty} \frac{\sum_{i=1}^{n} (q_i - p_i)}{p_{n+1}}, \quad \overline{d}(A) = \limsup_{n\to\infty} \frac{\sum_{i=1}^{n} (q_i - p_i)}{q_n}.$$

$$\underline{\delta}(A) = \liminf_{n\to\infty} \frac{\sum_{i=1}^{n} \ln \frac{q_i}{p_i}}{\ln p_{n+1}}, \quad \overline{\delta}(A) = \limsup_{n\to\infty} \frac{\sum_{i=1}^{n} \ln \frac{q_i}{p_i}}{\ln q_n}.$$

Proof. For all $n \in \mathbb{N}$ let a_n, b_n be such that

1. $|a_n| \le d$, $|b_n| \le d$,
2. both $p_n + a_n$ and $q_n + b_n$ are integers,
3. $A = \bigcup_{n=1}^{\infty} \langle p_n + a_n + 1, q_n + b_n \rangle \cap \mathbb{N}$.

First, let us notice that it is known that if for an increasing sequence of positive integers $\{p_n\}$ the series $\sum_{n=1}^{\infty} \frac{1}{p_n}$ converges then $\lim_{n\to\infty} \frac{n}{p_n} = 0$ ([1], 80. Theorem, p.124) and trivially also $\lim_{n\to\infty} \frac{n}{p_n + r} = 0$ for any fixed $r \in \mathbb{R}$.

Now a simple analysis shows that for each $n \in \mathbb{N}$

$$\frac{\sum_{i=1}^{n} (q_i - p_i - 2d)}{p_{n+1} + d} \le \frac{\sum_{i=1}^{n} ((q_i + b_i) - (p_i + a_i))}{p_{n+1} + a_{n+1}} \le \frac{\sum_{i=1}^{n} (q_i - p_i + 2d)}{p_{n+1} - d}$$

and, using Lemma 1,

$$\liminf_{n\to\infty} \frac{\sum_{i=1}^{n} (q_i - p_i - 2d)}{p_{n+1} + d} \le \liminf_{n\to\infty} \frac{\sum_{i=1}^{n} ((q_i + b_i) - (p_i + a_i))}{p_{n+1} + a_{n+1}} = \underline{d}(A)$$
$$\le \liminf_{n\to\infty} \frac{\sum_{i=1}^{n} (q_i - p_i + 2d)}{p_{n+1} - d}$$

or, rewritten,

$$\liminf_{n\to\infty} \left(\frac{\sum_{i=1}^{n} (q_i - p_i)}{p_{n+1} + d} - \frac{2dn}{p_{n+1} + d} \right) \le \underline{d}(A)$$
$$\le \liminf_{n\to\infty} \left(\frac{\sum_{i=1}^{n} (q_i - p_i)}{p_{n+1} - d} + \frac{2dn}{p_{n+1} - d} \right).$$

As both $\lim\limits_{n\to\infty} \frac{n}{p_{n+1}+d}$ and $\lim\limits_{n\to\infty} \frac{n}{p_{n+1}-d}$ equal 0 an application of the Sandwich Theorem completes the proof for $\underline{d}(A)$. In a very similar way one can prove the corresponding statement for $\overline{d}(A)$.

Let s be the first positive integer i such that $p_i - d > 0$. The following inequalities hold for every $i = s, s+1, \dots$.

$$\text{(I)} \qquad \ln\frac{q_i - d}{p_i + d} = \ln\frac{q_i}{p_i} - \int\limits_{p_i}^{p_i+d} \frac{1}{x}\,dx - \int\limits_{q_i-d}^{q_i} \frac{1}{x}\,dx \geq \ln\frac{q_i}{p_i} - \frac{2d}{p_i - d}$$

and

$$\text{(II)} \qquad \ln\frac{q_i + d}{p_i - d} = \ln\frac{q_i}{p_i} + \int\limits_{p_i-d}^{p_i} \frac{1}{x}\,dx + \int\limits_{q_i}^{q_i+d} \frac{1}{x}\,dx \leq \ln\frac{q_i}{p_i} - \frac{2d}{p_i - d}.$$

Again, a simple analysis shows that

$$\liminf_{n\to\infty} \frac{\sum\limits_{i=s}^{n} \ln\frac{q_i-d}{p_i+d}}{\ln(p_{n+1}+d)} \leq \liminf_{n\to\infty} \frac{\sum\limits_{i=s}^{n} \ln\frac{q_i+b_i}{p_i+a_i}}{\ln(p_{n+1}+a_{n+1})} = \underline{\delta}(A)$$
$$\leq \liminf_{n\to\infty} \frac{\sum\limits_{s=1}^{n} \ln\frac{q_i+d}{p_i-d}}{\ln(p_{n+1}-d)}$$

and, using (I) and (II),

$$\liminf_{n\to\infty} \left(\frac{\sum\limits_{i=s}^{n} \ln\frac{q_i}{p_i}}{\ln(p_{n+1}+d)} - 2d\frac{\sum\limits_{i=s}^{n} \frac{1}{p_i-d}}{\ln(p_{n+1}+d)} \right) \leq \underline{\delta}(A)$$
$$\leq \liminf_{n\to\infty} \left(\frac{\sum\limits_{i=s}^{n} \ln\frac{q_i}{p_i}}{\ln(p_{n+1}-d)} + 2d\frac{\sum\limits_{i=s}^{n} \frac{1}{p_i-d}}{\ln(p_{n+1}-d)} \right).$$

As the series $\sum\limits_{i=1}^{n} \frac{1}{p_i-d}$ is convergent and $\lim\limits_{n\to\infty} \frac{\ln(p_{n+1}+d)}{\ln(p_{n+1}-d)} = 1$ an application of the Sandwich Theorem completes the proof for $\underline{\delta}(A)$. In a very similar way one can prove the corresponding statement for $\overline{\delta}(A)$. □

The following simple lemma will be used in later calculations.

Lemma 2. *Let* $(a_i)_{1\leq i\leq n}$ *and* $(b_i)_{1\leq i\leq n}$ *such that* $1 \leq a_1$, $a_i < b_i$, $i = 1, 2, \dots, n$, $b_i \leq a_{i+1}$, $i = 1, 2, \dots, n-1$, *and* $b_n \leq w$. *Then* $\ln w \geq \sum\limits_{i=1}^{n} \ln\frac{b_i}{a_i}$.

Proof. The statement of the Lemma is a straightforward consequence of the following relations

$$w \geq b_n = \frac{b_n}{a_n}\frac{a_n}{b_{n-1}}\frac{b_{n-1}}{a_{n-1}}\cdots\frac{b_2}{a_2}\frac{a_2}{b_1}\frac{b_1}{a_1}a_1 = a_1\left(\prod_{i=1}^{n}\frac{b_i}{a_i}\right)\left(\prod_{i=1}^{n-1}\frac{a_{i+1}}{b_i}\right) \geq \prod_{i=1}^{n}\frac{b_i}{a_i}.$$

□

The following class of sets plays an important role in our consideration.

$$\mathbb{A} = \left\{A(a,b) = \bigcup_{n=s}^{\infty}\langle a^n b^n + 1, a^{n+1}b^n\rangle \cap \mathbb{N};\ 1 < a < b\right\}$$

where $s = \min\{n \in \mathbb{N};\ a^n b^n + 1 \leq a^{n+1}b^n\}$.

Theorem 1. *Let* $1 < a < b$ *and* $A = A(a,b) \in \mathbb{A}$. *Then*

(i) $$R(A) \cap \langle a,b\rangle = \emptyset,$$

(ii) $$\underline{d}(A) = \frac{a-1}{ab-1}, \qquad \overline{d}(A) = \frac{b(a-1)}{ab-1},$$

(iii) $$\delta(A) = \frac{\ln a}{\ln a + \ln b}.$$

Proof. (i) Let $x \in A$, $y \in A$ and $x < y$. First, let there be a $n \in \mathbb{N}$ such that both x and y belong to the block $\langle a^n b^n + 1, a^{n+1}b^n\rangle$. Then

$$\frac{y}{x} \leq \frac{a^{n+1}b^n}{a^n b^n + 1} < \frac{a^{n+1}b^n}{a^n b^n} = a.$$

On the other hand, let $x \in \langle a^n b^n + 1, a^{n+1}b^n\rangle$ and $y \in \langle a^m b^m + 1, a^{m+1}b^m\rangle$ for $m > n$. Then

$$\frac{y}{x} \geq \frac{a^m b^m + 1}{a^{n+1}b^n} > \frac{a^m b^m}{a^{n+1}b^n} \geq b.$$

In both cases $\frac{y}{x}$ does not belong to $\langle a,b\rangle$.

(ii) Calculate, using Lemma 1,

$$\begin{aligned}\underline{d}(A) &= \liminf_{n\to\infty}\frac{\sum_{i=s}^{n}(a^{i+1}b^i - a^i b^i)}{a^{n+1}b^{n+1}} = (a-1)a^s b^s \liminf_{n\to\infty}\frac{\sum_{i=0}^{n-s} a^i b^i}{a^{n+1}b^{n+1}} \\ &= \frac{a-1}{ab-1}\lim_{n\to\infty}\frac{a^{n+1}b^{n+1} - a^s b^s}{a^{n+1}b^{n+1}} = \frac{a-1}{ab-1}.\end{aligned}$$

The corresponding value of $\overline{d}(A)$ can be calculated in a very similar way.

(iii) Again, using Lemma 1, we have

$$\underline{\delta}(A) = \liminf_{n\to\infty} \frac{\sum_{i=s}^{n} \ln \frac{a^{i+1}b^i}{a^i b^i}}{\ln(a^{n+1}b^{n+1})} = \lim_{n\to\infty} \frac{(n-s+1)\ln a}{(n+1)(\ln a + \ln b)} = \frac{\ln a}{\ln a + \ln b}.$$

□

Remark 1. A simple analysis of equalities (ii) in Theorem 1 in comparison to the results (S1), (S2), (S3) shows

(A1) $$\{d(A);\ A \in \mathbb{D}\} = \{0\} = \{d(A);\ A \in \mathbb{A}\}.$$

(A2) $$\{\underline{d}(A);\ A \in \mathbb{D}\} = \langle 0, \tfrac{1}{2}) = \{\underline{d}(A);\ A \in \mathbb{A}\}.$$

(A3) $$\{\overline{d}(A);\ A \in \mathbb{D}\} = \langle 0, 1) = \{\overline{d}(A);\ A \in \mathbb{A}\}.$$

A similar analysis of equality (iii) in Theorem 1 leads to the following.

Conjecture. *The following equalities hold*

$$\{\underline{\delta}(A);\ A \in \mathbb{D}\} = \{\underline{\delta}(A);\ A \in \mathbb{A}\} = \{\overline{\delta}(A);\ A \in \mathbb{D}\} =$$
$$= \{\overline{\delta}(A);\ A \in \mathbb{A}\} = \{\delta(A);\ A \in \mathbb{D}\} = \{\delta(A);\ A \in \mathbb{A}\} = \left\langle 0, \frac{1}{2} \right) .$$

The purpose of the rest of this paper is to prove this conjecture. All the corresponding results will be corollaries to the following.

Theorem 2. *Let* $1 < a < b$ *and* $A = A(a,b) \in \mathbb{A}$. *Then the set* A *is maximal element in the set* $\{X \subset \mathbb{N};\ R(X) \cap \langle a, b \rangle = \emptyset\}$ *with respect to the partial order induced by any of* $\underline{\delta}$, $\overline{\delta}$, δ.

Proof. Let $X \subset \mathbb{N}$ be an infinite set such that $R(X) \cap \langle a, b \rangle = \emptyset$. Then X can be written in the form

$$X = \bigcup_{n=1}^{\infty} \langle p_n + 1, q_n \rangle \cap \mathbb{N}; \quad 0 \le p_1 < q_1 \le p_2 < q_2 \le \cdots \text{ are integers.}$$

For the proof it is sufficient to show (taking into account Theorem 1 (iii))

$$\overline{\delta}(X) \le \overline{\delta}(A(a,b)) = \delta(A(a,b)) = \frac{\ln a}{\ln a + \ln b}.$$

Thus we can also suppose

(0) $$\overline{\delta}(X) > 0.$$

The proof will be carried in several steps.

Step 1. In this step we will prove

(1) $$\text{if } p_n \ge \frac{1}{b-a} \text{ then } q_n < a(p_n + 1).$$

Proof of (1). Suppose $q_n \geq a(p_n+1)$. Then $\frac{q_n}{p_n+1} \geq a$ and also $\frac{p_n+1}{p_n+1} < a$ and, as $R(X) \cap \langle a,b\rangle = \emptyset$, there exists $m \in \langle p_n+1, q_n\rangle \cap \mathbb{N}$ such that $\frac{m}{p_n+1} < a$ and $\frac{m+1}{p_n+1} > b$. Consequently $\frac{m+1}{p_n+1} - \frac{m}{p_n+1} > b-a$ which implies $p_n < \frac{1}{b-a}$, a contradiction.

Step 2. In this step we will prove

$$\text{if } p_n \geq \frac{ab}{b-a} \text{ then } X \cap \langle a(p_n+1), bq_n\rangle = \emptyset. \tag{2}$$

Proof of (2). Let $p_n \geq \frac{ab}{b-a}$. Then also $p_n \geq \frac{1}{b-a}$ and, by the previous step, $q_n < a(p_n+1)$. Suppose on the contrary that there exists $x \in X \cap \langle a(p_n+1), bq_n\rangle$. Then $\frac{x}{p_n+1} \geq \frac{a(p_n+1)}{p_n+1} = a$, $\frac{x}{q_n} \leq \frac{bq_n}{q_n} = b$ and, as $R(X) \cap \langle a,b\rangle = \emptyset$, there exists $m \in \langle p_n+1, q_n\rangle \cap \mathbb{N}$ such that $\frac{x}{m} > b$ and $\frac{x}{m+1} < a$. Consequently

$$b-a < \frac{x}{m} - \frac{x}{m+1} = \frac{x}{m(m+1)} < \frac{x}{m^2} < \frac{bq_n}{(p_n+1)^2} < \frac{ba(p_n+1)}{(p_n+1)^2} = \frac{ab}{p_n+1}$$

which implies $p_n < \frac{ab}{b-a}$, a contradiction.

Step 3. In this step we will introduce some useful notation. Denote by k_0 the smallest integer k such that $p_k \geq \frac{ab}{b-a}$ and let $K_0 = \{k_0,\ k_0+1,\ k_0+2,\ \dots\}$. From (2) we have for every $k \in K_0$

$$\langle a(p_k+1), bq_k\rangle \cap \mathbb{N} \subset \mathbb{N} - X \tag{3}$$

and so we can define a function $\varphi:\ K_0 \to K_0$ by

$$\langle a(p_k+1), bq_k\rangle \subset (q_{\varphi(k)}, p_{\varphi(k)+1}+1). \tag{4}$$

The range of this function $\varphi(K_0) = \{l_1 < l_2 < \dots\}$ is infinite, denote $K_n = \varphi^{-1}(l_n)$ for each $n \in \mathbb{N}$. Evidently $K_0 = \bigcup_{n=1}^{\infty} K_n$ and for every $m < n$, $x \in K_m$ and $y \in K_n$ it is $x < y$. Let us call a **big gap in** X any interval of the form $(q_{\varphi(k)}, p_{\varphi(k)+1}+1)$ where $k \in K_n$ for $n \in \mathbb{N}$. Finally, let us introduce two sequences $\{u_n\}_{n=1}^{\infty}$, $\{v_n\}_{n=1}^{\infty}$ by

$$u_n = p_{\min\ K_n}, \quad v_n = q_{\max\ K_n} \quad \text{for} \quad n = 1,2,3,\dots\ .$$

The above definitions imply that both $a(u_n+1)$ and bv_n belong to the same big gap in X and, consequently,

$$v_n < a(u_n+1) \quad \text{for} \quad n = 1,2,3,\dots\ . \tag{5}$$

Step 4. In this step we will present and prove (if necessary) some simple relations and statement which will be used in the final calculation.

An easy analysis proves

$$(6)\qquad \sum_{a=p+1}^{q} \frac{1}{a} \le \ln \frac{q}{p} \quad \text{for all positive integers } p < q.$$

From Lemma 1 and (0) it can be seen that

$$(7)\qquad \text{the series} \quad \sum_{i=1}^{\infty} \ln \frac{q_i}{p_i} \quad \text{is divergent.}$$

As $a(u_{i+1}+1)$ belongs to the big gap next to the big gap in which $b(u_i+1)$ lies we have $u_{i+1}+1 \ge \frac{b}{a}(u_i+1)$ for every $i \in \mathbb{N}$. Therefore the series $\sum_{i=1}^{\infty} \frac{1}{u_i}$ is convergent and, consequently,

$$(8)\qquad \sum_{i=1}^{\infty} \ln \frac{u_i+1}{u_i} < \infty, \ \lim_{i\to\infty} \ln(1+\frac{1}{u_i}) = 0.$$

Step 5. For the rest of proof suppose that m is a sufficiently large fixed positive integer and denote by n the greatest (fixed from this moment) positive integer k for which $bv_k \le m$. Thus, using (5), we have

$$(9)\qquad bv_n \le m < bv_{n+1} < ab(u_{n+1}+1).$$

The considerations in Step 3 imply that the intervals $\langle p_i+1, q_i\rangle$ for $i = 1, 2, \dots, \max K_n$ and $\langle a(u_j+1), bv_j\rangle$ for $j = 1, 2, \dots, n$ are mutually disjoint and, by definition of numbers m and n, they are all contained in the interval $\langle 1, m\rangle$. Thus, by Lemma 2, we have

$$(10)\qquad \ln m \ge \sum_{i=1}^{\max K_n} \ln \frac{q_i}{p_i} + \sum_{j=1}^{n} \ln \frac{b}{a}\frac{v_j}{u_j+1}.$$

For similar reason we have

$$(11)\qquad \sum_{j=1}^{\max K_n} \ln \frac{q_j}{p_j} \le \sum_{j=1}^{k_0-1} \ln \frac{q_j}{p_j} + \sum_{i=1}^{n} \ln \frac{v_i}{u_i}.$$

The last inequality together with (5) imply

$$(12)\qquad \sum_{j=1}^{\max K_n} \ln \frac{q_j}{p_j} \le \sum_{j=1}^{k_0-1} \ln \frac{q_j}{p_j} + n \ln a + \sum_{i=1}^{n} \ln \left(1 + \frac{1}{u_i}\right).$$

Step 6. Denote $c = \sum_{j=1}^{k_0-1} \ln \frac{q_j}{p_j}$. Now we are able to estimate

$$
\begin{aligned}
s(m) &= \frac{\sum_{a\in X, a\le m} \frac{1}{a}}{\ln m} = \frac{\sum_{a\in X, a\le v_n} \frac{1}{a} + \sum_{a\in X, u_{n+1}<a\le m} \frac{1}{a}}{\ln m} \\
&\le \frac{\sum_{i=1}^{\max K_n} \sum_{a=p_i+1}^{q_i} \frac{1}{a} + \sum_{a=u_{n+1}+1}^{m} \frac{1}{a}}{\ln m} \le \frac{\sum_{i=1}^{\max K_n} \ln \frac{q_i}{p_i} + \ln \frac{m}{u_{n+1}}}{\ln m} \quad \text{by (6)} \\
&\le \frac{\sum_{i=1}^{\max K_n} \ln \frac{q_i}{p_i} + \ln \frac{ab(u_{n+1}+1)}{u_{n+1}}}{\sum_{i=1}^{\max K_n} \ln \frac{q_i}{p_i} + n \ln \frac{b}{a} + \sum_{i=1}^{n} \ln \frac{v_i}{u_i} + \sum_{i=1}^{n} \ln \frac{u_i}{u_i+1}} \quad \text{by (9), (10)} \\
&= \frac{1 + \frac{\ln ab + \ln(1+\frac{1}{u_{n+1}})}{\sum_{i=1}^{\max K_n} \ln \frac{q_i}{p_i}}}{1 + \frac{n \ln \frac{b}{a}}{\sum_{i=1}^{\max K_n} \ln \frac{q_i}{p_i}} + \frac{\sum_{i=1}^{n} \ln \frac{v_i}{u_i}}{\sum_{i=1}^{\max K_n} \ln \frac{q_i}{p_i}} + \frac{\sum_{i=1}^{n} \ln \frac{u_i}{u_i+1}}{\sum_{i=1}^{\max K_n} \ln \frac{q_i}{p_i}}} \\
&\le \frac{1 + \frac{\ln ab + \ln(1+\frac{1}{u_{n+1}})}{\sum_{i=1}^{\max K_n} \ln \frac{q_i}{p_i}}}{1 + \frac{n \ln \frac{b}{a}}{c + n \ln a + \sum_{i=1}^{n} \ln(1+\frac{1}{u_i})} + \frac{\left(\sum_{i=1}^{\max K_n} \ln \frac{q_i}{p_i}\right) - c}{\sum_{i=1}^{\max K_n} \ln \frac{q_i}{p_i}} + \frac{\sum_{i=1}^{n} \ln \frac{u_i}{u_i+1}}{\sum_{i=1}^{\max K_n} \ln \frac{q_i}{p_i}}} \quad \text{by (11), (12)} \\
&= S(m).
\end{aligned}
$$

Step 7. In this step we will complete the proof by limit process. Let $m \to \infty$ (and consequently $n \to \infty$). Then, using (7) and (8), we have

$$
\begin{aligned}
\overline{\delta}(X) = \limsup_{m\to\infty} s(m) \le \limsup_{m\to\infty} S(m) &= \lim_{m\to\infty} \frac{1+0}{1 + \frac{\ln b - \ln a}{\ln a} + 1 + 0} \\
&= \frac{\ln a}{\ln a + \ln b}.
\end{aligned}
$$

□

The following corollary is a direct consequence of the previous theorem and it shows that the relations between (R)-density and logarithmic densities are completely different from those between (R)-density and asymptotic densities.

Corollary. *The following relations hold*

$$\left\{\underline{\delta}(A);\ A \in \mathbb{D}\right\} = \left\{\overline{\delta}(A);\ A \in \mathbb{D}\right\} = \left\{\delta(A);\ A \in \mathbb{D}\right\} = \left\langle 0, \frac{1}{2}\right\rangle .$$

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