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A new exceptional polynomial for the integer transfinite diameter of $[0, 1]$

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Résumé. En améliorant l'algorithme utilisé par Habsieger et Salvy pour obtenir des polynômes à coefficients entiers de plus petite norme infinie sur $[0, 1]$, nous étendons leur table de polynômes jusqu'au degré 100. Au degré 95 nous trouvons un nouveau polynôme exceptionnel qui a des racines complexes. Notre méthode fait appel à des polynômes de Müntz-Legendre généralisés. Nous améliorons un peu la majoration du diamétre transfini entier de $[0, 1]$ et nous donnons une démonstration élémentaire de la minoration des exposants de certains polynômes critiques.

Abstract. Using refinement of an algorithm given by Habsieger and Salvy to find integer polynomials with smallest sup norm on $[0, 1]$ we extend their table of polynomials up to degree 100. For the degree 95 we find a new exceptionnal polynomial which has complex roots. Our method uses generalized Müntz-Legendre polynomials. We improve slightly the upper bound for the integer transfinite diameter of $[0, 1]$ and give elementary proofs of lower bounds for the exponents of some critical polynomials.

1. Introduction

For a positive integer k, let $Z_k[x]$ be the set of polynomials P of degree k with integer coefficients. Following Borwein and Erdelyi [2] we define the integer Chebyshev polynomials on the interval $[0, 1]$ as the polynomials P_k such that

$$\|P_k\|_\infty = \min_{P \in Z_k[x] \setminus \{0\}} \|P\|_\infty$$

where $\|P\|_\infty = \max_{t \in [0,1]} |P(t)|$. The integer transfinite diameter of $[0, 1]$ is defined by

$$t_Z([0, 1]) = \lim_{k \to \infty} \|P_k\|_\infty^{1/k}. \tag{1}$$

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The exact value of $t_Z([0,1])$ is not known. The best known result has been obtained by Pritsker who gives $t_Z([0,1]) \in (0.4213, 0.4232)$. For a general exposition see [3], [2] or [7].

The integer Chebyshev polynomials have been studied extensively by many authors such as Aparicio [1], Flammang, Rhin and Smyth [4], Borwein and Erdelyi [2], Habsieger and Salvy [5]. In [5] the authors gave a complete list of integer Chebyshev polynomials for the degrees 1 to 75. They found for the degree $k = 70$ a factor of P_k which has not all its zeros in $[0,1]$. This gives a negative answer to a question of [2]:

Do the integer Chebyshev polynomials of $[0,1]$ have all their zeros in $[0,1]$?

This exceptionnal polynomial is

$$\begin{aligned} 4921x^{10} - 24605x^9 + 53804x^8 - 67586x^7 + 53866x^6 - 28388x^5 \\ + 9995x^4 - 2317x^3 + 338x^2 - 28x + 1 \end{aligned}$$

which has four non real zeros. The question remained open to know whether there are other such exceptional polynomials.

As suggested by Habsieger and Salvy we use a new algorithm to extend their table and we give a list of integer Chebyshev polynomials up to degree 100. Nevertheless it seems that it is not possible to reach the degree 200 with our algorithm as suggested in [2]. For the degree 95 we have found a new exceptional polynomial which is

$$\begin{aligned} 43609x^{12} - 261654x^{11} + 704777x^{10} - 1125390x^9 + 1184854x^8 - 865270x^7 \\ + 448776x^6 - 166327x^5 + 43659x^4 - 7905x^3 + 936x^2 - 65x + 2. \end{aligned}$$

We will give in section 4 a good polynomial in $Z[x]$, of degree 10^8, which proves the following

Theorem 1.1.

$$t_Z([0,1]) < 0.423164171. \tag{2}$$

Remark. This improves slightly Pritsker's result.

One of the important tools used by the previous authors is the Müntz-Legendre polynomials. We will generalize these polynomials. This will provide us lower bounds for the exponent of critical polynomials : an irreducible polynomial T in $Z[x]$ is a critical polynomial for the interval $[0,1]$ if there exists a positive constant $C(T)$ such that for $n \geq n_0$ every integer Chebyshev polynomial of degree n is divisible by the polynomial T^k with $k \geq C(T)n$. For instance we prove that if T_n is an integer Chebyshev polynomial such that $\|T_n\|_\infty^{1/n} < 0.423164171$ then $T_n = x^k(1-x)^k S_{n-2k}$

with

$$k > 0.2976126n. \tag{3}$$

for n large enough. This bound is not as good as Pritsker's bound $k > 0.31n$, but the method of proof is more elementary.

This paper is organised as follows : section 2 will be devoted to the generalized Müntz-Legendre polynomials (GML) on $[0, 1]$. As already remarked in [5] it is interesting to deal with integer Chebyshev polynomials on $[0, 1/4]$ because $t_Z([0, 1]) = (t_Z([0, 1/4]))^{1/2}$. Then section 3 is devoted to the computations on the interval $[0, 1/4]$. In section 4 we summarize briefly the algorithm that we use (a more extensive version is available in [8] or [9]), and we give a list of integer Chebyshev polynomials on $[0, 1]$ from degree 76 to 100.

2. GML polynomials on $[0,\ 1]$

We consider, for $0 < a < b$, the scalar product of two real continuous functions on the interval $[a, b]$ defined by $(f, g)_\varphi = \int_a^b f(x)g(x)\varphi(x)dx$ where φ is a real positive function on the interval $[a, b]$. We note

$$\|f\|^2_{2,[a,b]} = (f, f)_\varphi; \quad \|f\|_{\infty,[a,b]} = \max_{a\leq x\leq b}\left(|f(x)|\sqrt{\varphi(x)}\right).$$

In the papers by Flammang, Rhin, Smyth [4] and Borwein, Erdelyi [2], the authors study the integer transfinite diameter of $[0,\ 1]$ by using the Müntz-Legendre polynomials which belong to the real vector space $V_{n,k}$ generated by $(x^n,\ x^{n-1},\ \cdots,\ x^k)$ $(n \geq k \geq 0)$. They apply the Gram-Schmidt process to the basis $(x^n,\ x^{n-1},\ \cdots,\ x^k)$ and the usual scalar product $(f,\ g) = \int_0^1 f(x)g(x)dx$ and obtain the orthogonal Müntz-Legendre polynomials

$$H_i(x) = \sum_{j=i}^{n}(-1)^{n-j}\binom{n+1+j}{n-i}\binom{n-i}{n-j}x^j \qquad (k \leq i \leq n).$$

Let $R_n[x]$ be the set of polynomials of degree n with real coefficients. For a polynomial $P(x) = x^k R(x) \in R_n[x]$ belonging to $V_{n,k}$ and $R(x) \in R_{n-k}[x]$, we get

$$|R(0)| \leq \sqrt{2k+1}\binom{n+1+k}{n-k}\|P(x)\|_{\infty,\ [0,1]}.$$

Let F be a fixed non zero polynomial of $R[x]$. We now consider the vector space $V_{n,k}$ generated by $(x^nF, x^{n-1}F, \cdots, x^kF)$. By the Gram-Schmidt process with a scalar product $(\ ,\ )_\varphi$ on the interval $[a, b]$, we get the set $(L_n, L_{n-1}, \cdots, L_k)$ of orthogonal generalized Müntz-Legendre polynomials.

Suppose then that $P = Fx^kR(x)$ belongs to $V_{n,k}$ then we get the inequality [8]

$$|a_k| \le \frac{\sqrt{b-a}}{\|L_k\|_{2,[a,b]}}\|P\|_{\infty,[a,b]}. \tag{4}$$

We consider now the case of the interval $[0,1]$ with $F = (1-x)^q$ and $\varphi(x) = 1$. Then we have, if $q > 0$, $k \le n$ are integers,

Lemma 2.1. *On the interval* $[0,1]$ *for* $F = (1-x)^q$, *the orthogonal GML polynomials are*

$$L_i(x) = \sum_{j=i}^{n} (-1)^{n-j} \binom{n+2q+j+1}{n-i+2q} \binom{n-i}{n-j} x^j (1-x)^q \qquad (k \le i \le n).$$

Proof. It is sufficient to prove that for $n \ge h \ge i+1$, $(L_i, x^h(1-x)^q) = 0$.

We know that

$$\int_0^1 x^{j_1+j_2}(1-x)^{2q}dx = \frac{(2q)!(j_1+j_2)!}{(2q+j_1+j_2+1)!},$$

so

$$\begin{aligned}
&(L_i,\, x^h(1-x)^q)\\
&\quad = \int_0^1 \sum_{j=i}^{n} (-1)^{n-j} \binom{n+2q+j+1}{n-i+2q} \binom{n-i}{n-j} x^{j+h}(1-x)^{2q}dx\\
&\quad = \sum_{j=i}^{n} (-1)^{n-j} \binom{n+2q+j+1}{n-i+2q} \binom{n-i}{n-j} \frac{(2q)!(j+h)!}{(2q+j+h+1)!}\\
&\quad = (2q)! \sum_{j=i}^{n} (-1)^{n-j} \frac{(n+2q+j+1)!}{(n-i+2q)!(i+j+1)!} \binom{n-i}{n-j} \frac{(j+h)!}{(j+h+2q+1)!}\\
&\quad = \frac{(2q)!}{(n-i+2q)!} \sum_{j=i}^{n} (-1)^{n-j} \binom{n-i}{n-j}\\
&\qquad \times (n+2q+j+1)\cdots(h+1+j+2q+1)(j+h)\cdots(j+i+2)\\
&\quad = \left[\frac{d^{h-i-1}}{dx^{h-i-1}} \frac{1}{x^{2q+1}} \left(\frac{d^{n-h}}{dx^{n-h}}(x-1)^{n-i}x^{n+i+2q+1}\right)\right]_{x=1} = 0.
\end{aligned}$$

□

Proposition 2.2. *Let* $Q(x) \in R_{n+q}[x]$, $Q(x) = a_kx^k(x-1)^q + a_{k+1}x^{k+1}(x-1)^q + \cdots + a_nx^n(x-1)^q$. *We have*

$$|a_k| \le \|Q\|_\infty \sqrt{(2k+1)\binom{n+1+k}{n-k}\binom{n+k+2q+1}{n-k+2q}}.$$

Proof. $Q(x)$ can be written $Q(x) = \lambda_k L_k + \lambda_{k+1} L_{k+1} + \cdots + \lambda_n L_n$ with

$$a_k = (-1)^{n-k} \lambda_k \binom{n+2q+k+1}{n-k+2q}.$$

By the inequality (4)

$$|a_k| = |\lambda_k| \binom{n+2q+k+1}{n-k+2q} \le \frac{\|Q\|_\infty}{\sqrt{(L_k,\ L_k)}} \binom{n+2q+k+1}{n-k+2q}.$$

We have

$$\begin{aligned}(L_i,\ L_i) &= \sum_{j=i}^{n} (-1)^{n-j} \binom{n+2q+j+1}{n-i+2q} \binom{n-i}{n-j} \left(L_i,\ x^j(1-x)^q\right) \\ &= (-1)^{n-i} \binom{n+2q+i+1}{n-i+2q} \left(L_i,\ x^i(1-x)^q\right).\end{aligned}$$

We put $S_i = \left(L_i,\ x^i(1-x)^q\right)$, so

$$\begin{aligned}S_i &= \sum_{j=i}^{n} (-1)^{n-j} \binom{n+2q+j+1}{n-i+2q} \binom{n-i}{n-j} \frac{(i+j)!(2q)!}{(i+j+2q+1)!} \\ &= \frac{(2q)!}{(n-i+2q)!} \\ &\quad \times \left(\sum_{j=i}^{n} (-1)^{n-j} \binom{n-i}{n-j} \frac{(n+2q+j+1)\cdots(i+j+2q+2)}{i+j+1} \right) \\ &= \frac{1}{(n-i+2q)\cdots(2q+1)} \\ &\quad \times \left(\sum_{j=i}^{n} (-1)^{n-j} \binom{n-i}{n-j} \frac{(n+2q+j+1)\cdots(i+j+2q+2)}{i+j+1} \right).\end{aligned}$$

We consider S_i as a rational function in the indeterminate q where $n,\ i$ are rational parameters. The degree of the denominator $(2q+n-i)\cdots(2q+1)$ is equal to $n-i$. The degree of each numerator is $(n+2q+j+1)-(i+j+2q+1) = n-i$. So if we keep the denominator in the form $(2q+n-i)\cdots(2q+1)$, then the numerator is of degree less or equal to $n-i$. We will show that S_i is a constant in $Q(q)$. For that, we show that the denominator divides the numerator. Let $q = -l/2$ which is a zero of the denominator with $1 \le l \le n-i$. We compute

$$\sum_{j=i}^{n}(-1)^{n-j}\binom{n-i}{n-j}\frac{(n-l+j+1)\cdots(i+j-l+2)}{i+j+1}.$$

The numerator of this sum is equal to zero because it is equal to

$$\left[\frac{d^{l-1}}{dx^{l-1}}\frac{1}{x}\left(\frac{d^{n-i-l}}{dx^{n-i-l}}(x-1)^{n-i}x^{n+i-l+1}\right)\right]_{x=1}=0,$$

so $(2q+n-i)\cdots(2q+1)$ divides the numerator of S_i, i.e. S_i is a constant. So, to compute S_i, we can take $q=-(n+i+2)/2$, and then

$$\begin{aligned}
S_i &= S_i\left(-\frac{n+i+2}{2}\right)\\
&=(-1)^{n-i}\frac{1}{(-2i-2)\cdots(-n-i-1)}\frac{-1\cdots(i-n)}{2i+1}\\
&=(-1)^{n-i}\frac{(n-i)!}{(n+i+1)\cdots(2i+2)(2i+1)}\\
&=(-1)^{n-i}\frac{(n-i)!(2i+1)!}{(2i+1)(n+i+1)!}.
\end{aligned}$$

Then

$$(L_i,\ L_i)=\binom{n+2q+i+1}{n-i+2q}\frac{(n-i)!(2i+1)!}{(2i+1)(n+i+1)!}$$

and

$$\begin{aligned}
|a_k| &\le \binom{n+2q+k+1}{n-k+2q}\|Q\|_\infty\\
&\quad\times\left(\sqrt{\binom{n+2q+k+1}{n-k+2q}\frac{(n-k)!(2k+1)!}{(2k+1)(n+k+1)!}}\right)^{-1}\\
&\le \|Q\|_\infty\sqrt{\binom{n+2q+k+1}{n-k+2q}\frac{(2k+1)(n+k+1)!}{(n-k)!(2k+1)!}}\\
&=\|Q\|_\infty\sqrt{(2k+1)\binom{n+2q+k+1}{n-k+2q}\binom{n+k+1}{n-k}}.
\end{aligned}$$

□

Then we show how we can deduce a result of type (3) using the Proposition 2.2, i.e. if $P_n\in Z_n[x]$ and $\|P_n\|_\infty^{\frac{1}{n}}\le 0.423164171$ then $P_n(x)=x^k(1-x)^kS_{n-2k}(x)$ where $S_{n-2k}\in Z_{n-2k}[x]$ and $k\ge 0.2907588n$ for n large enough.

In fact, let k the minimum of the exponents of x and $1-x$, then P_n is the form $P_n=x^k(1-x)^kS_{n-2k}(x)$ and

$$|a_k| \le \|P_n\|_\infty \sqrt{(2k+1)\binom{n+1}{n-2k}\binom{n+2k+1}{n}}$$

where $a_k = S_{n-2k}(0)$. If $a_k \neq 0$ then

$$\|P_n\|_\infty \sqrt{(2k+1)\frac{(n+1)!}{(n-2k)!(2k+1)!}\frac{(n+2k+1)!}{n!(2k+1)!}} \ge 1$$

and

$$\|P_n\|_\infty \sqrt{\left(\frac{(n+1)(n+2k+1)}{(2k+1)}\right)\frac{(n+2k)!}{(n-2k)!(2k)!^2}} \ge 1.$$

We remark that

$$((n-2k)+2k+2k)^{n+2k} \ge \frac{(n+2k)!}{(n-2k)!(2k)!(2k)!}(n-2k)^{n-2k}(2k)^{4k}$$

i.e.

$$\frac{(n+2k)^{n+2k}}{(n-2k)^{n-2k}(2k)^{4k}} \ge \frac{(n+2k)!}{(n-2k)!(2k)!^2}.$$

If we put $\alpha = \frac{k}{n}$, and

$$f(\alpha) = \frac{(1+2\alpha)^{1+2\alpha}}{(1-2\alpha)^{1-2\alpha}(2\alpha)^{4\alpha}}$$

then

$$f(\alpha) \ge \left(\frac{(n+2k)!}{(n-2k)!(2k)!^2}\right)^{\frac{1}{n}}.$$

By the inequality $\|P_n\|_\infty^{1/n} \le 0.423164171$ we have

$$0.423164171(f(\alpha))^{\frac{1}{2}}\left(\frac{(n+1)(1+2\alpha+\frac{1}{n})}{2\alpha+\frac{1}{n}}\right)^{1/2n} \ge 1.$$

Since

$$\left(\frac{(n+1)(1+2\alpha+\frac{1}{n})}{2\alpha+\frac{1}{n}}\right)^{1/2n} = 1+o(1)$$

it is enough to take α such that $0.423164171(f(\alpha))^{\frac{1}{2}} \ge 1$, so $\alpha \ge 0.2907588$, i.e. $k \ge 0.2907588n$ for n large enough.

Remark. Using Proposition 2.2 for $t_Z([0,1/4])$ we will prove the better relation (3) in the next section.

3. GML polynomials on $[0, \frac{1}{4}]$

We also know that $A_2 = 1 - 2x$ [5] is a critical polynomial on $[0, 1]$, with the change of variable $u = x(1-x)$, we obtain $A_2^2 = 4u - 1$ which is a critical polynomial for the integer Chebyshev polynomial on $[0, 1/4]$. We consider here the vector space generated by the basis $(x^n(1-4x)^q, x^{n-1}(1-4x)^q, \cdots, x^k(1-4x)^q)$ and the scalar product $(f, g)_2 = \int_0^{\frac{1}{4}} f(x)g(x)dx$. As for the lemma 2.1, we have

Lemma 3.1. *We put*

$$M_i(x) = \sum_{j=i}^{n}(-1)^{n-j}\binom{n+2q+j+1}{n-i+2q}\binom{n-i}{n-j}4^j x^j(1-4x)^q = L_i(4x),$$

then $(M_i)_{k\le i\le n}$ is an orthogonal GML family.

Proof. We put $y = 4x$ and it is easy to verify that

$$\left(M_i,\ x^h(1-4x)^q\right)_2 = \frac{1}{4^{h+1}}\left(L_i,\ y^h(1-y)^q\right) = 0.$$

$\square$

3.1. The factor x.

If we want to determine a lower bound for the exponent of the factor x when the exponent of the factor $1-4x$ is equal to q, we take a polynomial of degree $n+q$, $Q = a_k x^k(1-4x)^q + a_{k+1}x^{k+1}(1-4x)^q + \cdots + a_n x^n(1-4x)^q$, then $Q = \lambda_k M_k + \lambda_{k+1}M_{k+1} + \cdots + \lambda_n M_n$ with

$$|a_k| = 4^k|\lambda_k|\binom{n+k+2q+1}{n-k+2q}$$

then

$$\lambda_k^2(M_k,\ M_k)_2 \le (Q,\ Q)_2 \le \frac{1}{4}\|Q\|^2_{\infty,\ [0,\frac{1}{4}]}.$$

We have

Proposition 3.2.

$$|a_k| \le 4^k\|Q\|_{\infty,\ [0,\frac{1}{4}]}\sqrt{(2k+1)\binom{n+k+2q+1}{n-k+2q}\binom{n+k+1}{n-k}}.$$

Proof. By taking $y = 4x$, we check that

$$(M_i,\ M_i)_2 = \int_0^{\frac{1}{4}} M_i^2(x) = \int_0^{\frac{1}{4}} L_i^2(4x) = \frac{1}{4}\int_0^1 L_i^2(y)\,dy = \frac{1}{4}(L_i,\ L_i)$$

so

$$(M_i,\ M_i)_2 = \frac{1}{4}\binom{n+i+2q+1}{n-i+2q}\left((2i+1)\binom{n+i+1}{n-i}\right)^{-1}.$$

Then

$$|a_k|\left(\binom{n+k+2q+1}{n-k+2q}4^k\right)^{-1}$$

$$\leq \frac{1}{2}\|Q\|_{\infty,\ [0,\frac{1}{4}]}\sqrt{4(2k+1)\binom{n+k+1}{n-k}\binom{n+k+2q+1}{n-k+2q}^{-1}}.$$

This proves the proposition.

$\square$

Lemma 3.3. *Let $P_n = a_k x^k(1-4x)^q + \cdots + a_{n-q}x^{n-q}(1-4x)^q \in Z_n[x]$. Let $\beta = \frac{q}{n}$ be given, then $k \geq \alpha n + o(n)$ where α is the smallest positive root of $4^\alpha \|P_n\|_{\infty,[0,\frac{1}{4}]}^{\frac{1}{n}} f(\alpha,\ \beta) = 1$ where*

$$f(\alpha,\ \beta) = \left(\frac{(1+\alpha+\beta)^{1+\alpha+\beta}(1+\alpha-\beta)^{1+\alpha-\beta}}{(1-\alpha+\beta)^{1-\alpha+\beta}(1-\alpha-\beta)^{1-\alpha-\beta}(2\alpha)^{4\alpha}}\right)^{\frac{1}{2}}.$$

Proof. We have

$$((n-k+q)+2k)^{n+k+q} \geq \binom{n+k+q}{n-k+q}(n-k+q)^{n-k+q}(2k)^{2k}$$

$$((n-k-q)+2k)^{n+k-q} \geq \binom{n+k-q}{n-k-q}(n-k-q)^{n-k-q}(2k)^{2k}.$$

If we put $\alpha = \frac{k}{n}$ and $\beta = \frac{q}{n}$, we have

$$f(\alpha,\ \beta)^{2n} \geq \binom{n+k+q}{n-k+q}\binom{n+k-q}{n-k-q}.$$

Then

$$1 \leq |a_k| \leq 4^k\|P_n\|_{\infty,[0,\frac{1}{4}]}\sqrt{\frac{(n+k-q+1)(n+k+q+1)}{2k+1}}f(\alpha,\ \beta)^n$$

i.e.

$$1 \leq 4^\alpha\|P_n\|_{\infty,[0,\frac{1}{4}]}^{\frac{1}{n}}\left(\frac{n(1+\alpha-\beta+\frac{1}{n})(1+\alpha+\beta+\frac{1}{n})}{2\alpha+\frac{1}{n}}\right)^{\frac{1}{2n}} f(\alpha,\ \beta).$$

Since

$$\left(\frac{n(1+\alpha-\beta+\frac{1}{n})(1+\alpha+\beta+\frac{1}{n})}{2\alpha+\frac{1}{n}}\right)^{\frac{1}{2n}} = 1+o(1),$$

we have

$$1 \leq 4^\alpha\|P_n\|_{\infty,[0,\frac{1}{4}]}^{\frac{1}{n}} f(\alpha,\ \beta).$$

$\square$

If we take $\beta = 0$, i.e. the exponent of $1-4x$ is equal to zero, we have

$$|a_k| \le 4^k \|P_n\|_{\infty,[0,\frac{1}{4}]}\sqrt{2k+1}\binom{n+k+1}{n-k}$$

and we obtain $k \ge 2 \times 0.2907588n$ which corresponds to the result on the interval $[0,\ 1]$ in the section 2.

3.2. The factor $1-4x$.

To find a lower bound for the exponent of the factor $1-4x$ if the exponent of x is fixed, we consider the vector space generated by the basis $x^q(1-4x)^n$, $x^q(1-4x)^{n-1}$, $\cdots$, $x^q(1-4x)^k$ and the scalar product $(f,\ g)_2$. We put

$$N_i(x) = \frac{L_i(4x)}{4^q} = \sum_{j=i}^{n}(-1)^{n-j}\binom{n+2q+j+1}{n-i+2q}\binom{n-i}{n-j}x^q(1-4x)^j.$$

We have so $\left(N_i,\ (1-4x)^h x^q\right)_2 = 0$ for $n \ge h \ge i+1$, and

$$(N_i,\ N_i)_2 = \frac{1}{4^{2q+1}}\binom{n+i+2q+1}{n-i+2q}\left((2i+1)\binom{n+i+1}{n-i}\right)^{-1}.$$

We take a polynomial $Q \in Z_{n+q}[x]$, $Q = x^q(1-4x)^k R(x)$ where $4^{n-k}R \in Z[x]$ and $\deg R = n-k$. If we write $R(x) = b_0 + b_1(1-4x) + \cdots + b_{n-k}(1-4x)^{n-k}$ then $4^{n-k}R(1/4) = 4^{n-k}b_0$. So we write $Q = a_n x^q(1-4x)^n + a_{n-1}x^q(1-4x)^{n-1} + \cdots + a_k x^q(1-4x)^k$ and $Q = \mu_n N_n + \mu_{n-1}N_{n-1} + \cdots + \mu_k N_k$. Then $\mu_k^2(N_k,\ N_k)_2 \le (Q,\ Q)_2 \le \frac{1}{4}\|Q\|^2_{\infty,[0,\frac{1}{4}]}$,

$$|a_k| = |\mu_k|\binom{n+k+2q+1}{n-k+2q}$$

$$|a_k|\binom{n+k+2q+1}{n-k+2q}^{-1}$$

$$\le \left(\frac{1}{4}\|Q\|^2_{\infty,[0,\frac{1}{4}]}4^{2q+1}(2k+1)\binom{n+k+1}{n-k}\binom{n+k+2q+1}{n-k+2q}^{-1}\right)^{\frac{1}{2}}$$

and

$$4^{n-k}|a_k| \le 4^{n-k+q}\|Q\|_{\infty,[0,\frac{1}{4}]}\sqrt{(2k+1)\binom{n+k+2q+1}{n-k+2q}\binom{n+k+1}{n-k}}.$$

We obtain

Proposition 3.4. *Let* $P_n = a_k x^q(1-4x)^k + \cdots + a_{n-q}x^q(1-4x)^{n-q} \in Z_n[x]$, *then*

$$|a_k| \le 4^q \|P_n\|_{\infty,[0,\frac{1}{4}]} \sqrt{(2k+1)\binom{n+k+q+1}{n-k+q}\binom{n+k-q+1}{n-k-q}}.$$

We can estimate the exponent k of the factor $1-4x$, because it is clear that $4^{n-k-q}a_k$ is an integer. As in lemma 3.3, we can prove

Lemma 3.5. *We put* P_n *and* a_i *as above. If* $\beta = \frac{q}{n}$ *is given, then* $k \ge \alpha n + o(n)$, *where* α *is the smallest positive root of*

$$4^{1-\alpha}\|P_n\|^{\frac{1}{n}}_{\infty,[0,\frac{1}{4}]} f(\alpha,\ \beta) = 1$$

where $f(\alpha,\ \beta)$ *is defined in lemma 3.3.*

So, with lemma 3.3, by taking the exponent of $1-4x$ equal to zero, we obtain that the exponent of x is at least equal to $0.5815176996n$, and with this result we obtain that the exponent of $1-4x$ is at least equal to $0.0949670642n$ by lemma 3.5. We put this last result in lemma 3.3, we thus get that the exponent of x is at least equal to $0.5947009759n$, we continue and obtain the following results :

TABLE 1

	exponent of x	*of* $1-4x$		*exponent of* x	*of* $1-4x$
1	0.5815176996	*0*	6	*0.5952039244*	0.0966191046
2	*0.5815176996*	0.0949670642	7	0.5952244705	*0.0966191046*
3	0.5947009759	*0.0949670642*	8	*0.5952244705*	0.0966217252
4	*0.5947009759*	0.0965550944	9	0.5952253121	*0.0966217252*
5	0.5952039244	*0.0965550944*	10	*0.5952253121*	0.0966218326

By these results, we obtain the proof of the relation (3), i.e. let $P_n \in Z_n[x]$, if $\|P_n\|^{\frac{1}{n}}_{\infty,[0,\frac{1}{4}]} \le 0.179067916$ (because $t_Z\left([0,\ \frac{1}{4}]\right) = t_Z([0,\ 1])^2$), then $P_n(x) = x^k(1-4x)^q S_{n-k-q}(x)$ where $S_{n-k-q} \in Z_{n-k-q}[x]$ and $k \ge 0.5952253n$, $q \ge 0.0966218n$.

4. Application for $\mathbf{t}_Z\left([\mathbf{0},\mathbf{1}]\right)$ and a new exceptional polynomial

In the paper by Habsieger and Salvy [5], we find a table of integer Chebyshev polynomials of degree less than or equal to 75 on the interval $[0,1]$. They found the factors : $A_1 = x(1-x)$, $A_2 = 1-2x$, $A_3 =$

$5x^2-5x+1$ $A_4=6x^2-6x+1$, $A_5=29x^4-58x^3+40x^2-11x+1$, $A_6=(13x^3-20x^2+9x-1)(13x^3-19x^2+8x-1)$, $A_7=(31x^4-63x^3+44x^2-12x+1)(31x^4-61x^3+41x^2-11x+1)$, $A_8=4921x^{10}-24605x^9+53804x^8-67586x^7+53866x^6-28388x^5+9995x^4-2317x^3+338x^2-28x+1$.

By the change of variable $u=x(1-x)$, we obtain the factors on $[0,\frac{1}{4}]$: $A_1=u$, $A_2^2=4u-1$, $A_3=5u-1$, $A_4=6u-1$, $A_5=29u^2-11u+1$, $A_6=169u^3-94u^2+17u-1$, $A_7=961u^4-712u^3+194u^2-23u+1$, $A_8=4921u^5-4594u^4+1697u^3-310u^2+28u-1$.

In this section, we first consider the interval $[0,\frac{1}{4}]$. For the search of the factors of integer Chebyshev polynomials, we use the method which we detail in [8] and [9] with the following steps:

1. Find a good upper bound for $\|P_n\|_{\infty,[a,b]}$: In this step we use the LLL algorithm [6] which furnishes a LLL-reduced basis $(f_i)_{1\leq i\leq m}$ of H where H is a lattice with basis (e_i), i.e. such that the norm (euclidean norm) of vectors f_i is small and in particular such that the vector f_1 is not so far from being the smallest non zero vector in H. Then we can find a good upper bound.

2. Use this bound to deduce polynomials that are necessary factors of P_n : Now we use the generalized Müntz-Legendre method with this bound to give an upper bound for the exponents of critical polynomials as in section 2 and section 3. More precisely we comput explicitely the bound of the coefficient a_k when F is an explicit polynomial having even more than one irreducible factor (such as A_3, A_4).

3. Perform an exhaustive search for the missing factors : We have so a system of inequalities $|F(x_i)Q(x_i)|\leq c_n$ where F is determined by the step 2, c_n is the good upper bound in step 1, $Q(x)$ is a polynomial of degree $k=n-\deg F$ whose unknown coefficients are to be determined and the x_i are control points in the interval $[a,b]$ which are different from the roots of $F(x)$. This system defines a polyhedron of which we must determine the integer points. We solve this system with a method adapted from the simplex method and the LLL algorithm. We thus obtain a polynomial P_n which is appropriate.

We thus find a new factor

$$A_{10}=33u^2-12u+1=33x^4-66x^3+45x^2-12x+1$$

and also the factor

$$\begin{aligned}A_9&=941u^4-703u^3+193u^2-23u+1\\&=941x^8-3764x^7+6349x^6-5873x^5+3243x^4\\&\quad-1089x^3+216x^2-23x+1\end{aligned}$$

and we extend the table up to degree 50 (i.e. on the interval $[0,1]$ we can obtain the integer Chebyshev polynomials of degree up to 100 if we consider only the polynomials Q of even degree such that $Q(x)=Q(1-x)$).

Otherwise, by lemma 1 of Habsieger and Salvy [5], we can also search the factors of integer Chebyshev polynomials of odd degree on the interval $[0,\ 1]$ by transferring the search on the interval $[0,\ \frac{1}{4}]$, i.e. we compute $P_n(u)$ such that

$$\|P_n(u)\sqrt{1-4u}\|_{\infty,[0,\frac{1}{4}]} = \inf_{Q\in Z_n[u]} \|Q(u)\sqrt{1-4u}\|_{\infty,[0,\frac{1}{4}]}$$

and we replace u by $x(1-x)$, the table 2 is thus obtained. In this table, for the integer Chebyshev polynomial of degree 95, we have

$$\begin{aligned} A_{11} &= 43609x^{12} - 261654x^{11} + 704777x^{10} - 1125390x^{9} \\ &\quad + 1184854x^{8} - 865270x^{7} + 448776x^{6} - 166327x^{5} \\ &\quad + 43659x^{4} - 7905x^{3} + 936x^{2} - 65x + 2 \end{aligned}$$

which is a new exceptional polynomial, i.e. it has four non real zeros. For the polynomials of degree less or equal to 75, we find of course the same ones as those of the table of Habsieger and Salvy.

We will now explain how our computation let us obtain a polynomial of degree 10^8 which will imply the Theorem. Let $B_1 = 969581u^8 - 1441511u^7 + 928579u^6 - 338252u^5 + 76143u^4 - 10836u^3 + 951u^2 - 47u + 1$, $B_2 = 49u^2 - 20u + 2$, $B_3 = 34u^2 - 12u + 1$, $B_4 = 193u^3 - 104u^2 + 18u - 1$, $B_5 = 199u^3 - 105u^2 + 18u - 1$, $B_6 = 182113u^7 - 233968u^6 + 127434u^5 - 38125u^4 + 6763u^3 - 711u^2 + 41u - 1$, where the polynomials B_i appear during the computation of the table 2. Using a classical semi-infinite linear programming [8], we get the smallest bound for $\max_{0<u<1/4} |A_1^{\alpha_1} A_2^{\alpha_2} \cdots A_{10}^{\alpha_{10}} A_{11}^{\alpha_{11}} B_1^{\beta_1} \cdots B_6^{\beta_6}|$ with the condition $\sum_{i=1}^{11} \alpha_i \deg A_i + \sum_{j=1}^{6} \beta_j \deg B_j = 10^8$. Then we obtain the polynomial H of degree 10^8 with

$$\begin{array}{lll} \alpha_1 = 64117551, & \alpha_2 = 12048122, & \alpha_3 = 8256155, \\ \alpha_4 = 492953, & \alpha_5 = 2698672, & \alpha_6 = 839171, \\ \alpha_7 = 363750, & \alpha_8 = 47433, & \alpha_9 = 749597, \\ \alpha_{10} = 115864, & \alpha_{11} = 0, & \\ \beta_1 = 120310, & \beta_2 = 60103, & \beta_3 = 132344, \\ \beta_4 = 65765, & \beta_5 = 34290, & \beta_6 = 85792. \end{array}$$

TABLE 2

n	*polynomial*	$t_Z([0,1]) \leq$	n	*polynomial*	$t_Z([0,1]) \leq$
76	$A_1^{24}A_2^{10}A_3^4A_4A_5A_{10}$	0.4276	89	$A_1^{29}A_2^{11}A_3^5A_5A_6$	0.4270
77	$A_1^{25}A_2^9A_3^3A_5A_9$	0.4279	90	$A_1^{28}A_2^{10}A_3^3A_5A_6A_9$	0.4269
78	$A_1^{26}A_2^{10}A_3^4A_5^2$	0.4279	91	$A_1^{30}A_2^{13}A_3^4A_4A_5^2$	0.4277
79	$A_1^{26}A_2^9A_3^3A_5A_9$	0.4272	92	$A_1^{30}A_2^{12}A_3^3A_4A_5A_9$	0.4273
80	$A_1^{27}A_2^{10}A_3^4A_5^2$	0.4275	93	$A_1^{30}A_2^{13}A_3^5A_4A_5A_{10}$	0.4271
81	$A_1^{26}A_2^9A_3^5A_5A_6$	0.4276	94	$A_1^{29}A_2^{10}A_3^4A_5A_6A_9$	0.4267
82	$A_1^{27}A_2^{12}A_3^3A_4A_5^2$	0.4275	95	$A_1^{31}A_2^{11}A_3^3A_5A_{11}$	0.4273
83	$A_1^{27}A_2^{11}A_3^4A_5A_6$	0.4279	96	$A_1^{32}A_2^{14}A_3^4A_4A_5^2$	0.4267
84	$A_1^{27}A_2^{10}A_3^5A_5A_6$	0.4281	97	$A_1^{31}A_2^{11}A_3^4A_5^2A_9$	0.4273
85	$A_1^{26}A_2^9A_3^3A_5A_6A_9$	0.4272	98	$A_1^{32}A_2^{12}A_3^6A_5A_6$	0.4269
86	$A_1^{28}A_2^{10}A_3^5A_5A_6$	0.4273	99	$A_1^{32}A_2^{13}A_3^6A_4A_5A_{10}$	0.4271
87	$A_1^{27}A_2^9A_3^3A_5A_6A_9$	0.4273	100	$A_1^{34}A_2^{12}A_3^4A_5A_9$	0.4271
88	$A_1^{29}A_2^{10}A_3^4A_5A_9$	0.4273			

So

$$t_Z([0,\ 1/4]) \leq \max_{0<x<1/4} |H(x)|^{10^{-8}} < 0.179067916.$$

This proves the theorem. Surprisingly we see that the exceptional polynomial A_{11} does not appear as a factor of H!

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