Journal de Théorie des Nombres
de Bordeaux **15** (2003), 593–626

Well-poised hypergeometric service for diophantine problems of zeta values

par Wadim ZUDILIN

Résumé. On montre comment les concepts classiques de séries et intégrales hypergéométriques bien équilibrées devient crucial dans l'étude des propriétés arithmétiques des valeurs de la fonction zêta de Riemann. Par ces arguments, on obtient (1) un groupe de permutations pour les formes linéaires en 1 et $\zeta(4) = \pi^4/90$ donnant une majoration conditionnelle de la mesure d'irrationalité de $\zeta(4)$; (2) une récurrence d'ordre deux pour $\zeta(4)$ semblable à celles introduites par Apéry pour $\zeta(2)$ et $\zeta(3)$, ainsi que des récurrences d'ordre réduit pour les formes linéaires en des valeurs de la fonction zêta aux entiers impairs ; (3) un gros groupe de permutations pour une famille d'intégrales multiples généralisant les intégrales dites de Beukers pour $\zeta(2)$ et $\zeta(3)$.

Abstract. It is explained how the classical concept of well-poised hypergeometric series and integrals becomes crucial in studying arithmetic properties of the values of Riemann's zeta function. By these well-poised means we obtain: (1) a permutation group for linear forms in 1 and $\zeta(4) = \pi^4/90$ yielding a conditional upper bound for the irrationality measure of $\zeta(4)$; (2) a second-order Apéry-like recursion for $\zeta(4)$ and some low-order recursions for linear forms in odd zeta values; (3) a rich permutation group for a family of certain Euler-type multiple integrals that generalize so-called Beukers' integrals for $\zeta(2)$ and $\zeta(3)$.

1. Introduction

In this work, we deal with the values of Riemann's zeta function (*zeta values*)

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}$$

at integral points $s = 2, 3, 4, \dots$. Lindemann's proof of the transcendence of π as well as Euler's formula for even zeta values, summarized by the

Manuscrit reçu le 5 juin 2002.

inclusions $\zeta(2n) \in \mathbb{Q}\pi^{2n}$ for $n = 1, 2, \dots$, yield the irrationality (and transcendence) of $\zeta(2), \zeta(4), \zeta(6), \dots$. The story for odd zeta values is not so complete, we know only that:

- $\zeta(3)$ is irrational (R. Apéry [Ap], 1978);
- infinitely many of the numbers $\zeta(3), \zeta(5), \zeta(7), \dots$ are irrational (T. Rivoal [Ri1], [BR], 2000);
- at least one of the four numbers $\zeta(5), \zeta(7), \zeta(9), \zeta(11)$ is irrational[1] (this author [Zu3], [Zu4], 2001).

The last two results are due to a certain well-poised hypergeometric[2] construction, and a similar approach can be put forward for proving Apéry's theorem (see [Ri3] and [Zu5] for details).

After remarkable Apéry's proof [Ap] of the irrationality of both $\zeta(2)$ and $\zeta(3)$, there have appeared several other explanations of why it is so; we are not able to indicate here the complete list of such publications and mention the most known approaches:

- orthogonal polynomials [Be1], [Hat] and Padé-type approximations [Be2], [So1], [So3];
- multiple Euler-type integrals [Be1], [Hat], [RV2];
- hypergeometric-type series [Gu], [Ne1];
- modular interpretation [Be3].

G. Rhin and C. Viola have developed a new group-structure arithmetic method to obtain nice estimates for irrationality measures of $\zeta(2)$ and $\zeta(3)$ (see [RV1], [RV2], [Vi]). The permutation groups in [RV1], [RV2] for multiple integrals can be translated into certain hypergeometric series and integrals, and this translation [Zu4] leads one to classical permutation groups (due to F. J. W. Whipple and W. N. Bailey) for very-well-poised hypergeometric series.

The aim of this paper is to demonstrate potentials of the well-poised hypergeometric service (series and integrals) in solving quite different problems concerning zeta values. Here we concentrate on the following features:

- hypergeometric permutation groups for $\zeta(4)$ (Sections 3–5) and for linear forms in odd/even zeta values (Section 8);
- a conditional estimate for the irrationality measure of $\zeta(4)$ via the group-structure arithmetic method (Section 6);
- an Apéry-like difference equation and a continued fraction for $\zeta(4)$ (Section 2) and similar difference equations for linear forms in odd zeta values (Section 7);

[1]The first record of this type, at least one of the nine numbers $\zeta(5), \zeta(7), \dots, \zeta(21)$ is irrational, is due to T. Rivoal [Ri2].

[2]We refer the reader to [Ba], Section 2.5, or to formula (69) for a formal definition, to [An] for a nice historical exposition, and to Sections 2–8 below for number-theoretic applications.

- Euler-type multiple integrals represented very-well-poised hypergeometric series and, as a consequence, linear forms in odd/even zeta values (Section 8).

All these features can be considered as a part of the general hypergeometric construction proposed recently by Yu. Nesterenko [Ne2], [Ne3].

Hypergeometric sums and integrals of Sections 3–6 are prompted by Bailey's integral transform (Proposition 2 below), and it is a pity that the permutation group for $\zeta(4)$ (containing 51840 elements!) leads to an estimate for the irrationality measure of $\zeta(4)$ under a certain (denominator) conjecture only. We indicate this conjecture (supported by our numerical calculations) in Section 6. The particular case of the construction is presented in Section 2; this case can be regarded as a toy-model of that follows, and its main advantage is a certain nice recursion satisfied by linear forms in 1 and $\zeta(4)$.

Section 7 is devoted to difference equations for higher zeta values; such recursions make possible to predict a true arithmetic (i.e., denominators) of linear forms in zeta values.

The subject of Section 8 is motivated by multiple integrals

$$J_{k,n} := \underset{[0,1]^k}{\int \cdots \int} \frac{x_1^n(1-x_1)^n x_2^n(1-x_2)^n \cdots x_k^n(1-x_k)^n}{(1-(1-(\cdots(1-(1-x_k)x_{k-1})\cdots)x_2)x_1)^{n+1}} \, \mathrm{d}x_1 \, \mathrm{d}x_2 \cdots \mathrm{d}x_k$$

that were conjecturally $\mathbb{Q}$-linear forms in odd/even zeta values depending on parity of k (see [VaD]). D. Vasilyev [VaD] required several clever but cumbersome tricks to prove the conjecture for $k = 4$ and $k = 5$. However, one can see no obvious generalization of Vasilyev's scheme and, in [Zu4], we have made another conjecture, yielding the old one, about the coincidence of the multiple integrals with some very-well-poised hypergeometric series. We now prove the conjecture of [Zu4] in more general settings and explain how this result leads to a permutation group for a family of multiple integrals.

Acknowledgements. I am grateful to F. Amoroso and F. Pellarin for their kind invitation to contribute to this volume of *Actes des 12èmes rencontres arithmétiques de Caen* (June 29–30, 2001). I am kindly thankful to T. Rivoal for his comments and useful discussions on the subject and to G. Rhin for pointing out the reference [Co], where the recurrence for $\zeta(4)$ was first discovered by means of Apéry's original method. Special gratitude is due to E. Mamchits for his valuable help in computing the group $\mathfrak{G}$ of Section 5 for linear forms in $1, \zeta(4)$.

2. Difference equation for $\zeta(4)$

In his proof of the irrationality of $\zeta(3)$, Apéry consider the sequences u_n and v_n of rationals satisfying the difference equation

$$(n+1)^3u_{n+1} - (2n+1)(17n^2+17n+5)u_n + n^3u_n = 0, \qquad u_0 = 1, \quad u_1 = 5, \qquad v_0 = 0, \quad v_1 = 6. \tag{1}$$

A priori, the recursion (1) implies the obvious inclusions $n!^3u_n, n!^3v_n \in \mathbb{Z}$, but a miracle happens and one can check (at least experimentally) the inclusions

$$u_n \in \mathbb{Z}, \quad D_n^3 v_n \in \mathbb{Z}$$

for each $n = 1, 2, \dots$; here and later, by D_n we denote the least common multiple of the numbers $1, 2, \dots, n$ (and $D_0 = 1$ for completeness), thanks to the prime number theorem

$$\lim_{n\to\infty} \frac{\log D_n}{n} = 1. \tag{2}$$

The sequence

$$u_n\zeta(3) - v_n, \qquad n = 0, 1, 2, \dots,$$

is also a solution of the difference equation (1), and it exponentially tends to 0 as $n \to \infty$ (even after multiplying it by D_n^3). A similar approach has been used for proving the irrationality of $\zeta(2)$ (see [Ap], [Po]), and several other Apéry-like difference equations have been discovered later (see, e.g., [Be4]). Surprisingly, a second-order recursion exists for $\zeta(4)$ and we are now able to present and prove it by hypergeometric means.

Remark. During preparation of this article, we have known that the difference equation for $\zeta(4)$, in slightly different normalization, had been stated independently by V. Sorokin [So4] by means of certain explicit Padé-type approximations. Later we have learned that the same but again differently normalized recursion had been already known [Co] in 1981 thanks to H. Cohen and G. Rhin (and Apéry's original 'accélération de la convergence' method). We underline that our approach presented below differs from that of [Co] and [So4]. We also mention that no second-order recursion for $\zeta(5)$ and/or higher zeta values is known.

Consider the difference equation

$$(n+1)^5u_{n+1} - b(n)u_n - 3n^3(3n-1)(3n+1)u_{n-1} = 0, \tag{3}$$

where

$$\begin{aligned} b(n) &= 3(2n+1)(3n^2+3n+1)(15n^2+15n+4) \\ &= 270n^5 + 675n^4 + 702n^3 + 378n^2 + 105n + 12, \end{aligned} \tag{4}$$

with the initial data

$$u_0 = 1, \quad u_1 = 12, \qquad v_0 = 0, \quad v_1 = 13 \tag{5}$$

for its two independent solutions u_n and v_n.

Theorem 1. *For each $n = 0, 1, 2, \dots$, the numbers u_n and v_n are positive rationals satisfying the inclusions*

$$6D_n u_n \in \mathbb{Z}, \qquad 6D_n^5 v_n \in \mathbb{Z}, \tag{6}$$

and there holds the limit relation

$$\lim_{n\to\infty} \frac{v_n}{u_n} = \frac{\pi^4}{90} = \zeta(4). \tag{7}$$

Application of Poincaré's theorem then yields the asymptotic relations

$$\lim_{n\to\infty} \frac{\log u_n}{n} = \lim_{n\to\infty} \frac{\log v_n}{n} = 3\log(3 + 2\sqrt{3}\,) = 5.59879212\dots$$

and (see [Zu1], Proposition 2)

$$\lim_{n\to\infty} \frac{\log|u_n\zeta(4) - v_n|}{n} = 3\log|3 - 2\sqrt{3}\,| = -2.30295525\dots,$$

since the characteristic polynomial $\lambda^2 - 270\lambda - 27$ of the equation (3) has zeros $135\pm 78\sqrt{3} = (3\pm 2\sqrt{3}\,)^3$. Thus, we can consider v_n/u_n as convergents of a continued fraction for $\zeta(4)$ and making the equivalent transform of the fraction ([JT], Theorems 2.2 and 2.6) we obtain

Theorem 2. *There holds the following continued-fraction expansion:*

$$\zeta(4) = \frac{13|}{|b(0)} + \frac{1^7\cdot 2\cdot 3\cdot 4|}{|\quad b(1)} + \frac{2^7\cdot 5\cdot 6\cdot 7|}{|\quad b(2)} + \dots + \frac{n^7(3n-1)(3n)(3n+1)|}{|\quad b(n)} + \cdots,$$

where the polynomial $b(n)$ is defined in (4).

Unfortunately, the linear forms

$$6D_n^5(u_n\zeta(4) - v_n) \in \mathbb{Z}\zeta(4) + \mathbb{Z}$$

do not tend to 0 as $n \to \infty$.[3]

A motivation of a hypergeometric construction considered below leans on the two series

$$-\sum_{t=1}^{\infty} \frac{\mathrm{d}}{\mathrm{d}t}\biggl(\frac{(t-1)\cdots(t-n)}{t(t+1)\cdots(t+n)}\biggr)^2 \in \mathbb{Q}\zeta(3) + \mathbb{Q}, \qquad n = 0, 1, 2, \dots \tag{8}$$

(Gutnik's form of Apéry's sequence [Gu], [Ne1]), and

$$n!^2 \sum_{t=1}^{\infty} (2t+n) \frac{(t-1)\cdots(t-n)\cdot(t+n+1)\cdots(t+2n)}{(t(t+1)\cdots(t+n))^4} \in \mathbb{Q}\zeta(3) + \mathbb{Q}, \qquad n = 0, 1, 2, \dots \tag{9}$$

[3] For a simple explanation why $\zeta(4)$ is irrational, see [Han].

(Ball's sequence), and on the coincidence of these series proved by T. Rivoal [Ri2], [Ri3] with a help of the difference equation (1). These arguments make possible to give a new 'elementary' proof of the irrationality of $\zeta(3)$ (see [Zu5] for details).

Consider the rational function

$$R_n(t) := (-1)^n(2t+n)\biggl(\frac{(t-1)\cdots(t-n)\cdot(t+n+1)\cdots(t+2n)}{(t(t+1)\cdots(t+n))^2}\biggr)^2 \tag{10}$$

and the corresponding series

$$F_n := -\sum_{t=1}^{\infty} R_n'(t). \tag{11}$$

In some sense, the series (11) is a mixed generalization of both (8) and (9).

Lemma 1. *There holds the equality*

$$F_n = U_n\zeta(5) + U_n'\zeta(4) + U_n''\zeta(3) + U_n'''\zeta(2) - V_n, \tag{12}$$

where $U_n, D_nU_n', D_n^2U_n'', D_n^3U_n''', D_n^5V_n \in \mathbb{Z}$.

Proof. The polynomials

$$P_n^{(1)}(t) := \frac{(t-1)\cdots(t-n)}{n!}, \qquad P_n^{(2)}(t) := \frac{(t+n+1)\cdots(t+2n)}{n!} \tag{13}$$

are integral-valued and, as it is well known,

$$\frac{D_n^j}{j!}\,\frac{\mathrm{d}^jP_n(t)}{\mathrm{d}t^j}\bigg|_{t=-k} \in \mathbb{Z} \qquad \text{for} \quad k\in\mathbb{Z} \quad \text{and} \quad j=0,1,2,\dots, \tag{14}$$

where $P_n(t)$ is any of the polynomials (13).

The rational function

$$Q_n(t) := \frac{n!}{t(t+1)\cdots(t+n)} \tag{15}$$

has also 'nice' arithmetic properties. Namely,

$$a_k := Q_n(t)(t+k)\big|_{t=-k} = \begin{cases} (-1)^k\binom{n}{k} \in \mathbb{Z} & \text{if } k=0,1,\dots,n, \\ 0 & \text{for other } k\in\mathbb{Z}, \end{cases} \tag{16}$$

that allow to write the following partial-fraction expansion:

$$Q_n(t) = \sum_{l=0}^{n} \frac{a_l}{t+l}.$$

Hence, for $j=1,2,\dots$ we obtain

$$\frac{D_n^j}{j!}\,\frac{\mathrm{d}^j}{\mathrm{d}t^j}\big(Q_n(t)(t+k)\big)\big|_{t=-k}=\frac{D_n^j}{j!}\,\frac{\mathrm{d}^j}{\mathrm{d}t^j}\sum_{l=0}^{n}a_l\biggl(1-\frac{l-k}{t+l}\biggr)\bigg|_{t=-k}=(-1)^{j-1}D_n^j\sum_{\substack{l=0\\ l\ne k}}^{n}\frac{1}{(l-k)^j}\in\mathbb{Z}. \tag{17}$$

Therefore the inclusions (14), (16), (17) and the Leibniz rule for differentiating a product imply that the numbers

$$A_{jk}=A_{jk}^{(n)}:=\frac{1}{(4-j)!}\,\frac{\mathrm{d}^{4-j}}{\mathrm{d}t^{4-j}}\big(R_n(t)(t+k)^4\big)\big|_{t=-k} \tag{18}$$

$$=\frac{1}{(4-j)!}\,\frac{\mathrm{d}^{4-j}}{\mathrm{d}t^{4-j}}\big((-1)^n(2t+n)\cdot P_n^{(1)}(t)\cdot P_n^{(2)}(t)\cdot(Q_n(t)(t+k))^4\big)\big|_{t=-k}$$

satisfy the inclusions

$$D_n^{4-j}\cdot A_{jk}^{(n)}\in\mathbb{Z}\qquad\text{for}\quad k=0,1,\dots,n\quad\text{and}\quad j=1,2,3,4. \tag{19}$$

Now, writing down the partial-fraction expansion of the rational function (10),

$$R_n(t)=\sum_{j=1}^{4}\sum_{k=0}^{n}\frac{A_{jk}^{(n)}}{(t+k)^j}, \tag{20}$$

we obtain that the quantity

$$\begin{aligned}F_n&=\sum_{t=1}^{\infty}\sum_{j=1}^{4}\sum_{k=0}^{n}\frac{jA_{jk}}{(t+k)^{j+1}}=\sum_{j=1}^{4}\sum_{k=0}^{n}\sum_{l=k+1}^{\infty}\frac{jA_{jk}}{l^{j+1}}\\&=\sum_{j=1}^{4}j\sum_{k=0}^{n}A_{jk}\biggl(\sum_{l=1}^{\infty}-\sum_{l=1}^{k}\biggr)\frac{1}{l^{j+1}}\end{aligned}$$

has the desired form (12) with

$$U_n=4\sum_{k=0}^{n}A_{4k}^{(n)},\ U_n'=3\sum_{k=0}^{n}A_{3k}^{(n)},\ U_n''=2\sum_{k=0}^{n}A_{2k}^{(n)},\ U_n'''=\sum_{k=0}^{n}A_{1k}^{(n)}, \tag{21}$$

$$V_n=\sum_{j=1}^{4}j\sum_{k=0}^{n}A_{jk}^{(n)}\sum_{l=1}^{k}\frac{1}{l^{j+1}}. \tag{22}$$

Finally, using the inclusions (19) and

$$D_n^{j+1}\cdot\sum_{l=1}^{k}\frac{1}{l^{j+1}}\in\mathbb{Z}\qquad\text{for}\quad k=0,1,\dots,n,\quad j=1,2,3,4,$$

we deduce that $U_n, D_nU_n', D_n^2U_n'', D_n^3U_n''', D_n^5V_n \in \mathbb{Z}$ as required. $\square$

Now, with a help of Zeilberger's algorithm of creative telescoping ([PWZ], Chapter 6) we get the rational function (*certificate*) $S_n(t) := s_n(t)R_n(t)$, where

$$
\begin{aligned}
(23)\qquad s_n(t) :=\ & \frac{1}{(2t+n)(t+2n-1)^2(t+2n)^2} \\
&\times \bigl(-(122n^2+115n+29)(t+2(5n-1))t^7 \\
&\quad - (4796n^4+2336n^3-859n^2-459n+16)t^6 \\
&\quad - 2(4333n^5-43n^4-2645n^3-734n^2+86n+7)t^5 \\
&\quad - (3965n^6-13782n^5-14109n^4-2207n^3+878n^2 \\
&\qquad + 142n+7)t^4 \\
&\quad + 2(5906n^7+17354n^6+10901n^5+329n^4-1340n^3 \\
&\qquad - 289n^2-15n+2)t^3 \\
&\quad + (22774n^8+42602n^7+20740n^6-2935n^5-4922n^4 \\
&\qquad - 1162n^3+13n^2+44n+4)t^2 \\
&\quad + 2n(8249n^8+13764n^7+5775n^6-2178n^5-2468n^4 \\
&\qquad - 568n^3+94n^2+64n+8)t \\
&\quad + n^2(4549n^8+7531n^7+2923n^6-1975n^5-2056n^4 \\
&\qquad - 424n^3+196n^2+112n+16)\bigr)
\end{aligned}
$$

satisfying the following property.

Lemma 2. *For each $n = 1, 2, \dots$, there holds the identity*

$$
\begin{aligned}
(24)\quad (n+1)^5R_{n+1}(t) - b(n)R_n(t) - 3n^3(3n-1)(3n+1)R_{n-1}(t) \\
= S_n(t+1) - S_n(t),
\end{aligned}
$$

where the polynomial $b(n)$ is given in (4).

Proof. Divide both sides of (24) by $R_n(t)$ and verify the identity

$$
\begin{aligned}
&-(n+1)^5 \cdot \frac{(2t+n+1)(t-n-1)^2(t+2n+1)^2(t+2n+2)^2}{(2t+n)(t+n+1)^6} \\
&\qquad - 3(2n+1)(15n^2+15n+4)(3n^2+3n+1) \\
&\qquad + 3n^3(3n-1)(3n+1) \cdot \frac{(2t+n-1)(t+n)^6}{(2t+n)(t-n)^2(t+2n-1)^2(t+2n)^2} \\
&\quad = s_n(t+1)\frac{(2t+n+2)t^6(t+2n+1)^2}{(2t+n)(t-n)^2(t+n+1)^6} - s_n(t),
\end{aligned}
$$

where $s_n(t)$ is given in (23). $\square$

Lemma 3. *The quantity* (11) *satisfies the difference equation* (3) *for* $n = 1, 2, \dots$.

Proof. Since $R_n(t) = O(t^{-3})$ and $S_n'(t) = O(t^{-2})$ as $t \to \infty$ for $n \ge 1$, differentiating identity (24) and summing the result over $t = 1, 2, \dots$ we arrive at the equality

$$(n+1)^5 F_{n+1} - b(n) F_n - 3n^3(3n-1)(3n+1) F_{n-1} = S_n'(1).$$

It remains to note that, for $n \ge 1$, both functions $R_n(t)$ and $S_n(t) = s_n(t) R_n(t)$ have second-order zero at $t = 1$. Thus $S_n'(1) = 0$ for $n = 1, 2, \dots$ and we obtain the desired recurrence (3) for the quantity (11). $\square$

Lemma 4. *The coefficients* $U_n, U_n', U_n'', U_n''', V_n$ *in the representation* (12) *satisfy the difference equation* (3) *for* $n = 1, 2, \dots$.

Proof. Write the partial-fraction expansion (20) in the form

$$R_n(t) = \sum_{j=1}^{4} \sum_{k=-\infty}^{+\infty} \frac{A_{jk}^{(n)}}{(t+k)^j},$$

where the formulae (18) remain valid for all $k \in \mathbb{Z}$ and $j = 1, 2, 3, 4$. Multiply both sides of (24) by $(t+k)^4$, take $(4-j)$th derivative of the result, substitute $t = -k$ and sum over all $k \in \mathbb{Z}$; this procedure yields that, for each $j = 1, 2, 3, 4$, the numbers (21) written as

$$U_n = 4 \sum_{k=-\infty}^{+\infty} A_{4k}^{(n)}, \ U_n' = 3 \sum_{k=-\infty}^{+\infty} A_{3k}^{(n)}, \ U_n'' = 2 \sum_{k=-\infty}^{+\infty} A_{2k}^{(n)}, \ U_n''' = \sum_{k=-\infty}^{+\infty} A_{1k}^{(n)}$$

satisfy the difference equation (3). Finally, the sequence

$$V_n = U_n \zeta(5) + U_n' \zeta(4) + U_n'' \zeta(3) + U_n''' \zeta(2) - F_n$$

also satisfies the recursion (3). $\square$

Since

$$R_0(t) = \frac{2}{t^3}, \quad R_1(t) = -\frac{4}{t^4} + \frac{4}{(t+1)^4} + \frac{12}{t^3} + \frac{12}{(t+1)^3} - \frac{13}{t^2} + \frac{13}{(t+1)^2},$$

in accordance with (21), (22) we obtain

$$U_0' = 6, \quad U_0 = U_0'' = U_0''' = V_0 = 0,$$

$$U_1' = 72, \quad V_1 = 78, \quad U_1 = U_1'' = U_1''' = 0,$$

hence as a consequence of Lemma 4 we arrive at the following result.

Lemma 5. *There holds the equality*

$$F_n = U_n' \zeta(4) - V_n,$$

where $D_n U_n' \in \mathbb{Z}$ *and* $D_n^5 V_n \in \mathbb{Z}$.

The sequences $u_n := U'_n/6$ and $v_n := V_n/6$ satisfy the difference equation (3) and initial conditions (5); the fact $|F_n| \to 0$ as $n \to \infty$, which yields the limit relation (7), will be proved in Section 4. This completes our proof of Theorem 1.

The conclusion (6) of Theorem 1 is far from being precise; in fact, (experimentally) there hold the inclusions

$$u_n \in \mathbb{Z}, \qquad D_n^4 v_n \in \mathbb{Z},$$

and, moreover, there exists the sequence of positive integers Φ_n, $n = 0, 1, 2, \dots$, such that

$$\Phi_n^{-1} u_n \in \mathbb{Z}, \qquad \Phi_n^{-1} D_n^4 v_n \in \mathbb{Z}.$$

This sequence can be determined as follows: if ν_p is the order of prime p in $(3n)!/n!^3$, then

$$\Phi_n := \prod_p p^{\lfloor \nu_p/2 \rfloor};$$

here and below $\lfloor x \rfloor$ and $\{x\} := x - \lfloor x \rfloor$ denote respectively the integral and fractional parts of a real number x. For primes $p > \sqrt{3n}$ we obtain the explicit (simple) formula

$$\lfloor \nu_p/2 \rfloor = \begin{cases} 1 & \text{if } \{n/p\} \in [\frac{2}{3}, 1), \\ 0 & \text{otherwise,} \end{cases}$$

hence

$$\lim_{n\to\infty} \frac{\log \Phi_n}{n} = \psi(1) - \psi\Big(\frac{2}{3}\Big) = 0.74101875\dots,$$

where $\psi(x) := \Gamma'(x)/\Gamma(x)$. Thus, we obtain that the linear forms

$$\Phi_n^{-1} D_n^4 (u_n \zeta(4) - v_n) \overset{?}{\in} \mathbb{Z}\zeta(4) + \mathbb{Z} \tag{25}$$

do not tend to 0 as $n \to \infty$.

3. Well-poised hypergeometric construction

Consider the set of eight positive integral parameters

$$\boldsymbol{h} = (h_0, h_{-1}; h_1, h_2, h_3, h_4, h_5, h_6),$$
$$\text{where} \quad h_{-1} = 2 + 3h_0 - (h_1 + h_2 + h_3 + h_4 + h_5 + h_6), \tag{26}$$

satisfying the conditions

$$h_0 - h_{-1} < h_j < \frac{1}{2} h_0, \qquad j = 1, 2, 3, 4, 5, 6, \tag{27}$$

and assign to $\boldsymbol{h}$ the rational function

$$
\begin{aligned}
(28)\quad R(t) = R(\boldsymbol{h};t) &:= (-1)^{h_0}\gamma(\boldsymbol{h})\cdot(h_0+2t)\cdot\frac{\prod_{j=-1}^{6}\Gamma(h_j+t)}{\prod_{j=-1}^{6}\Gamma(1+h_0-h_j+t)} \\
&= (-1)^{h_0}\cdot(h_0+2t) \\
&\qquad\times\Gamma(1+h_0-h_1-h_2)\,\frac{\Gamma(h_1+t)}{\Gamma(1+h_0-h_2+t)} \\
&\qquad\times\Gamma(1+h_0-h_1-h_5)\,\frac{\Gamma(h_5+t)}{\Gamma(1+h_0-h_1+t)} \\
&\qquad\times\Gamma(1+h_0-h_2-h_4)\,\frac{\Gamma(h_2+t)}{\Gamma(1+h_0-h_4+t)} \\
&\qquad\times\Gamma(1+h_0-h_3-h_6)\,\frac{\Gamma(h_6+t)}{\Gamma(1+h_0-h_3+t)} \\
&\qquad\times\frac{1}{\Gamma(h_3)}\,\frac{\Gamma(h_3+t)}{\Gamma(1+t)} \\
&\qquad\times\frac{1}{\Gamma(h_{-1}-h_0+h_4)}\,\frac{\Gamma(h_4+t)}{\Gamma(1+h_0-h_{-1}+t)} \\
&\qquad\times\frac{1}{\Gamma(h_5)}\,\frac{\Gamma(h_0+t)}{\Gamma(1+h_0-h_5+t)} \\
&\qquad\times\frac{1}{\Gamma(h_{-1}-h_0+h_6)}\,\frac{\Gamma(h_{-1}+t)}{\Gamma(1+h_0-h_6+t)}.
\end{aligned}
$$

In the last representation we pick out the rational functions

$$
\begin{aligned}
\Gamma(b-a)\,\frac{\Gamma(a+t)}{\Gamma(b+t)} &= \frac{(b-a-1)!}{(t+a)(t+a+1)\cdots(t+b-1)} \quad \text{if } a<b, \\
\frac{1}{\Gamma(1+a-b)}\,\frac{\Gamma(a+t)}{\Gamma(b+t)} &= \frac{(t+b)(t+b+1)\cdots(t+a-1)}{(a-b)!} \quad \text{if } a\ge b,
\end{aligned}
$$

of the form (15), (13), having some nice arithmetic properties ([Zu4], Section 7).

It is easy to verify that, due to (26), for the rational function (28) the difference of numerator and denominator degrees is equal to 3, hence

$$
(29)\qquad R(t) = O(t^{-3}) \qquad \text{as} \quad t\to\infty.
$$

The series

$$
(30)\qquad
\begin{gathered}
F(\boldsymbol{h}) := -\sum_{t=t_0}^{\infty}\frac{\mathrm{d}}{\mathrm{d}t}R(\boldsymbol{h};t) \\
\text{with any } t_0\in\mathbb{Z},\quad 1-\min_{1\le j\le 6}\{h_j\}\le t_0\le 1-\max\{0,h_0-h_{-1}\},
\end{gathered}
$$

produces a linear form in 1 and $\zeta(4)$.

Lemma 6. *The quantity $F(\boldsymbol{h})$ is a linear form in* 1 *and* $\zeta(4)$ *with rational coefficients.*

Proof. Order the parameters $h_1,\dots,h_6$ as $h_1^*\le\dots\le h_6^*$ and consider the partial-fraction expansion of the rational function (28):

$$R(t)=\sum_{j=1}^{4}\sum_{k=h_{j+2}^*}^{h_0-h_{j+2}^*}\frac{A_{jk}}{(t+k)^j}, \tag{31}$$

where

$$A_{jk}=\frac{1}{(4-j)!}\frac{\mathrm{d}^{4-j}}{\mathrm{d}t^{4-j}}\big(R(t)(t+k)^4\big)\big|_{t=-k}\in\mathbb{Q} \tag{32}$$
$$\text{for}\quad k=h_{j+2}^*,\dots,h_0-h_{j+2}^*\quad\text{and}\quad j=1,2,3,4.$$

Then we obtain

$$\begin{aligned}F(\boldsymbol{h})&=\sum_{t=1-h_1^*}\sum_{j=1}^{4}\sum_{k=h_{j+2}^*}^{h_0-h_{j+2}^*}\frac{jA_{jk}}{(t+k)^{j+1}}=\sum_{j=1}^{4}\sum_{k=h_{j+2}^*}^{h_0-h_{j+2}^*}jA_{jk}\bigg(\sum_{l=1}^{\infty}-\sum_{l=1}^{k-h_1^*}\bigg)\frac{1}{l^{j+1}}\\&=\sum_{j=1}^{4}A_j\zeta(j+1)-A_0,\end{aligned}$$

with

$$A_j=j\sum_{k=h_{j+2}^*}^{h_0-h_{j+2}^*}A_{jk},\quad j=1,2,3,4,\qquad A_0=\sum_{j=1}^{4}\sum_{k=h_{j+2}^*}^{h_0-h_{j+2}^*}jA_{jk}\sum_{l=1}^{k-h_1^*}\frac{1}{l^{j+1}},$$

and the well-poised origin of the series (30) (namely, the property $R(-t-h_0)=-R(t)$, hence $A_{jk}=(-1)^{j-1}A_{j,h_0-k}$ by (32), cf. [Zu4], Section 8, with $r=2$ and $q=6$) yields $A_2=A_4=0$, while the residue sum theorem accompanied with (29) implies $A_1=0$ (cf. [Ne1], Lemma 1). $\square$

Remark. The question of denominators of the rational numbers A_3 and A_0 that appear as the coefficients in $F(\boldsymbol{h})$ can be solved by application of Nesterenko's denominator theorem [Ne3] (announced by Yu. Nesterenko in his Caen's talk). Namely, consider the set

$$\begin{aligned}\mathcal{N}:=\{&h_3-1,h_{-1}-h_0+h_4-1,h_5-1,h_{-1}-h_0+h_6-1,h_0-2h_1,\\&h_0-2h_2,h_0-h_1-h_2,h_0-h_1-h_3,h_0-h_1-h_4,h_0-h_1-h_6,\\&h_0-h_2-h_3,h_0-h_2-h_5,h_0-h_2-h_6,h_0-h_3-h_5,h_0-h_4-h_5,\\&h_0-h_4-h_6,h_0-h_1^*-h_3^*,h_0-h_1^*-h_3^*,h_0-h_1^*-h_4^*,h_0-h_1^*-h_5^*,\\&h_0-h_1^*-h_6^*\},\end{aligned}$$

then,

$$D_{m_1}D_{m_2}D_{m_3}D_{m_4}D_{m_5}\cdot F(\boldsymbol{h})\in\mathbb{Z}\zeta(4)+\mathbb{Z}, \tag{33}$$

where $m_1 \ge \dots \ge m_5$ are the five successive maxima of the set $\mathcal{N}$.

Unfortunately, we have not succeeded in using the inclusion (33) for arithmetic applications; actually, our experimental calculations show that the stronger inclusion for the linear forms $F(\boldsymbol{h})$, indicated at the beginning of Section 6, holds.

Using standard arguments, the property (29) and the fact that $R(t)$ has second-order zeros at integers $t = 1 - h_1^*, \dots, -\max\{0, h_0 - h_{-1}\}$, one deduces the following hypergeometric-integral representation of the series (30).

Lemma 7 (cf. [Ne1], Lemma 2). *There holds the equality*

$$
\begin{aligned}
(34)\quad F(\boldsymbol{h}) &= \frac{1}{2\pi i}\int_{t_1-i\infty}^{t_1+i\infty} R(\boldsymbol{h};t)\left(\frac{\pi}{\sin \pi t}\right)^2 \mathrm{d}t \\
&= \frac{(-1)^{h_{-1}}\gamma(\boldsymbol{h})}{\pi i}\int_{t_1-i\infty}^{t_1+i\infty} \frac{\Gamma(h_0+t)\,\Gamma(1+\frac12 h_0+t)\,\Gamma(h_{-1}+t)\,\Gamma(h_1+t)}{\Gamma(\frac12 h_0+t)\,\Gamma(1+h_0-h_1+t)} \\
&\qquad \times \frac{\Gamma(h_2+t)\cdots\Gamma(h_6+t)\cdot\Gamma(h_{-1}-h_0-t)\,\Gamma(-t)}{\Gamma(1+h_0-h_2+t)\cdots\Gamma(1+h_0-h_6+t)}\,\mathrm{d}t,
\end{aligned}
$$

with any $t_1 \in \mathbb{R}$, $1 - h_1^* < t_1 < -\max\{0, h_0 - h_{-1}\}$.

The series (30) as well as the corresponding hypergeometric integral (34) are known in the theory of hypergeometric functions and integrals as very-well-poised objects, i.e., one can split their top and bottom parameters in pairs such that

$$h_0 + 1 = (1 + \tfrac12 h_0) + \tfrac12 h_0 = h_{-1} + (1 + h_0 - h_{-1}) = \dots = h_6 + (1 + h_0 - h_6)$$

and the second parameter has the special form $1 + \frac12 h_0$.

Remark. As it is easily seen, the sequence F_n of Section 2 corresponds (after a suitable shift of the summation parameter t) to the choice

$$(35)\qquad h_0 = h_{-1} = 3n + 2, \qquad h_1 = h_2 = h_3 = h_4 = h_5 = h_6 = n + 1$$

of the parameters $\boldsymbol{h}$. Hence the equalities $U_n = U_n'' = U_n''' = 0$ in the representation (12) can be deduced from Lemma 6.

4. Asymptotics

We take the new set of positive parameters

$$(36)\qquad \boldsymbol{\eta} = (\eta_0, \eta_{-1}; \eta_1, \dots, \eta_6)$$

satisfying the conditions

$$(37)\qquad 4\eta_0 = \sum_{j=-1}^{6} \eta_j, \qquad \eta_0 - \eta_{-1} < \eta_j < \frac{1}{2}\eta_0, \quad j = 1, 2, 3, 4, 5, 6,$$

and for each $n = 0, 1, 2, \dots$ relate them to the old parameters by the formulae

$$h_0 = \eta_0 n + 2, \quad h_{-1} = \eta_{-1} n + 2, \qquad h_j = \eta_j n + 1, \quad j = 1, 2, \dots, 6. \tag{38}$$

Then Lemma 6 yields that the quantities $F_n = F_{n,\boldsymbol{\eta}} := F(\boldsymbol{h})$ are linear forms in 1 and $\zeta(4)$ with rational coefficients, say

$$F_n = F_{n,\boldsymbol{\eta}} = u_n \zeta(4) - v_n, \qquad n = 0, 1, 2, \dots,$$

and the goal of this section is to determine the asymptotic behaviour of these linear forms as well as their coefficients u_n and v_n as $n \to \infty$.

To the set (36) assign the polynomial

$$\prod_{j=-1}^{6} (\tau - \eta_j) - \prod_{j=-1}^{6} (\tau - \eta_0 + \eta_j) \tag{39}$$

and the function

$$\begin{aligned} f_0(\tau) := \sum_{j=-1}^{6} &\eta_j \log(\eta_j - \tau) \\ &- (\eta_0 - \eta_{-1}) \log(\tau - \eta_0 + \eta_{-1}) - \sum_{j=1}^{6} (\eta_0 - \eta_j) \log(\eta_0 - \eta_j - \tau) \\ &+ (\eta_0 - \eta_1 - \eta_2) \log(\eta_0 - \eta_1 - \eta_2) + (\eta_0 - \eta_1 - \eta_5) \log(\eta_0 - \eta_1 - \eta_5) \\ &+ (\eta_0 - \eta_2 - \eta_4) \log(\eta_0 - \eta_2 - \eta_4) + (\eta_0 - \eta_3 - \eta_6) \log(\eta_0 - \eta_3 - \eta_6) \\ &- \eta_3 \log \eta_3 - (\eta_{-1} - \eta_0 + \eta_4) \log(\eta_{-1} - \eta_0 + \eta_4) \\ &- \eta_5 \log \eta_5 - (\eta_{-1} - \eta_0 + \eta_6) \log(\eta_{-1} - \eta_0 + \eta_6) \end{aligned}$$

defined in the cut τ-plane $\mathbb{C} \setminus (-\infty, \max\{0, \eta_0 - \eta_{-1}\}] \cup [\eta_1^*, +\infty)$, where $\eta_1^* \le \eta_2^* \le \dots \le \eta_6^*$ denotes the ordered version of the set $\eta_1, \eta_2, \dots, \eta_6$.

The first condition in (37) implies that (39) is a fifth-degree polynomial; moreover, the symmetry under substitution $\tau \mapsto \eta_0 - \tau$ and the second condition in (37) yield that this polynomial has zeros

$$\frac{\eta_0}{2}, \ \frac{\eta_0}{2} \pm s_0, \ \text{and} \ \frac{\eta_0}{2} \pm i s_1,$$
$$\text{where} \quad \frac{\eta_0}{2} - s_0 \in \bigl(\max\{0, \eta_0 - \eta_{-1}\}, \eta_1^*\bigr), \ s_1 \in (0, +\infty).$$

The last four zeros can be easily determined by solving a certain biquadratic (in terms of $\eta_0/2 - \tau$) equation. Set

$$\tau_0 := \frac{\eta_0}{2} - s_0 \qquad \text{and} \qquad \tau_1 := \frac{\eta_0}{2} + i s_1. \tag{40}$$

Proposition 1. *The following limit relations hold:*

$$C_0 := -\lim_{n\to\infty}\frac{\log|F_n|}{n} = -f_0(\tau_0), \tag{41}$$

$$C_1 := \limsup_{n\to\infty}\frac{\log|u_n|}{n} = \limsup_{n\to\infty}\frac{\log|v_n|}{n} = \operatorname{Re} f_0(\tau_1). \tag{42}$$

Proof. The proof is based on application of the saddle-point method to the integral representation of Lemma 7 for the quantities F_n and a similar integral representation (see formula (48) below) for the coefficients u_n; the fact that both limits in (42) are equal follows immediately from the limit relation

$$\lim_{n\to\infty}\frac{v_n}{u_n} = \lim_{n\to\infty}\frac{u_n\zeta(4)-F_n}{u_n} = \zeta(4)\neq 0$$

since $-C_0<0<C_1$ under the conditions (37).

Without loss of generality, we will restrict ourselves to the 'most symmetric' case (35), i.e.,

$$\eta_0=\eta_{-1}=3 \qquad\text{and}\qquad \eta_1=\cdots=\eta_6=1, \tag{43}$$

that corresponds to the linear forms in $1,\zeta(4)$ constructed in Section 2.

In the case (43), the zeros (40) of the corresponding polynomial (39) are as follows:

$$\tau_0=\frac{3}{2}-3^{1/4}\cos\frac{\pi}{12}=\frac{3}{2}-\sqrt{\frac{3}{4}+\frac{\sqrt{3}}{2}}=0.22877012\ldots,$$

$$\tau_1=\frac{3}{2}+i3^{1/4}\sin\frac{\pi}{12}=\frac{3}{2}+\sqrt{\frac{3}{4}-\frac{\sqrt{3}}{2}}=1.5+i0.34062501\ldots.$$

By Lemma 7,

$$\begin{aligned}
F_n&=\frac{(-1)^n}{2\pi i}\int_{t_1-i\infty}^{t_1+i\infty}(3n+2+2t)\frac{\Gamma(3n+2+t)^2\Gamma(n+1+t)^6\Gamma(-t)^2}{\Gamma(2n+2+t)^6}\,\mathrm{d}t\\
&=\frac{(-1)^n}{2\pi i}\int_{t_1-i\infty}^{t_1+i\infty}\frac{(3n+2+2t)(3n+1+t)^2(3n+t)^2(n+t)^6}{(2n+1+t)^6(2n+t)^6}\\
&\qquad\qquad\times\frac{\Gamma(3n+t)^2\Gamma(n+t)^6\Gamma(-t)^2}{\Gamma(2n+t)^6}\,\mathrm{d}t,
\end{aligned}$$

with any $t_1\in\mathbb{R}$, $-n<t_1<0$. Using the asymptotic formula

$$\log\Gamma(z)=\left(z-\frac{1}{2}\right)\log z-z+\log\sqrt{2\pi}+O\big(|z|^{-1}\big)$$

for $z\in\mathbb{C}$ with $\operatorname{Re} z=\text{const}>0$, taking $t_1=-n\tau_0$ and changing variables $t=-n\tau$, after necessary transformations we obtain

$$F_n=\frac{2\pi(-1)^n}{in^2}\int_{\tau_0-i\infty}^{\tau_0+i\infty}\frac{(3-2\tau)(3-\tau)^3(1-\tau)^3}{\tau(2-\tau)^9}e^{nf(\tau)}\big(1+O(n^{-1})\big)\,\mathrm{d}\tau \tag{44}$$

as $n \to \infty$, where

$$f(\tau) := 2(3-\tau)\log(3-\tau) + 6(1-\tau)\log(1-\tau) + 2\tau\log\tau - 6(2-\tau)\log(2-\tau).$$

Since

$$f'(\tau) = \log\frac{\tau^2(2-\tau)^6}{(3-\tau)^2(1-\tau)^6} \tag{45}$$

and τ_0 is a zero of the polynomial (39) (which is $(\tau-3)^2(\tau-1)^6 - \tau^2(\tau-2)^6$ in the restricted case), we conclude that $f'(\tau_0) = 0$ and τ_0 is the unique maximum of the function $\operatorname{Re} f(\tau)$ on the contour. Thus the integral (44) is determined by the contribution of the saddle-point τ_0 (see [Br], Section 5.7):

$$F_n = \frac{(-1)^n(2\pi)^{3/2}}{n^{5/2}} \cdot \frac{(3-2\tau_0)(3-\tau_0)^3(1-\tau_0)^3}{\tau_0(2-\tau_0)^9} \cdot |f''(\tau_0)|^{-1/2} \cdot e^{nf(\tau_0)} \times \bigl(1 + O(n^{-1})\bigr),$$

hence

$$\begin{aligned} \lim_{n\to\infty}\frac{\log|F_n|}{n} &= f(\tau_0) = f(\tau_0) - \tau_0 f'(\tau_0) =: f_0(\tau_0) \\ &= \log\frac{(3-\tau_0)^6(1-\tau_0)^6}{(2-\tau_0)^{12}} = 3\log(2\sqrt{3}-3) =: -C_0. \end{aligned} \tag{46}$$

This proves the limit relation (41).

In the neighbourhood of $t = -k$, where $k = n+1, \dots, 2n+1$, the function $R(t)$ has the expansion

$$R(t) = \frac{A_{4k}}{(t+k)^4} + \frac{A_{3k}}{(t+k)^3} + \frac{A_{2k}}{(t+k)^2} + \frac{A_{1k}}{t+k} + O(1)$$

by (31). On the other hand,

$$\left(\frac{\sin\pi t}{\pi}\right)^2 = \left(\frac{\sin\pi(t+k)}{\pi}\right)^2 = (t+k)^2 + O\bigl((t+k)^4\bigr)$$

about $t = -k$ for $k \in \mathbb{Z}$. Therefore,

$$\operatorname{Res}_{t=-k}\left(\left(\frac{\sin\pi t}{\pi}\right)^2 R(t)\right) = \begin{cases} A_{3k} & \text{if } k = n+1, \dots, 2n+1, \\ 0 & \text{for other } k \in \mathbb{Z}, \end{cases}$$

and if $\mathcal{L}$ is a closed clockwise contour surrounding points $t = -n-1, \dots, -2n-1$, then

$$\begin{aligned} \text{(47)} \quad \frac{1}{3}u_n = \sum_{k=n+1}^{2n+1} A_{3k} &= -\frac{1}{2\pi i}\oint_{\mathcal{L}} \left(\frac{\sin \pi t}{\pi}\right)^2 R(t)\,\mathrm{d}t \\ &= -\frac{(-1)^n}{2\pi i}\oint_{\mathcal{L}} \left(\frac{\sin \pi t}{\pi}\right)^4 (3n+2+2t) \\ &\qquad \times \frac{\Gamma(3n+2+t)^2\Gamma(n+1+t)^6\Gamma(-t)^2}{\Gamma(2n+2+t)^6}\,\mathrm{d}t. \end{aligned}$$

Taking the rectangle with vertices $\pm it_2 \pm N$, for some fixed real $t_2 > 0$ and any $N > 2n+1$, as the contour $\mathcal{L}$ and using the estimates

$$\left|\frac{\sin \pi t}{t}\right| \le \frac{e^{\pi t_2}}{\pi}, \qquad R(t) = O(N^{-3}) \quad \text{as } N \to \infty$$

on the lateral sides of the rectangle, from (47) we deduce that

$$\begin{aligned} u_n = -\frac{3(-1)^n}{2\pi i}&\left(\int_{it_2-N}^{it_2+N} + \int_{-it_2+N}^{-it_2-N}\right)\left(\frac{\sin \pi t}{\pi}\right)^4 \\ &\times (3n+2+2t)\frac{\Gamma(3n+2+t)^2\Gamma(n+1+t)^6\Gamma(-t)^2}{\Gamma(2n+2+t)^6}\,\mathrm{d}t + O(N^{-2}), \end{aligned}$$

where the constant in $O(N^{-2})$ depends on t_2 only. Tending $N \to \infty$ and making the substitution $t \mapsto -t - h_0 = -t - (3n+2)$ in the first integral, we obtain

$$\begin{aligned} \text{(48)} \quad u_n = -\frac{3(-1)^n}{\pi i}&\int_{-it_2+\infty}^{-it_2-\infty}\left(\frac{\sin \pi t}{\pi}\right)^4 (3n+2+2t) \\ &\times \frac{\Gamma(3n+2+t)^2\Gamma(n+1+t)^6\Gamma(-t)^2}{\Gamma(2n+2+t)^6}\,\mathrm{d}t \end{aligned}$$

(cf. [Zu2], Lemma 3.1). Finally, take $t_2 = -ns_1 = -n\operatorname{Im}\tau_1$, change the variable $t = -n\tau$ and apply the asymptotic formula

$$\begin{aligned} \log\Gamma(z) = \left(z - \frac{1}{2}\right)\log z - z + \log\sqrt{2\pi} + O\big(|z|^{-1}\big) + O(e^{-2\pi|\operatorname{Im} z|}) \\ \text{for} \quad z \in \mathbb{C}, \quad |\operatorname{Im} z| \ge y_0 > 0 \end{aligned}$$

(see [Br], Section 6.5, and [Zu2], Lemma 3.2), to get from (48) the expansion

$$\begin{aligned} u_n = \frac{12\pi(-1)^n}{in^2}&\int_{is_1-\infty}^{is_1+\infty}\frac{(3-2\tau)(3-\tau)^3(1-\tau)^3}{\tau(2-\tau)^9}e^{nf(\tau)} \\ &\times \left(\frac{\sin \pi n\tau}{\pi}\right)^4\big(1 + O(n^{-1}) + O(e^{-2\pi ns_1})\big)\,\mathrm{d}\tau. \end{aligned}$$

Since

$$\left|\left(\frac{\sin\pi n\tau}{\pi}\right)^4 - \frac{e^{-4\pi i n\tau}}{(2\pi)^4}\right| = \left|\frac{e^{-4\pi i n\tau}}{(2\pi)^4}\right| \times \left| -4e^{2\pi i n\tau} + 6e^{4\pi i n\tau} - 4e^{6\pi i n\tau} + e^{8\pi i n\tau}\right| < 15e^{-2\pi n s_1}\cdot\left|\frac{e^{-4\pi i n\tau}}{(2\pi)^4}\right|$$

for $\tau\in\mathbb{C}$ with $\operatorname{Im}\tau = s_1 > 0$, we obtain

$$u_n = \frac{3(-1)^n}{4\pi^3 i n^2}\int_{is_1-\infty}^{is_1+\infty}\frac{(3-2\tau)(3-\tau)^3(1-\tau)^3}{\tau(2-\tau)^9}e^{n(f(\tau)-4\pi i\tau)} \times \big(1+O(n^{-1})+O(e^{-2\pi n s_1})\big)\,\mathrm{d}\tau. \tag{49}$$

By (45) and the definition of the point τ_1 (that is the zero of the polynomial (39)), hence $f'(\tau_1) - 4\pi i\tau_1 = 0$, we conclude that $\tau = \tau_1$ is the unique maximum of the function $\operatorname{Re}(f(\tau) - 4\pi i\tau)$ on the line $\operatorname{Im}\tau = s_1$. Therefore, the saddle-point method says that the asymptotics of the integral in (49) is determined by the contribution of the point $\tau = \tau_1$ that yields the desired limit relation

$$\limsup_{n\to\infty}\frac{\log|u_n|}{n} = \operatorname{Re} f(\tau_1) = \operatorname{Re}(f(\tau_1) - \tau_1 f'(\tau_1)) =: \operatorname{Re} f_0(\tau_1) = \log\frac{|3-\tau_1|^6|1-\tau_1|^6}{|2-\tau_1|^{12}} = 3\log(2\sqrt{3}+3) =: C_1.$$

The proof of Proposition 1 is complete. □

Remark. The limit relation (46) yields that $|F_n|\to 0$ as $n\to\infty$, and this is the fact that we have promised to prove for Theorem 1 (see the paragraph after Lemma 5). To be honest, the fact, that the asymptotics of the linear forms and their coefficients in the case (35) is determined by the zeros $(3\pm 2\sqrt{3}\,)^3$ of a quadratic polynomial with integral coefficients, gave us the idea to look for a second-order difference equation.

5. Group structure for $\zeta(4)$

This section can be viewed as a continuation of the story in [Zu4], Sections 4–6, where we explain the Rhin–Viola group structures for $\zeta(2)$ and $\zeta(3)$ by means of classical hypergeometric identities.

Proposition 2 (Bailey's integral transform [Ba], Section 6.8, formula (1)). *There holds the identity*

$$(50)\quad \frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{\Gamma(a+t)\,\Gamma(1+\frac12 a+t)\,\Gamma(b+t)\,\Gamma(c+t)\,\Gamma(d+t)}{\Gamma(\frac12 a+t)\,\Gamma(1+a-c+t)\,\Gamma(1+a-d+t)}$$
$$\times\frac{\Gamma(e+t)\,\Gamma(f+t)\,\Gamma(g+t)\,\Gamma(h+t)\,\Gamma(b-a-t)\,\Gamma(-t)}{\Gamma(1+a-e+t)\,\Gamma(1+a-f+t)\,\Gamma(1+a-g+t)\,\Gamma(1+a-h+t)}\,\mathrm{d}t$$
$$=\frac{\Gamma(c)\,\Gamma(d)\,\Gamma(e)\,\Gamma(f+b-a)\,\Gamma(g+b-a)\,\Gamma(h+b-a)}{\Gamma(k+c-a)\,\Gamma(k+d-a)\,\Gamma(k+e-a)\,\Gamma(1+a-g-h)\times\Gamma(1+a-f-h)\,\Gamma(1+a-f-g)}$$
$$\times\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{\Gamma(k+t)\,\Gamma(1+\frac12 k+t)\,\Gamma(b+t)\,\Gamma(k+c-a+t)\,\Gamma(k+d-a+t)}{\Gamma(\frac12 k+t)\,\Gamma(1+a-c+t)\,\Gamma(1+a-d+t)\,\Gamma(1+a-e+t)}$$
$$\times\frac{\Gamma(k+e-a+t)\,\Gamma(f+t)\,\Gamma(g+t)\,\Gamma(h+t)\,\Gamma(b-k-t)\,\Gamma(-t)}{\Gamma(1+k-f+t)\,\Gamma(1+k-g+t)\,\Gamma(1+k-h+t)}\,\mathrm{d}t,$$

where $k=1+2a-c-d-e$, and the parameters are connected by the relation

$$2+3a=b+c+d+e+f+g+h.$$

By Lemma 7 the transform (50) rearranges the parameters $\boldsymbol{h}$ as follows:

$$(51)\quad \mathfrak{b}=\mathfrak{b}_{123}\colon \boldsymbol{h}\mapsto(1+2h_0-h_1-h_2-h_3,h_{-1};1+h_0-h_2-h_3,$$
$$1+h_0-h_1-h_3,1+h_0-h_1-h_2,h_4,h_5,h_6).$$

Consider the set of 27 complementary parameters $\boldsymbol{e}$,

$$(52)\quad \begin{gathered} e_{jk}=h_0-h_j-h_k,\ 1\le j<k\le 6,\quad e_{0k}=h_k-1,\ 1\le k\le 6,\\ \overline{e}_{0k}=h_{-1}-h_0+h_k-1=1+2h_0-(h_1+\dots+h_6)+h_k,\\ 1\le k\le 6,\end{gathered}$$

and set

$$H(\boldsymbol{e}):=F(\boldsymbol{h}).$$

Then Bailey's transform can be written as follows:

$$(53)\qquad H(\boldsymbol{e})=\frac{\Gamma(e_{01}+1)\,\Gamma(e_{02}+1)\,\Gamma(e_{12}+1)\,\Gamma(\overline{e}_{05}+1)}{\Gamma(e_{23}+1)\,\Gamma(e_{13}+1)\,\Gamma(e_{03}+1)\,\Gamma(e_{46}+1)}H(\mathfrak{b}\boldsymbol{e}),$$

where $\mathfrak{b}$ from (51) is the following second-order permutation of the parameters (52):

$$(54)\qquad \mathfrak{b}=(e_{01}\ e_{23})(e_{02}\ e_{13})(e_{03}\ e_{12})(\overline{e}_{04}\ e_{56})(\overline{e}_{05}\ e_{46})(\overline{e}_{06}\ e_{45}).$$

We can also write the transform (53) in the form

$$(55)\qquad \frac{H(\boldsymbol{e})}{\Pi_1(\boldsymbol{e})}=\frac{H(\mathfrak{b}\boldsymbol{e})}{\Pi_1(\mathfrak{b}\boldsymbol{e})},\qquad \text{where}\quad \Pi_1(\boldsymbol{e}):=e_{01}!\,e_{02}!\,e_{12}!\,\overline{e}_{05}!\,.$$

Further, the $\boldsymbol{h}$-trivial group (i.e., the group of permutations of the parameters $h_1, h_2, \dots, h_6$) is generated by second-order permutations of h_k, $1 \le k \le 5$, and h_6. The action of these five permutations on the set (52) is as follows:

$$
\begin{aligned}
&\mathfrak{h}_1 = (h_1\ h_6) = (e_{01}\ e_{06})(\overline{e}_{01}\ \overline{e}_{06})(e_{12}\ e_{26})(e_{13}\ e_{36})(e_{14}\ e_{46})(e_{15}\ e_{56}),\\
&\mathfrak{h}_2 = (h_2\ h_6) = (e_{02}\ e_{06})(\overline{e}_{02}\ \overline{e}_{06})(e_{12}\ e_{16})(e_{23}\ e_{36})(e_{24}\ e_{46})(e_{25}\ e_{56}),\\
&\mathfrak{h}_3 = (h_3\ h_6) = (e_{03}\ e_{06})(\overline{e}_{03}\ \overline{e}_{06})(e_{13}\ e_{16})(e_{23}\ e_{26})(e_{34}\ e_{46})(e_{35}\ e_{56}),\\
&\mathfrak{h}_4 = (h_4\ h_6) = (e_{04}\ e_{06})(\overline{e}_{04}\ \overline{e}_{06})(e_{14}\ e_{16})(e_{24}\ e_{26})(e_{34}\ e_{36})(e_{45}\ e_{56}),\\
&\mathfrak{h}_5 = (h_5\ h_6) = (e_{05}\ e_{06})(\overline{e}_{05}\ \overline{e}_{06})(e_{15}\ e_{16})(e_{25}\ e_{26})(e_{35}\ e_{36})(e_{45}\ e_{46}),
\end{aligned}
\tag{56}
$$

and the quantity

$$
\frac{\Gamma(e_{03}+1)\,\Gamma(\overline{e}_{04}+1)\,\Gamma(e_{05}+1)\,\Gamma(\overline{e}_{06}+1)}{\Gamma(e_{12}+1)\,\Gamma(e_{15}+1)\,\Gamma(e_{24}+1)\,\Gamma(e_{36}+1)} \cdot H(\boldsymbol{e}) \tag{57}
$$

(due to the definition (28)) is stable under the action of (56). Setting

$$
\mathcal{E} = \mathcal{E}(\boldsymbol{e}) := \{e_{01}, e_{02}, e_{04}, e_{06}, \overline{e}_{01}, \overline{e}_{02}, \overline{e}_{03}, \overline{e}_{05}, e_{12}, e_{15}, e_{24}, e_{36}\} \tag{58}
$$

and combining the above stability results we arrive at the following fact.

Lemma 8. *The quantity*

$$
\frac{H(\boldsymbol{e})}{\Pi(\boldsymbol{e})}, \qquad \textit{where} \quad \Pi(\boldsymbol{e}) := \prod_{e_{jk} \in \mathcal{E}} e_{jk}!,
$$

is stable under the action of the group

$$
\mathfrak{G} := \langle \mathfrak{b}, \mathfrak{h}_1, \mathfrak{h}_2, \mathfrak{h}_3, \mathfrak{h}_4, \mathfrak{h}_5 \rangle.
$$

Moreover, the quantities h_{-1} and

$$
\Sigma(\boldsymbol{e}) := \sum_{e_{jk} \in \mathcal{E}} e_{jk}
$$

are also $\mathfrak{G}$-stable.

Proof. Routine calculations show the stability of $H(\boldsymbol{e})/\Pi(\boldsymbol{e})$ under the action of $\mathfrak{b}, \mathfrak{h}_1, \mathfrak{h}_2, \mathfrak{h}_3, \mathfrak{h}_4, \mathfrak{h}_5$ with a help of (55) and (57). Hence $H(\boldsymbol{e})/\Pi(\boldsymbol{e})$ is stable under the action of the $\boldsymbol{e}$-permutation group generated by these six permutations (54), (56).

The stability of h_{-1} under the action of (56) is obvious, and $\mathfrak{b}$ does not change the parameter h_{-1} by (51). Finally,

$$
\Sigma(\boldsymbol{e}) = 12h_0 - 4(h_1 + h_2 + h_3 + h_4 + h_5 + h_6) = 4h_{-1} - 8
$$

that yields the stability of $\Sigma(\boldsymbol{e})$ under the action of $\mathfrak{G}$. The proof is complete. $\square$

With the help of a `C++` program we have discovered that the group $\mathfrak{G}$ consists of 51840 elements, hence the left factor $\mathfrak{G}/\mathfrak{S}_6$ includes $51840/6! = 72$ left cosets; here $\mathfrak{S}_6$ is identified with the $\boldsymbol{h}$-trivial group $\langle \mathfrak{h}_1, \mathfrak{h}_2, \mathfrak{h}_3, \mathfrak{h}_4, \mathfrak{h}_5 \rangle$. It is interesting to mention that the group $\mathfrak{G}_0$ acting trivially on the set (58) consists of just 4 elements: $\mathfrak{g}_0 = \mathrm{id}$,

$$\begin{aligned}
\mathfrak{g}_1 &= (\mathfrak{h}_3\,\mathfrak{h}_1\,\mathfrak{h}_2\,\mathfrak{h}_5\,\mathfrak{b}\,\mathfrak{h}_1\,\mathfrak{h}_4\,\mathfrak{h}_5\,\mathfrak{b}\,\mathfrak{h}_1)^3 \\
&= (e_{01}\ \overline{e}_{02})(e_{02}\ \overline{e}_{01})(e_{03}\ \overline{e}_{06})(e_{04}\ \overline{e}_{05})(e_{05}\ \overline{e}_{04})(e_{06}\ \overline{e}_{03}) \\
&\qquad (e_{13}\ e_{26})(e_{14}\ e_{25})(e_{15}\ e_{24})(e_{16}\ e_{23})(e_{34}\ e_{56})(e_{35}\ e_{46}), \\
\mathfrak{g}_2 &= (\mathfrak{h}_1\,\mathfrak{h}_2\,\mathfrak{h}_4\,\mathfrak{h}_2\,\mathfrak{b}\,\mathfrak{h}_3\,\mathfrak{h}_5\,\mathfrak{h}_1\,\mathfrak{h}_2)^3 \\
&= (e_{01}\ e_{24})(e_{02}\ \overline{e}_{03})(e_{03}\ e_{46})(e_{04}\ \overline{e}_{05})(e_{05}\ e_{26})(e_{06}\ \overline{e}_{01}) \\
&\qquad (\overline{e}_{02}\ e_{15})(\overline{e}_{04} e_{13})(\overline{e}_{06}\ e_{35})(e_{12}\ e_{36})(e_{14}\ e_{56})(e_{25}\ e_{34}), \\
\mathfrak{g}_3 &= \mathfrak{h}_1\,\mathfrak{h}_2\,\mathfrak{b}\,\mathfrak{h}_3\,\mathfrak{h}_1\,\mathfrak{h}_5\,\mathfrak{h}_2\,\mathfrak{h}_3\,\mathfrak{b} = \mathfrak{g}_1\,\mathfrak{g}_2 \\
&= (e_{01}\ e_{15})(e_{02}\ e_{06})(e_{03}\ e_{35})(e_{05}\ e_{13})(\overline{e}_{01}\ \overline{e}_{03})(\overline{e}_{02}\ e_{24}) \\
&\qquad (\overline{e}_{04}\ e_{26})(\overline{e}_{06}\ e_{46})(e_{12}\ e_{36})(e_{14}\ e_{34})(e_{16}\ e_{23})(e_{25}\ e_{56}).
\end{aligned}$$

Remark. In the most symmetric case (35) all complementary parameters (52) are equal to n that means that any permutation from $\mathfrak{G}$ does not change the quantity $F(\boldsymbol{h})$. This fact explains why do we dub this case as 'most symmetric'.

6. Denominators of linear forms

As we have mentioned in Remark to Lemma 6, 'trivial' arithmetic (33) of the linear forms $H(\boldsymbol{e}) = F(\boldsymbol{h})$ does not lead us to a qualitative result for $\zeta(4)$. We are able to estimate the irrationality measure of $\zeta(4)$ under the following condition, which we have checked numerically for several values of $\boldsymbol{h}$ satisfying (26) and (27).

Denominator Conjecture. *There holds the inclusion*[4]

$$D_{m_1} D_{m_2} D_{m_3} D_{m_4} \cdot \Phi^{-1}(\boldsymbol{e}) \cdot H(\boldsymbol{e}) \in \mathbb{Z}\zeta(4) + \mathbb{Z},$$

where $m_1 \ge m_2 \ge m_3 \ge m_4$ *are the four successive maxima of the set* $\mathbf{e}$ *in* (52) *and*

$$\Phi(\boldsymbol{e}) := \prod_{p > \sqrt{h_{-1}}} p^{\nu_p}$$

with

$$\nu_p := \left\lfloor \frac{1}{2} \left\lfloor \frac{1}{4} \sum_{e_{jk} \in \mathcal{E}} \frac{e_{jk}}{p} \right\rfloor - \frac{1}{8} \sum_{e_{jk} \in \mathcal{E}} \left\lfloor \frac{e_{jk}}{p} \right\rfloor \right\rfloor = \left\lfloor \frac{1}{2} \left\lfloor \frac{h_{-1} - 2}{p} \right\rfloor - \frac{1}{8} \sum_{e_{jk} \in \mathcal{E}} \left\lfloor \frac{e_{jk}}{p} \right\rfloor \right\rfloor.$$

[4] In the most symmetric case (35) this conjecture reduces to the conjecture (25) of Section 2.

If this conjecture is true, then taking any element $\mathfrak{g} \in \mathfrak{G}$ and writing conclusion of Lemma 8 as

$$D_{m_1}D_{m_2}D_{m_3}D_{m_4}H(\boldsymbol{e}) = D_{m_1}D_{m_2}D_{m_3}D_{m_4}\Phi^{-1}(\mathfrak{g}\boldsymbol{e})H(\mathfrak{g}\boldsymbol{e}) \cdot \frac{\Pi(\boldsymbol{e})\Phi(\mathfrak{g}\boldsymbol{e})}{\Pi(\mathfrak{g}\boldsymbol{e})}$$

we deduce that, for any prime $p > \sqrt{h_{-1}}$,

$$\begin{aligned}\operatorname{ord}_p\bigl(D_{m_1}D_{m_2}D_{m_3}D_{m_4}H(\boldsymbol{e})\bigr) &\ge \operatorname{ord}_p \frac{\Pi(\boldsymbol{e})\Phi(\mathfrak{g}\boldsymbol{e})}{\Pi(\mathfrak{g}\boldsymbol{e})} \\ &= \sum_{e_{jk}\in\mathcal{E}}\left\lfloor\frac{e_{jk}}{p}\right\rfloor - \sum_{e'_{jk}\in\mathfrak{g}\mathcal{E}}\left\lfloor\frac{e'_{jk}}{p}\right\rfloor + \left\lfloor\frac{1}{2}\left\lfloor\frac{h_{-1}-2}{p}\right\rfloor - \frac{1}{8}\sum_{e'_{jk}\in\mathfrak{g}\mathcal{E}}\left\lfloor\frac{e'_{jk}}{p}\right\rfloor\right\rfloor,\end{aligned} \tag{59}$$

where $\mathfrak{g}\mathcal{E} = \mathcal{E}(\mathfrak{g}\boldsymbol{e})$ and $\operatorname{ord}_p(u\zeta(4) - v) := \min\{\operatorname{ord}_p u, \operatorname{ord}_p v\}$ for rational numbers u, v. Finally, setting

$$\Lambda(\boldsymbol{e}) = \prod_{p>\sqrt{h_{-1}}} p^{\lambda_p}$$

with

$$\lambda_p := \max_{\mathfrak{g}\in\mathfrak{G}}\left(\sum_{e_{jk}\in\mathcal{E}}\left\lfloor\frac{e_{jk}}{p}\right\rfloor - \sum_{e'_{jk}\in\mathfrak{g}\mathcal{E}}\left\lfloor\frac{e'_{jk}}{p}\right\rfloor + \left\lfloor\frac{1}{2}\left\lfloor\frac{h_{-1}-2}{p}\right\rfloor - \frac{1}{8}\sum_{e'_{jk}\in\mathfrak{g}\mathcal{E}}\left\lfloor\frac{e'_{jk}}{p}\right\rfloor\right\rfloor\right),$$

from (59) we obtain the inclusion

$$D_{m_1}D_{m_2}D_{m_3}D_{m_4} \cdot \Lambda^{-1}(\boldsymbol{e}) \cdot H(\boldsymbol{e}) \in \mathbb{Z}\zeta(4) + \mathbb{Z}. \tag{60}$$

Now, to each $n = 0, 1, 2, \dots$ assign the parameters $\boldsymbol{h}$ in accordance with (38) and set

$$\begin{aligned} &e_{jk} = \eta_0 - \eta_j - \eta_k, \quad 1 \le j < k \le 6, \qquad e_{0k} = \eta_k, \quad 1 \le k \le 6, \\ &\overline{e}_{0k} = \eta_{-1} - \eta_0 + \eta_k = 2\eta_0 - (\eta_1 + \dots + \eta_6) + \eta_k, \quad 1 \le k \le 6, \end{aligned}$$

so that the set of complementary parameters $\boldsymbol{e}\cdot n$ corresponds to the set $\boldsymbol{h}$. Then, in the above notation, we can write the inclusion (60) as

$$D_{m_1n}D_{m_2n}D_{m_3n}D_{m_4n} \cdot \Lambda^{-1}(\boldsymbol{e}n) \cdot H(\boldsymbol{e}n) \in \mathbb{Z}\zeta(4) + \mathbb{Z}.$$

The asymptotic behaviour of the linear forms $H(\boldsymbol{e}n) \in \mathbb{Q}\zeta(4) + \mathbb{Q}$ and their coefficients as $n \to \infty$ is determined by Proposition 1; in addition,

$$\lim_{n\to\infty} \frac{\log(D_{m_1n}D_{m_2n}D_{m_3n}D_{m_4n})}{n} = m_1 + m_2 + m_3 + m_4$$

by the consequence (2) of the prime number theorem, while the arithmetic lemma of Chudnovsky–Rukhadze–Hata (see, e.g., [Zu2], Lemma 4.4) yields

$$\lim_{n\to\infty} \frac{\log \Lambda(\boldsymbol{e}n)}{n} = \int_0^1 \lambda(x)\,\mathrm{d}\psi(x),$$

where

$$\lambda(x) := \max_{\mathfrak{g}\in\mathfrak{G}}\biggl(\sum_{e_{jk}\in\mathcal{E}}\lfloor e_{jk}x\rfloor - \sum_{e'_{jk}\in\mathfrak{g}\mathcal{E}}\lfloor e'_{jk}x\rfloor + \biggl\lfloor\frac{1}{2}\lfloor\eta_{-1}x\rfloor - \frac{1}{8}\sum_{e'_{jk}\in\mathfrak{g}\mathcal{E}}\lfloor e'_{jk}x\rfloor\biggr\rfloor\biggr)$$

is a 1-periodic function.

Recalling the notation of Proposition 1 and combining its results with saying above, as in [RV2], the proof of Theorem 5.1, we arrive at the following statement.

Proposition 3. *Under the denominator conjecture, let*

$$C_0 = -f_0(\tau_0), \qquad C_1 = \operatorname{Re} f_0(\tau_1),$$

$$C_2 = m_1 + m_2 + m_3 + m_4 - \int_0^1 \lambda(x)\,\mathrm{d}\psi(x).$$

If $C_0 > C_2$, then the irrationality exponent of $\zeta(4)$ satisfies the estimate

$$\mu(\zeta(4)) \le \frac{C_0 + C_1}{C_0 - C_2}.$$

Recall that the *irrationality exponent* $\mu = \mu(\alpha)$ of a real irrational number α is the least possible exponent such that for any $\varepsilon > 0$ the inequality

$$\left|\alpha - \frac{p}{q}\right| \le \frac{1}{q^{\mu+\varepsilon}}$$

has only finitely many solutions in integers p, q with $q > 0$.

With a help of Proposition 3 we are able to state the following conditional result.

Theorem 3. *The irrationality exponent of $\zeta(4)$ satisfies the estimate*

$$\mu(\zeta(4)) \le 25.38983113\ldots \tag{61}$$

provided that the denominator conjecture holds.

Proof. Taking $\boldsymbol{\eta} = (68, 57; 22, 23, 24, 25, 26, 27)$ we obtain

$$\tau_0 = 11.83684636\ldots, \qquad C_0 = -f_0(\tau_0) = 37.85606933\ldots,$$
$$\tau_1 = 34 + i6.34312459\ldots, \qquad C_1 = \operatorname{Re} f_0(\tau_1) = 104.96178579\ldots,$$

and

$$\begin{aligned} C_2 &= m_1 + m_2 + m_3 + m_4 - \int_0^1 \lambda(x)\,\mathrm{d}\psi(x) \\ &= 27 + 26 + 25 + 24 - 69.76893283\ldots = 32.23106716\ldots . \end{aligned}$$

Thus, application of Proposition 3 yields the desired estimate (61). $\square$

The estimate (61) can be compared with the 'best known' estimate

$$\mu(\zeta(4)) \le 204.94259587\dots,$$

which follows from the general result of Yu. Aleksentsev [Al] on approximations of π by algebraic numbers.[5]

7. Further difference equations for zeta values

A natural very-well-poised generalization of Ball's sequence (9),

$$(62)\quad F_{k,n} := n!^{k-1}\sum_{t=1}^{\infty}(2t+n)\frac{(t-1)\cdots(t-n)\cdot(t+n+1)\cdots(t+2n)}{t^{k+1}(t+1)^{k+1}\cdots(t+n)^{k+1}} \times(-1)^{(k-1)(t+n+1)}$$

$$\in\begin{cases}\mathbb{Q}\zeta(k)+\mathbb{Q}\zeta(k-2)+\dots+\mathbb{Q}\zeta(2)+\mathbb{Q} & \text{for } k\ge 2 \text{ even},\\ \mathbb{Q}\zeta(k)+\mathbb{Q}\zeta(k-2)+\dots+\mathbb{Q}\zeta(3)+\mathbb{Q} & \text{for } k\ge 2 \text{ odd},\end{cases}$$

where $n=1,2,\dots$, gives rise for searching difference equations satisfied by both linear forms $F_{k,n}$ and their rational coefficients. Applying Zeilberger's algorithm of creative telescoping in the manner of Section 2 we deduce the following result for the linear forms

$$(63)\qquad F_{5,n}=u_n\zeta(5)+w_n\zeta(3)-v_n.$$

Theorem 4. *The numbers u_n, w_n, v_n in the representation* (63) *are positive rationals satisfying the third-order difference equation*

$$(64)\quad (n+1)(n+2)^5 b_0(n)u_{n+2}-b_1(n)u_{n+1}-b_2(n)u_n + 2(2n+1)n^5 b_0(n+1)u_{n-1}=0,$$

with

$$u_0=2,\quad w_0=0,\quad v_0=0,\qquad u_1=18,\quad w_1=66,\quad v_1=98,$$
$$u_2=938,\quad w_2=\frac{6125}{2},\quad v_2=\frac{74463}{16},$$

[5] In fact, the result of [Al] is proved for approximations of π by algebraic numbers of sufficiently large degree.

where

$$b_0(n) = 41218n^3 + 48459n^2 + 20010n + 2871,$$

$$b_1(n) = 2(n+1)(3874492n^8 + 33613836n^7 + 123666762n^6 + 250134420n^5 + 301587620n^4 + 220011738n^3 + 94372815n^2 + 21917736n + 2131500),$$

$$b_2(n) = 2(48802112n^9 + 350188128n^8 + 1080631646n^7 + 1882848690n^6 + 2045758212n^5 + 1442754107n^4 + 663248761n^3 + 192486369n^2 + 32136756n + 2360484).$$

The characteristic polynomial $\lambda^3 - 188\lambda^2 - 2368\lambda + 4$ of the difference equation (64) determines the asymptotic behaviour of the linear forms (63) and their coefficients as $n \to \infty$.

A similar (but quite cumbersome) fourth-order recursion with characteristic polynomial $\lambda^4 - 828\lambda^3 - 132246\lambda^2 + 260604\lambda - 27$ has been discovered by us for the linear forms $F_{7,n}$ and their coefficients. These recursions allow us to verify the inclusions

$$D_n^5 F_{5,n} \in \mathbb{Z}\zeta(5) + \mathbb{Z}\zeta(3) + \mathbb{Z}, \qquad D_n^7 F_{7,n} \in \mathbb{Z}\zeta(7) + \mathbb{Z}\zeta(5) + \mathbb{Z}\zeta(3) + \mathbb{Z}$$

up to $n = 1000$, although we are able to prove that

$$D_n^{k+1}\widetilde{\Phi}_n^{-1} F_{k,n} \in \mathbb{Z}\zeta(k) + \mathbb{Z}\zeta(k-2) + \dots + \mathbb{Z}\zeta(3) + \mathbb{Z} \quad \text{for } k \text{ odd}, \tag{65}$$

where

$$\widetilde{\Phi}_n := \prod_{\substack{p<n \\ \{n/p\}\in[\frac{2}{3},1)}} p, \qquad \lim_{n\to\infty} \frac{\log \widetilde{\Phi}_n}{n} = \psi(1) - \psi\Big(\frac{2}{3}\Big) - \frac{1}{2} = 0.24101875\dots, \tag{66}$$

using our arithmetic results [Zu2], Lemmas 4.2–4.4.

Another story deals with the quantities

$$\widetilde{F}_n := \frac{1}{2}\sum_{t=1}^{\infty} \frac{\mathrm{d}^2}{\mathrm{d}t^2}\bigg((2t+n)\bigg(\frac{(t-1)\cdots(t-n)\cdot(t+n+1)\cdots(t+2n)}{(t(t+1)\cdots(t+n))^2}\bigg)^3\bigg) = \widetilde{u}_n\zeta(7) + \widetilde{w}_n\zeta(5) - \widetilde{v}_n,$$

where $\widetilde{u}_n, \widetilde{w}_n, \widetilde{v}_n$ are positive rationals. We have discovered a (quite cumbersome) fourth-order difference equation satisfied by $\widetilde{u}_n, \widetilde{w}_n, \widetilde{v}_n$; its characteristic polynomial is

$$\lambda^4 + 9264\lambda^3 - 12116166\lambda^2 - 752300\lambda - 19683 \qquad (19683 = 3^9).$$

As we have proved in [Zu2], Proposition 4.1, the following inclusions hold:

$$D_n^8 \cdot \widetilde{\Phi}_n^{-3} \cdot \widetilde{F}_n \in \mathbb{Z}\zeta(7) + \mathbb{Z}\zeta(5) + \mathbb{Z},$$

where $\widetilde{\Phi}_n$ is given in (66), while our calculations up to $n = 1000$ with a help of the recursion mentioned above show that

$$D_n^7 \cdot \widetilde{\Phi}_n^{-2} \cdot \widetilde{F}_n \in \mathbb{Z}\zeta(7) + \mathbb{Z}\zeta(5) + \mathbb{Z}.$$

What is a trick that makes arithmetic as it is?

8. Multiple-integral representation of very-well-poised hypergeometric series

In [Zu4], Section 9, we conjecture, for integer $k \geq 2$, the coincidence of the very-well-poised hypergeometric series (62) and the multiple integral

$$(67)\quad J_{k,n} := \int\!\!\cdots\!\!\int_{[0,1]^k} \frac{x_1^n(1-x_1)^n x_2^n(1-x_2)^n \cdots x_k^n(1-x_k)^n}{Q_k(x_1,x_2,\dots,x_k)^{n+1}}\, \mathrm{d}x_1\, \mathrm{d}x_2 \cdots \mathrm{d}x_k,$$

where $Q_0 := 1$ and

$$(68)\quad \begin{aligned} Q_k = Q_k(x_1,x_2,\dots,x_k) &:= 1 - (1 - (\cdots(1 - (1 - x_k)x_{k-1})\cdots)x_2)x_1 \\ &= 1 - x_1 Q_{k-1}(x_2,\dots,x_k) \\ &= Q_{k-1}(x_1,\dots,x_{k-1}) + (-1)^k x_1 x_2 \cdots x_k \end{aligned}$$

for $k \geq 1$. The integrals $J_{2,n}$ and $J_{3,n}$ have been studied by F. Beukers [Be1] in the connection with Apéry's proof of the irrationality of $\zeta(2)$ and $\zeta(3)$. In [Zu4], we prove the coincidence of $F_{3,n}$ and $J_{3,n}$ with the help of Bailey's identity ([Ba], Section 6.3, formula (2)) and Nesterenko's integral theorem ([Ne2], Theorem 2), and use similar arguments for showing that $F_{2,n} = J_{2,n}$. For general integer $k \geq 2$, the integrals (67) are introduced by O. Vasilenko [VaO] who states several results for $J_{k,0}$. The cases $k = 4, 5$ and an arbitrary integer n in (67) are developed by D. Vasilyev [VaD]; in particular, he conjectures the inclusions

$$D_n^k J_{k,n} \in \mathbb{Z}\zeta(k) + \mathbb{Z}\zeta(k-2) + \cdots + \mathbb{Z}\zeta(3) + \mathbb{Z} \quad \text{for } k \text{ odd}$$

(cf. (65)), and proves them if $k = 5$.

There is a regular way to obtain difference equations for the quantities (67); it is a part of the general *WZ-theory* developed by H. Wilf and D. Zeilberger [WZ]. However, difference equations for $J_{4,n}$ and $J_{5,n}$ by these means are out of calculative abilities of our computer, so we cannot use a 'routine matter' to verify the identity $F_{k,n} = J_{k,n}$ even when $k = 4, 5$.

The aim of this section is to deduce the desired coincidence of (62) and (67) from a general analytic result on a multiple-integral representation of very-well-poised hypergeometric series.[6]

[6] As it is mentioned by G. E. Andrews in [An], Section 16, "an entire survey paper could be written just on integrals connected with well-poised series". The following theorem would extend this survey a little bit.

Consider two objects: very-well-poised hypergeometric series

$$
\begin{aligned}
(69)\quad F_k(\boldsymbol{h}) = F_k(h_0; h_1, \dots, h_k) &:= \frac{\Gamma(1+h_0)\prod_{j=1}^{k}\Gamma(h_j)}{\prod_{j=1}^{k}\Gamma(1+h_0-h_j)} \\
&\times {}_{k+2}F_{k+1}\left(\begin{matrix} h_0, 1+\frac{1}{2}h_0, & h_1, & \dots, & h_k \\ \frac{1}{2}h_0, & 1+h_0-h_1, & \dots, & 1+h_0-h_k \end{matrix} \,\middle|\, (-1)^{k+1}\right) \\
&= \sum_{\mu=0}^{\infty}(h_0+2\mu)\frac{\prod_{j=0}^{k}\Gamma(h_j+\mu)}{\prod_{j=0}^{k}\Gamma(1+h_0-h_j+\mu)}(-1)^{(k+1)\mu},
\end{aligned}
$$

and multiple integrals

$$
\begin{aligned}
(70)\qquad J_k(\boldsymbol{a}, \boldsymbol{b}) &:= J_k\left(\begin{matrix} a_0, a_1, \dots, a_k \\ b_1, \dots, b_k \end{matrix}\right) \\
&:= \int\cdots\int_{[0,1]^k} \frac{\prod_{j=1}^{k} x_j^{a_j-1}(1-x_j)^{b_j-a_j-1}}{Q_k(x_1, x_2, \dots, x_k)^{a_0}}\,\mathrm{d}x_1\,\mathrm{d}x_2\cdots\mathrm{d}x_k.
\end{aligned}
$$

Theorem 5. *For each $k \ge 1$, there holds the identity*

$$
\begin{aligned}
(71)\quad &\frac{\prod_{j=1}^{k+1}\Gamma(1+h_0-h_j-h_{j+1})}{\Gamma(h_1)\,\Gamma(h_{k+2})}\cdot F_{k+2}(h_0; h_1, \dots, h_{k+2}) \\
&\qquad = J_k\left(\begin{matrix} h_1, & h_2, & h_3, & \dots, & h_{k+1} \\ & 1+h_0-h_3, & 1+h_0-h_4, & \dots, & 1+h_0-h_{k+2} \end{matrix}\right),
\end{aligned}
$$

provided that

$$(72)\qquad 1+\operatorname{Re} h_0 > \frac{2}{k+1}\cdot\sum_{j=1}^{k+2}\operatorname{Re} h_j,$$

$$(73)\qquad \operatorname{Re}(1+h_0-h_{j+1}) > \operatorname{Re} h_j > 0 \qquad \textit{for} \quad j = 2, \dots, k+1,$$

$$(74)\qquad h_1, h_{k+2} \ne 0, -1, -2, \dots .$$

Remark. Condition (72) is required for the absolute convergence of the series (69) in the unit circle (and, in particular, at the point $(-1)^{k+1}$), while condition (73) ensures the convergence of the corresponding multiple integral (70). The restriction (74) can be removed by the theory of analytic continuation if we write $\Gamma(h_j+\mu)/\Gamma(h_j)$ for $j = 1, k+2$ as Pochhammer's symbol $(h_j)_\mu$ when summing in (69).

In the case of *integral* parameters $\boldsymbol{h}$, the quantities (69) are known to be $\mathbb{Q}$-linear forms in even/odd zeta values depending on parity of $k \ge 4$ (see [Zu4], Section 9). Therefore, if positive integral parameters $\boldsymbol{a}$ and $\boldsymbol{b}$ satisfy the additional condition

$$(75)\qquad b_1 + a_2 = b_2 + a_3 = \cdots = b_{k-1} + a_k,$$

then the quantities (70) are $\mathbb{Q}$-linear forms in even/odd zeta values. Specialization $a_j = n+1$ and $b_j = 2n+2$ gives one the desired coincidence of (62) and (67). The choice $a_j = rn+1$ and $b_j = (r+1)n+2$ in (70) (or, equivalently, $h_0 = (2r+1)n+2$ and $h_j = rn+1$ for $j = 1,\dots,k+2$ in (69)) with the integer $r \ge 1$ depending on a given odd integer k presents almost the same linear forms in odd zeta values as considered by T. Rivoal in [Ri1] for proving his remarkable result on infiniteness of irrational numbers in the set $\zeta(3), \zeta(5), \zeta(7), \dots$.

In addition, we have to mention, under hypothesis (75), the obvious stability of the quantity

$$\begin{aligned}\frac{F_{k+2}(h_0; h_1, \dots, h_{k+2})}{\prod_{j=1}^{k+2}\Gamma(h_j)} &= \frac{J_k(\boldsymbol{a}, \boldsymbol{b})}{\prod_{j=2}^{k+1}\Gamma(h_j)\cdot\prod_{j=1}^{k+1}\Gamma(1+h_0-h_j-h_{j+1})}\\ &= \frac{J_k(\boldsymbol{a}, \boldsymbol{b})}{\prod_{j=1}^{k}\Gamma(a_j)\cdot\Gamma(b_1+a_2-a_0-a_1)\cdot\prod_{j=1}^{k}\Gamma(b_j-a_j)}\end{aligned}$$

under the action of the ($\boldsymbol{h}$-trivial) group $\mathfrak{G}_k$ of order $(k+2)!$ containing all permutations of the parameters $h_1, \dots, h_{k+2}$. This fact can be applied for number-theoretic applications as in [RV1], [RV2] and Sections 5, 6 above. In the cases $k=2$ and $k=3$ the change of variables $(x_{k-1}, x_k) \mapsto (1-x_k, 1-x_{k-1})$ in (70) produces an additional transformation $\mathfrak{b}$ of both (70) and (69); for $k \ge 4$ this transformation is not yet available since condition (75) is broken. The groups $\langle\mathfrak{G}_2, \mathfrak{b}\rangle$ and $\langle\mathfrak{G}_3, \mathfrak{b}\rangle$ of orders 120 and 1920 respectively are known: see [Ba], Sections 3.6 and 7.5, for a hypergeometric-series origin and [RV1], [RV2] for a multiple-integral explanation. G. Rhin and C. Viola make a use of these groups to discover nice estimates for the irrationality measures of $\zeta(2)$ and $\zeta(3)$. Finally, we want to note that the group $\mathfrak{G}_k$ can be easily interpretated as the permutation group of the parameters

$$e_{0l} = h_l - 1, \quad 1 \le l \le k+2, \qquad e_{jl} = h_0 - h_j - h_l, \quad 1 \le j < l \le k+2,$$

as in Section 5 (see [Zu4], Section 9, for details).

Lemma 9. *Theorem* 5 *is true if* $k=1$.

Proof. Thanks to a limiting case of Dougall's theorem,

$$F_3(h_0; h_1, h_2, h_3) = \frac{\Gamma(h_1)\,\Gamma(h_2)\,\Gamma(h_3)\,\Gamma(1+h_0-h_1-h_2-h_3)}{\Gamma(1+h_0-h_1-h_2)\,\Gamma(1+h_0-h_1-h_3)\times\Gamma(1+h_0-h_2-h_3)} \tag{76}$$

(see, e.g., [Ba], Section 4.4, formula (1)), provided that $1+\operatorname{Re} h_0 > \operatorname{Re}(h_1+h_2+h_3)$ and h_j is not a non-positive integer for $j = 1,2,3$. On the other hand, the integral on the right of (71) has Euler type, that is

$$J_1\begin{pmatrix} h_1, & h_2 \\ & 1+h_0-h_3 \end{pmatrix} = \int_0^1 \frac{x^{h_2-1}(1-x)^{h_0-h_2-h_3}}{(1-x)^{h_1}}\,\mathrm{d}x = \frac{\Gamma(h_2)\,\Gamma(1+h_0-h_1-h_2-h_3)}{\Gamma(1+h_0-h_1-h_3)},$$

provided that $1+\operatorname{Re} h_0 > \operatorname{Re}(h_1+h_2+h_3)$ and $\operatorname{Re} h_2 > 0$. Therefore, multiplying equality (76) by the required product of gamma-functions we deduce identity (71) if $k=1$. □

Remark. If we arrange about $J_0(a_0)$ to be 1, the claim of Theorem 5 remains valid if $k=0$ thanks to another consequence of Dougall's theorem ([Ba], Section 4.4, formula (3)).

Lemma 10 ([Ne2], Section 3.2). *Let $a_0, a, b \in \mathbb{C}$ and $t_0 \in \mathbb{R}$ be numbers satisfying the conditions*

$$\operatorname{Re} a_0 > t_0 > 0, \quad \operatorname{Re} a > t_0 > 0, \quad \textit{and} \quad \operatorname{Re} b > \operatorname{Re} a_0 + \operatorname{Re} a.$$

Then for any non-zero $z \in \mathbb{C} \setminus (1,+\infty)$ the following identity holds:

$$\int_0^1 \frac{x^{a-1}(1-x)^{b-a-1}}{(1-zx)^{a_0}}\,\mathrm{d}x = \frac{\Gamma(b-a)}{\Gamma(a_0)} \cdot \frac{1}{2\pi i}\int_{-t_0-i\infty}^{-t_0+i\infty} \frac{\Gamma(a_0+t)\,\Gamma(a+t)\,\Gamma(-t)}{\Gamma(b+t)}\,(-z)^t\,\mathrm{d}t, \tag{77}$$

where $(-z)^t = |z|^t e^{it\arg(-z)}$, $-\pi < \arg(-z) < \pi$ for $z \in \mathbb{C} \setminus [0,+\infty)$ and $\arg(-z) = \pm\pi$ for $z \in (0,1]$. The integral on the right-hand side of (77) converges absolutely. In addition, if $|z| \le 1$, both integrals in (77) can be identified with the absolutely convergent Gauss hypergeometric series

$$\frac{\Gamma(a)\,\Gamma(b-a)}{\Gamma(b)} \cdot {}_2F_1\left(\left.\begin{matrix} a_0, a \\ b \end{matrix}\,\right|\, z\right) = \frac{\Gamma(b-a)}{\Gamma(a_0)} \sum_{\nu=0}^{\infty} \frac{\Gamma(a_0+\nu)\,\Gamma(a+\nu)}{\nu!\,\Gamma(b+\nu)} z^\nu.$$

Set $\varepsilon_k = 0$ for k even and $\varepsilon_k = 1$ or -1 for k odd.

Lemma 11. *For each integer $k \ge 2$, there holds the relation*

$$\begin{aligned} J_k&\begin{pmatrix} a_0, a_1, \ldots, a_{k-1}, a_k \\ b_1, \ldots, b_{k-1}, b_k \end{pmatrix} \\ &= \frac{\Gamma(b_k-a_k)}{\Gamma(a_0)} \cdot \frac{1}{2\pi i}\int_{-t_0-i\infty}^{-t_0+i\infty} \frac{\Gamma(a_0+t)\,\Gamma(a_k+t)\,\Gamma(-t)}{\Gamma(b_k+t)} \\ &\qquad \times e^{\varepsilon_k \pi i t} \cdot J_{k-1}\begin{pmatrix} a_0+t, a_1+t, \ldots, a_{k-1}+t \\ b_1+t, \ldots, b_{k-1}+t \end{pmatrix}\mathrm{d}t, \end{aligned}$$

provided that $\operatorname{Re} a_0 > t_0 > 0$, $\operatorname{Re} a_k > t_0 > 0$, $\operatorname{Re} b_k > \operatorname{Re} a_0 + \operatorname{Re} a_k$, and the integral on the left converges.

Proof. We start with mentioning that the first recursion in (68) and inductive arguments yield the inequality

$$0 < Q_k(x_1, x_2, \dots, x_k) < 1 \text{ for } (x_1, x_2, \dots, x_k) \in (0,1)^k \text{ and } k \ge 1. \tag{78}$$

By the second recursion in (68), $Q_k = Q_{k-1} \cdot (1 - zx_k)$ for $k \ge 2$, where

$$z = \frac{(-1)^{k+1} x_1 \cdots x_{k-1}}{Q_{k-1}(x_1, \dots, x_{k-1})}.$$

For each $(x_1, \dots, x_{k-1}) \in (0,1)^{k-1}$, the number z is real with the property $z < 0$ for k even and $0 < z < 1$ for k odd, since in the last case we have

$$z = \frac{x_1 \cdots x_{k-1}}{Q_{k-1}(x_1, \dots, x_{k-2}, x_{k-1})} = \frac{x_1 \cdots x_{k-1}}{Q_{k-2}(x_1, \dots, x_{k-2}) + x_1 \cdots x_{k-1}} < 1$$

by (78). Therefore, splitting the integral (70) over $[0,1]^k = [0,1]^{k-1} \times [0,1]$ and applying Lemma 10 to the integral

$$\int_0^1 \frac{x_k^{a_k - 1}(1 - x_k)^{b_k - a_k - 1}}{(1 - zx_k)^{a_0}} \, \mathrm{d}x_k$$

we arrive at the desired relation. $\square$

Proof of Theorem 5. The case $k = 1$ is considered in Lemma 9. Therefore we will assume that $k \ge 2$, identity (71) holds for $k - 1$, and, in addition, that

$$1 + \operatorname{Re} h_0 > \frac{2}{k} \cdot \sum_{j=1}^{k+1} \operatorname{Re} h_j, \qquad \operatorname{Re} h_{k+2} < 1. \tag{79}$$

The restrictions (79) can be easily removed from the final result by the theory of analytic continuation.

By the inductive hypothesis, for $t \in \mathbb{C}$ with $\operatorname{Re} t < 0$, we deduce that

$$\begin{aligned}
&J_{k-1}\begin{pmatrix} h_1 + t, & h_2 + t, & h_3 + t, & \dots, & h_k + t \\ & 1 + h_0 - h_3 + t, & 1 + h_0 - h_4 + t, & \dots, & 1 + h_0 - h_{k+1} + t \end{pmatrix} \\
&\quad = \frac{\prod_{j=1}^{k} \Gamma(1 + h_0 - h_j - h_{j+1})}{\Gamma(h_1 + t)\,\Gamma(h_{k+1} + t)} \cdot F_{k+1}(h_0 + 2t; h_1 + t, \dots, h_{k+1} + t) \\
&\quad = \frac{\prod_{j=1}^{k} \Gamma(1 + h_0 - h_j - h_{j+1})}{\Gamma(h_1 + t)\,\Gamma(h_{k+1} + t)} \cdot \frac{1}{2\pi i} \int_{-s_0 - i\infty}^{-s_0 + i\infty} (h_0 + 2t + 2s) \\
&\qquad \times \frac{\Gamma(h_0 + 2t + s) \prod_{j=1}^{k+1} \Gamma(h_j + t + s)\,\Gamma(-s)}{\prod_{j=1}^{k+1} \Gamma(1 + h_0 - h_j + t + s)} \, e^{\varepsilon_{k-1}\pi i s} \, \mathrm{d}s,
\end{aligned} \tag{80}$$

where the real number $s_0 > 0$ satisfies the conditions

$$\begin{aligned}
&\operatorname{Re}(h_0 + 2t) > s_0, \quad \operatorname{Re}(1 + \tfrac{1}{2}h_0 + t) > s_0, \\
&\operatorname{Re}(h_j + t) > s_0 \quad \text{for } j = 1, \dots, k+1,
\end{aligned}$$

and the absolute convergence of the last Barnes-type integral follows from [Ne2], Lemma 3. Shifting the variable $t + s \mapsto s$ in (80) (with a help of the equality $e^{\varepsilon_k \pi i t} \cdot e^{\varepsilon_{k-1}\pi i s} = e^{\varepsilon_{k-1}\pi i (t+s)} \cdot e^{\varepsilon_1 \pi i t}$), applying Lemma 11, and interchanging double integration (thanks to the absolute convergence of the integrals) we conclude that

$$\begin{aligned}
(81)\quad & J_k\begin{pmatrix} h_1, & h_2, & h_3, & \dots, & h_k, & h_{k+1} \\ & 1+h_0-h_3, & 1+h_0-h_4, & \dots, & 1+h_0-h_{k+1}, & 1+h_0-h_{k+2} \end{pmatrix} \\
&= \frac{\prod_{j=1}^{k+1} \Gamma(1+h_0-h_j-h_{j+1})}{\Gamma(h_1)} \\
&\quad \times \frac{1}{2\pi i} \int_{-s_1-i\infty}^{-s_1+i\infty} (h_0+2s) \frac{\prod_{j=1}^{k+1} \Gamma(h_j+s)}{\prod_{j=1}^{k+1} \Gamma(1+h_0-h_j+s)} e^{\varepsilon_{k-1}\pi i s} \\
&\quad \times \frac{1}{2\pi i} \int_{-t_0-i\infty}^{-t_0+i\infty} \frac{\Gamma(-s+t)\,\Gamma(h_0+s+t)\,\Gamma(-t)}{\Gamma(1+h_0-h_{k+2}+t)} e^{\varepsilon_1 \pi i t}\,\mathrm{d}t\,\mathrm{d}s,
\end{aligned}$$

where $s_1 = s_0 + t_0$. Since $\operatorname{Re} h_{k+2} < 1$ and $h_{k+2} \ne 0, -1, -2, \dots$, the last Barnes-type integral has the following closed form by Lemma 10:

$$\begin{aligned}
&\frac{1}{2\pi i} \int_{-t_0-i\infty}^{-t_0+i\infty} \frac{\Gamma(-s+t)\,\Gamma(h_0+s+t)\,\Gamma(-t)}{\Gamma(1+h_0-h_{k+2}+t)} e^{\pm\pi i t}\,\mathrm{d}t \\
&\quad = \frac{\Gamma(-s)}{\Gamma(1-h_{k+2}-s)} \int_0^1 \frac{x^{h_0+s-1}(1-x)^{-h_{k+2}-s}}{(1-x)^{-s}}\,\mathrm{d}x \\
&\quad = \frac{\Gamma(-s)}{\Gamma(1-h_{k+2}-s)} \cdot \frac{\Gamma(h_0+s)\,\Gamma(1-h_{k+2})}{\Gamma(1+h_0-h_{k+2}+s)} \\
&\quad = \frac{\Gamma(h_0+s)\,\Gamma(h_{k+2}+s)\,\Gamma(-s)}{\Gamma(h_{k+2})\,\Gamma(1+h_0-h_{k+2}+s)} \cdot \frac{\sin\pi(h_{k+2}+s)}{\sin\pi h_{k+2}} \\
&\quad = \frac{\Gamma(h_0+s)\,\Gamma(h_{k+2}+s)\,\Gamma(-s)}{\Gamma(h_{k+2})\,\Gamma(1+h_0-h_{k+2}+s)} \\
&\qquad \times \left(e^{\pi i s} \cdot \frac{1 - i\cot\pi h_{k+2}}{2} + e^{-\pi i s} \cdot \frac{1 + i\cot\pi h_{k+2}}{2} \right).
\end{aligned}$$

Substituting this final expression in (81) we obtain

$$
\begin{aligned}
J_k&\begin{pmatrix} h_1, & h_2, & h_3, & \dots, & h_k, & h_{k+1} \\ & 1+h_0-h_3, & 1+h_0-h_4, & \dots, & 1+h_0-h_{k+1}, & 1+h_0-h_{k+2} \end{pmatrix} \\
&= \frac{\prod_{j=1}^{k+1}\Gamma(1+h_0-h_j-h_{j+1})}{\Gamma(h_1)\,\Gamma(h_{k+2})} \\
&\qquad \times \biggl(\frac{1-i\cot\pi h_{k+2}}{4\pi i}\int_{-s_1-i\infty}^{-s_1+i\infty}(h_0+2s) \\
&\qquad\qquad \times \frac{\prod_{j=0}^{k+2}\Gamma(h_j+s)\,\Gamma(-s)}{\prod_{j=1}^{k+2}\Gamma(1+h_0-h_j+s)} e^{(\varepsilon_{k-1}+1)\pi i s}\,\mathrm{d}s \\
&\qquad + \frac{1+i\cot\pi h_{k+2}}{4\pi i}\int_{-s_1-i\infty}^{-s_1+i\infty}(h_0+2s) \\
&\qquad\qquad \times \frac{\prod_{j=0}^{k+2}\Gamma(h_j+s)\,\Gamma(-s)}{\prod_{j=1}^{k+2}\Gamma(1+h_0-h_j+s)} e^{(\varepsilon_{k-1}-1)\pi i s}\,\mathrm{d}s\biggr).
\end{aligned}
$$

If k is even, we take $\varepsilon_{k-1}=-1$ in the first integral and $\varepsilon_{k-1}=1$ in the second one. Therefore the both integrals are equal to

$$
\begin{aligned}
\int_{-s_1-i\infty}^{-s_1+i\infty}(h_0+2s)\frac{\prod_{j=0}^{k+2}\Gamma(h_j+s)\,\Gamma(-s)}{\prod_{j=1}^{k+2}\Gamma(1+h_0-h_j+s)} e^{\varepsilon_k\pi i s}\,\mathrm{d}s \\
= 2\pi i\cdot F_{k+2}(h_0;h_1,\dots,h_{k+2})
\end{aligned}
$$

that gives the desired identity (71). The proof of Theorem 5 is complete. □

Another family of multiple integrals

$$
S(z) := \int\cdots\int_{[0,1]^k} \frac{\prod_{j=1}^{k} x_j^{a_j-1}(1-x_j)^{b_j-a_j-1}}{\prod_{i=1}^{m}(1-zx_1x_2\cdots x_{r_i})^{c_i}}\,\mathrm{d}x_1\,\mathrm{d}x_2\cdots\mathrm{d}x_k, \qquad 1\le r_1<r_2<\dots<r_m=k, \tag{82}
$$

is known due to works of V. Sorokin [So2], [So3]. Recently, S. Zlobin [Zl1], [Zl2] has proved (in more general settings) that the integrals (70) can be reduced to the form (82) with $z=1$. Therefore, Theorem 5 gives one a way to reduce the integrals $S(1)$ to the very-well-poised hypergeometric series (69) under certain conditions on the parameters a_j, b_j, c_i, and r_i in (82). In addition, Zlobin [Zl1] shows that, for integral parameters in (82) satisfying natural restrictions of convergence, the integral $S(z)$ is a $\mathbb{Q}[z^{-1}]$-linear combination of modified multiple polylogarithms

$$\sum_{n_1 \ge n_2 \ge \dots \ge n_l \ge 1} \frac{z^{n_1}}{n_1^{s_1} n_2^{s_2} \cdots n_l^{s_l}} \qquad \text{with} \quad s_j \ge 1, \ s_j \in \mathbb{Z}, \ j = 1, \dots, l,$$

where $0 \le s_1 + s_2 + \dots + s_l \le k$ and $0 \le l \le m$.

Following a spirit of this section, we would like to finish the paper with the following

Problem. *Find a multiple integral over* $[0,1]^5$ *that represents the series defined in* (30) (*or, equivalently, the integral* (34)) *of Section* 3.

References

[Al] Yu. M. Aleksentsev, *On the measure of approximation for the number π by algebraic numbers.* Mat. Zametki [Math. Notes] **66**:4 (1999), 483–493.

[An] G. E. Andrews, *The well-poised thread: An organized chronicle of some amazing summations and their implications.* The Ramanujan J. **1**:1 (1997), 7–23.

[Ap] R. Apéry, *Irrationalité de $\zeta(2)$ et $\zeta(3)$.* Astérisque **61** (1979), 11–13.

[Ba] W. N. Bailey, *Generalized hypergeometric series.* Cambridge Math. Tracts **32** (Cambridge Univ. Press, Cambridge, 1935); 2nd reprinted edition Stechert-Hafner, New York–London, 1964.

[BR] K. Ball, T. Rivoal, *Irrationalité d'une infinité de valeurs de la fonction zêta aux entiers impairs.* Invent. Math. **146**:1 (2001), 193–207.

[Be1] F. Beukers, *A note on the irrationality of $\zeta(2)$ and $\zeta(3)$.* Bull. London Math. Soc. **11**:3 (1979), 268–272.

[Be2] F. Beukers, *Padé approximations in number theory.* Lecture Notes in Math. **888**, Springer-Verlag, Berlin, 1981, 90–99.

[Be3] F. Beukers, *Irrationality proofs using modular forms.* Astérisque **147–148** (1987), 271–283.

[Be4] F. Beukers, *On Dwork's accessory parameter problem.* Math. Z. **241**:2 (2002), 425–444.

[Br] N. G. de Bruijn, *Asymptotic methods in analysis.* North-Holland Publ., Amsterdam, 1958.

[Co] H. Cohen, *Accélération de la convergence de certaines récurrences linéaires*, Séminaire de Théorie des nombres de Bordeaux (Année 1980–81), exposé 16, 2 pages.

[Gu] L. A. Gutnik, *On the irrationality of certain quantities involving $\zeta(3)$.* Uspekhi Mat. Nauk [Russian Math. Surveys] **34**:3 (1979), 190; Acta Arith. **42**:3 (1983), 255–264.

[Han] J. Hancl, *A simple proof of the irrationality of π^4.* Amer. Math. Monthly **93** (1986), 374–375.

[Hat] M. Hata, *Legendre type polynomials and irrationality measures.* J. Reine Angew. Math. **407**:1 (1990), 99–125.

[JT] W. B. Jones, W. J. Thron, *Continued fractions. Analytic theory and applications.* Encyclopaedia Math. Appl. Section: Analysis **11**, Addison-Wesley, London, 1980.

[Ne1] Yu. V. Nesterenko, *A few remarks on $\zeta(3)$.* Mat. Zametki [Math. Notes] **59**:6 (1996), 865–880.

[Ne2] Yu. V. Nesterenko, *Integral identities and constructions of approximations to zeta values.* J. Théor. Nombres Bordeaux **15** (2003), ?–?.

[Ne3] Yu. V. Nesterenko, *Arithmetic properties of values of the Riemann zeta function and generalized hypergeometric functions* in preparation (2002).

[PWZ] M. Petkovšek, H. S. Wilf, D. Zeilberger, $A = B$. A. K. Peters, Ltd., Wellesley, MA, 1997.

[Po] A. van der Poorten, *A proof that Euler missed... Apéry's proof of the irrationality of $\zeta(3)$.* An informal report, Math. Intelligencer **1**:4 (1978/79), 195–203.

[RV1] G. RHIN, C. VIOLA, *On a permutation group related to* $\zeta(2)$. Acta Arith. **77**:1 (1996), 23–56.

[RV2] G. RHIN, C. VIOLA, *The group structure for* $\zeta(3)$. Acta Arith. **97**:3 (2001), 269–293.

[Ri1] T. RIVOAL, *La fonction zêta de Riemann prend une infinité de valeurs irrationnelles aux entiers impairs*. C. R. Acad. Sci. Paris Sér. I Math. **331**:4 (2000), 267–270.

[Ri2] T. RIVOAL, *Propriétés diophantiennes des valeurs de la fonction zêta de Riemann aux entiers impairs*. Thèse de Doctorat, Univ. de Caen, 2001.

[Ri3] T. RIVOAL, *Séries hypergéométriques et irrationalité des valeurs de la fonction zêta*. J. Théor. Nombres Bordeaux **15** (2003), 351–365.

[So1] V. N. SOROKIN, *Hermite–Padé approximations for Nikishin's systems and irrationality of* $\zeta(3)$. Uspekhi Mat. Nauk [Russian Math. Surveys] **49**:2 (1994), 167–168.

[So2] V. N. SOROKIN, *A transcendence measure of* π^2. Mat. Sb. [Russian Acad. Sci. Sb. Math.] **187**:12 (1996), 87–120.

[So3] V. N. SOROKIN, *Apéry's theorem*. Vestnik Moskov. Univ. Ser. I Mat. Mekh. [Moscow Univ. Math. Bull.] **53**:3 (1998), 48–52.

[So4] V. N. SOROKIN, *One algorithm for fast calculation of* π^4. Preprint (Russian Academy of Sciences, M. V. Keldysh Institute for Applied Mathematics, Moscow, 2002), 59 pages; `http://www.wis.kuleuven.ac.be/applied/intas/Art5.pdf`.

[VaO] O. N. VASILENKO, *Certain formulae for values of the Riemann zeta-function at integral points*. Number theory and its applications, Proceedings of the science-theoretic conference (Tashkent, September 26–28, 1990), 27 (Russian).

[VaD] D. V. VASILYEV, *On small linear forms for the values of the Riemann zeta-function at odd points*. Preprint no. 1 (558) (Nat. Acad. Sci. Belarus, Institute Math., Minsk, 2001).

[Vi] C. VIOLA, *Birational transformations and values of the Riemann zeta-function*. J. Théor. Nombres Bordeaux **15** (2003), ?–?

[WZ] H. S. WILF, D. ZEILBERGER, *An algorithmic proof theory for hypergeometric* (*ordinary and "q"*) *multisum/integral identities*. Invent. Math. **108**:3 (1992), 575–633.

[Zl1] S. A. ZLOBIN, *Integrals expressible as linear forms in generalized polylogarithms*. Mat. Zametki [Math. Notes] **71**:5 (2002), 782–787.

[Zl2] S. A. ZLOBIN, *On some integral identities*. Uspekhi Mat. Nauk [Russian Math. Surveys] **57**:3 (2002), 153–154.

[Zu1] W. ZUDILIN, *Difference equations and the irrationality measure of numbers*. Collection of papers: Analytic number theory and applications, Trudy Mat. Inst. Steklov [Proc. Steklov Inst. Math.] **218** (1997), 165–178.

[Zu2] W. ZUDILIN, *Irrationality of values of Riemann's zeta function*. Izv. Ross. Akad. Nauk Ser. Mat. [Russian Acad. Sci. Izv. Math.] **66**:3 (2002), 49–102.

[Zu3] W. V. ZUDILIN, *One of the numbers* $\zeta(5), \zeta(7), \zeta(9), \zeta(11)$ *is irrational*. Uspekhi Mat. Nauk [Russian Math. Surveys] **56**:4 (2001), 149–150.

[Zu4] W. ZUDILIN, *Arithmetic of linear forms involving odd zeta values*. J. Théor. Nombres Bordeaux, to appear.

[Zu5] W. ZUDILIN, *An elementary proof of Apéry's theorem*. E-print `math.NT/0202159` (February 2002).

Wadim ZUDILIN
Department of Mechanics and Mathematics
Moscow Lomonosov State University
Vorobiovy Gory, GSP-2
119992 Moscow
Russia
URL: `http://wain.mi.ras.ru/index.html`
E-mail : `wadim@ips.ras.ru`