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ON A TWO-POINT BOUNDARY VALUE PROBLEM FOR SECOND ORDER FUNCTIONAL DIFFERENTIAL EQUATIONS, II

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In the present note, we continue the investigation of the question of solvability of the boundary value problem

$$u''(t) = F(u)(t), \tag{1}$$

$$u(a) = 0, \quad u(b) = 0, \tag{2}$$

which was begun in [1]. Notation introduced there remains valid. Moreover, we assume that:

$L_2([a,b])$ is the space of quadratically summable functions $p:]a,b[\to R$ with the norm $\|p\|_{L_2} = \int_a^b p^2(s)\,ds$;

$\sigma_{ab} : L([a,b]) \to L([a,b])$ is an operator defined by

$$\sigma_{ab}(p)(t) = \frac{1}{\sigma(p)(t)} \int_a^t \sigma(p)(s)\,ds \cdot \int_t^b \sigma(p)(s)\,ds.$$

Theorem 1. *Let on the set $C_0'([a,b])$ the inequalities*

$$\int_a^b \frac{[v(s)(F(v)(s) - g(s)v'(s)]_-}{\sigma(g)(s)}\,ds \le \alpha_0\left(\left\|\frac{v'}{\sqrt{\sigma(g)}}\right\|_{L_2}^2\right) + \\ + \alpha_1\left(\|v\|_{C'}\right) \cdot \beta_1\left(\left\|\frac{v'}{\sqrt{\sigma(g)}}\right\|_{L_2}^{2\gamma}\right),$$

$$\left\|\Big[(F(v) - gv')\operatorname{sgn} v\Big]_-\right\|_L \le \alpha_2\left(\|v\|_{C'}\right) + \beta_2\left(\|v'\|_{L_2}^{2\delta}\right)$$

be fulfilled, where $g \in L([a,b])$, $\gamma + \delta = 1$, and continuous, nondecreasing functions $\alpha_i, \beta_j : R_+ \to R_+$, $i = \overline{0,2}$, $j = 1,2$, satisfy the conditions

$$\limsup_{x\to+\infty} \frac{1}{x}\,\alpha_0(x) < 1,$$

$$\lim_{x\to+\infty} \frac{1}{x}\,\alpha_i(x) = 0, \quad \beta_i(x) = O(x), \quad i = 1,2.$$

Then the problem (1), (2) *has at least one solution.*

Corollary 1. *Let on the set $C_0'([a,b])$ the inequality*

$$\Big[F(v)(t) - p(t)v(t) - g(t)v'(t) - l(v)(t)\Big]\operatorname{sgn} v(t) \ge -q\Big(t, \|v\|_{C'}\Big) \tag{3}$$

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be fulfilled, where $l \in \mathcal{L}([a,b])$*,* $p,\ g \in L([a,b])$*,* $q \in K_1([a,b] \times R, R_+)$ *is nondecreasing in the second argument, and*

$$\lim_{x \to +\infty} \frac{1}{x} \int_a^b q(s,x)\, ds = 0. \tag{4}$$

Moreover, let

$$\frac{1}{\int_a^b \sigma(g)(s)\, ds} \cdot \int_a^b \sigma_{ab}(g)(s)[p(s)]_-\, ds + \frac{1}{2}\, \|\widehat{l}\| < 1,$$

where $\widehat{l} : L([a,b]) \to L([a,b])$ *is the operator defined by the equality*

$$\widehat{l}(v)(t) = \frac{1}{\sqrt{\sigma(g)(t)}} \sqrt{\sigma_{ab}(g)(t)}\, l(v)(t).$$

Then the problem (1), (2) *has at least one solution.*

Mention two corollaries of Theorem 1 for the equation

$$u''(t) = p(t)u(t) + g(t)u'(t) + h(t)u(\tau(t)) + G(u)(t), \tag{5}$$

where $p,\ g,\ h \in L([a,b])$, $\tau \in M([a,b],[a,b])$ and $G \in K([a,b])$.

Corollary 2. *Let on the set* $C_0'([a,b])$ *the inequality*

$$G(v)(t) \operatorname{sgn} v(t) \geq -q\big(t, \|v\|_{C'}\big) \tag{6}$$

be fulfilled, where $q \in K_1([a,b] \times R, R_+)$ *is nondecreasing in the second argument and satisfies* (4)*. Let, moreover,*

$$\frac{1}{\int_a^b \sigma(g)(s)\, ds} \cdot \int_a^b \sigma_{ab}(g)(s)[p(s) + h(s)]_-\, ds + \\ + \frac{1}{\sqrt{\int_a^b \sigma(g)(s)\, ds}} \cdot \int_a^b \sqrt{\sigma_{ab}(g)(s)} \sqrt{\frac{|\int_s^{\tau(s)} \sigma(g)(\xi)\, d\xi|}{\sigma(g)(s)}}\, |h(s)|\, ds < 1. \tag{7}$$

Then the problem (5), (2) *has at least one solution.*

Remark 1. Note that, unlike Corollaries 1–3 in [1], the restriction imposed on the sign of the function h is not required here.

In the case where

$$h(t) \geq 0 \quad \text{for} \quad a < t < b, \tag{8}$$

the condition (7) can be somewhat improved. More exactly, we have

Corollary 3. *Let on the set* $C_0'([a,b])$ *the inequality* (6) *be fulfilled, where* $q \in K_1([a,b] \times R, R_+)$ *is nondecreasing in the second argument and satisfies* (4)*. Let, moreover, the inequality* (8) *holds, and*

$$\frac{1}{\int_a^b \sigma(g)(s)\, ds} \cdot \int_a^b \sigma_{ab}(g)(s)[p(s)]_-\, ds + \frac{1}{4} \int_a^b \frac{h(s)}{\sigma(g)(s)} \left| \int_s^{\tau(s)} \sigma(g)(\xi)\, d\xi \right| ds < 1.$$

Then the problem (5), (2) *has at least one solution.*

Theorem 2. *Let on the set* $C_0'([a,b])$ *the inequality* (3) *be fulfilled, where* $l \in \mathcal{L}([a,b])$ *is a positive operator,* $p,\ g \in L([a,b])$, *and* $q \in K([a,b] \times R, R_+)$ *is nondecreasing in the second argument and satisfies* (4). *Let, moreover,*

$$\int_a^b \sigma_{ab}(g)(s)[p(s)]_- \, ds < \int_a^b \sigma(g)(s)\, ds$$

and

$$\int_a^b \frac{l(1)(s)}{\sigma(g)(s)}\, ds < \frac{16}{\int_a^b \sigma(g)(s)\, ds} \left(1 - \frac{1}{\int_a^b \sigma(g)(s)\, ds} \cdot \int_a^b \sigma_{ab}(g)(s)[p(s)]_- \, ds\right).$$

Then the problem (1), (2) *has at least one solution.*

References

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