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Birational transformations and values of the Riemann zeta-function

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Résumé. Dans sa preuve du théorème d'Apéry sur l'irrationalité de $\zeta(3)$, Beukers [B] a introduit des intégrales doubles et triples de fonctions rationnelles donnant de bonnes suites d'approximations rationnelles de $\zeta(2)$ et $\zeta(3)$. La méthode de Beukers a été, par la suite, améliorée par Dvornicich et Viola, par Hata, et par Rhin et Viola. Nous présentons ici un survol de nos résultats récents ([RV2] et [RV3]) sur les mesures d'irrationalité de $\zeta(2)$ et $\zeta(3)$ obtenus par de nouvelles méthodes algébriques mettant en jeu les actions de transformations birationnelles et de groupes de permutations sur des intégrales doubles et triples du type de celles introduites par Beukers. Dans les deux dernières parties, nous donnons une méthode constructive pour obtenir les transformations birationnelles appropriées pour les intégrales triples à partir des transformations correspondantes pour les intégrales doubles. Cette méthode est également appliquée pour obtenir l'action de transformations birationnelles sur des intégrales quadruples du type de celles introduites par Vasilyev.

Abstract. In his proof of Apéry's theorem on the irrationality of $\zeta(3)$, Beukers [B] introduced double and triple integrals of suitable rational functions yielding good sequences of rational approximations to $\zeta(2)$ and $\zeta(3)$. Beukers' method was subsequently improved by Dvornicich and Viola, by Hata, and by Rhin and Viola. We give here a survey of our recent results ([RV2] and [RV3]) on the irrationality measures of $\zeta(2)$ and $\zeta(3)$ based upon a new algebraic method involving birational transformations and permutation groups acting on double and triple integrals of Beukers' type. In the last two sections we give a constructive method to obtain the relevant birational transformations for triple integrals from the analogous transformations for double integrals, and we also apply such a method to get birational transformations acting on quadruple integrals of Vasilyev's type.

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1. The irrationality measures of $\zeta(2)$ and $\zeta(3)$

Let $\mu(\alpha)$ denote the least irrationality measure of an irrational number α, i.e., the least exponent λ such that for any $\varepsilon > 0$ there exists a constant $q_0 = q_0(\varepsilon) > 0$ for which

$$\left| \alpha - \frac{p}{q} \right| > q^{-\lambda-\varepsilon}$$

for all integers p and q with $q > q_0$. We recall that an irrationality measure of a number α is usually obtained through the construction of a convenient sequence (r_n/s_n) of rational approximations to α, by applying the following well known

PROPOSITION *Let $\alpha \in \mathbb{R}$, and let (r_n), (s_n) be sequences of integers satisfying*

$$\lim_{n\to\infty} \frac{1}{n} \log |r_n - s_n\alpha| = -R$$

and

$$\limsup_{n\to\infty} \frac{1}{n} \log |s_n| \le S$$

for some positive numbers R and S. Then $\alpha \notin \mathbb{Q}$, and

$$\mu(\alpha) \le \frac{S}{R} + 1.$$

The irrationality measures of $\zeta(2) = \pi^2/6$ and of $\zeta(3)$, where ζ denotes the Riemann zeta-function, were extensively studied in recent years. Since Apéry's paper [A], several improvements in the search for sequences of good rational approximations to $\zeta(2)$ and to $\zeta(3)$ were obtained. Thus, the irrationality measures successively proved for these constants are as follows:

$$\begin{aligned} \mu(\zeta(2)) &< 11.85078\ldots && \text{(Apéry [A])}, \\ &< 10.02979\ldots && \text{(Dvornicich and Viola [DV])}, \\ &< 7.5252 && \text{(Hata [H1])}, \\ &< 7.398536\ldots && \text{(Rhin and Viola [RV1])}, \\ &< 6.3489 && \text{(Hata [H2])}, \\ &< 5.687 && \text{(Hata [H2], Addendum)}, \\ &< 5.441242\ldots && \text{(Rhin and Viola [RV2])}, \end{aligned} \tag{1.1}$$

and

$$\begin{aligned}
\mu(\zeta(3)) &< 13.41782\ldots && \text{(Apéry [A])},\\
&< 12.74359\ldots && \text{(Dvornicich and Viola [DV])},\\
&< 8.830283\ldots && \text{(Hata [H1])},\\
&< 7.377956\ldots && \text{(Hata [H3])},\\
&< 5.513890\ldots && \text{(Rhin and Viola [RV3])}.
\end{aligned} \tag{1.2}$$

Soon after Apéry's results [A], Beukers [B] found through a different method the same sequences of rational approximations to $\zeta(2)$ and $\zeta(3)$ previously obtained by Apéry. Beukers' method was based on the arithmetical study of the integrals

$$\int_0^1\int_0^1 \left(\frac{x(1-x)y(1-y)}{1-xy}\right)^n \frac{\mathrm{d}x\,\mathrm{d}y}{1-xy} \tag{1.3}$$

for $\zeta(2)$, and

$$\int_0^1\int_0^1\int_0^1 \left(\frac{x(1-x)y(1-y)z(1-z)}{1-(1-xy)z}\right)^n \frac{\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z}{1-(1-xy)z} \tag{1.4}$$

for $\zeta(3)$, and of their asymptotic behaviours as $n \to \infty$. The integrals (1.3) and (1.4) yield the same inequalities $\mu(\zeta(2)) < 11.85078\ldots$ and $\mu(\zeta(3)) < 13.41782\ldots$ found by Apéry. We remark that all the successive improvements (1.1) and (1.2) on Apéry's irrationality measures were obtained through the study of arithmetic and analytic properties of suitable variants of Beukers' integrals (1.3) and (1.4).

2. Beukers' and Hata's methods

Beukers' method [B] employs two different representations for each of the integrals (1.3) or (1.4). Consider (1.3) for simplicity. By n-fold partial integration one gets

$$\begin{aligned}
\int_0^1 x^n(1-x)^n \frac{\mathrm{d}x}{(1-xy)^{n+1}} &= -\frac{1}{ny}\int_0^1 \frac{\mathrm{d}}{\mathrm{d}x}\big(x^n(1-x)^n\big)\frac{\mathrm{d}x}{(1-xy)^n} = \ldots\\
&= \frac{(-1)^n}{n!\,y^n}\int_0^1 \frac{\mathrm{d}^n}{\mathrm{d}x^n}\big(x^n(1-x)^n\big)\frac{\mathrm{d}x}{1-xy}.
\end{aligned}$$

Therefore, if we denote by $L_n(x)$ the n-th Legendre polynomial defined by

$$L_n(x) = \frac{1}{n!}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\big(x^n(1-x)^n\big),$$

whence

$$L_n(x)=\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{n+k}{k}x^k\in\mathbb{Z}[x],$$

we obtain

(2.1)

$$\int_0^1\!\!\int_0^1\left(\frac{x(1-x)y(1-y)}{1-xy}\right)^n\frac{\mathrm{d}x\,\mathrm{d}y}{1-xy}\;=\;(-1)^n\int_0^1\!\!\int_0^1 L_n(x)(1-y)^n\,\frac{\mathrm{d}x\,\mathrm{d}y}{1-xy}\,.$$

Hence the integral (1.3) can be also expressed in the form

$$\int_0^1\!\!\int_0^1 P_n(x,y)\,\frac{\mathrm{d}x\,\mathrm{d}y}{1-xy}\,,\tag{2.2}$$

where $P_n(x,y)\in\mathbb{Z}[x,y]$ is a suitable polynomial (instead of a rational function as in (1.3)) satisfying $\deg_x P_n=\deg_y P_n=n$. Using the series expansion

$$\frac{1}{1-xy}\;=\;\sum_{k=0}^{\infty}x^k y^k,$$

one sees that (2.2) equals $a_n+b_n\zeta(2)$ with $b_n\in\mathbb{Z}$ and $d_n^2a_n\in\mathbb{Z}$, where $d_n=\text{l.c.m.}\{1,\dots,n\}$. Thus the right side of (2.1), being of type (2.2), shows that the integral has the required arithmetic form $a_n+b_n\zeta(2)$, whereas the left side of (2.1) is suitable to obtain asymptotic estimates of $a_n+b_n\zeta(2)$ and $|b_n|$ as $n\to\infty$. By combining such estimates with the asymptotic estimate $d_n=\exp(n+o(n))$ given by the Prime Number Theorem, and applying the Proposition in Section 1, one obtains the irrationality measure

$$\mu(\zeta(2))\;\le\;\frac{5\log\dfrac{\sqrt{5}+1}{2}+2}{5\log\dfrac{\sqrt{5}+1}{2}-2}+1\;=\;11.85078\ldots$$

A slightly more complicated argument, again based on repeated partial integration, yields the analogue of the formula (2.1) for the integral (1.4), namely

$$\begin{aligned}\int_0^1\!\!\int_0^1\!\!\int_0^1\left(\frac{x(1-x)y(1-y)z(1-z)}{1-(1-xy)z}\right)^n&\frac{\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z}{1-(1-xy)z}\\&=\int_0^1\!\!\int_0^1\!\!\int_0^1 L_n(x)L_n(y)\,\frac{\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z}{1-(1-xy)z}\,.\end{aligned}\tag{2.3}$$

As above, the right side of this formula is easily seen to be $a_n^* + 2b_n^*\zeta(3)$ with $b_n^* \in \mathbb{Z}$ and $d_n^3 a_n^* \in \mathbb{Z}$, while the left side of (2.3) yields the asymptotic estimates of $a_n^* + 2b_n^*\zeta(3)$ and $|b_n^*|$. Thus one gets the irrationality measure

$$\mu(\zeta(3)) \le \frac{4\log(\sqrt{2}+1)+3}{4\log(\sqrt{2}+1)-3} + 1 = 13.41782\ldots$$

In [H2] and [H3], Hata improved the above irrationality measures through the study of variants of the integrals (1.3) and (1.4), where the exponents of the factors x, $1-x$, y, $1-y$, $1-xy$ appearing in the rational function in (1.3), or of the factors x, $1-x$, y, $1-y$, z, $1-z$, $1-(1-xy)z$ in (1.4), are not all equal. For instance, in [H2] Hata considered the integral

$$\int_0^1\!\!\int_0^1 \left(\frac{x^{15}(1-x)^{15}y^{14}(1-y)^{14}}{(1-xy)^{12}}\right)^n \frac{\mathrm{d}x\,\mathrm{d}y}{1-xy}, \tag{2.4}$$

and transformed (2.4), by $12n$-fold partial integration with respect to x, into an integral of type (2.2) with $\deg_x P_n = 18n$ and $\deg_y P_n = 16n$, thus showing that (2.4) equals $a_n + b_n\zeta(2)$ with $a_n \in \mathbb{Q}$ and $b_n \in \mathbb{Z}$. More generally, k-fold and $(12n-k)$-fold partial integrations with respect to x and y respectively, for any k such that $0 \le k \le 12n$, transform (2.4) into an integral of type (2.2) divided by $\binom{12n}{k}$, where now $P_n(x,y) = F_k(x)G_k(y)$ for suitable Legendre-type polynomials F_k and G_k. On choosing k appropriately, Hata got a good control of the denominator of the rational part a_n of the integral (2.4) through the p-adic valuation of the above $\binom{12n}{k}$ and of the binomial coefficients occurring as coefficients of F_k and G_k. Thus Hata proved the irrationality measure $\mu(\zeta(2)) < 6.3489$. In his addendum to the paper [H2], he subsequently applied to the integral (2.4) a change of variables introduced in [RV1] showing that (2.4) equals

$$\int_0^1\!\!\int_0^1 \left(\frac{x^{17}(1-x)^{14}y^{15}(1-y)^{14}}{(1-xy)^{13}}\right)^n \frac{\mathrm{d}x\,\mathrm{d}y}{1-xy}. \tag{2.5}$$

By combining (2.4) and (2.5), Hata proved that $\mu(\zeta(2)) < 5.687$.

3. Permutation groups for double integrals

The equality of the integrals (2.4) and (2.5) is a special instance of a far more general phenomenon discovered by Rhin and Viola, and described in [RV2] for double integrals, and in [RV3] for triple integrals of Beukers' type. This phenomenon depends upon the actions of suitable birational transformations on the double or triple integrals considered.

In this section we outline the main results of [RV2]. Let

$$I(h,i,j,k,l) = \int_0^1\!\!\int_0^1 \frac{x^h(1-x)^i y^k(1-y)^j}{(1-xy)^{i+j-l}}\,\frac{\mathrm{d}x\,\mathrm{d}y}{1-xy}\,, \tag{3.1}$$

where h, i, j, k, l are any non-negative integers (this condition clearly ensures that $I(h,i,j,k,l)$ is finite). Let $\tau : (x,y) \mapsto (X,Y)$ be the birational transformation defined by the equations

$$\tau : \begin{cases} X = \dfrac{1-x}{1-xy} \\ Y = 1-xy. \end{cases} \tag{3.2}$$

It is easily seen that τ has period 5 and maps the open unit square $(0,1)^2$ onto itself. Moreover, we have

$$\frac{X(1-X)Y(1-Y)}{1-XY} = \frac{x(1-x)y(1-y)}{1-xy} \tag{3.3}$$

and

$$\frac{\mathrm{d}X\,\mathrm{d}Y}{1-XY} = \frac{\mathrm{d}x\,\mathrm{d}y}{1-xy}\,, \tag{3.4}$$

so that both the rational function (3.3) and the measure (3.4) are invariant under the action of τ.

If we apply the birational transformation τ to $I(h,i,j,k,l)$, i.e., if we make in the integral (3.1) the change of variables

$$\tau^{-1} : \begin{cases} x = 1-XY \\ y = \dfrac{1-Y}{1-XY} \end{cases}$$

and then replace X, Y with x, y respectively, by virtue of (3.4) we obtain the integral $I(i,j,k,l,h)$. Thus, it is natural to associate with the action of τ on $I(h,i,j,k,l)$ the cyclic permutation

$$\boldsymbol{\tau} = (h\ \ i\ \ j\ \ k\ \ l).$$

Also, if we apply to $I(h,i,j,k,l)$ the transformation

$$\sigma_2 : \begin{cases} X = y \\ Y = x, \end{cases}$$

i.e., if we interchange the variables x, y in (3.1), we get the integral $I(k,j,i,h,l)$. Hence with the action of σ_2 on $I(h,i,j,k,l)$ we associate the permutation

$$\boldsymbol{\sigma}_2 = (h\ \ k)(i\ \ j).$$

We use the subscript 2 for σ_2 and $\boldsymbol{\sigma}_2$, and later on for φ_2, $\boldsymbol{\varphi}_2$ and $\boldsymbol{\Phi}_2$ (see (3.19) below), to indicate that such transformations and permutations are related with double integrals. In Section 4, the analogues for triple integrals will be denoted by a similar notation with the subscript 3.

The permutation group

$$\boldsymbol{T} = \langle \boldsymbol{\tau}, \boldsymbol{\sigma}_2 \rangle$$

generated by $\boldsymbol{\tau}$ and $\boldsymbol{\sigma}_2$ is clearly isomorphic to the dihedral group $\mathfrak{D}_5$ of order 10, and the value of $I(h,i,j,k,l)$ is invariant under the action of the group $\boldsymbol{T}$ on h, i, j, k, l. Note that

$$I(15n, 15n, 14n, 14n, 17n)$$

is the integral (2.4), and

$$I(17n, 14n, 14n, 15n, 15n)$$

is (2.5), so that the equality of (2.4) and (2.5) is a special case of the formula

$$I(h,i,j,k,l) = I(l,k,j,i,h)$$

obtained by applying to h, i, j, k, l the permutation $\boldsymbol{\tau}\boldsymbol{\sigma}_2 \in \boldsymbol{T}$ (here and in the sequel, a product $\boldsymbol{\alpha}\boldsymbol{\beta}$ of permutations is meant to be the permutation obtained by applying first $\boldsymbol{\beta}$ and then $\boldsymbol{\alpha}$).

Besides the integers

$$h,\ i,\ j,\ k,\ l, \tag{3.5}$$

we consider the five auxiliary integers

$$j+k-h, \quad k+l-i, \quad l+h-j, \quad h+i-k, \quad i+j-l, \tag{3.6}$$

the last of which is the exponent of the factor $1-xy$ in the rational function appearing in the integral (3.1). Also, it is natural to extend the actions of the above permutations $\boldsymbol{\tau}$ and $\boldsymbol{\sigma}_2$ on any linear combination of h, i, j, k, l by linearity. Thus $\boldsymbol{\tau}(j+k-h) = \boldsymbol{\tau}(j) + \boldsymbol{\tau}(k) - \boldsymbol{\tau}(h) = k+l-i$, etc. Hence $\boldsymbol{\tau}$ and $\boldsymbol{\sigma}_2$ act on the ten integers (3.5) and (3.6) as follows:

$$\boldsymbol{\tau} = (h\ \ i\ \ j\ \ k\ \ l)(j+k-h\ \ \ k+l-i\ \ \ l+h-j\ \ \ h+i-k\ \ \ i+j-l) \tag{3.7}$$

and

$$\boldsymbol{\sigma}_2 = (h\ \ k)(i\ \ j)(j+k-h\ \ \ h+i-k)(k+l-i\ \ \ l+h-j). \tag{3.8}$$

Let $d_0 = 1$ and, as above, $d_n = \text{l.c.m.}\{1, \dots, n\}$ for any integer $n \geq 1$. We also need some further notation: we denote by max, max$'$, max$''$, ... the successive maxima in a finite sequence of real numbers. Precisely, if $\mathcal{A} = (a_1, \dots, a_n)$ is any finite sequence of real numbers (with $n \geq 3$) and $i_1, \dots, i_n$ is a reordering of $1, \dots, n$ such that

$$a_{i_1} \geq a_{i_2} \geq a_{i_3} \geq \cdots \geq a_{i_n},$$

we let

$$\max \mathcal{A} = a_{i_1}, \qquad \max{}' \mathcal{A} = a_{i_2}, \qquad \max{}'' \mathcal{A} = a_{i_3}. \tag{3.9}$$

For the integral (3.1), let $\mathcal{S}$ be the sequence of the integers (3.6):

$$\mathcal{S} = (j+k-h,\ k+l-i,\ l+h-j,\ h+i-k,\ i+j-l),$$

and let

$$M = \max \mathcal{S}, \qquad N = \max{}' \mathcal{S}. \tag{3.10}$$

We incidentally remark that our assumption $h,\ i,\ j,\ k,\ l \geq 0$ easily implies that at most two of the integers (3.6) can be strictly negative (in which case, the two negative integers must be cyclically consecutive in (3.6)). Therefore $\max \mathcal{S} \geq \max' \mathcal{S} \geq \max'' \mathcal{S} \geq 0$. One can show that

$$I(h,i,j,k,l) = a + b\zeta(2) \quad \text{with } b \in \mathbb{Z} \text{ and } d_M d_N a \in \mathbb{Z}, \tag{3.11}$$

where M and N are defined by (3.10). In [RV2], Theorem 2.2, this result is established in a slightly stronger form, i.e., with a more precise definition for N. In fact, depending on the numerical values for $h,\ i,\ j,\ k,\ l$, the N defined in [RV2] can be either $\max' \mathcal{S}$, or $\max'' \mathcal{S} < \max' \mathcal{S}$. However, the definition (3.10) for N turns out to be appropriate in practice, since the N defined in [RV2] equals $\max' \mathcal{S}$ for all the "good" numerical choices of $h,\ i,\ j,\ k,\ l$ (i.e., those eventually yielding good irrationality measures of $\zeta(2)$). The proof of (3.11) given in [RV2] is independent of a representation of type (2.2) for the integral (3.1), and relies on the invariance of the value of $I(h,i,j,k,l)$ under the actions of the permutations $\boldsymbol{\tau}$ and $\boldsymbol{\sigma}_2$, and on a method of descent based on suitable linear decompositions of the rational function appearing in (3.1). Moreover, the same method of descent shows that the integer b in (3.11) can be expressed as a double contour integral, in the form

$$b = -\left(\frac{1}{2\pi\sqrt{-1}}\right)^2 \oint\limits_{C} \oint\limits_{C_x} \frac{x^h(1-x)^i y^k (1-y)^j}{(1-xy)^{i+j-l}} \frac{\mathrm{d}x\,\mathrm{d}y}{1-xy}, \tag{3.12}$$

where $C = \{x \in \mathbb{C} : |x| = \varrho_1\}$ and $C_x = \{y \in \mathbb{C} : |y - 1/x| = \varrho_2\}$ for any $\varrho_1, \varrho_2 > 0$.

It is worth remarking that if $i+j-l > \min\{h,i,j,k\}$, the integral (3.1) cannot be transformed by partial integration into an integral of type (2.2) to which Hata's method applies. Thus it is essential to dispense with the partial integration method and with Legendre-type polynomials, and to apply directly to (3.1) the method of descent alluded to above.

As is obvious from (3.7) and (3.8), the permutation group $\boldsymbol{T} = \langle \boldsymbol{\tau}, \boldsymbol{\sigma}_2 \rangle$ is intransitive over the set of the ten integers (3.5) and (3.6). We will now enlarge our permutation group, under the further assumption that the integers (3.6) are all non-negative, by introducing a new permutation $\boldsymbol{\varphi}_2$ (defined below in (3.19)), which mixes up (3.5) with (3.6), so that the larger permutation group $\boldsymbol{\Phi}_2 = \langle \boldsymbol{\varphi}_2, \boldsymbol{\tau}, \boldsymbol{\sigma}_2 \rangle$ thus obtained is transitive over the set of (3.5) and (3.6). Moreover, the group $\boldsymbol{\Phi}_2$ has the advantage of bringing naturally into play the factorials of the integers (3.5) and (3.6), and the p-adic valuation of such factorials yields strong arithmetic information on the denominator of the $a \in \mathbb{Q}$ in (3.11). The permutation $\boldsymbol{\varphi}_2$ is related to Euler's integral representation of Gauss's hypergeometric function, and to the invariance of this function under the interchange of the two parameters appearing in the numerator of the hypergeometric series.

We recall that Gauss's hypergeometric function is defined as follows:

$$ {}_2F_1(\alpha,\beta;\gamma;y) = \sum_{n=0}^{\infty} \frac{(\alpha)_n\,(\beta)_n}{(\gamma)_n}\,\frac{y^n}{n!}\,, \tag{3.13}$$

where α, β and γ are complex parameters, $\gamma \neq 0,-1,-2,\dots$, and y is a complex variable satisfying $|y| < 1$. The Pochhammer symbols in (3.13) are defined by

$$(\alpha)_0 = 1, \qquad (\alpha)_n = \alpha(\alpha+1)\dots(\alpha+n-1) \quad (n = 1,2,\dots),$$

and similarly for $(\beta)_n$ and $(\gamma)_n$. Euler's integral representation, valid for $\operatorname{Re}\gamma > \operatorname{Re}\beta > 0$, is

$$ {}_2F_1(\alpha,\beta;\gamma;y) \;=\; \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)} \int_0^1 \frac{x^{\beta-1}(1-x)^{\gamma-\beta-1}}{(1-xy)^{\alpha}}\,\mathrm{d}x, \tag{3.14}$$

where Γ denotes the Euler gamma-function, and gives the analytic continuation of ${}_2F_1(\alpha,\beta;\gamma;y)$ outside the unit disc $|y| < 1$. Since, by (3.13),

$$ {}_2F_1(\alpha,\beta;\gamma;y) \;=\; {}_2F_1(\beta,\alpha;\gamma;y),$$

if $\operatorname{Re}\gamma > \max\{\operatorname{Re}\alpha, \operatorname{Re}\beta\}$ and $\min\{\operatorname{Re}\alpha, \operatorname{Re}\beta\} > 0$ we get from (3.14)

$$\int_0^1 \frac{x^{\beta-1}(1-x)^{\gamma-\beta-1}}{(1-xy)^{\alpha}}\,\mathrm{d}x \;=\; \frac{\Gamma(\beta)\Gamma(\gamma-\beta)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)} \int_0^1 \frac{x^{\alpha-1}(1-x)^{\gamma-\alpha-1}}{(1-xy)^{\beta}}\,\mathrm{d}x. \tag{3.15}$$

We henceforth assume the non-negative integers (3.5) to be such that (3.6) are also non-negative. Then we may take in (3.15)

$$\alpha = i+j-l+1, \qquad \beta = h+1, \qquad \gamma = h+i+2,$$

whence

$$\int_0^1 \frac{x^{h}(1-x)^{i}}{(1-xy)^{i+j-l+1}}\,\mathrm{d}x \;=\; \frac{h!\,i!}{(i+j-l)!(l+h-j)!} \int_0^1 \frac{x^{i+j-l}(1-x)^{l+h-j}}{(1-xy)^{h+1}}\,\mathrm{d}x.$$

Multiplying by $y^k(1-y)^j$ and integrating in $0 \le y \le 1$ we obtain

$$ I(h,i,j,k,l) \;=\; \frac{h!\,i!}{(i+j-l)!(l+h-j)!}\, I(i+j-l,\, l+h-j,\, j,\, k,\, l). \tag{3.16}$$

If we divide (3.16) by $h!\,i!\,j!\,k!\,l!$ we get

$$\frac{I(h,i,j,k,l)}{h!\,i!\,j!\,k!\,l!} = \frac{I(i+j-l,\, l+h-j,\, j,\, k,\, l)}{(i+j-l)!\,(l+h-j)!\,j!\,k!\,l!}\,. \tag{3.17}$$

Let φ_2 be the integral transformation changing

$$\frac{I(h,i,j,k,l)}{h!\,i!\,j!\,k!\,l!} \tag{3.18}$$

into

$$\frac{I(i+j-l,\, l+h-j,\, j,\, k,\, l)}{(i+j-l)!\,(l+h-j)!\,j!\,k!\,l!}\,,$$

and let $\boldsymbol{\varphi}_2$ be the corresponding permutation, mapping the integers (3.5) respectively to $i+j-l$, $l+h-j$, j, k, l, and extended to any linear combination of the integers (3.5) by linearity. Thus $\boldsymbol{\varphi}_2$ is the permutation acting on the ten integers (3.5) and (3.6) as follows:

$$\boldsymbol{\varphi}_2 = (h \;\; i+j-l)(i \;\; l+h-j)(j+k-h \;\; k+l-i). \tag{3.19}$$

By (3.17), the value of the quotient (3.18) is clearly invariant under the action on the integers (3.5) and (3.6) of the permutation group

$$\boldsymbol{\Phi}_2 = \langle \boldsymbol{\varphi}_2, \boldsymbol{\tau}, \boldsymbol{\sigma}_2 \rangle$$

generated by $\boldsymbol{\varphi}_2$, $\boldsymbol{\tau}$ and $\boldsymbol{\sigma}_2$.

We want to determine the structure of the permutation group $\boldsymbol{\Phi}_2$ acting on (3.5) and (3.6), and in particular its order $|\boldsymbol{\Phi}_2|$. For this purpose, we show that $\boldsymbol{\Phi}_2$ can be viewed as a permutation group acting on five integers only, i.e., on the sums

$$h+i,\;\; i+j,\;\; j+k,\;\; k+l,\;\; l+h. \tag{3.20}$$

To prove this, it clearly suffices to show that $\boldsymbol{\varphi}_2$, $\boldsymbol{\tau}$ and $\boldsymbol{\sigma}_2$ permute the integers (3.20), and that if a permutation $\boldsymbol{\varrho} \in \boldsymbol{\Phi}_2$ acts identically on the integers (3.20), it acts identically also on the integers (3.5), and therefore also on (3.6). It is plain that the actions of the above permutations on the integers (3.20) are as follows:

$$\begin{aligned} \boldsymbol{\varphi}_2 &= (i+j \;\; l+h), \\ \boldsymbol{\tau} &= (h+i \;\; i+j \;\; j+k \;\; k+l \;\; l+h), \\ \boldsymbol{\sigma}_2 &= (h+i \;\; j+k)(k+l \;\; l+h). \end{aligned} \tag{3.21}$$

Moreover, if $\boldsymbol{\varrho} \in \boldsymbol{\Phi}_2$ acts identically on the integers (3.20) we get, by linearity,

$$\begin{aligned} 2\boldsymbol{\varrho}(h) = \boldsymbol{\varrho}(2h) &= \boldsymbol{\varrho}\big((h+i)-(i+j)+(j+k)-(k+l)+(l+h)\big) \\ &= \boldsymbol{\varrho}(h+i)-\boldsymbol{\varrho}(i+j)+\boldsymbol{\varrho}(j+k)-\boldsymbol{\varrho}(k+l)+\boldsymbol{\varrho}(l+h) \\ &= (h+i)-(i+j)+(j+k)-(k+l)+(l+h) \\ &= 2h, \end{aligned}$$

and similarly

$$2\boldsymbol{\varrho}(i) = \boldsymbol{\varrho}(2i) = \boldsymbol{\varrho}(i+j) - \boldsymbol{\varrho}(j+k) + \boldsymbol{\varrho}(k+l) - \boldsymbol{\varrho}(l+h) + \boldsymbol{\varrho}(h+i) = 2i,$$

and so on. Hence $\boldsymbol{\varrho}$ acts identically on the integers (3.5). Since the symmetric group $\mathfrak{S}_5$ of the 5! permutations of five elements (i.e., of the integers (3.20)) is generated by a cyclic permutation of the five elements and a transposition, from (3.21) we conclude that

$$\boldsymbol{\Phi}_2 = \langle \boldsymbol{\varphi}_2, \boldsymbol{\tau}, \boldsymbol{\sigma}_2 \rangle = \langle \boldsymbol{\varphi}_2, \boldsymbol{\tau} \rangle \tag{3.22}$$

is isomorphic to $\mathfrak{S}_5$. Therefore $|\boldsymbol{\Phi}_2| = 120$.

We now consider again the actions (3.7), (3.8) and (3.19) of the above permutations on the ten integers (3.5) and (3.6). If we apply to (3.18) any permutation $\boldsymbol{\varrho} \in \boldsymbol{\Phi}_2$, we get the transformation formula

$$\frac{I(h,i,j,k,l)}{h!\,i!\,j!\,k!\,l!} = \frac{I\big(\boldsymbol{\varrho}(h), \boldsymbol{\varrho}(i), \boldsymbol{\varrho}(j), \boldsymbol{\varrho}(k), \boldsymbol{\varrho}(l)\big)}{\boldsymbol{\varrho}(h)!\,\boldsymbol{\varrho}(i)!\,\boldsymbol{\varrho}(j)!\,\boldsymbol{\varrho}(k)!\,\boldsymbol{\varrho}(l)!}\,. \tag{3.23}$$

Thus we associate with $\boldsymbol{\varrho}$ the quotient

$$\frac{h!\,i!\,j!\,k!\,l!}{\boldsymbol{\varrho}(h)!\,\boldsymbol{\varrho}(i)!\,\boldsymbol{\varrho}(j)!\,\boldsymbol{\varrho}(k)!\,\boldsymbol{\varrho}(l)!} \tag{3.24}$$

resulting from the transformation formula (3.23) for $I(h,i,j,k,l)$. If $\boldsymbol{\varrho}, \boldsymbol{\varrho}' \in \boldsymbol{\Phi}_2$ lie in the same left coset of $\boldsymbol{T}$ in $\boldsymbol{\Phi}_2$, i.e., if $\boldsymbol{\varrho} = \boldsymbol{\varrho}'\boldsymbol{\varrho}''$ with $\boldsymbol{\varrho}'' \in \boldsymbol{T}$, the integers $\boldsymbol{\varrho}(h)$, $\boldsymbol{\varrho}(i)$, $\boldsymbol{\varrho}(j)$, $\boldsymbol{\varrho}(k)$, $\boldsymbol{\varrho}(l)$ obviously coincide with $\boldsymbol{\varrho}'(h)$, $\boldsymbol{\varrho}'(i)$, $\boldsymbol{\varrho}'(j)$, $\boldsymbol{\varrho}'(k)$, $\boldsymbol{\varrho}'(l)$, up to a permutation. Hence the quotient (3.24) for $\boldsymbol{\varrho}$ equals the analogous quotient for $\boldsymbol{\varrho}'$. Thus with each left coset of $\boldsymbol{T}$ in $\boldsymbol{\Phi}_2$ we may associate the corresponding quotient (3.24), where $\boldsymbol{\varrho}$ is any representative of the coset considered. Also, for any $\boldsymbol{\varrho} \in \boldsymbol{\Phi}_2$ we simplify the quotient (3.24) by removing the factorials of the integers appearing both in the numerator and in the denominator. If, after this simplification, the resulting quotient of factorials has v factorials in the numerator and v in the denominator, we say that $\boldsymbol{\varrho}$ is a permutation of level v (or that the left coset $\boldsymbol{\varrho T}$ is of level v). In other words, $\boldsymbol{\varrho}$ is of level v if the intersection of the set $\{\boldsymbol{\varrho}(h), \boldsymbol{\varrho}(i), \boldsymbol{\varrho}(j), \boldsymbol{\varrho}(k), \boldsymbol{\varrho}(l)\}$ with the set of the integers (3.6) contains v elements.

Since $|\boldsymbol{\Phi}_2| = 120$ and $|\boldsymbol{T}| = 10$, there are 12 left cosets of $\boldsymbol{T}$ in $\boldsymbol{\Phi}_2$. In [RV2], pp. 39–40, we show that the 12 left cosets can be classified as follows:

1 coset of level 0,

5 cosets of level 2,

5 cosets of level 3,

1 coset of level 5.

Moreover, the 5 quotients of factorials associated with the 5 left cosets of level 2 are all distinct, and similarly the 5 quotients of factorials associated with the 5 left cosets of level 3 are all distinct.

For the rest of this section we define

$$M = \max \mathcal{T}, \qquad N = \max{}' \mathcal{T}, \tag{3.25}$$

where $\mathcal{T}$ is the sequence of the ten integers (3.5) and (3.6). Then, *a fortiori*, (3.11) holds with M and N given by (3.25). Also, for any $\boldsymbol{\varrho} \in \boldsymbol{\Phi}_2$, $\boldsymbol{\varrho}(j) + \boldsymbol{\varrho}(k) - \boldsymbol{\varrho}(h) = \boldsymbol{\varrho}(j+k-h)$ is one of the integers (3.5) or (3.6), and similarly for $\boldsymbol{\varrho}(k) + \boldsymbol{\varrho}(l) - \boldsymbol{\varrho}(i)$, etc. Hence we have, by (3.11),

$$I\big(\boldsymbol{\varrho}(h), \boldsymbol{\varrho}(i), \boldsymbol{\varrho}(j), \boldsymbol{\varrho}(k), \boldsymbol{\varrho}(l)\big) \;=\; a_{\boldsymbol{\varrho}} + b_{\boldsymbol{\varrho}}\zeta(2) \qquad \text{with } b_{\boldsymbol{\varrho}} \in \mathbb{Z} \text{ and } d_M d_N a_{\boldsymbol{\varrho}} \in \mathbb{Z}, \tag{3.26}$$

where M and N are defined by (3.25), and where $a_{\boldsymbol{\varrho}}$ and $b_{\boldsymbol{\varrho}}$ depend only on the left coset $\boldsymbol{\varrho}\boldsymbol{T}$.

For fixed h, i, j, k, l, we now abbreviate

$$I_n \;=\; I(hn, in, jn, kn, ln) \;=\; a_n + b_n\zeta(2) \qquad (n = 1, 2, \dots), \tag{3.27}$$

with $a_n \in \mathbb{Q}$ and $b_n \in \mathbb{Z}$. Pick any $\boldsymbol{\varrho} \in \boldsymbol{\Phi}_2$ of level 2: for instance, take $\boldsymbol{\varrho} = \boldsymbol{\varphi}_2$. By (3.16), the transformation formula for I_n corresponding to the permutation $\boldsymbol{\varphi}_2$ is

$$I_n = \frac{(hn)!\,(in)!}{\big((i+j-l)n\big)!\,\big((l+h-j)n\big)!}\, I'_n, \tag{3.28}$$

where

$$I'_n \;=\; I\big((i+j-l)n,\; (l+h-j)n,\; jn,\; kn,\; ln\big) \;=\; a'_n + b'_n\zeta(2), \tag{3.29}$$

with $a'_n \in \mathbb{Q}$ and $b'_n \in \mathbb{Z}$. By (3.27), (3.28) and (3.29), and by the irrationality of $\zeta(2)$, we obtain

$$\big((i+j-l)n\big)!\,\big((l+h-j)n\big)!\,a_n \;=\; (hn)!\,(in)!\,a'_n. \tag{3.30}$$

On multiplying (3.30) by $d_{Mn}d_{Nn}$, with M and N given by (3.25), we get

$$\big((i+j-l)n\big)!\,\big((l+h-j)n\big)!\,A_n \;=\; (hn)!\,(in)!\,A'_n, \tag{3.31}$$

where $A_n = d_{Mn}d_{Nn}a_n$ and $A'_n = d_{Mn}d_{Nn}a'_n$ are integers, by (3.11) and (3.26).

For a prime p, let

$$\omega = \{n/p\} = n/p - [n/p]$$

denote the fractional part of n/p. Using the p-adic valuation of the factorials appearing in (3.31), it is easy to see ([RV2], pp. 44–45) that any prime $p > \sqrt{Mn}$ for which

$$[(i+j-l)\omega] + [(l+h-j)\omega] < [h\omega] + [i\omega] \tag{3.32}$$

divides $A_n = d_{Mn}d_{Nn}a_n$.

The above discussion applies to each transformation formula for I_n corresponding to a left coset of $\boldsymbol{T}$ in $\boldsymbol{\Phi}_2$ of level 2. The 5 left cosets of level 2 yield the 5 inequalities for ω obtained by applying the powers of the permutation $\boldsymbol{\tau}$ to $(i+j-l,\, l+h-j,\, h,\, i)$ in (3.32). Let Ω be the set of real numbers $\omega \in [0,1)$ satisfying at least one of such 5 inequalities. We conclude that any prime $p > \sqrt{Mn}$, for which $\{n/p\} \in \Omega$, divides the integer $A_n = d_{Mn}d_{Nn}a_n$. We incidentally remark that Ω is clearly the union of finitely many intervals $[\alpha_q, \beta_q)$ with rational endpoints $\alpha_q, \beta_q \in (0,1)$.

A similar analysis applies to the 5 transformation formulae for I_n corresponding to the left cosets of level 3, and yields a subset $\Omega' \subset \Omega$ such that p^2 divides $A_n = d_{Mn}d_{Nn}a_n$ for any prime $p > \sqrt{Mn}$ satisfying $\{n/p\} \in \Omega'$. Finally, the transformation formula corresponding to the left coset of level 5 gives no further prime factors of A_n besides the above ([RV2], pp. 46–49).

Let

$$\Delta_n = \prod_{\substack{p>\sqrt{Mn} \\ \{n/p\}\in\Omega}} p\,, \qquad \Delta'_n = \prod_{\substack{p>\sqrt{Mn} \\ \{n/p\}\in\Omega'}} p \qquad (n = 1, 2, \ldots).$$

From the above discussion we get $\Delta_n\Delta'_n \mid A_n$. Also, it is easy to show that $\Delta_n \mid d_{Mn}$ and $\Delta'_n \mid d_{Nn}$. Hence, if we define

$$D_n = \frac{d_{Mn}d_{Nn}}{\Delta_n\Delta'_n}\,, \tag{3.33}$$

we have $D_n \in \mathbb{Z}$ and

$$D_n a_n = \frac{A_n}{\Delta_n\Delta'_n} \in \mathbb{Z}.$$

Therefore, by (3.27),

$$D_n I_n = D_n a_n + D_n b_n \zeta(2) \in \mathbb{Z} + \mathbb{Z}\zeta(2), \tag{3.34}$$

and the asymptotic estimates for D_n, I_n and $|b_n|$ as $n \to \infty$ in (3.34) yield an irrationality measure of $\zeta(2)$, by applying the Proposition in Section 1.

Since $d_{Mn}d_{Nn} = \exp((M+N)n + o(n))$ by the Prime Number Theorem, from (3.33) we get

$$\lim_{n\to\infty} \frac{1}{n} \log D_n \;=\; M + N - \lim_{n\to\infty} \frac{1}{n} \log \Delta_n - \lim_{n\to\infty} \frac{1}{n} \log \Delta'_n.$$

Standard arguments (see, e.g., [Vi], pp. 463–464) show that

$$\lim_{n\to\infty} \frac{1}{n} \log \Delta_n = \int_{\Omega} \mathrm{d}\psi(x), \qquad \lim_{n\to\infty} \frac{1}{n} \log \Delta'_n = \int_{\Omega'} \mathrm{d}\psi(x),$$

where $\psi(x) = \Gamma'(x)/\Gamma(x)$ is the logarithmic derivative of the Euler gamma-function. Therefore

$$\lim_{n\to\infty} \frac{1}{n} \log D_n \;=\; M + N - \Bigg(\int\limits_{\Omega} \mathrm{d}\psi(x) \;+ \int\limits_{\Omega'} \mathrm{d}\psi(x) \Bigg). \tag{3.35}$$

Note that if the integral I_n given by (3.27) is Beukers' integral (1.3), i.e., if $h = i = j = k = l$, the five inequalities of type (3.32) become $2[h\omega] < 2[h\omega]$ and therefore are false, whence $\Omega = \varnothing$. Thus

$$\int\limits_{\Omega} \mathrm{d}\psi(x) \;+ \int\limits_{\Omega'} \mathrm{d}\psi(x) \tag{3.36}$$

vanishes for the integral (1.3). A judicious choice of the parameters in (3.27), i.e., one where h, i, j, k, l are not all equal but their differences are not too large, forces to lose a certain amount on the asymptotics for I_n and $|b_n|$ in comparison with the integral (1.3), but allows to gain much more on the divisor D_n of $d_{Mn}d_{Nn}$, by virtue of the arithmetical correction (3.36) in the asymptotic formula (3.35).

Let

$$f(x,y) = \frac{x^h(1-x)^i y^k (1-y)^j}{(1-xy)^{i+j-l}} \,.$$

If we assume the integers (3.5) and (3.6) to be all strictly positive, it is easy to see that the function $f(x,y)$ has exactly two stationary points (x_0, y_0) and (x_1, y_1) satisfying $x(1-x)y(1-y) \neq 0$, with $0 < x_0, y_0 < 1$ and $x_1, y_1 < 0$, $x_1 y_1 > 1$. Plainly

$$\lim_{n\to\infty} \frac{1}{n} \log I_n = \log f(x_0, y_0), \tag{3.37}$$

and the double contour integral representation (3.12) for b_n easily yields

$$\limsup_{n\to\infty} \frac{1}{n} \log |b_n| \leq \log |f(x_1, y_1)|. \tag{3.38}$$

If we denote

$$c_0 = -\log f(x_0, y_0), \quad c_1 = \log |f(x_1, y_1)|,$$

$$c_2 \;=\; M + N - \Bigg(\int\limits_{\Omega} \mathrm{d}\psi(x) \;+ \int\limits_{\Omega'} \mathrm{d}\psi(x) \Bigg),$$

from (3.34), (3.35), (3.37), (3.38) and the Proposition in Section 1 we get the irrationality measure

$$\mu(\zeta(2)) \;\leq\; \frac{c_1 + c_2}{c_0 - c_2} + 1 \;=\; \frac{c_0 + c_1}{c_0 - c_2}\,, \tag{3.39}$$

provided that $c_0 > c_2$.

The choice $h = i = 12$, $j = k = 14$, $l = 13$ gives $M = j + k - h = 16$, $N = k + l - i = 15$,

$$\int_{\Omega} \mathrm{d}\psi(x) = 9.29787398\ldots, \qquad \int_{\Omega'} \mathrm{d}\psi(x) = 1.76783442\ldots,$$

whence $c_2 = 19.93429159\ldots$. Also $\begin{cases} c_0 = 31.27178857\ldots \\ c_1 = 30.41828189\ldots \end{cases}$. From (3.39) we obtain

$$\mu(\zeta(2)) < 5.441242\ldots.$$

4. Permutation groups for triple integrals

The theory outlined in Section 3 has its analogue for triple integrals of Beukers' type, and this has been worked out in [RV3]. In this section we briefly summarize the results of [RV3]. Again we introduce a permutation group arising on the one hand from the action on the triple integrals of a three-dimensional birational transformation ϑ (defined by (4.2) below), analogous to the two-dimensional transformation τ given by (3.2), and on the other hand from the hypergeometric integral transformation given by the relation (3.15). Thus, the hypergeometric part of our method for triple integrals still relies on the formula (3.15) involving simple integrals, while a process to derive the three-dimensional transformation ϑ from the two-dimensional transformation τ is definitely less obvious, and is described in Section 5. Moreover, as we have seen in Section 3, the rational function appearing in the double integral $I(h,i,j,k,l)$ contains 5 factors, the transformation τ has period 5, and the permutation group $\boldsymbol{\Phi}_2 = \langle \boldsymbol{\varphi}_2, \boldsymbol{\tau}, \boldsymbol{\sigma}_2 \rangle$ is isomorphic to the symmetric group $\mathfrak{S}_5$ of permutations of 5 elements. The situation for triple integrals that we treat in this section is more complicated, since no number has the role played by 5 for double integrals. In fact, the rational function in the triple integral (4.1) considered in this section contains 7 factors, but, in order to ensure that the integral has the required arithmetic expression, only 6 exponents of such factors are taken to be independent, by virtue of the linear condition (4.6) below. Also, the transformation ϑ has period 8, and the permutation group arising from ϑ and from the hypergeometric integral transformation can be naturally embedded in the alternating group $\mathfrak{A}_{10}$ of the even permutations of 10 elements.

For integer parameters h, j, k, l, q, r, s, we consider the integral

$$\int_0^1\!\!\int_0^1\!\!\int_0^1 \frac{x^h(1-x)^l y^k (1-y)^s z^j (1-z)^q}{(1-(1-xy)z)^{q+h-r}} \, \frac{\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z}{1-(1-xy)z}\,. \tag{4.1}$$

We assume $h, j, k, l, q, r, s \geq 0$ and $h \leq k+r$, since these inequalities are easily seen to be necessary and sufficient for the integral (4.1) to be finite. Let $\vartheta : (x,y,z) \mapsto (X,Y,Z)$ be the birational transformation defined by

$$\vartheta : \begin{cases} X = (1-y)z \\ Y = \dfrac{(1-x)(1-z)}{1-(1-xy)z} \\ Z = \dfrac{y}{1-(1-y)z} \,. \end{cases} \tag{4.2}$$

It is easy to check that ϑ has period 8 and maps the open unit cube $(0,1)^3$ onto itself. Moreover, under the action of ϑ we have

$$\frac{X(1-X)Y(1-Y)Z(1-Z)}{1-(1-XY)Z} = \frac{x(1-x)y(1-y)z(1-z)}{1-(1-xy)z}$$

and

$$\frac{\mathrm{d}X\,\mathrm{d}Y\,\mathrm{d}Z}{1-(1-XY)Z} = -\frac{\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z}{1-(1-xy)z} \,. \tag{4.3}$$

If we apply the transformation ϑ to the above integral, i.e., if we make in (4.1) the change of variables

$$\vartheta^{-1} : \begin{cases} x = \dfrac{(1-Y)(1-Z)}{1-(1-XY)Z} \\ y = (1-X)Z \\ z = \dfrac{X}{1-(1-X)Z} \,, \end{cases}$$

and then replace X, Y, Z with x, y, z respectively, by (4.3) the integral (4.1) becomes

$$\int_0^1\!\!\int_0^1\!\!\int_0^1 \frac{x^j(1-x)^{k+r-h}y^l(1-y)^h z^k(1-z)^r}{(1-(1-x)z)^{j+q-l-s}(1-(1-xy)z)^{r+l-q}} \frac{\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z}{1-(1-xy)z} \,. \tag{4.4}$$

Thus it is natural to define $m = k+r-h$, whence $m \geq 0$ and

$$h + m = k + r, \tag{4.5}$$

and to assume the linear condition

$$j + q = l + s, \tag{4.6}$$

which eliminates from (4.4) the undesired extra factor $1-(1-x)z$. Also, by (4.6) we have $r+l-q = r+j-s$, so that the integral (4.4) can be written in the form

$$\int_0^1\int_0^1\int_0^1 \frac{x^j(1-x)^m y^l(1-y)^h z^k(1-z)^r}{(1-(1-xy)z)^{r+j-s}} \frac{\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z}{1-(1-xy)z}.$$

This integral is obtained from (4.1) by applying to the parameters the cyclic permutation $(h\ j\ k\ l\ m\ q\ r\ s)$. Therefore, if for any non-negative integers h, j, k, l, m, q, r, s satisfying (4.5) and (4.6) we define

$$(4.7)\quad I(h,j,k,l,m,q,r,s) = \int_0^1\int_0^1\int_0^1 \frac{x^h(1-x)^l y^k(1-y)^s z^j(1-z)^q}{(1-(1-xy)z)^{q+h-r}} \frac{\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z}{1-(1-xy)z},$$

where m is a "hidden" parameter, we see that the transformation ϑ given by (4.2) changes the integral (4.7) into $I(j,k,l,m,q,r,s,h)$. Thus we associate with the action of ϑ on $I(h,j,k,l,m,q,r,s)$ the cyclic permutation

$$\boldsymbol{\vartheta} = (h\ \ j\ \ k\ \ l\ \ m\ \ q\ \ r\ \ s).$$

If we apply to the integral (4.7) the transformation

$$\sigma_3 : \begin{cases} X = y \\ Y = x \\ Z = z, \end{cases}$$

i.e., if we interchange the variables x, y in (4.7), by (4.5) we get the integral $I(k,j,h,s,r,q,m,l)$. Hence with the action of σ_3 on $I(h,j,k,l,m,q,r,s)$ we associate the permutation

$$\boldsymbol{\sigma}_3 = (h\ \ k)(l\ \ s)(m\ \ r).$$

The permutation group

$$\boldsymbol{\Theta} = \langle \boldsymbol{\vartheta}, \boldsymbol{\sigma}_3 \rangle$$

generated by $\boldsymbol{\vartheta}$ and $\boldsymbol{\sigma}_3$ is clearly isomorphic to the dihedral group $\mathfrak{D}_8$ of order 16, and the value of $I(h,j,k,l,m,q,r,s)$ is invariant under the action of $\boldsymbol{\Theta}$ on h, j, k, l, m, q, r, s.

In analogy with (3.5) and (3.6), besides the integers

$$(4.8)\qquad h,\ j,\ k,\ l,\ m,\ q,\ r,\ s$$

we consider the eight auxiliary integers

$$
\begin{aligned}
h' &:= h+l-j = h+q-s,\\
j' &:= j+m-k = j+r-h,\\
k' &:= k+q-l = k+s-j,\\
l' &:= l+r-m = l+h-k,\\
m' &:= m+s-q = m+j-l,\\
q' &:= q+h-r = q+k-m,\\
r' &:= r+j-s = r+l-q,\\
s' &:= s+k-h = s+m-r,
\end{aligned} \tag{4.9}
$$

where the double expression for each of the integers (4.9) is obtained by applying (4.5) or (4.6). Also, we extend the actions of the permutations $\boldsymbol{\vartheta}$ and $\boldsymbol{\sigma}_3$ on any linear combination of the integers (4.8) by linearity, and we can do this because

$$
\begin{aligned}
\boldsymbol{\vartheta}(h)+\boldsymbol{\vartheta}(m) &= j+q = l+s = \boldsymbol{\vartheta}(k)+\boldsymbol{\vartheta}(r),\\
\boldsymbol{\vartheta}(j)+\boldsymbol{\vartheta}(q) &= k+r = m+h = \boldsymbol{\vartheta}(l)+\boldsymbol{\vartheta}(s),\\
\boldsymbol{\sigma}_3(h)+\boldsymbol{\sigma}_3(m) &= k+r = h+m = \boldsymbol{\sigma}_3(k)+\boldsymbol{\sigma}_3(r),\\
\boldsymbol{\sigma}_3(j)+\boldsymbol{\sigma}_3(q) &= j+q = s+l = \boldsymbol{\sigma}_3(l)+\boldsymbol{\sigma}_3(s),
\end{aligned} \tag{4.10}
$$

so that $\boldsymbol{\vartheta}$ and $\boldsymbol{\sigma}_3$ preserve the relations (4.5) and (4.6). In particular, $\boldsymbol{\vartheta}$ and $\boldsymbol{\sigma}_3$ act on the sixteen integers (4.8) and (4.9) as follows:

$$\boldsymbol{\vartheta} = (h\ \ j\ \ k\ \ l\ \ m\ \ q\ \ r\ \ s)(h'\ \ j'\ \ k'\ \ l'\ \ m'\ \ q'\ \ r'\ \ s')$$

and

$$\boldsymbol{\sigma}_3 = (h\ \ k)(l\ \ s)(m\ \ r)(h'\ \ k')(l'\ \ s')(m'\ \ r').$$

Let

$$\mathcal{U} = (h', j', k', l', m', q', r', s')$$

be the sequence of the integers (4.9), and let

$$M = \max \mathcal{U}, \qquad N = \max{}' \mathcal{U}, \qquad Q = \max{}'' \mathcal{U},$$

where we use again the notation (3.9). It is easy to see that $M \geq N \geq Q \geq 0$. In [RV3], Theorem 2.1, we prove that

$$I(h,j,k,l,m,q,r,s) = a+2b\zeta(3) \quad \text{with } b\in\mathbb{Z} \text{ and } d_M d_N d_Q a \in \mathbb{Z}. \tag{4.11}$$

As with (3.11), the proof of (4.11) is based on the invariance of the value of the integral (4.7) under the actions of $\boldsymbol{\vartheta}$ and $\boldsymbol{\sigma}_3$, and on a suitable method of descent. We also have the analogue of (3.12), since the integer b in (4.11) is given by[1]

$$(4.12)\quad b = \left(\frac{1}{2\pi\sqrt{-1}}\right)^3 \times \oint_C \oint_{C_x} \oint_{C_{x,y}} \frac{x^h(1-x)^l y^k (1-y)^s z^j (1-z)^q}{(1-(1-xy)z)^{q+h-r}} \frac{\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z}{1-(1-xy)z},$$

where $C = \{x \in \mathbb{C} : |x| = \varrho_1\}$, $C_x = \{y \in \mathbb{C} : |y - 1/x| = \varrho_2\}$ and $C_{x,y} = \{z \in \mathbb{C} : |z - (1-xy)^{-1}| = \varrho_3\}$ for any $\varrho_1, \varrho_2, \varrho_3 > 0$.

The linear relations (4.5) and (4.6) are essential for the validity of (4.11), since if $j + q > l + s$, in general the integral (4.1) is a linear combination of 1, $\zeta(2)$ and $\zeta(3)$ with rational coefficients (see [RV3], Remark 2.2).

We now assume the non-negative integers (4.8) to be such that (4.9) are also non-negative. If in the formula (3.15) we change y into $-yz/(1-z)$, and take

$$\alpha = q + h - r + 1, \qquad \beta = h + 1, \qquad \gamma = h + l + 2,$$

we easily obtain

$$\int_0^1 \frac{x^h(1-x)^l}{(1-(1-xy)z)^{q+h-r+1}}\,\mathrm{d}x = (1-z)^{r-q}\,\frac{h!\,l!}{q'!\,r'!} \int_0^1 \frac{x^{q'}(1-x)^{r'}}{(1-(1-xy)z)^{h+1}}\,\mathrm{d}x.$$

Multiplying by $y^k(1-y)^s z^j (1-z)^q$ and integrating in $0 \le y \le 1$, $0 \le z \le 1$, we get

$$I(h,j,k,l,m,q,r,s) = \frac{h!\,l!}{q'!\,r'!}\, I(q',j,k,r',m,r,q,s).$$

Therefore

$$\frac{I(h,j,k,l,m,q,r,s)}{h!\,j!\,k!\,l!\,m!\,q!\,r!\,s!} = \frac{I(q',j,k,r',m,r,q,s)}{q'!\,j!\,k!\,r'!\,m!\,r!\,q!\,s!}.$$

Let φ_3 be the integral transformation changing

$$(4.13)\qquad \frac{I(h,j,k,l,m,q,r,s)}{h!\,j!\,k!\,l!\,m!\,q!\,r!\,s!}$$

into

$$\frac{I(q',j,k,r',m,r,q,s)}{q'!\,j!\,k!\,r'!\,m!\,r!\,q!\,s!},$$

and let $\boldsymbol{\varphi}_3$ be the corresponding permutation, mapping the integers (4.8) respectively to q', j, k, r', m, r, q, s, and extended to any linear combination

[1] I take this opportunity to correct a misprint in [RV3] occurring in the proof of the above integral representation (4.12) for b. On p. 277 of [RV3], in the line between the formulae (3.2) and (3.3), the inequality $\varrho_1\varrho_2\varrho_3 > 1$ should be replaced by $(\varrho_1\varrho_2 - 1)\varrho_3 > 1$.

of the integers (4.8) by linearity. In fact, in analogy with (4.10) we have

$$\boldsymbol{\varphi}_3(h)+\boldsymbol{\varphi}_3(m)=q'+m=k+q=\boldsymbol{\varphi}_3(k)+\boldsymbol{\varphi}_3(r),$$
$$\boldsymbol{\varphi}_3(j)+\boldsymbol{\varphi}_3(q)=j+r=r'+s=\boldsymbol{\varphi}_3(l)+\boldsymbol{\varphi}_3(s),$$

so that $\boldsymbol{\varphi}_3$ also preserves (4.5) and (4.6). Hence $\boldsymbol{\varphi}_3$ acts on (4.8) and (4.9) as follows:

$$\boldsymbol{\varphi}_3=(h\ \ q')(l\ \ r')(q\ \ r)(m'\ \ s').$$

We can also apply the hypergeometric transformation with respect to z. If in (3.15) we change x into z and y into $1-xy$, and take

$$\alpha=q+h-r+1,\qquad \beta=j+1,\qquad \gamma=j+q+2,$$

we have

$$\int_0^1 \frac{z^j(1-z)^q}{(1-(1-xy)z)^{q+h-r+1}}\,\mathrm{d}z \;=\; \frac{j!\,q!}{q'!\,j'!}\int_0^1 \frac{z^{q'}(1-z)^{j'}}{(1-(1-xy)z)^{j+1}}\,\mathrm{d}z.$$

Multiplying by $x^h(1-x)^l y^k(1-y)^s$ and integrating, we get

$$I(h,j,k,l,m,q,r,s) \;=\; \frac{j!\,q!}{q'!\,j'!}\,I(h,q',k,l,m,j',r,s),$$

whence

$$\frac{I(h,j,k,l,m,q,r,s)}{h!\,j!\,k!\,l!\,m!\,q!\,r!\,s!} \;=\; \frac{I(h,q',k,l,m,j',r,s)}{h!\,q'!\,k!\,l!\,m!\,j'!\,r!\,s!}\,.$$

We denote by χ this hypergeometric integral transformation, and by $\boldsymbol{\chi}$ the corresponding permutation acting on (4.8) and (4.9). Again, $\boldsymbol{\chi}$ preserves (4.5) and (4.6), since

$$\boldsymbol{\chi}(h)+\boldsymbol{\chi}(m)=h+m=k+r=\boldsymbol{\chi}(k)+\boldsymbol{\chi}(r),$$
$$\boldsymbol{\chi}(j)+\boldsymbol{\chi}(q)=q'+j'=l+s=\boldsymbol{\chi}(l)+\boldsymbol{\chi}(s).$$

Therefore

$$\boldsymbol{\chi}=(j\ \ q')(q\ \ j')(h'\ \ r')(k'\ \ m').$$

The value of the quotient (4.13) is clearly invariant under the action on the integers (4.8) and (4.9) of the permutation group

$$\boldsymbol{\Phi}_3=\langle\boldsymbol{\varphi}_3,\boldsymbol{\chi},\boldsymbol{\vartheta},\boldsymbol{\sigma}_3\rangle$$

generated by $\boldsymbol{\varphi}_3$, $\boldsymbol{\chi}$, $\boldsymbol{\vartheta}$ and $\boldsymbol{\sigma}_3$.

The structure of $\boldsymbol{\Phi}_3$ can be analysed by an argument similar in principle to the discussion on the structure of $\boldsymbol{\Phi}_2$ made in Section 3, although more complicated. $\boldsymbol{\Phi}_3$ can be viewed as a permutation group acting on ten integers, i.e., on the sums

$$h+l,\ j+m,\ k+q,\ l+r,\ m+s,\ q+h,\ r+j,\ s+k,\ j+q,\ k+r,$$

which, for brevity, we denote by $u_1, \dots, u_{10}$ respectively. In fact, the actions of $\boldsymbol{\varphi}_3$, $\boldsymbol{\chi}$, $\boldsymbol{\vartheta}$ and $\boldsymbol{\sigma}_3$ on such integers are the following:

$$\begin{aligned} \boldsymbol{\varphi}_3 &= (u_3\ \ u_{10})(u_7\ \ u_9), \\ \boldsymbol{\chi} &= (u_2\ \ u_3)(u_6\ \ u_7), \\ \boldsymbol{\vartheta} &= (u_1\ \ u_2\ \ u_3\ \ u_4\ \ u_5\ \ u_6\ \ u_7\ \ u_8)(u_9\ \ u_{10}), \\ \boldsymbol{\sigma}_3 &= (u_1\ \ u_8)(u_2\ \ u_7)(u_3\ \ u_6)(u_4\ \ u_5), \end{aligned} \tag{4.14}$$

and if a permutation $\boldsymbol{\varrho} \in \boldsymbol{\Phi}_3$ acts identically on the set

$$U = \{u_1, \dots, u_{10}\}$$

we get, by linearity,

$$\begin{aligned} 2\boldsymbol{\varrho}(h) = \boldsymbol{\varrho}(2h) &= \boldsymbol{\varrho}(u_1 - u_4 + u_{10} - u_3 + u_6) \\ &= \boldsymbol{\varrho}(u_1) - \boldsymbol{\varrho}(u_4) + \boldsymbol{\varrho}(u_{10}) - \boldsymbol{\varrho}(u_3) + \boldsymbol{\varrho}(u_6) \\ &= u_1 - u_4 + u_{10} - u_3 + u_6 \\ &= 2h, \end{aligned}$$

and similarly $2\boldsymbol{\varrho}(j) = \boldsymbol{\varrho}(2j) = \boldsymbol{\varrho}(u_2) - \boldsymbol{\varrho}(u_5) + \boldsymbol{\varrho}(u_9) - \boldsymbol{\varrho}(u_4) + \boldsymbol{\varrho}(u_7) = 2j$ (where we use $u_9 = j + q = l + s$), etc. Hence $\boldsymbol{\varrho}$ acts identically on the integers (4.8), and therefore also on (4.9). Since (4.14) are even permutations of U, we have an embedding of the group $\boldsymbol{\Phi}_3$ in the alternating group $\mathfrak{A}_{10}$ of all the even permutations of U.

The group $\langle \boldsymbol{\varphi}_3, \boldsymbol{\vartheta} \rangle$ is clearly transitive over U, and hence so is $\boldsymbol{\Phi}_3$. Moreover, $\boldsymbol{\Phi}_3$ is imprimitive over U, with blocks of imprimitivity given by the elements of the partition

$$\mathcal{P} = \big\{\{u_1, u_5\},\ \{u_2, u_6\},\ \{u_3, u_7\},\ \{u_4, u_8\},\ \{u_9, u_{10}\}\big\}$$

of the set U. In fact, the permutations $\boldsymbol{\varphi}_3^*$, $\boldsymbol{\chi}^*$, $\boldsymbol{\vartheta}^*$ and $\boldsymbol{\sigma}_3^*$ of $\mathcal{P}$ defined by

$$\begin{aligned} \boldsymbol{\varphi}_3^* &= \big(\{u_3, u_7\}\ \{u_9, u_{10}\}\big), \\ \boldsymbol{\chi}^* &= \big(\{u_2, u_6\}\ \{u_3, u_7\}\big), \\ \boldsymbol{\vartheta}^* &= \big(\{u_1, u_5\}\ \{u_2, u_6\}\ \{u_3, u_7\}\ \{u_4, u_8\}\big), \\ \boldsymbol{\sigma}_3^* &= \big(\{u_1, u_5\}\ \{u_4, u_8\}\big)\big(\{u_2, u_6\}\ \{u_3, u_7\}\big) \end{aligned}$$

are clearly induced by $\boldsymbol{\varphi}_3$, $\boldsymbol{\chi}$, $\boldsymbol{\vartheta}$ and $\boldsymbol{\sigma}_3$ respectively, whence the mapping $\boldsymbol{\varphi}_3 \mapsto \boldsymbol{\varphi}_3^*$, $\boldsymbol{\chi} \mapsto \boldsymbol{\chi}^*$, $\boldsymbol{\vartheta} \mapsto \boldsymbol{\vartheta}^*$, $\boldsymbol{\sigma}_3 \mapsto \boldsymbol{\sigma}_3^*$ extends to a homomorphism $\boldsymbol{\Phi}_3 \xrightarrow{*} \mathfrak{S}_5$ of the group $\boldsymbol{\Phi}_3$ into the symmetric group $\mathfrak{S}_5$ of all the permutations of $\mathcal{P}$. Such a homomorphism is easily seen to be surjective. Hence, if we denote its kernel by K, we have an exact sequence of multiplicative groups:

$$1 \to K \hookrightarrow \boldsymbol{\Phi}_3 \xrightarrow{*} \mathfrak{S}_5 \to 1. \tag{4.15}$$

The structures of the two flanking groups in (4.15) yield information on the structure of $\boldsymbol{\Phi}_3$, and in particular determine the order $|\boldsymbol{\Phi}_3|$. It is easy to show that K is isomorphic to the additive group $(\mathbb{Z}/2\mathbb{Z})^4$ (see [RV3], p. 283). Thus we have $|K|=2^4$, $|\mathfrak{S}_5|=5!$, whence

$$|\boldsymbol{\Phi}_3| = 2^4\cdot 5! = 1920.$$

Also, a further argument shows that

$$\boldsymbol{\Phi}_3 = \langle \boldsymbol{\varphi}_3, \boldsymbol{\chi}, \boldsymbol{\vartheta}, \boldsymbol{\sigma}_3\rangle = \langle \boldsymbol{\varphi}_3, \boldsymbol{\vartheta}\rangle,$$

in analogy with (3.22).

The rest of this discussion is similar to the one in Section 3. With any permutation $\boldsymbol{\varrho}\in\boldsymbol{\Phi}_3$ we associate the quotient

$$\frac{h!\,j!\,k!\,l!\,m!\,q!\,r!\,s!}{\boldsymbol{\varrho}(h)!\,\boldsymbol{\varrho}(j)!\,\boldsymbol{\varrho}(k)!\,\boldsymbol{\varrho}(l)!\,\boldsymbol{\varrho}(m)!\,\boldsymbol{\varrho}(q)!\,\boldsymbol{\varrho}(r)!\,\boldsymbol{\varrho}(s)!}, \tag{4.16}$$

and if $\boldsymbol{\varrho}$, $\boldsymbol{\varrho}'$ lie in the same left coset of $\boldsymbol{\Theta}$ in $\boldsymbol{\Phi}_3$, the quotient (4.16) for $\boldsymbol{\varrho}$ equals the analogous quotient for $\boldsymbol{\varrho}'$. Also, we say that $\boldsymbol{\varrho}$ is a permutation of level v, or that the left coset $\boldsymbol{\varrho\Theta}$ is of level v, if, after simplifying (4.16), we have v factorials in the numerator and v in the denominator.

Since $|\boldsymbol{\Phi}_3| = 1920$ and $|\boldsymbol{\Theta}|=16$, there are 120 left cosets of $\boldsymbol{\Theta}$ in $\boldsymbol{\Phi}_3$, yielding 120 distinct quotients of factorials, which can be classified as follows (see [RV3], pp. 286–287):

1 coset of level 0,
12 cosets of level 2,
32 cosets of level 3,
30 cosets of level 4,
32 cosets of level 5,
12 cosets of level 6,
1 coset of level 8.

We now define

$$M=\max\mathcal{V},\qquad N=\max{}'\,\mathcal{V},\qquad Q=\max{}''\,\mathcal{V},$$

where

$$\mathcal{V}=(h,j,k,l,m,q,r,s,h',j',k',l',m',q',r',s')$$

is the sequence of the integers (4.8) and (4.9). By (4.11) we have, for any $\boldsymbol{\varrho}\in\boldsymbol{\Phi}_3$,

$$I\big(\boldsymbol{\varrho}(h),\boldsymbol{\varrho}(j),\boldsymbol{\varrho}(k),\boldsymbol{\varrho}(l),\boldsymbol{\varrho}(m),\boldsymbol{\varrho}(q),\boldsymbol{\varrho}(r),\boldsymbol{\varrho}(s)\big) = a_{\boldsymbol{\varrho}} + 2b_{\boldsymbol{\varrho}}\zeta(3)$$
$$\text{with } b_{\boldsymbol{\varrho}}\in\mathbb{Z} \text{ and } d_M d_N d_Q a_{\boldsymbol{\varrho}}\in\mathbb{Z},$$

where $a_{\boldsymbol{\varrho}}$ and $b_{\boldsymbol{\varrho}}$ depend only on the left coset $\boldsymbol{\varrho\Theta}$.

For fixed h, j, k, l, m, q, r, s, let

$$I_n = I(hn,jn,kn,ln,mn,qn,rn,sn) = a_n + 2b_n\zeta(3)\quad (n=1,2,\dots), \tag{4.17}$$

with $a_n \in \mathbb{Q}$ and $b_n \in \mathbb{Z}$. From (4.11) we get $d_{Mn}d_{Nn}d_{Qn}a_n \in \mathbb{Z}$. As in Section 3, each transformation formula for I_n corresponding to a left coset of $\boldsymbol{\Theta}$ in $\boldsymbol{\Phi}_3$ of level > 0 yields information on the p-adic valuation of the integer $A_n = d_{Mn}d_{Nn}d_{Qn}a_n$, thus allowing to eliminate divisors of A_n of the types p, p^2 or p^3 for suitable primes p. However, the resulting arithmetic discussion clearly depends on the level of the left coset associated with the transformation formula considered. In Section 5 of [RV3] we show that the choice

$$h = 16,\ j = 17,\ k = 19,\ l = 15,\ m = 12,\ q = 11,\ r = 9,\ s = 13, \tag{4.18}$$

satisfying $h + m = k + r$ and $j + q = l + s$ as required, with $M = k = 19$, $N = q' = 18$, $Q = j = 17$, yields a suitable set of 15 (among the 30) left cosets of level 4, such that the corresponding 15 transformation formulae for I_n suffice for the study of the p-adic valuation of A_n. Using such 15 transformation formulae, by a treatment similar to the one described in Section 3 we define a divisor D_n of $d_{Mn}d_{Nn}d_{Qn}$ such that $D_n a_n \in \mathbb{Z}$ and

$$\lim_{n\to\infty} \frac{1}{n} \log D_n = 29.81231469\ldots < M + N + Q = 54. \tag{4.19}$$

By (4.17) we have

$$D_n I_n = D_n a_n + 2D_n b_n \zeta(3) \in \mathbb{Z} + 2\mathbb{Z}\zeta(3), \tag{4.20}$$

and the asymptotic estimates for D_n, I_n and $|b_n|$ in (4.20) yield an irrationality measure of $\zeta(3)$, again by applying the Proposition in Section 1.

Let

$$f(x, y, z) = \frac{x^h(1-x)^l y^k (1-y)^s z^j (1-z)^q}{(1-(1-xy)z)^{q+h-r}}.$$

If the integers (4.8) and (4.9) are all strictly positive, $f(x, y, z)$ has exactly two stationary points (x_0, y_0, z_0) and (x_1, y_1, z_1) satisfying $x(1-x)y(1-y)z(1-z) \neq 0$, with $0 < x_0, y_0, z_0 < 1$ and $x_1, y_1, z_1 < 0$, $x_1 y_1 > 1$, $z_1 < (1 - x_1 y_1)^{-1}$. Then, in analogy with (3.37) and (3.38), for the integral (4.17) we have

$$\lim_{n\to\infty} \frac{1}{n} \log I_n = \log f(x_0, y_0, z_0) \tag{4.21}$$

and

$$\limsup_{n\to\infty} \frac{1}{n} \log |b_n| \le \log |f(x_1, y_1, z_1)|. \tag{4.22}$$

With the values (4.18) we get

$$\log f(x_0, y_0, z_0) = -47.15472079\ldots \tag{4.23}$$

and

$$\log |f(x_1, y_1, z_1)| = 48.46940964\ldots. \tag{4.24}$$

Thus, using (4.19), (4.21), (4.22), (4.23), (4.24) in (4.20), and applying the Proposition in Section 1, we find that the choice (4.18) yields the irrationality measure

$$\mu(\zeta(3)) < 5.513890\ldots.$$

5. Invariants

As we have seen in Section 4, the main new tool in our treatment of triple integrals, in comparison with the theory for double integrals outlined in Section 3, is represented by the three-dimensional birational transformation ϑ defined by (4.2). In this section we give a constructive method to derive ϑ from the two-dimensional birational transformation τ defined by (3.2). Our method employs suitable rational functions and measures invariant under the action of a two-dimensional involution $\widetilde{\tau}$ related with τ.

Denote the permutation $\boldsymbol{\tau\sigma}_2$ by $\widetilde{\boldsymbol{\tau}}$. First, we remark that $\widetilde{\boldsymbol{\tau}} = \boldsymbol{\tau\sigma}_2 = (h\ \ l)(i\ \ k)$ is a product of transpositions, and hence has period 2. Accordingly, the corresponding birational transformation $\widetilde{\tau} = \sigma_2\tau : (x,y) \mapsto (X,Y)$ given by

$$\widetilde{\tau} : \begin{cases} X = 1 - xy \\ Y = \dfrac{1-x}{1-xy} \end{cases} \tag{5.1}$$

is an involution, i.e., has also period 2. Since $\boldsymbol{\tau} = \boldsymbol{\tau\sigma}_2^2 = \widetilde{\boldsymbol{\tau}}\boldsymbol{\sigma}_2$, for the permutation group $\boldsymbol{T}$ we obviously have

$$\boldsymbol{T} = \langle \boldsymbol{\tau}, \boldsymbol{\sigma}_2 \rangle = \langle \widetilde{\boldsymbol{\tau}}, \boldsymbol{\sigma}_2 \rangle.$$

Thus the involutions $\widetilde{\tau}$ and σ_2, together with the hypergeometric integral transformation φ_2, suffice to obtain the results of Section 3.

Using an involution η (in any dimension) has the advantage that one can easily construct several functions invariant under the action of η. For if the transformation $\eta : (x_1,\ldots,x_n) \mapsto (X_1,\ldots,X_n)$ defined by the equations

$$\eta : \begin{cases} X_1 = f_1(x_1,\ldots,x_n) \\ \quad\vdots \qquad\qquad \vdots \\ X_n = f_n(x_1,\ldots,x_n) \end{cases}$$

is an involution, we have $\eta^{-1} = \eta$, whence

$$\eta^{-1} : \begin{cases} x_1 = f_1(X_1,\ldots,X_n) \\ \quad\vdots \qquad\qquad \vdots \\ x_n = f_n(X_1,\ldots,X_n). \end{cases}$$

Therefore, if $F(u,v)$ is any symmetric function of two variables and $G(t_1,\dots,t_n)$ is any function of n variables, we get

$$\begin{aligned}
&F\bigl(G(X_1,\dots,X_n),\, G(f_1(X_1,\dots,X_n),\dots,f_n(X_1,\dots,X_n))\bigr)\\
&= F\bigl(G(f_1(x_1,\dots,x_n),\dots,f_n(x_1,\dots,x_n)),\, G(x_1,\dots,x_n)\bigr)\\
&= F\bigl(G(x_1,\dots,x_n),\, G(f_1(x_1,\dots,x_n),\dots,\ f_n(x_1,\dots,x_n))\bigr),
\end{aligned}$$

and we conclude that

$$F\bigl(G(x_1,\dots,x_n),\, G(f_1(x_1,\dots,x_n),\dots,f_n(x_1,\dots,x_n))\bigr)$$

is invariant under the action of η. For instance, taking $F(u,v)=uv$, $n=2$ and $G(t_1,t_2)=t_2$, from the second equation in (5.1) we see that the rational function

$$\frac{(1-X)Y}{1-XY} = \frac{(1-x)y}{1-xy} \tag{5.2}$$

is invariant under the action of the involution $\widetilde{\tau}$.

We want to get three-dimensional birational transformations, acting on the Beukers type integral (4.1), from the two-dimensional birational transformations previously used for the integral (3.1). Therefore, it is natural to seek a rational function $Z(x,y,z)$ such that the transformation $(x,y,z)\mapsto(X,Y,Z)$ obtained by associating the involution (5.1) with the equation $Z=Z(x,y,z)$:

$$\begin{cases} X = 1-xy \\ Y = \dfrac{1-x}{1-xy} \\ Z = Z(x,y,z) \end{cases}$$

maps the open unit cube $(0,1)^3$ onto itself and satisfies

$$\frac{\mathrm{d}X\,\mathrm{d}Y\,\mathrm{d}Z}{1-(1-XY)Z} = \mp\frac{\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z}{1-(1-xy)z}\,. \tag{5.3}$$

Thus we demand a solution $Z(x,y,z)$ of the differential equation (5.3), with X and Y given by (5.1), such that

$$0 < Z(x,y,z) < 1 \quad \text{for any } 0 < x,y,z < 1. \tag{5.4}$$

Since X and Y are independent of z, the jacobian determinant factorizes:

$$\frac{\mathrm{d}(X,Y,Z)}{\mathrm{d}(x,y,z)} = \frac{\mathrm{d}(X,Y)}{\mathrm{d}(x,y)}\,\frac{\partial Z}{\partial z}\,.$$

Thus

$$\mathrm{d}X\,\mathrm{d}Y\,\mathrm{d}Z = \left(\frac{\mathrm{d}(X,Y)}{\mathrm{d}(x,y)}\,\mathrm{d}x\,\mathrm{d}y\right)\left(\frac{\partial Z}{\partial z}\,\mathrm{d}z\right) = (\mathrm{d}X\,\mathrm{d}Y)\partial Z, \tag{5.5}$$

where we denote

$$\partial Z = \frac{\partial Z}{\partial z}\,\mathrm{d}z,$$

the differential of $Z(x,y,z)$ with respect to z only. Since the measure $\mathrm{d}x\,\mathrm{d}y/(1-xy)$ is invariant (up to the sign) under the action of (5.1), if we multiply and divide by $1-XY$ the left side of (5.3) and by $1-xy$ the right side, by virtue of (5.5) we get the differential equation

$$\frac{(1-XY)\,\partial Z}{1-(1-XY)Z} \;=\; \pm\frac{(1-xy)\,\mathrm{d}z}{1-(1-xy)z},$$

whence

$$\log(1-(1-XY)Z) \;=\; \pm\log(1-(1-xy)z)+\log C,$$

with $C=C(x,y)$ independent of z. Therefore

$$1-(1-XY)Z \;=\; C(1-(1-xy)z)^{\pm 1}.$$

The condition (5.4) becomes

$$0<Z=\frac{1-C(1-(1-xy)z)^{\pm 1}}{1-XY}<1,$$

i.e.,

$$XY < C(1-(1-xy)z)^{\pm 1} < 1$$

for $0<x,y,z<1$. In particular, taking $z=0$ and $z=1$, we require

$$\begin{cases} XY \le C \le 1 \\ XY \le C(xy)^{\pm 1} \le 1, \end{cases}$$

whence, choosing either sign for the exponent,

$$XY \le xy \quad \text{for any } \ 0<x,y<1. \tag{5.6}$$

Since (5.1) is an involution, if we had (5.6) we should also have the same inequality with (x,y) and (X,Y) interchanged, whence $XY=xy$. But this is false, since the function xy is not invariant under the action of $\widetilde{\tau}$. Hence the differential equation (5.3), with X and Y given by (5.1), has no solutions satisfying (5.4).

However, since the jacobian determinant

$$\frac{\mathrm{d}(X,Y,Z)}{\mathrm{d}(x,y,z)}$$

is invariant (up to the sign) under the interchange of x with $1-x$, or of X with $1-X$, etc., or under any permutation of $\{x,y,z\}$ or of $\{X,Y,Z\}$, we can apply the method described above to solve the differential equation obtained by twisting (5.3) with any of the above mentioned interchanges or permutations. For instance, we may demand a solution $Z(x,y,z)$ satisfying (5.4) of the twisted differential equation

$$\frac{\mathrm{d}X\,\mathrm{d}Y\,\mathrm{d}Z}{1-(1-(1-X)Z)Y} = \mp\frac{\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z}{1-(1-(1-x)z)y}, \tag{5.7}$$

again with X and Y given by (5.1). Since (5.2) is invariant under the action of (5.1), the measure

$$\frac{\mathrm{d}x\,\mathrm{d}y}{(1-x)y} = \frac{\mathrm{d}x\,\mathrm{d}y}{1-xy}\,\frac{1-xy}{(1-x)y}$$

is also invariant (up to the sign). Hence (5.7) can be written in the form

$$\frac{(1-X)Y\,\partial Z}{1-Y+(1-X)YZ} = \pm\frac{(1-x)y\,\mathrm{d}z}{1-y+(1-x)yz},$$

whence

$$\log(1-Y+(1-X)YZ) = \pm\log(1-y+(1-x)yz)+\log C,$$

with $C=C(x,y)$. Therefore

$$1-Y+(1-X)YZ = C(1-y+(1-x)yz)^{\pm 1}.$$

From (5.4) we obtain

$$0<Z=\frac{C(1-y+(1-x)yz)^{\pm 1}-(1-Y)}{(1-X)Y}<1, \tag{5.8}$$

i.e.,

$$1-Y<C(1-y+(1-x)yz)^{\pm 1}<1-Y+(1-X)Y=1-XY$$

for $0<x,y,z<1$. Taking $z=0$ and $z=1$ we require

$$\begin{cases} 1-Y\le C(1-y)^{\pm 1}\le 1-XY \\ 1-Y\le C(1-xy)^{\pm 1}\le 1-XY. \end{cases}$$

These inequalities yield

$$1-Y\le C(1-y) = C(1-xy)\,\frac{1-y}{1-xy}\le(1-XY)\,\frac{1-y}{1-xy} \tag{5.9}$$

if we choose the exponent $+1$, or

$$1-Y\le\frac{C}{1-xy} = \frac{C}{1-y}\,\frac{1-y}{1-xy}\le(1-XY)\,\frac{1-y}{1-xy} \tag{5.10}$$

if we choose the exponent -1. In either case we get

$$\frac{1-Y}{1-XY}\le\frac{1-y}{1-xy}.$$

By (5.2), the function

$$1-\frac{(1-x)y}{1-xy} = \frac{1-y}{1-xy}$$

is also invariant under the action of (5.1). Thus

$$1 - Y = (1 - XY)\,\frac{1-y}{1-xy}\,,$$

and the conditions (5.9) or (5.10) yield

$$C = \frac{x}{1-xy} \tag{5.11}$$

or

$$C = x(1-y), \tag{5.12}$$

respectively. By (5.8) and (5.11) or (5.12), the differential problem (5.1)-(5.4)-(5.7) has the solution

$$Z = z \tag{5.13}$$

if we choose the $-$ sign in (5.7), i.e. the exponent $+1$ in (5.8), or the solution

$$Z = \frac{(1-y)(1-z)}{1-(1-(1-x)z)y} \tag{5.14}$$

if we choose the $+$ sign in (5.7). If in the transformations $(x,y,z) \mapsto (X,Y,Z)$ obtained by associating (5.1) with (5.13) or with (5.14) we interchange x with $1-x$, X with $1-X$, y with z, and Y with Z, we get the involutions $(x,y,z) \mapsto (X,Y,Z)$ given by

$$\begin{cases} X = (1-x)z \\ Y = y \\ Z = \dfrac{x}{1-(1-x)z} \end{cases}$$

or by

$$\begin{cases} X = (1-x)z \\ Y = \dfrac{(1-y)(1-z)}{1-(1-xy)z} \\ Z = \dfrac{x}{1-(1-x)z}\,, \end{cases} \tag{5.15}$$

which map $(0,1)^3$ onto itself and, by virtue of (5.7), satisfy

$$\frac{\mathrm{d}X\,\mathrm{d}Y\,\mathrm{d}Z}{1-(1-XY)Z} = -\frac{\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z}{1-(1-xy)z}$$

or

$$\frac{\mathrm{d}X\,\mathrm{d}Y\,\mathrm{d}Z}{1-(1-XY)Z} = \frac{\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z}{1-(1-xy)z}\,.$$

respectively, as required. Denoting the involution (5.15) by $\widetilde{\vartheta}$, the transformation ϑ given by (4.2) is obtained as $\vartheta = \widetilde{\vartheta}\sigma_3$, i.e., by interchanging x and y in (5.15).

We remark that the success of the twist method described above relies on the invariance of $\mathrm{d}x\,\mathrm{d}y/(1-xy)$ (up to the sign), $(1-x)y/(1-xy)$ and $(1-y)/(1-xy)$ under the action of the involution $\widetilde{\tau}$ defined by (5.1).

6. Vasilyev's integral

If we interchange x with $1-x$ in (1.3) and (1.4), we get the denominators $1-(1-x)y$ and $1-(1-(1-x)y)z$. Therefore, the integral

$$I_n(k) = \int\limits_{(0,1)^k} \left(\frac{x_1(1-x_1)\cdots x_k(1-x_k)}{Q_k(x_1,\ldots,x_k)}\right)^n \frac{\mathrm{d}x_1\cdots\mathrm{d}x_k}{Q_k(x_1,\ldots,x_k)},$$

where the polynomial $Q_k(x_1,\ldots,x_k)$ is recursively defined by

$$Q_0 = 1, \qquad Q_k = 1 - Q_{k-1}x_k \quad \text{for } k \geq 1,$$

is a natural generalization of Beukers' integrals (1.3) and (1.4).

The above integral $I_n(k)$ was studied by Vasilyev [Va], who proved that

$$4\,d_n^4 I_n(4) = A_n + B_n\zeta(2) + C_n\zeta(4) \tag{6.1}$$

and

$$d_n^5 I_n(5) = A'_n + B'_n\zeta(3) + C'_n\zeta(5), \tag{6.2}$$

with A_n, B_n, C_n, A'_n, B'_n, $C'_n \in \mathbb{Z}$, and that the linear forms (6.1) and (6.2) tend exponentially to zero as $n \to \infty$. Moreover, Vasilyev conjectured that, for any $h > 2$,

$$d_n^{2h+1} I_n(2h+1) = A_{0,n}^{(h)} + \sum_{j=1}^{h} A_{j,n}^{(h)}\,\zeta(2j+1),$$

with $A_{j,n}^{(h)} \in \mathbb{Z}$ $(j = 0,1,\ldots,h)$.

In this section we apply again our twist method, introduced in Section 5, to obtain four-dimensional birational transformations which can be used to investigate the arithmetical properties of a suitable generalization of the integral $I_n(4)$, where the exponents of the factors x_1, $1-x_1$, $\ldots$, x_4, $1-x_4$, $Q_4(x_1,x_2,x_3,x_4)$ are not all equal.

By analogy with (1.3) and (1.4), we write the integral $I_n(4)$ in the form

$$I_n(4) = \int\limits_{(0,1)^4} \left(\frac{x(1-x)y(1-y)z(1-z)t(1-t)}{1-(1-(1-xy)z)t}\right)^n \frac{\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}t}{1-(1-(1-xy)z)t}.$$

Accordingly, we seek birational transformations $(x,y,z,t) \mapsto (X,Y,Z,T)$ mapping the open unit hypercube $(0,1)^4$ onto itself and satisfying

$$\frac{\mathrm{d}X\,\mathrm{d}Y\,\mathrm{d}Z\,\mathrm{d}T}{1-(1-(1-XY)Z)T} = \mp\frac{\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}t}{1-(1-(1-xy)z)t}.$$

We start from the involution $\sigma_3\vartheta^2 : (x,y,z) \mapsto (X,Y,Z)$ given by the equations

$$\sigma_3\vartheta^2 : \begin{cases} X = 1-(1-xy)z \\ Y = \dfrac{xy}{1-(1-xy)z} \\ Z = \dfrac{1-x}{1-xy}, \end{cases} \tag{6.3}$$

which maps $(0,1)^3$ onto itself and preserves (up to the sign) the measure

$$\frac{\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z}{1-(1-xy)z}.$$

Also, it is easily seen that under the action of (6.3) we have the invariants

$$\frac{(1-X)YZ}{1-(1-XY)Z} = \frac{(1-x)yz}{1-(1-xy)z} \tag{6.4}$$

and

$$\frac{1-(1-Y)Z}{1-(1-XY)Z} = \frac{1-(1-y)z}{1-(1-xy)z}. \tag{6.5}$$

We seek a solution $T(x,y,z,t)$ of the twisted differential equation

$$\frac{\mathrm{d}X\,\mathrm{d}Y\,\mathrm{d}Z\,\mathrm{d}T}{1-(1-(1-(1-X)T)Y)Z} = \mp\frac{\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}t}{1-(1-(1-(1-x)t)y)z}, \tag{6.6}$$

with X, Y and Z given by (6.3), such that

$$0 < T(x,y,z,t) < 1 \quad \text{for any } 0 < x,y,z,t < 1. \tag{6.7}$$

By (6.4), the measure

$$\frac{\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z}{(1-x)yz} = \frac{\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z}{1-(1-xy)z}\,\frac{1-(1-xy)z}{(1-x)yz}$$

is invariant (up to the sign) under the action of (6.3). Thus, following the method of Section 5, (6.6) can be written as

$$\frac{(1-X)YZ\,\partial T}{1-Z+YZ-(1-X)YZT} = \pm\frac{(1-x)yz\,\mathrm{d}t}{1-z+yz-(1-x)yzt},$$

where

$$\partial T = \frac{\partial T}{\partial t}\,\mathrm{d}t.$$

Therefore

$$1 - Z + YZ - (1 - X)YZT \;=\; C(1 - z + yz - (1 - x)yzt)^{\pm 1},$$

with $C = C(x, y, z)$. From (6.7) we get

$$0 < T = \frac{1 - Z + YZ - C(1 - z + yz - (1 - x)yzt)^{\pm 1}}{(1 - X)YZ} < 1,$$

i.e.,

$$1 - Z + XYZ < C(1 - z + yz - (1 - x)yzt)^{\pm 1} < 1 - Z + YZ$$

for $0 < x, y, z, t < 1$. Taking $t = 0$ and $t = 1$ we require

$$\begin{cases} 1 - Z + XYZ \leq C(1 - z + yz)^{\pm 1} \leq 1 - Z + YZ \\ 1 - Z + XYZ \leq C(1 - z + xyz)^{\pm 1} \leq 1 - Z + YZ. \end{cases}$$

Choosing either sign for the exponent, we easily obtain the condition

$$\frac{1 - Z + XYZ}{1 - Z + YZ} \leq \frac{1 - z + xyz}{1 - z + yz}\,.$$

By (6.5), this rational function is invariant under the action of (6.3). Hence by the method of Section 5 we get $C = x/(1 - (1 - xy)z)$ and

$$T = t \tag{6.8}$$

if we choose the exponent $+1$, or $C = x(1 - (1 - y)z)$ and

$$T = \frac{(1 - t)(1 - (1 - y)z)}{1 - (1 - (1 - (1 - x)t)y)z} \tag{6.9}$$

if we choose the exponent -1. If in the transformations $(x, y, z, t) \mapsto (X, Y, Z, T)$ obtained by associating (6.3) with (6.8) or with (6.9) we interchange x with $1 - x$, X with $1 - X$, and apply the permutations $(y\ \ z\ \ t)$ and $(Y\ \ Z\ \ T)$, by virtue of (6.6) we conclude that the involutions $(x, y, z, t) \mapsto (X, Y, Z, T)$ given by

$$\begin{cases} X = (1 - (1 - x)z)t \\[1ex] Y = y \\[1ex] Z = \dfrac{(1 - x)z}{1 - (1 - (1 - x)z)t} \\[2ex] T = \dfrac{x}{1 - (1 - x)z} \end{cases}$$

or by

$$\begin{cases} X = (1-(1-x)z)t \\ Y = \dfrac{(1-y)(1-(1-z)t)}{1-(1-(1-xy)z)t} \\ Z = \dfrac{(1-x)z}{1-(1-(1-x)z)t} \\ T = \dfrac{x}{1-(1-x)z} \end{cases}$$

map $(0,1)^4$ onto itself and satisfy

$$\frac{\mathrm{d}X\,\mathrm{d}Y\,\mathrm{d}Z\,\mathrm{d}T}{1-(1-(1-XY)Z)T} = -\frac{\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}t}{1-(1-(1-xy)z)t}$$

or

$$\frac{\mathrm{d}X\,\mathrm{d}Y\,\mathrm{d}Z\,\mathrm{d}T}{1-(1-(1-XY)Z)T} = \frac{\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}t}{1-(1-(1-xy)z)t},$$

respectively.

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