NORMAL SPACES AND THE LUSIN-MENCHOFF PROPERTY

PAVEL PYRIH, Praha[1]

(Received March 14, 1996)

Abstract. We study the relation between the Lusin-Menchoff property and the F_σ-"semiseparation" property of a fine topology in normal spaces. Three examples of normal topological spaces having the F_σ-"semiseparation" property without the Lusin-Menchoff property are given. A positive result is obtained in the countable compact space.

Keywords: fine topology, finely separated sets, Lusin-Menchoff property, normal space

MSC 1991: 54A10, 26A03, 31C40

1. Introduction

All topological spaces considered should be Hausdorff. Let (X, ϱ) be a topological space. Any topology τ finer than ϱ is called a *fine topology.* We use the terms finely open, finely closed, ... with respect to a fine topology (similarly for another topology). We say that A, $B \subset X$ are *finely separated* if there are disjoint finely open sets $\mathcal{G}_A$ and $\mathcal{G}_B$ such that $A \subset \mathcal{G}_A$, $B \subset \mathcal{G}_B$.

An important tool in the study of fine topologies is the Lusin-Menchoff property. We say that a fine topology τ on (X, ϱ) has the *Lusin-Menchoff property* (with respect to ϱ) if for each pair of disjoint subsets F and $\mathcal{F}$ of X, F closed, $\mathcal{F}$ finely closed, there are disjoint subsets G and $\mathcal{G}$ of X, G open, $\mathcal{G}$ finely open, such that $\mathcal{F} \subset G$, $F \subset \mathcal{G}$ ([2], p. 85).

In [5] we proved the following

Theorem 1.1. *Let a fine topology have the Lusin-Menchoff property. Suppose a and b are finely closed sets. Suppose A and B are sets of type F_σ with $a \subset A$,*

[1] Research supported by the grant No. GAUK 186/96 of the Charles University.

$b \subset B$, A disjoint with b, and B disjoint with a. Then there are disjoint finely open sets α and β such that $a \subset \alpha$ and $b \subset \beta$.

Let $a \subset A \subset X$ and $b \subset B \subset X$ where A and B are of type F_σ, A is disjoint with b, and B is disjoint with a. In this situation we say that a and b are F_σ-"semiseparated". Theorem 1.1 says (assuming the Lusin-Menchoff property) that F_σ- "semiseparated" finely closed sets are finely separated.

We can formulate a simple corollary.

Corollary 1.2. *Let a fine topology have the Lusin-Menchoff property and the F_σ-"semiseparation" property (it means that any two finely closed sets can be F_σ-"semiseparated"). Then the fine topology is normal.*

A natural question arises:

Question 1.3. Let a fine topology be normal and have the F_σ-"semiseparation" property. Does this imply that the fine topology has the Lusin-Menchoff property?

In the following examples we show that the answer is no, even with stronger assumptions (see Propositions 2.3, 3.4 and 4.3). A positive result is obtained in the countable compact space (see Proposition 5.1).

2. The train topology

Definition 2.1. Let $X = \mathbb{R}^2$. We define the train topology by the neighbourhood basis of any point. The origin has the neighbourhood basis consisting of sets of the kind

$$U = \{(x,y) \in \mathbb{R}^2 \colon x^2 + y^2 < \varepsilon^2\} \cup \{(x,y) \in \mathbb{R}^2 \colon |y| < 1, x > K\}$$

(the second set is the "long train") for any ε, $K > 0$. Other points have the neighbourhood basis of Euclidean open sets.

We can easily see the following

Observation 2.2. The properties of the train topology:

(i) the Euclidean topology is strongly finer than the train topology;
(ii) the family of G_δ sets in the train topology contains all Euclidean open sets;
(iii) the train topology is not normal (the origin and $\{(x,y) \in \mathbb{R}^2 \colon y = 1\}$ are train closed sets which are not train separated).

Proposition 2.3. *There exists a fine topology which is normal, has the F_σ-"semiseparation" property and has not the Lusin-Menchoff property.*

Proof. Let the original topology on $\mathbb{R}^2$ be the train topology and let the fine topology be the Euclidean one. Then the F_σ-"semiseparation" property of the fine topology follows from Observation 2.2 (ii). The set $F = \{(x, y) \in \mathbb{R}^2 : y = 1\}$ is closed in the train topology, $\mathcal{F} = \{(0,0)\}$ is a Euclidean closed set and any train open cover of $\mathcal{F}$ meets any Euclidean open cover of F. The train topology has not the Lusin-Menchoff property with respect to the Euclidean topology on $\mathbb{R}^2$. □

3. The cuckoo topology

Definition 3.1. Let $e_n \to 0$, $c_n \to \infty$ be disjoint non zero points, $X = \mathbb{R} \setminus \{e_n\}$. We define the cuckoo topology by the neighbourhood basis of any point. The origin has the neighbourhood basis consisting of sets of the kind $\{x \in X : |x| < \varepsilon\} \cup \{x \in X : |x| > K\}$ for any $\varepsilon, K > 0$. The points c_n (the cuckooes) have the neighbourhood basis of the form $\{x \in X : |x - c_n| < \varepsilon\} \cup \{x \in X : |x - e_n| < \varepsilon\}$ (the "home" united with the punctured "egg" given near the origin = "bird") for $\varepsilon > 0$. Other points of X have the neighbourhood basis of all Euclidean open sets.

We can easily see the following

Observation 3.2. The properties of the cuckoo topology:

(i) the Euclidean topology is strongly finer than the cuckoo topology;
(ii) the family of G_δ sets in the cuckoo topology contains all Euclidean open sets;
(iii) the cuckoo topology is compact (near infinity and near "eggs" e_n the situation is simple, due to the definition of the cuckoo topology);
(iv) the Euclidean topology on X is normal.

Proposition 3.3. *The cuckoo topology on X is normal.*

Proof. Let F, G be disjoint cuckoo closed sets. Then
(i) near the origin and finitely many e_n the cuckoo topology is topologically like the Euclidean topology near infinity;
(ii) if $c_n \in F$, then some neighbourhood of c_n (containing an "egg" near e_n) is disjoint with G;
(iii) if $0 \in F$, then some cuckoo neighbourhood of the origin is disjoint with G.
In all situations we can easily find the cuckoo open sets separating F and G. □

Proposition 3.4. *There exists a normal fine topology having the F_σ-"semiseparation" property with respect to a normal and compact original topology such that the fine topology has not the Lusin-Menchoff property with respect to the original topology.*

P r o o f. Let the fine and the original topologies be the Euclidean and the cuckoo topology on X (Definition 3.1), respectively. Then due to Observation 3.2 and Proposition 3.3 it is enough to show that the Lusin-Menchoff property does not hold. We take a cuckoo closed set $F = \{0\}$ and a Euclidean closed set $\mathcal{F} = \{c_n\}_{n=1}^{\infty}$. Any Euclidean open cover of F meets some "egg" in any cuckoo cover of $\mathcal{F}$. The Lusin-Menchoff property does not hold. □

4. The jump topology

Definition 4.1. Let $a_n \to 0$ be nonzero points of $X = [0,1]$. We define the *jump topology* on X by the *jump metric* $\mathrm{jump}(x,y) = d(\varphi(x),\varphi(y))$, where $\varphi\colon X \to \mathbb{R}^2$, $\varphi(a_n) = (a_n, 1)$, $\varphi(x) = (x, 0)$ elsewhere (at a_n the function φ "jumps" to 1) and d is the Euclidean metric in $\mathbb{R}^2$.

We can easily see the following

O b s e r v a t i o n 4.2. The properties of the jump topology:

(i) the jump topology is finer than the Euclidean topology;
(ii) the jump topology is metric;
(iii) the jump closed sets are F_σ sets in the Euclidean topology;
(iv) the jump topology has the F_σ-"semiseparation" property.

Proposition 4.3. *There exists a metric fine topology having the F_σ-"semiseparation" property with respect to a compact metric original topology such that the fine topology has not the Lusin-Menchoff property with respect to the original topology.*

P r o o f. Let the fine and the original topologies be the jump and the Euclidean topology on X (Definition 4.1), respectively. Then due to Observation 4.2 it is enough to show that the Lusin-Menchoff property does not hold. We take a jump closed set $\mathcal{F} = \{a_n\}_{n=1}^{\infty}$ and a Euclidean closed set $F = \{0\}$. Any Euclidean open cover of $\mathcal{F}$ meets any jump cover of F. The Lusin-Menchoff property does not hold. □

5. The countable compact topology

We see that for a compact fine topology both topologies coincide. Hence we weaken the compactness to the following notion. We say that a topological space is *countable compact* if from any countable open cover we can select a finite subcover. We can easily prove

Proposition 5.1. *Let a fine topology be countable compact and have the F_σ-"semiseparation" property with respect to a normal original topology. Then the fine topology has the Lusin-Menchoff property.*

Proof. Let F be a closed set disjoint with a finely closed $\mathcal{F}$. Due to the F_σ-*"semiseparation"* property we find $\{F_n\}$ such that $\mathcal{F} \subset \bigcup F_n$, F_n disjoint with F. Due to normality of the original topology, for any couple F, F_n we find a disjoint couple of open sets G_n and H_n such that $F_n \subset G_n$ and $F \subset H_n$. Due to the countable compactness of the fine topology we find m such that $\mathcal{F} \subset G = \bigcup_{n=1}^{m} F_n$. The set $\mathcal{G} = \bigcap_{n=1}^{m} H_n$ is an open cover of F, the set G is an open cover of $\mathcal{F}$. The sets G and $\mathcal{G}$ show that the Lusin-Menchoff property holds. □

Remark 5.2. Other material on this subject can be found in [1], [2], [3], [4], [5], [6].

References

[1] *Laczkovich, M.*: Separation properties of some subclasses of Baire 1 functions. Acta Math. Acad. Sci. Hungar. *26* (1975), 405–421.

[2] *Lukeš, J., Malý, J., Zajíček, L.*: Fine Topology Methods in Real Analysis and Potential Theory. Lecture Notes in Mathematics 1189, Springer-Verlag, Berlin, 1986.

[3] *Lukeš, J., Zajíček, L.*: The insertion of G_δ sets and fine topologies. Comment. Math. Univ. Carolin. *18* (1977), 101–104.

[4] *Malý, J.*: A note on separation of sets by approximately continuous functions. Comment. Math. Univ. Carolin. *20* (1979), 579–588.

[5] *Pyrih, P.*: Separation of finely closed sets by finely open sets. Real Anal. Exchange *21* (1995/96), no. 1, 345–348.

[6] *Tall, F. D.*: Normal subspaces of the density topology. Pacific J. Math. *75* (1978), 579–588.

Author's address: *Pavel Pyrih*, Department of Mathematical Analysis, Charles University, Sokolovská 83, 186 00 Prague 8, Czech Republic, e-mail: pyrih@karlin.mff.cuni.cz.