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**SOLUTIONS WITH SUBGROUP SYMMETRY FOR
SINGULAR EQUATIONS IN BIFURCATION THEORY**

(submitted by A. M. Elizarov)

ABSTRACT. In the article, on the base of abstract theory (B. V. Loginov, 1979) the nonlinear eigenvalue problems for nonlinearly perturbed Helmholtz equations having application to low temperature plasma theory and to some problems of differential geometry are considered. Other possible often technically more difficult applications (for instance, periodical solutions in heat convection theory) are completely determined by the group symmetry of original equations and do not depend on their concrete essence. In the general case of finite group symmetry with known composition law, a computer program for determination of all subgroups is given, in particular, for dihedral and also planar and spatial crystallographic groups.

1. INTRODUCTION

In applied problems of critical phenomena, solutions that are invariant with respect to subgroups of the symmetry group of the original bifurcation problem are interesting. The general theory of construction and investigation of branching equations for bifurcational symmetry breaking

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problems is given in [3,4,9]. It is supposed that the nonlinear equation

$$By - A(\lambda)y = R(y, \lambda), \quad \|R(y, \lambda)\| = o(\|y\|) \quad (1)$$

($B, A(\lambda)$ are linear operators from E_1 to E_2 , E_1 and E_2 are Banach spaces, $\lambda \in R^1$) admits the motion group of the Euclidean space R^s , $s > 1$. In neighborhoods of critical values λ_0 of the parameter λ that are eigenvalues of the problem $(B - A(\lambda))\varphi = 0$, periodical solutions with crystallographic group symmetry (the semi-direct product $G = G_1 \rtimes \tilde{G}^1$ of the s -parametrical continuous shift group $G_1 = G_1(\alpha_1, \dots, \alpha_s)$ and the group \tilde{G}^1 of the elementary cell of periodicity constructed on the basic translations) arise, which are mutually transformed by the action of the group \tilde{G}^1 .

Basic elements of the zero-subspace $N = N(B - A(\lambda_0))$ have the form of Bloch functions

$$\varphi_r = \varphi_{\mathbf{l}_r} = \exp[i\langle \mathbf{l}_r, q \rangle], \quad q = (x_1, \dots, x_s), \quad r = 1, \dots, n, \quad (2)$$

where the inverse lattice vectors \mathbf{l}_r are given by the dispersion relation, which determines critical values λ_0 of the bifurcation parameter and connects the integer multiples of periods $|\mathbf{a}_k|$, $k = 1, \dots, s$, along the basic translations \mathbf{a}_k with physical dimensionless parameters of the applied problem. An arbitrary s -periodic function can be represented in the form of Fourier series on the inverse lattice $F(q) = \sum_{\mathbf{l} \in \Lambda'} f_{\mathbf{l}} e^{i\langle \mathbf{l}, q \rangle}$ and the basic elements of zero-subspace $N(B - A(\lambda_0))$ should be determined as this Fourier series components. By the theorem on inheritance of the group symmetry of equation (1), the corresponding branching equation (BEq) $0 = f(\xi, \varepsilon) : \Xi^n \rightarrow \Xi^n$ admits the s -parametrical rotation group $\underbrace{SO(2) \times \dots \times SO(2)}_{s \text{ times}}$, which is homomorphic to the shift group $G_1(\alpha)$,

and the discrete rotation-reflection group \tilde{G}_1 determined by the vectors \mathbf{l}_r and elements $\varphi_{\mathbf{l}_r}$,

$$f(\mathcal{A}_g \xi, \varepsilon) = \mathcal{B}_g f(\xi, \varepsilon). \quad (3)$$

Here \mathcal{A}_g is the representation of the group G in Ξ^n , contragredient to its representation in N , and \mathcal{B}_g is its representation in the defect subspace $N^* = N^*(B - A(\lambda_0))$.

The problem on finding solutions of equation (1) which are invariant with respect to subgroups of the discrete symmetry group \tilde{G}^1 arises. The general scheme for its solving is given in [2, 13], and also in [3, 4, 9].

The initial problem is the discrete group \tilde{G}^1 and the structure $L(\tilde{G}^1)$ of all its subgroups. If $H_0 = \tilde{G}^1 \supset H_1 \supset H_2 \supset \dots \supset H_{\infty} = \{H_k\}_1^{\infty}$ is some chain of subgroups of the length ∞ then there exists the basis

$R_{\mathfrak{a}}$ in N with respect to which the representation \mathcal{A}_g for every subgroup H_i splits into irreducible representations. The set of all BEqs for H -invariant solutions forms the dual by inclusion structure L' to $L(\tilde{G}^1)$: BEq of solutions which are invariant with respect to the more slender subgroup contains the BEq of solutions which are invariant with respect to wider subgroup. For two chains $A = \{H_k\}_1^{\mathfrak{a}}$ and $g^{-1}Ag = \{g^{-1}H_k g\}_1^{\mathfrak{a}}$ of similar subgroups, the connection between the H_k -invariant element subspaces and respectively between the BEqs of H_k -invariant solutions is realized by the element g .

For the simple illustration of this abstract theory, here for the equations

$$\Delta u + \lambda^2 \sinh u = 0 \quad (4)$$

and

$$\Delta u + \lambda^2 \sin u = 0 \quad (5)$$

periodical solutions with hexagonal lattice of periodicity are found. Applications of these equations to low temperature plasma theory [6, 7] and to some problems of differential geometry [1, 14] are known. Complicated examples, for instance, periodical solutions in heat convection theory [10, 11], also can be investigated according to the same scheme. In the general case of finite group with known composition law, a computer program for the determination of all subgroups is given.

We use the terminology and notation from [3, 4, 9, 12].

2. BRANCHING EQUATION WITH HEXAGON GROUP SYMMETRY D_6 FOR THE EQUATIONS (4), (5)

The general form of BEq admitting the symmetry of hexagonal lattice $\mathbf{l}_1 = \ell \mathbf{i} + \sqrt{3}m \mathbf{j}$, $\mathbf{l}_3 = \frac{1}{2}[(\ell - 3m)\mathbf{i} + \sqrt{3}(\ell + m)\mathbf{j}]$, $\mathbf{l}_5 = \frac{1}{2}[-(\ell + 3m)\mathbf{i} + \sqrt{3}(\ell - m)\mathbf{j}]$, $\mathbf{l}_{2k} = \mathbf{l}_{2k-1}$, $k = 1, 2, 3$ (the integers ℓ and m have the same parity) for the first bifurcation point $\ell = m = 1$ with the basis (2) $\{\varphi_r = \exp[i\langle \mathbf{l}_r, \mathbf{q} \rangle]\}_1^6$ in the zero-subspace can be obtained [5, 9] by group analysis methods on the base of the inheritance theorem (3), where

$$\begin{aligned} \mathcal{B}_{g(\alpha)} = \mathcal{A}_{g(\alpha)} = & \text{diag}\{\exp(i\beta(\alpha_1 + \sqrt{3}\alpha_2)), \\ & \exp(-i\beta(\alpha_1 + \sqrt{3}\alpha_2)), \exp(-i\beta(\alpha_1 - \sqrt{3}\alpha_2)), \\ & \exp(i\beta(\alpha_1 - \sqrt{3}\alpha_2)), \exp(-2i\beta\alpha_1), \exp(2i\beta\alpha_1)\}, \end{aligned} \quad (6)$$

$\beta = \frac{\pi}{a}$ and $2a$ is the lattice width. The equality (3) means that the manifold $\mathcal{F} = \{\xi, f | f - f(\xi) = 0\}$ is an invariant manifold of the transformation group $\tilde{\xi} = \mathcal{A}_{g(\alpha)}\xi$, $\tilde{f} = \mathcal{A}_{g(\alpha)}f$ and can be expressed [8] through the complete system of functionally independent invariants $I_j = \frac{f_j}{\xi_j}$, $j = \overline{1, 6}$,

$I_7 = \xi_1\xi_2$, $I_8 = \xi_3\xi_4$, $I_9 = \xi_5\xi_6$, $I_{10} = \xi_2\xi_3\xi_6$. Thus the branching system allowing the hexagon group symmetry has the form [5, 9]

$$\begin{aligned}
f_1(\xi, \varepsilon) &= \sum_p a_{p;0}(\varepsilon) \xi_1(\xi_1\xi_2)^{p_1}(\xi_3\xi_4)^{p_2}(\xi_5\xi_6)^{p_3} \\
&+ \sum_{p; k \geq 1} (\xi_1\xi_2)^{p_1}(\xi_3\xi_4)^{p_2}(\xi_5\xi_6)^{p_3} \\
&\quad [a_{p;k}(\varepsilon) \xi_2^{k-1} \xi_3^k \xi_6^k + b_{p;k}(\varepsilon) \xi_1^{k+1} \xi_4^k \xi_5^k] = 0 \quad (7) \\
f_2(\xi, \varepsilon) &\equiv r^3 f_1(\xi, \varepsilon) = 0, f_3(\xi, \varepsilon) \equiv r f_1(\xi, \varepsilon) = 0, \\
f_4(\xi, \varepsilon) &\equiv r^4 f_1(\xi, \varepsilon) = 0, f_5(\xi, \varepsilon) \equiv s f_1(\xi, \varepsilon) = 0, \\
f_6(\xi, \varepsilon) &\equiv s r^3 f_1(\xi, \varepsilon) = 0,
\end{aligned}$$

where the permutation of the hexagon top numbers (i.e. the hexagon group D_6) is generated by the permutations $r = (135246)$ (the rotation on the angle $\frac{\pi}{6}$ counterclockwise) and $s = (15)(26)(3)(4)$ (the reflection around the axis joining the tops (3) and (4)).

The main part of the branching system (7) has the form

$$\begin{aligned}
\xi_1\varepsilon + A\xi_1^2\xi_2 + B\xi_1\xi_3\xi_4 + B\xi_1\xi_5\xi_6 + \dots &= 0 \\
\xi_2\varepsilon + A\xi_2^2\xi_1 + B\xi_2\xi_3\xi_4 + B\xi_2\xi_5\xi_6 + \dots &= 0 \\
\xi_3\varepsilon + A\xi_3^2\xi_4 + B\xi_1\xi_2\xi_3 + B\xi_3\xi_5\xi_6 + \dots &= 0 \\
\xi_4\varepsilon + A\xi_4^2\xi_3 + B\xi_1\xi_2\xi_4 + B\xi_4\xi_5\xi_6 + \dots &= 0 \\
\xi_5\varepsilon + A\xi_5^2\xi_6 + B\xi_5\xi_3\xi_4 + B\xi_5\xi_1\xi_2 + \dots &= 0 \\
\xi_6\varepsilon + A\xi_6^2\xi_5 + B\xi_6\xi_3\xi_4 + B\xi_6\xi_1\xi_2 + \dots &= 0
\end{aligned} \quad (8)$$

where $A = \pm \frac{\lambda_0^2}{2}$, $B = \pm \lambda_0^2$, $\lambda_0^2 = \frac{4\pi^2}{a^2}$ (the upper sign is related to the equation (4) and the lower one to (5))

In the article [5] the following statement is proved

Lemma 1. *In the case of hexagonal symmetry, let $n = \dim N(B) = 6$ and assume that the group of symmetries for the branching equation is given by (6) and the permutations D_6 . Then the subspace $N(B)$ decomposes into the direct sum of two one-dimensional and two two-dimensional irreducible D_6 -invariant subspaces with basic elements*

$$\begin{aligned}
N_1^{(1)} : \quad e_1 &= \frac{1}{\sqrt{6}} (\varphi_1 + \varphi_2 + \varphi_3 + \varphi_4 + \varphi_5 + \varphi_6) \\
&= \frac{\sqrt{2}}{\sqrt{3}} \left(\cos \frac{\pi}{a} (x + \sqrt{3}y) + \cos \frac{2\pi}{a} x + \cos \frac{\pi}{a} (x - \sqrt{3}y) \right), \\
N_2^{(1)} : \quad e_2 &= \frac{i}{\sqrt{6}} (\varphi_1 - \varphi_2 - \varphi_3 + \varphi_4 + \varphi_5 - \varphi_6) = \\
&= \frac{\sqrt{2}}{\sqrt{3}} \left(-\sin \frac{\pi}{a} (x + \sqrt{3}y) + \sin \frac{2\pi}{a} x - \sin \frac{\pi}{a} (x - \sqrt{3}y) \right), \quad (9)
\end{aligned}$$

$$\begin{aligned}
N_3^{(2)} : \quad & \begin{cases} e_3 = \frac{i}{2\sqrt{3}}(\varphi_1 - \varphi_2 + 2\varphi_3 - 2\varphi_4 + \varphi_5 - \varphi_6) = \\ & = -\frac{1}{\sqrt{3}} \left(\sin \frac{\pi}{a}(x + \sqrt{3}y)y + 2 \sin \frac{2\pi}{a}x + \sin \frac{\pi}{a}(x - \sqrt{3}y) \right), \\ e_4 = \frac{i}{2}(-\varphi_1 + \varphi_2 + \varphi_5 - \varphi_6) = \\ & = \sin \frac{\pi}{a}(x + \sqrt{3}y) - \sin \frac{\pi}{a}(x - \sqrt{3}y), \end{cases} \\
N_4^{(2)} : \quad & \begin{cases} e_5 = \frac{1}{2\sqrt{3}}(\varphi_1 + \varphi_2 - 2\varphi_3 - 2\varphi_4 + \varphi_5 + \varphi_6) = \\ & = \frac{1}{\sqrt{3}} \left(\cos \frac{\pi}{a}(x + \sqrt{3}y) - 2 \cos \frac{2\pi}{a}x + \cos \frac{\pi}{a}(x - \sqrt{3}y) \right), \\ e_6 = \frac{1}{2}(\varphi_1 + \varphi_2 - \varphi_5 - \varphi_6) = \\ & = \cos \frac{\pi}{a}(x + \sqrt{3}y) - \cos \frac{\pi}{a}(x - \sqrt{3}y). \end{cases}
\end{aligned}$$

In the same work [5] branching equations for solutions invariant with respect to normal divisors together with asymptotics of such solutions are written out.

3. SOLUTIONS WITH SUBGROUPS SYMMETRY

Here we find the solutions of (4), (5) which are invariant with respect to subgroups of the hexagonal group D_6 .

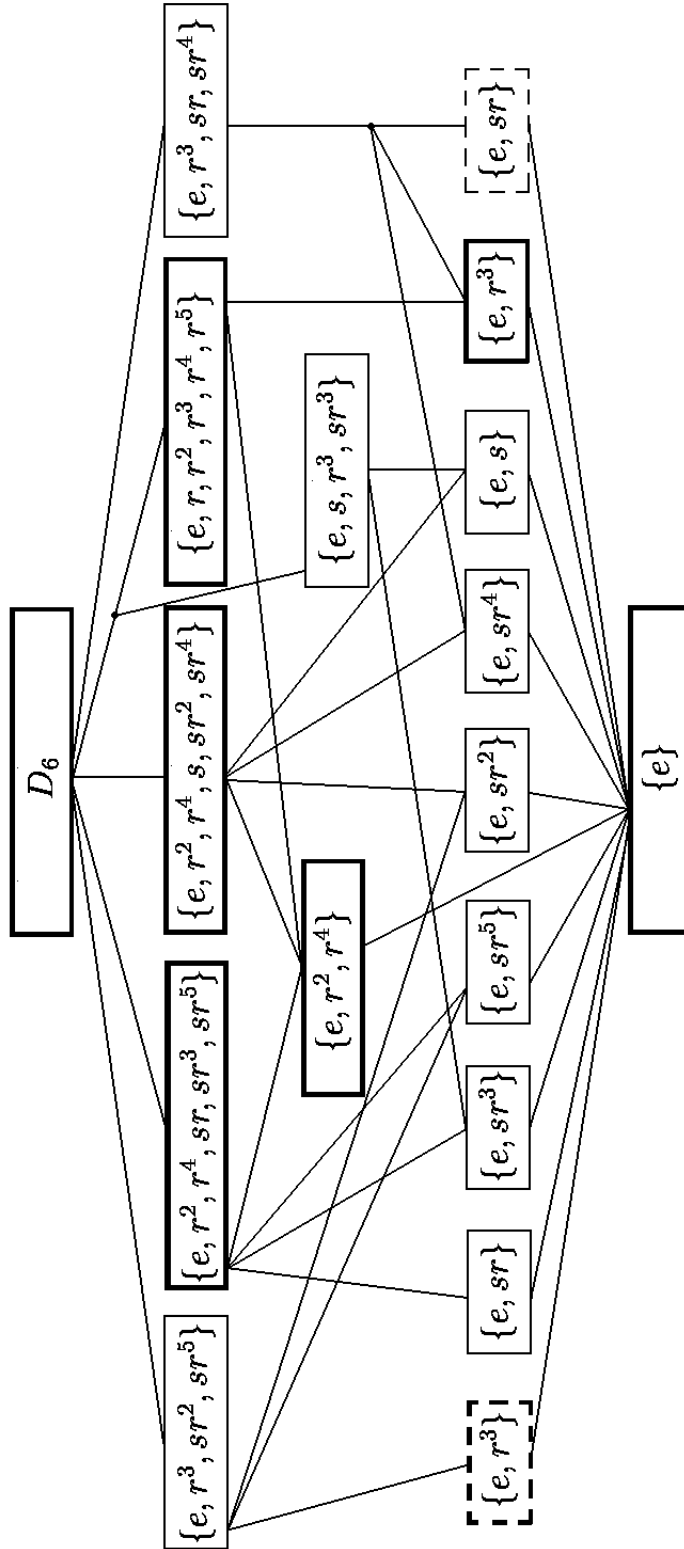
The initial one is the hexagon group

$$D_6 = \{e, r, r^2, r^3, r^4, r^5, s, sr, sr^2, sr^3, sr^4, sr^5\}$$

generated by the substitutions of $N(B)$ basic elements indexes and the structure $L(D_6)$ of all its subgroups.

In the structure $L(D_6)$ the following subgroup chains are selected

$$\begin{aligned}
A_1 : A_{1,0} = D_6 \supset A_{1,1} &= \{e, r^2, r^4, sr, sr^3, sr^5\} \\
&\supset A_{1,2} = \{e, sr\} \supset A_{1,3} = \{e\}; \\
A_2 : A_{2,0} = D_6 \supset A_{2,1} &= A_{1,1} \supset A_{2,2} = \{e, r^2, r^4\} \supset A_{2,3} = \{e\}; \\
A_3 = sr^5 A_1 sr^5 = sr A_4 sr : A_{3,0} &= D_6 \supset A_{3,1} = A_{1,1} \\
&\supset A_{3,2} = \{e, sr^3\} \supset A_{3,3} = \{e\}; \\
A_4 = r^4 A_1 r^2 : A_{4,0} = D_6 \supset A_{4,1} &= A_{1,1} \supset A_{4,2} = \{e, sr^5\} \supset A_{4,3} = \{e\}; \\
A_5 : A_{5,0} = D_6 \supset A_{5,1} &= \{e, s, r^3, sr^3\} \supset A_{5,2} = \{e, sr^3\} \supset A_{5,3} = \{e\}; \\
A_6 : A_{6,0} = D_6 \supset A_{6,1} &= A_{5,1} \supset A_{6,2} = \{e, s\} \supset A_{6,3} = \{e\}; \\
A_7 : A_{7,0} = D_6 \supset A_{7,1} &= \{e, r^2, r^4, s, sr^2, sr^4\} \\
&\supset A_{7,2} = \{e, r^2, r^4\} = A_{2,2} \supset A_{7,3} = \{e\}; \\
A_8 : A_{8,0} = D_6 \supset A_{8,1} &= A_{7,1} \supset A_{8,2} = \{e, sr^2\} \supset A_{8,3} = \{e\}; \\
A_9 = r^5 A_8 r = sr^2 A_{10} sr^2 : A_{9,0} &= D_6 \supset A_{9,1} = A_{7,1} \\
&\supset A_{9,2} = \{e, sr^4\} \supset A_{9,3} = \{e\}; \\
A_{10} = r^4 A_8 r^2 : A_{10,0} = D_6 \supset A_{10,1} &= A_{7,1} \\
&\supset A_{10,2} = \{e, s\} \supset A_{10,3} = \{e\};
\end{aligned}$$



$$\begin{aligned}
A_{11} : A_{11,0} &= D_6 \supset A_{11,1} = \{e, r, r^2, r^3, r^4, r^5\} \\
&\supset A_{11,2} = \{e, r^2, r^4\} \supset A_{11,3} = \{e\}; \\
A_{12} : A_{12,0} &= D_6 \supset A_{12,1} = A_{11,1} \supset A_{12,2} = \{e, r^3\} \supset A_{12,3} = \{e\}; \\
A_{13} : A_{13,0} &= D_6 \supset A_{13,1} = \{e, r^3, sr^2, sr^5\} \\
&\supset A_{13,2} = A_{4,2} \supset A_{13,3} = \{e\}; \\
A_{14} : A_{14,0} &= D_6 \supset A_{14,1} = A_{13,1} \supset A_{14,2} = A_{8,2} \supset A_{14,3} = \{e\}; \\
A_{15} : A_{15,0} &= D_6 \supset A_{15,1} = A_{13,1} \supset A_{15,2} = A_{12,2} \supset A_{15,3} = \{e\}; \\
A_{16} : A_{16,0} &= D_6 \supset A_{16,1} = \{e, r^3, sr, sr^4\} \supset A_{16,2} = A_{1,2} \supset A_{16,3} = \{e\}; \\
A_{17} : A_{17,0} &= D_6 \supset A_{17,1} = A_{16,1} \supset A_{17,2} = A_{9,2} \supset A_{17,3} = \{e\}; \\
A_{18} : A_{18,0} &= D_6 \supset A_{18,1} = A_{16,1} \supset A_{18,2} = A_{12,2} \supset A_{18,3} = \{e\}.
\end{aligned}$$

The subgroups

$$\begin{aligned}
&\{e, r^2, r^4, sr, sr^3, sr^5\}, \{e, r^2, r^4, s, sr^2, sr^4\}, \\
&\{e, r, r^2, r^3, r^4, r^5\}, \{e, sr^3\}, \{e, r^2, r^4\}
\end{aligned}$$

are normal divisors (on the figure they are shown by semiboldface lines)

For the brevity of presentation we consider here only the four first chains of subgroups.

3 A. Consider the chains A_1, A_3, A_4 . The projective operator $P(A_{1,1})$ transforms the $N(B)$ into one-dimensional subspace of $A_{1,1}$ -invariant elements $\text{span}\{\varphi_1^\times = \frac{1}{\sqrt{3}}(\varphi_1 + \varphi_2 + \varphi_3 + \varphi_4 + \varphi_5 + \varphi_6)\}$. Complete this one-dimensional subspace up to $N(B)$ by the basic elements of irreducible invariant subspaces

$$\begin{aligned}
\varphi_2^\times &= \frac{i}{\sqrt{3}}(\varphi_1 - \varphi_2 - \varphi_3 + \varphi_4 + \varphi_5 - \varphi_6), \\
\varphi_3^\times &= \frac{i}{\sqrt{3}}(\varphi_1 - \varphi_2 + 2\varphi_3 - 2\varphi_4 + \varphi_5 - \varphi_6), \\
\varphi_4^\times &= \frac{i}{\sqrt{3}}(-\varphi_1 + \varphi_2 + \varphi_5 - \varphi_6), \\
\varphi_5^\times &= \frac{1}{\sqrt{3}}(\varphi_1 + \varphi_2 - 2\varphi_3 - 2\varphi_4 + \varphi_5 + \varphi_6), \\
\varphi_6^\times &= \frac{1}{\sqrt{3}}(\varphi_1 + \varphi_2 - \varphi_5 - \varphi_6).
\end{aligned}$$

Then BEq of $A_{1,1}$ -invariant solutions is resulting from BEq in new base at $\eta_k = 0, k = \overline{2, 6}$.

$$\begin{aligned}
&\sum_p a_{p,0} \frac{\eta_1^{2(p_1+p_2+p_3)+1}}{\sqrt{3} 3^{p_1+p_2+p_3}} \\
&+ \sum_{p,k \geq 1} \frac{\eta_1^{2(p_1+p_2+p_3)}}{3^{p_1+p_2+p_3}} \left[a_{p,k} \frac{\eta_1^{3k-1}}{(\sqrt{3})^{3k-1}} + b_{p,k} \frac{\eta_1^{3k+1}}{(\sqrt{3})^{3k+1}} \right] \quad (10)
\end{aligned}$$

The main part of (10) is the following

$$\eta_1 \varepsilon + \frac{A}{3} \eta_1^3 + \frac{2B}{3} \eta_1^3 = 0$$

with solutions $\eta_{1,2} = \pm \sqrt{\frac{-3\varepsilon}{A+2B}}$, $\text{sign } \varepsilon = -\text{sign}(A+2B)$, i.e. $\varepsilon < 0$ ($\varepsilon > 0$) for the equation (4) ((5)).

The subspace of $N(B)$ elements which are invariant with respect to the $A_{1,2}$ -subgroup $\text{span}\{\varphi_1^\times = \frac{\varphi_1+\varphi_3}{\sqrt{3}}, \varphi_2^\times = \frac{\varphi_2+\varphi_4}{\sqrt{3}}, \varphi_5^\times = \frac{\varphi_5+\varphi_6}{\sqrt{3}}\}$ is transferred by transformation r^4 into the subspace $\text{span}\{\varphi_1^\times = \frac{\varphi_1+\varphi_2}{\sqrt{3}}, \varphi_3^\times = \frac{\varphi_3+\varphi_5}{\sqrt{3}}, \varphi_4^\times = \frac{\varphi_4+\varphi_6}{\sqrt{3}}\}$ of $A_{4,2} = r^4 A_{1,2} r^2$ -invariant elements and by transformation sr^5 into the subspace $\text{span}\{\varphi_1^\times = \frac{\varphi_1+\varphi_6}{\sqrt{3}}, \varphi_2^\times = \frac{\varphi_2+\varphi_5}{\sqrt{3}}, \varphi_3^\times = \frac{\varphi_3+\varphi_4}{\sqrt{3}}\}$ of $A_{3,2} = sr^5 A_{1,2} sr^5$ -invariant elements. The BEq solutions invariant with respect to $A_{1,2}$ has the form

$$\begin{aligned} & \sum_p a_{p;0} \frac{\eta_1}{\sqrt{3}} \frac{\eta_1^{p_1+p_2} \eta_2^{p_1+p_2} \eta_5^{2p_3}}{3^{p_1+p_2+p_3}} + \sum_{p,k \geq 1} \frac{\eta_1^{p_1+p_2} \eta_2^{p_1+p_2} \eta_5^{2p_3}}{3^{p_1+p_2+p_3}} \cdot \\ & \cdot \left[a_{p;k} \left(\frac{\eta_2}{\sqrt{3}} \right)^{k-1} \left(\frac{\eta_1}{\sqrt{3}} \right)^k \left(\frac{\eta_5}{\sqrt{3}} \right)^k + b_{p;k} \left(\frac{\eta_1}{\sqrt{3}} \right)^{k+1} \left(\frac{\eta_2}{\sqrt{3}} \right)^k \left(\frac{\eta_5}{\sqrt{3}} \right)^k \right] = 0 \\ & \sum_p a_{p;0} \frac{\eta_2}{\sqrt{3}} \frac{\eta_1^{p_1+p_3} \eta_2^{p_1+p_3} \eta_5^{2p_2}}{3^{p_1+p_2+p_3}} + \sum_{p,k \geq 1} \frac{\eta_1^{p_1+p_3} \eta_2^{p_1+p_3} \eta_5^{2p_2}}{3^{p_1+p_2+p_3}} \cdot \\ & \cdot \left[a_{p;k} \left(\frac{\eta_1}{\sqrt{3}} \right)^{k-1} \left(\frac{\eta_2}{\sqrt{3}} \right)^k \left(\frac{\eta_5}{\sqrt{3}} \right)^k + b_{p;k} \left(\frac{\eta_2}{\sqrt{3}} \right)^{k+1} \left(\frac{\eta_1}{\sqrt{3}} \right)^k \left(\frac{\eta_5}{\sqrt{3}} \right)^k \right] = 0 \\ & \sum_p a_{p;0} \frac{\eta_5}{\sqrt{3}} \frac{\eta_1^{p_2+p_3} \eta_2^{p_2+p_3} \eta_5^{2p_1}}{3^{p_1+p_2+p_3}} + \sum_{p,k \geq 1} \frac{\eta_1^{p_2+p_3} \eta_2^{p_2+p_3} \eta_5^{2p_1}}{3^{p_1+p_2+p_3}} \cdot \\ & \cdot \left[a_{p;k} \left(\frac{\eta_5}{\sqrt{3}} \right)^{k-1} \left(\frac{\eta_1}{\sqrt{3}} \right)^k \left(\frac{\eta_2}{\sqrt{3}} \right)^k + b_{p;k} \left(\frac{\eta_5}{\sqrt{3}} \right)^{k+1} \left(\frac{\eta_1}{\sqrt{3}} \right)^k \left(\frac{\eta_2}{\sqrt{3}} \right)^k \right] = 0 \end{aligned} \quad (11)$$

with the corresponding main part

$$\begin{aligned} \eta_1 \left(\varepsilon + \frac{A+B}{3} \eta_1 \eta_2 + \frac{B}{3} \eta_5^2 \right) &= 0 \\ \eta_2 \left(\varepsilon + \frac{A+B}{3} \eta_1 \eta_2 + \frac{B}{3} \eta_5^2 \right) &= 0 \\ \eta_5 \left(\varepsilon + \frac{2B}{3} \eta_1 \eta_2 + \frac{A}{3} \eta_5^2 \right) &= 0 \end{aligned} \quad (12)$$

Consequently the $A_{1,2}$ -invariant solutions of the equations (4), (5) are representing by the formula $\eta_1^* \varphi_1^\times + \eta_2^* \varphi_2^\times + \eta_5^* \varphi_5^\times$, where the vector $(\eta_1^*, \eta_2^*, \eta_5^*)$ passes the solution set of the system (12). Respectively the $A_{4,2} = r^4 A_{1,2} r^2$ -invariant solutions ($A_{3,2} = sr^5 A_{1,2} sr^5$ -invariant solutions) are the following

$$\begin{aligned} r^4(\eta_1^* \varphi_1^\times + \eta_2^* \varphi_2^\times + \eta_5^* \varphi_5^\times) &= \eta_1^* r^4 \varphi_1^\times + \eta_2^* r^4 \varphi_2^\times + \eta_5^* r^4 \varphi_5^\times = \frac{1}{\sqrt{3}} [\eta_1^* (\varphi_4 + \varphi_6) + \eta_2^* (\varphi_3 + \varphi_5) + \eta_5^* (\varphi_1 + \varphi_2)] \\ (sr^5(\eta_1^* \varphi_1^\times + \eta_2^* \varphi_2^\times + \eta_5^* \varphi_5^\times)) &= \eta_1^* sr^5 \varphi_1^\times + \eta_2^* sr^5 \varphi_2^\times + \eta_5^* sr^5 \varphi_5^\times = \frac{1}{\sqrt{3}} [\eta_1^* (\varphi_2 + \varphi_5) + \eta_2^* (\varphi_1 + \varphi_6) + \eta_5^* (\varphi_3 + \varphi_4)] \end{aligned}$$

3 B. Consider the normal divisors chain A_2 . At its investigation in accord to [5, 9] pass to the indicated in Lemma 1 basis of irreducible invariant subspaces.

For the construction of the equivalent BEq in the basis (9) it should be taken the substitution $\xi = C_1 \zeta$, where C_1' is the transformation matrix from $\{\varphi_j\}_1^6$ to $\{e_j\}_1^6$ obtained in the Lemma 1, where $\zeta = (\zeta_1, \zeta_2, \dots, \zeta_6)$ are the coordinates of the vector $\varphi \in N(B)$ in the basis $\{e_j\}_1^6$. Since $\xi_{2k} = \bar{\xi}_{2k-1}$, we can take $\xi_{2k-1} = \tau_{2k-1} + i\tau_{2k}$, $\xi_{2k} = \tau_{2k-1} - i\tau_{2k}$ and the transformation matrix $\tau = C\zeta$ is defined by the formula $C = C_0^{-1} \cdot C_1$, where C_0 is quasidiagonal matrix with blocks $\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}$. The corresponding transformation $\tau \leftrightarrow \zeta$ has the form

$$\begin{aligned} \tau_1 &= \frac{1}{\sqrt{6}}\zeta_1 & & + \frac{1}{2\sqrt{3}}\zeta_5 & + \frac{1}{2}\zeta_6 \\ \tau_2 &= \frac{1}{\sqrt{6}}\zeta_2 & + \frac{1}{2\sqrt{3}}\zeta_5 & - \frac{1}{2}\zeta_4 \\ \tau_3 &= \frac{1}{\sqrt{6}}\zeta_1 & & - \frac{1}{\sqrt{3}}\zeta_5 \\ \tau_4 &= -\frac{1}{2\sqrt{6}}\zeta_2 & + \frac{1}{\sqrt{3}}\zeta_3 & \\ \tau_5 &= \frac{1}{\sqrt{6}}\zeta_1 & & + \frac{1}{2\sqrt{3}}\zeta_5 & - \frac{1}{2}\zeta_6 \\ \tau_6 &= \frac{1}{\sqrt{6}}\zeta_2 & + \frac{1}{2\sqrt{3}}\zeta_3 & + \frac{1}{2}\zeta_4 \end{aligned} \quad (13)$$

Applying the formula $P(H_k) = \frac{1}{|H_k|} \sum_{g \in H_k} \hat{A}_g$, where \hat{A}_g are quasidiagonal matrices with blocks T_j of irreducible representations, for $A_{2,2}$ and $A_{2,1} = A_{1,1}$ one has the projector with respect to ζ variables:

$$P_{A_{2,1}} = \text{diag}(1, 1, 0, 0, 0, 0), \quad P_{A_{2,2}} = \text{diag}(1, 0, 0, 0, 0, 0)$$

The following statement is true. For $A_{2,2}$ -invariant solutions it should be taken $\zeta_3 = \dots = \zeta_6 = 0$ or according to (13) $\tau_1 = \tau_3 = \tau_5$, $\tau_2 = -\tau_4 = \tau_6$, for $A_{2,1}$ -invariant solutions one has $\zeta_1 \neq 0$, $\zeta = \dots = \zeta_6 = 0$ or $\tau_1 = \tau_3 = \tau_5$, $\tau_2 = \tau_4 = \tau_6$.

The relevant branching equations with respect to τ variables have the following solutions: $A_{2,2}$ -invariant solutions

$$\begin{aligned} \tau_1 = 0, \quad \tau_2 &= \pm \left(-\frac{\varepsilon}{A+2B} \right)^{\frac{1}{2}} = \pm \frac{\sqrt{2}}{\lambda_0 \sqrt{5}} (\mp \varepsilon)^{\frac{1}{2}}; \\ \tau_2 = 0, \quad \tau_1 &= \pm \left(-\frac{\varepsilon}{A+2B} \right)^{\frac{1}{2}} = \pm \frac{\sqrt{2}}{\lambda_0 \sqrt{5}} (\mp \varepsilon)^{\frac{1}{2}}; \\ \tau_1 \neq 0, \quad \tau_2 \neq 0, \quad \tau_1^2 + \tau_2^2 &= -\frac{\varepsilon}{A+2B} = \mp \frac{2\varepsilon}{5\lambda_0}; \end{aligned}$$

and $A_{2,1}$ -invariant solutions

$$\tau_1 = \tau_3 = \tau_5 = \pm \left(-\frac{\varepsilon}{A+2B} \right)^{\frac{1}{2}} = \pm \frac{\sqrt{2}}{\lambda_0 \sqrt{5}} (\mp \varepsilon)^{\frac{1}{2}}.$$

Here $\varepsilon < 0$ for equation (4) and $\varepsilon > 0$ for equation (5). The corresponding solutions for nonlinear problem (4), (5) are represented by convergent series in $\varepsilon^{\frac{1}{2}}$, their asymptotics has the form of linear combinations $\sum_{k=1}^6 \tau_k \hat{\varphi}_k$, where the omitted components τ_k are equal to zero,

and

$$\begin{aligned}\hat{\varphi}_1 &= 2 \cos \frac{\pi}{a}(x + \sqrt{3}y), & \hat{\varphi}_2 &= 2 \sin \frac{\pi}{a}(x + \sqrt{3}y), & \hat{\varphi}_3 &= 2 \cos \frac{\pi}{a}x, \\ \hat{\varphi}_4 &= 2 \sin \frac{\pi}{a}x, & \hat{\varphi}_5 &= 2 \cos \frac{\pi}{a}(x - \sqrt{3}y), & \hat{\varphi}_6 &= 2 \sin \frac{\pi}{a}(x - \sqrt{3}y).\end{aligned}$$

Conclusion. Using equations (4), (5) as an example, we have demonstrated the general scheme of solution construction with subgroup symmetry. For every chain of subgroups there exists the basis of the zero-subspace $N(B)$ in which the BEqs of subgroup invariant solutions form the dual chain. For two chains $A = \{H_k\}_1^\infty$ and $g^{-1}Ag = \{g^{-1}H_k g\}_1^\infty$ of similar subgroups the connection between the H_k -invariant element subspaces, and between the BEqs of H_k -invariant solutions, respectively, is realized by the element g .

This result is completely determined by the original singular nonlinear equation group symmetry and does not depend on the essence of the simulated concrete bifurcation phenomenon.

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