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Correction to: Linear fractional transformations of continued fractions with bounded partial quotients

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Résumé. Nous comblons un trou dans la démonstration d'un théorème de notre article cité dans le titre.

Abstract. We fill a gap in the proof of a theorem of our paper cited in the title.

The proof of Theorem 1.1 of our paper [1] has a gap, which we fill below. The gap is that in the subcase $c \neq 0$ and $qa - pc \neq 0$ the inequality (4.11) given in [1] does not follow from the inequality immediately preceding it, and may not be valid. We give here a corrected argument, which also clarifies the logic treating this subcase together with the subcase $c \neq 0$ and $qa - pc = 0$.

Theorem 1.1 asserts the inequality (4.1), which states that

$$L\left(\frac{a\theta + b}{c\theta + d}\right) \leq |\det(M)| L(\theta) + |c(c\theta + d)|. \tag{4.1}$$

It can be derived as follows. Define

$$\psi := \frac{a\theta + b}{c\theta + d}.$$

The case $c = 0$ is treated as in the paper. Suppose $c \neq 0$. We either have an infinite sequence of distinct approximations $x = q||q\psi||$ with

$$L(\psi) \geq \frac{1}{x} \geq L(\psi) - \epsilon,$$

with $\epsilon \to 0$, or else we have a single approximation with $L(\psi) = \frac{1}{x}$; we reduce the second case to the first by viewing it as an infinite sequence with the same value of q repeated over and over. By taking a suitable subsequence we may reduce to the case where either all of the approximations have $qa - pc = 0$ or all of them have $qa - pc \neq 0$, with $\epsilon \to 0$.

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In the first subcase, where all $qa - pc = 0$, the inequality (4.7) in [1] yields

$$L(\psi) - \epsilon \leq \frac{1}{x} \leq |c\theta + d| \leq |c(c\theta + d)|,$$

since $c \neq 0$. Letting $\epsilon \to 0$ yields (4.1).

In the second subcase, where all $qa - pc \neq 0$, we obtain as in [1] the inequality (4.10), which states that

$$|c\theta + d| \left| \frac{qa - pc}{q} \right| L(\theta) \geq \frac{1}{x} \geq L(\psi) - \epsilon. \tag{4.10}$$

We then continue by using the triangle inequality

$$\left| q(\frac{a}{c}) - p \right| \leq \left| q(\frac{a\theta + b}{c\theta + d}) - q(\frac{a}{c}) \right| + \left| q(\frac{a\theta + b}{c\theta + d}) - p \right|.$$

Here the first term on the right side is equal to $q|\det(M)|\frac{1}{|c(c\theta+d)|}$ while the second term is $\frac{x}{q}$ where $x := q||q\psi||$, provided we pick the integer p properly, and we then have

$$\left| \frac{qa - pc}{c} \right| \leq q|\det(M)| \frac{1}{|c(c\theta + d)|} + \frac{x}{q}.$$

Multiplying this by $\frac{c}{q}$ and substituting it in (4.10) gives the inequality

$$L(\psi) - \epsilon \leq |\det(M)| L(\theta) + |c(c\theta + d)| \frac{xL(\theta)}{q^2}, \tag{4.11$'$}$$

which replaces (4.11). Now

$$x \leq \frac{1}{L(\psi) - \epsilon},$$

which when used in the inequality (4.11′) yields

$$L(\psi) - \epsilon \leq |\det(M)| L(\theta) + \frac{|c(c\theta - d)|}{q^2} \cdot \frac{L(\theta)}{L(\psi) - \epsilon}.$$

Now there are two cases. First, if $L(\theta) \geq L(\psi)$, then the inequality (4.1) holds trivially, since $|\det(M)| \geq 1$. Second, if $L(\theta) < L(\psi)$ then, letting $\epsilon \to 0$ in the preceding inequality, the ratio $\frac{L(\theta)}{L(\psi)-\epsilon}$ becomes ≤ 1 in the limit, and since $q \geq 1$, (4.1) follows. This completes the proof.

References

[1] J.C. Lagarias, J.O. Shallit, *Linear Fractional Transformations of Continued Fractions with Bounded Partial Quotients.* J. Théor. Nombres Bordeaux **9** (1997), 267–279.

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