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WEDGE-TYPE BOUNDARY-CONTACT DYNAMIC PROBLEMS OF ELASTICITY

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Let D_1 and D_2 be finite domains in the three-dimensional Euclidean space $\mathbb{R}^3$ with compact boundaries ∂D_1, ∂D_2 ($\partial D_1 \in C^\infty$), and let there exist a surface $\overline{S}_0$ of the class C^∞ of dimension 2, which divides the domain D_2 into two subdomains $D_2^{(1)}$ and $D_2^{(2)}$ with the C^∞ boundaries $\partial D_2^{(1)}$ and $\partial D_2^{(2)}$ ($D_2^{(1)} \cap D_2^{(2)} = \varnothing$, $\overline{D}_2^{(1)} \cap \overline{D}_2^{(2)} = \overline{S}_0$). Then ∂S_0 is the boundary of the surface S_0 ($\partial S_0 \subset \partial D_2$), representing 1-dimensional closed cuspidal edge, where ∂S_0 is the crack edge.

Let the domains D_1 and D_2 have the contact on the 2-dimensional manifolds $\overline{S}_0^{(1)}$ and $\overline{S}_0^{(2)}$ of the class C^∞, i.e. $\partial D_1 \cap \partial D_2 = \overline{S}_0^{(1)} \cup \overline{S}_0^{(2)}$, $D_1 \cap D_2 = \varnothing$, $\overline{S}_0^{(1)} \cap \overline{S}_0^{(2)} = \varnothing$ and $S_1 = \partial D_1 \setminus (\overline{S}_0^{(1)} \cup \overline{S}_0^{(2)})$. Then $\partial D_2^{(1)} = S_2^{(1)} \cup \overline{S}_0^{(1)} \cup \overline{S}_0^{(2)}$, $\partial D_2^{(2)} = S_2^{(2)} \cup \overline{S}_0^{(2)} \cup \overline{S}_0$.

Suppose that the domains D_q, $q = 1, 2$, are filled with different anisotropic homogeneous elastic materials.

The basic dynamic equations of elasticity for anisotropic homogeneous elastic media are written as

$$A^{(q)}(\partial_x)u^{(q)}(x,t) - \frac{\partial^2 u^{(q)}(x,t)}{\partial t^2} = F^{(q)}(x,t), \ \ (x,t) \in D_q \times [0, +\infty), \ \ q = 1, 2,$$

where $u^{(q)} = (u_1^{(q)}, u_2^{(q)}, u_3^{(q)})$ is the displacement vector, $F^{(q)} = (F_1^{(q)}, F_2^{(q)}, F_3^{(q)})$ is the mass force to D_q, and $A^{(q)}(\partial_x)$ is the matrix differential operator

$$A^{(q)}(\partial_x) = \|A_{jk}^{(q)}(\partial_x)\|_{3\times 3}, \ \ A_{jk}^{(q)}(\partial_x) = a_{ijlk}^{(q)}\partial_i\partial_l, \ \ \partial_i = \frac{\partial}{\partial x_i}, \ \ q = 1, 2;$$

$a_{ijlk}^{(q)}$ are elastic constants satisfying the conditions

$$a_{ijlk}^{(q)} = a_{lkij}^{(q)} = a_{ijkl}^{(q)}.$$

Under repeated indices we understand the summation from 1 to 3.

It is assumed that the quadratic forms

$$a_{ijlk}^{(q)}\xi_{ij}\xi_{lk}, \ \ \xi_{ij} = \xi_{ji}, \ \ q = 1, 2,$$

with respect to the variables ξ_{ij} are positive definite.

We introduce the differential stress operator

$$T^{(q)} = T^{(q)}(\partial_y, n(y)) = \big\|T_{jk}^{(q)}(\partial_y, n(y))\big\|_{3\times 3}, \ \ T_{jk}^{(q)}(\partial_y, n(y)) = a_{ijlk}^{(q)} n_i(y)\partial_l, \ \ q = 1, 2,$$

$n(y) = (n_1(y), n_2(y), n_3(y))$ is the unit normal of the manifold ∂D_1 at a point $y \in \partial D_1$ (external with respect to D_1) and a point $y \in \partial D_2$ (internal with respect to D_2).

The operators $A^{(q)}(\partial_x)$, $q = 1, 2$, are strongly elliptic.

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Let $\mathbb{B}$ be a Banach space, $C_a^m([0,+\infty),\mathbb{B})$ denote the set of all m-times continuously differentiable $\mathbb{B}$-valued functions on $[0,+\infty)$ satisfying the conditions

$$\frac{\partial^l u(t)}{\partial t^l}=0,\;\; l=0,\dots,m,\quad \Big\|\frac{\partial^l u(t)}{\partial t^l}\Big\|_{\mathbb{B}}=O(e^{\alpha t})\;\;\forall\,\alpha>a>0,\;\; l=0,\dots,m.$$

Define $C_{0,a}^m([0,+\infty),\mathbb{B})$ as the set of all m-times continuously differentiable $\mathbb{B}$-valued functions on $[0,+\infty)$ satisfying the conditions

$$\frac{\partial^l u(t)}{\partial t^l}=0,\;\; l=0,\dots,m-2,\quad \Big\|\frac{\partial^l u(t)}{\partial t^l}\Big\|_{\mathbb{B}}=O(e^{at}),\;\; l=0,\dots,m.$$

(For the definition of these spaces, see [1].)

We have studied the solvability and asymptotics of solutions of the following wedge-type boundary-contact dynamic problems in the spaces $\overset{\circ}{C}{}_a^m([0,+\infty),W_p^1(D_q))$, $q=1,2$.

The boundary-contact dynamic problem with the Neumann boundary conditions:

$$\begin{cases}
A^{(q)}(\partial_x)u^{(q)}(x,t)-\dfrac{\partial^2 u^{(q)}(x,t)}{\partial t^2}=F^{(q)}(x,t), & (x,t)\in D_q\times[0,+\infty),\;\; q=1,2,\\
\big\{T^{(1)}u^{(1)}(y,t)\big\}^+=\varphi_1(y,t), & (y,t)\in S_1\times[0,+\infty),\\
\big\{T^{(2)}u^{(2)}(y,t)\big\}^+=\varphi_2(y,t), & (y,t)\in S_2^{(1)}\times[0,+\infty),\\
\big\{T^{(2)}u^{(2)}(y,t)\big\}^+=\varphi_3(y,t), & (y,t)\in S_2^{(2)}\times[0,+\infty),\\
\big\{u^{(1)}(y,t)\big\}^+-\big\{u^{(2)}(y,t)\big\}^+=f_i(y,t), & (y,t)\in S_0^{(i)}\times[0,+\infty),\;\; i=1,2,\\
\big\{T^{(1)}u^{(1)}(y,t)\big\}^+-\big\{T^{(2)}u^{(2)}(y,t)\big\}^+=h_i(y,t), & (y,t)\in S_0^{(i)}\times[0,+\infty),\;\; i=1,2,\\
u^{(q)}(x,0)=\dfrac{\partial u^{(q)}(x,0)}{\partial t}=0, & x\in D_q,\;\; q=1,2,
\end{cases}$$

where the symbol $\{\ \}^+$ denotes the trace on ∂D_q, $q=1,2$,

$$F^{(q)}\in C_{0,a}^M([0,+\infty),L_{\max\{p,2\}}(D_q)),\;\; q=1,2,$$
$$\varphi_1\in C_{0,a}^{M+2}([0,+\infty),B_{p,p}^{-1/p}(S_1)),\quad \varphi_2\in C_{0,a}^{M+2}([0,+\infty),B_{p,p}^{-1/p}(S_2^{(1)})),$$
$$\varphi_3\in C_{0,a}^{M+2}([0,+\infty),B_{p,p}^{-1/p}(S_2^{(2)})),\quad f_i\in C_{0,a}^{M+2}([0,+\infty),B_{p,p}^{1/p'}(S_0^{(i)})),\;\; i=1,2,$$
$$h_i\in C_{0,a}^{M+2}([0,+\infty),B_{p,p}^{-1/p}(S_0^{(i)})),\;\; i=1,2,\;\; p'=\frac{p}{p-1}\,,\;\; 1<p<\infty,\;\; M>m+4;$$

here $B_{p,p}^{-1/p}$ and $B_{p,p}^{1/p'}$ are the Besov spaces.

The boundary-contact dynamic problem with mixed boundary conditions:

$$\begin{cases}
A^{(q)}(\partial_x)u^{(q)}(x,t)-\dfrac{\partial^2 u^{(q)}(x,t)}{\partial t^2}=F^{(q)}(x,t), & (x,t)\in D_q\times[0,+\infty),\;\; q=1,2,\\
\big\{u^{(1)}(y,t)\big\}^+=\varphi_1(y,t), & (y,t)\in S_1\times[0,+\infty),\\
\big\{T^{(2)}u^{(2)}(y,t)\big\}^+=\varphi_2(y,t), & (y,t)\in S_2^{(1)}\times[0,+\infty),\\
\big\{T^{(2)}u^{(2)}(y,t)\big\}^+=\varphi_3(y,t), & (y,t)\in S_2^{(2)}\times[0,+\infty),\\
\big\{u^{(1)}(y,t)\big\}^+-\big\{u^{(2)}(y,t)\big\}^+=f_i(y,t), & (y,t)\in S_0^{(i)}\times[0,+\infty),\;\; i=1,2,\\
\big\{T^{(1)}u^{(1)}(y,t)\big\}^+-\big\{T^{(2)}u^{(2)}(y,t)\big\}^+=h_i(y,t), & (y,t)\in S_0^{(i)}\times[0,+\infty),\;\; i=1,2,\\
u^{(q)}(x,0)=\dfrac{\partial u^{(q)}(x,0)}{\partial t}=0, & x\in D_q,\;\; q=1,2,
\end{cases}$$

where

$$F^{(q)}\in C_{0,a}^M([0,+\infty),L_{\max\{p,2\}}(D_q)),\;\; q=1,2,$$
$$\varphi_1\in C_{0,a}^{M+2}([0,+\infty),B_{p,p}^{1/p'}(S_1)),\quad \varphi_2\in C_{0,a}^{M+2}([0,+\infty),B_{p,p}^{-1/p}(S_2^{(1)})),$$
$$\varphi_3\in C_{0,a}^{M+2}([0,+\infty),B_{p,p}^{-1/p}(S_2^{(2)})),\quad f_i\in C_{0,a}^{M+2}([0,+\infty),B_{p,p}^{1/p'}(S_0^{(i)})),\;\; i=1,2,$$
$$h_i\in C_{0,a}^{M+2}([0,+\infty),B_{p,p}^{-1/p}(S_0^{(i)})),\;\; i=1,2,\;\; p'=\frac{p}{p-1}\,,\;\; 1<p<\infty,\;\; M>m+4.$$

In the formulation of dynamic problems it is assumed that the crack and contact surfaces do not depend on the time parameter t.

Theorems on the existence and uniqueness of solutions of the considered boundary-contact dynamic problems are obtained by using the Laplace transformation, the potential theory and the general theory of pseudo-differential equations on a manifold with boundary.

The following theorems hold.

Theorem 1. *Let* $4/3 < p < 4$, $a > 0$, $m \geq 2$,

$$F^{(q)} \in C_{0,a}^{m+5}([0,+\infty), L_{\max\{p,2\}}(D_q)), \quad q = 1, 2,$$

$$\varphi_1 \in C_{0,a}^{m+7}([0,+\infty), B_{p,p}^{-1/p}(S_1)), \quad \varphi_2 \in C_{0,a}^{m+7}([0,+\infty), B_{p,p}^{-1/p}(S_2^{(1)})),$$

$$\varphi_3 \in C_{0,a}^{m+7}([0,+\infty), B_{p,p}^{-1/p}(S_2^{(2)})), \quad f_i \in C_{0,a}^{m+7}([0,+\infty), B_{p,p}^{1/p'}(S_0^{(i)})), \quad i = 1, 2,$$

$$h_i \in C_{0,a}^{m+7}([0,+\infty), B_{p,p}^{-1/p}(S_0^{(i)})), \quad i = 1, 2.$$

Then the boundary-contact dynamic problem with Neumann boundary conditions has a unique solution in the spaces $\overset{\circ}{C}{}_a^m([0,+\infty), W_p^1(D_q))$, $q = 1, 2$.

Theorem 2. *Let* $4/3 < \alpha < p < \beta < 4$, $a > 0$, $m \geq 2$,

$$F^{(q)} \in C_{0,a}^{m+5}([0,+\infty), L_{\max\{p,2\}}(D_q)), \quad q = 1, 2,$$

$$\varphi_1 \in C_{0,a}^{m+7}([0,+\infty), B_{p,p}^{1/p'}(S_1)), \quad \varphi_2 \in C_{0,a}^{m+7}([0,+\infty), B_{p,p}^{-1/p}(S_2^{(1)})),$$

$$\varphi_3 \in C_{0,a}^{m+7}([0,+\infty), B_{p,p}^{-1/p}(S_2^{(2)})), \quad f_i \in C_{0,a}^{m+7}([0,+\infty), B_{p,p}^{1/p'}(S_0^{(i)})), \quad i = 1, 2,$$

$$h_i \in C_{0,a}^{m+7}([0,+\infty), B_{p,p}^{-1/p}(S_0^{(i)})), \quad i = 1, 2.$$

Then the boundary-contact dynamic problem with mixed boundary conditions has a unique solution in the spaces $\overset{\circ}{C}{}_a^m([0,+\infty), W_p^1(D_q))$, $q = 1, 2$.

Note that α and β depend on the elastic constants as well as on the geometry of the contact boundaries $\partial S_0^{(1)}$, $\partial S_0^{(2)}$.

For sufficiently smooth data of these problems by using the asymptotic expansion of solutions of strongly elliptic pseudo-differential equations obtained in [2] and also that of potential-type functions (see [3]), we obtain a complete asymptotics of solutions near the contact boundaries and near the cuspidal edge (crack edge).

In the asymptotic expansion of solutions of these dynamic problems the time parameter t appears only in asymptotic coefficients. Therefore so formulated dynamic problems have mechanical meaning because the time parameter t appears in particular in the first coefficient. The fulfilment of the fracture criterion depends on the first coefficient. In this case, the so-called Griffits criterion can be formulated as a problem of finding a moment of time after which fracture begins.

The singularity of solutions of the boundary-contact problem with Neumann boundary conditions is $1/2$. The necessary and sufficient conditions for vanishing oscillation of solutions are found near the contact boundaries. In these asymptotic expansions the step is one.

The singularity of solutions of the boundary-contact problem with mixed boundary conditions near the cuspidal edge (crack edge) is $1/2$. The singularity of solutions near the contact boundaries has following properties:

1) The singularity γ of solutions depends on the elastic constants, and also on the geometry of the contact boundaries, and can take any values from the interval $(0, 1/2)$; the classes of isotropic (with elastic constants μ_q, λ_q, $q = 1, 2$) and transversally-isotropic (with elastic constants $c_{11}^{(q)}$, $c_{33}^{(q)}$, $c_{13}^{(q)}$, $c_{55}^{(q)}$, $c_{66}^{(q)}$ in the conditions $c_{11}^{(q)} = c_{33}^{(q)}$, $q = 1, 2$) bodies are found when oscillation of solutions vanishes near the contact boundaries. In

such cases, the effective formulas are obtained for calculation of singularities of solutions near the contact boundaries $\partial S_0^{(i)}$, $i = 1, 2$:

$$\gamma = \frac{1}{2} - \frac{1}{\pi} \operatorname{arctg} \sqrt{\frac{\mu_2}{\mu_1}} \quad \text{(the isotropic case)}$$

and

$$\gamma = \frac{1}{2} - \frac{1}{\pi} \operatorname{arctg} \sqrt[4]{\frac{c_{55}^{(2)} c_{66}^{(2)}}{c_{55}^{(1)} c_{66}^{(1)}}} \quad \text{(the transversally-isotropic case).}$$

Note that the first three terms have no logarithms. It should also be noted that these classes are found only for spatial problems, since oscillation of solutions does not vanish in plane problems.

In the transversally-isotropic case we assume that the neighborhood of the contact boundaries is parallel to the isotropic plane.

2) In the general case (in particular, in the transversally-isotropic case, where $c_{11}^{(q)} \neq c_{33}^{(q)}$, $q = 1, 2$) we have found a class of anisotropic bodies when the oscillation in the asymptotic expansion vanishes and singularities of solutions are calculated by a simple formula near $\partial S_0^{(i)}$, $i = 1, 2$,

$$\gamma_i = \frac{1}{2} - \sup_{\substack{\partial S_0^{(i)} \\ 1 \le j \le 3}} \frac{1}{\pi} \operatorname{arctg} \frac{1}{\sqrt{\alpha_j \beta_j}}, \quad i = 1, 2,$$

where $\alpha_j > 0$, $\beta_j > 0$, $j = 1, 2, 3$, are the eigenvalues of the principal homogeneous symbol of the Poincaré–Steklov operators.

3) If the domains are filled with the same material, the singularities of the first and second terms are $1/4$ and $3/4$, respectively; these terms are free from logarithms, and the oscillation does not vanish. In the asymptotic expansion the step is one-half.

Asymptotic properties of the same kind for the static problems of couple-stress elasticity were obtained in [4].

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