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# MATHEMATICAL PROBLEMS OF GENERALIZED THERMO-ELECTRO-MAGNETO-ELASTICITY THEORY

**Abstract.** The monograph is dedicated to the theoretical investigation of basic, mixed, and crack type three-dimensional initial-boundary value problems of the generalized thermo-electro-magneto-elasticity theory associated with Green–Lindsay's model. The essential feature of the generalized model under consideration is that heat propagation has a finite speed. We investigate the uniqueness of solutions to the dynamical initial-boundary value problems and analyse the corresponding boundary value problems of pseudo-oscillations which are obtained form the dynamical problems by the Laplace transform. The solvability of the boundary value problems under consideration are analyzed by the potential method in appropriate Sobolev–Slobodetskii ($W_p^s$), Bessel potential ($H_p^s$), and Besov ($B_{p,q}^s$) spaces. The smoothness properties and singularities of thermo-mechanical and electro-magnetic fields are investigated near the crack edges and the curves where the different types of boundary conditions collide.

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**Key words and phrases.** Generalized thermo-electro-magneto-elasticity, piezoelectricity, Green–Lindsay's model, initial and boundary value problems, mixed problems, crack problems, fundamental solution, transmission problems, potential method, pseudodifferential equations, asymptotic properties of solutions, stress singularity exponents.

**რეზიუმე.** მონოგრაფია ეძღვნება გრინ–ლინდსეის მოდელთან ასოცირებული განზოგადებული თერმო-ელექტრო-მაგნიტო-დრეკადობის ძირითადი, შერეული და ბზარის ტიპის სამგანზომილებიანი საწყის-სასაზღვრო ამოცანების გამოკვლევას. განხილული განზოგადებული მოდელის არსებით თავისებურებას წარმოადგენს სითბოს გავრცელების სასრული სიჩქარე. ნაშრომში შესწავლილია დინამიკის საწყის-სასაზღვრო ამოცანების ერთადერთობა და ჩატარებულია დინამიკის ამოცანებიდან ლაპლასის გარდაქმნით მიღებული ფსევდო-რხევის შესაბამისი სასაზღვრო ამოცანების ანალიზი. განხილული სასაზღვრო ამოცანების ამოხსნადობა შესწავლილია პოტენციალთა მეთოდით შესაბამის სობოლევ–სლობოდეცკის ($W_p^s$), ბესელის პოტენციალთა ($H_p^s$) და ბესოვის ($B_{p,q}^s$) სივრცეებში. გამოკვლეულია თერმო-მექანიკური და ელექტრომაგნიტური ველების სიგლუვის თვისებები და სინგულარობები ბზარის კიდეებისა და იმ წირების მახლობლობაში, რომელთა სხვადასხვა მხარეს დასმულია განსხვავებული ტიპის სასაზღვრო პირობები.

## 1. Introduction

Modern industrial and technological processes apply widely, on the one hand, composite materials with complex microstructure and, on the other hand, complex composed structures consisting of materials having essentially different physical properties (for example, piezoelectric, piezomagnetic, hemitropic materials, two- and multi-component mixtures, nano-materials, bio-materials, and solid structures constructed by composition of these materials, such as, e.g., Smart Materials and other meta-materials). Therefore the investigation and analysis of mathematical models describing the mechanical, thermal, electric, magnetic and other physical properties of such materials have a crucial importance for both fundamental research and practical applications. In particular, the investigation of correctness of corresponding mathematical models (namely, existence, uniqueness, smoothness, asymptotic properties and stability of solutions) and construction of appropriate adequate and efficient numerical algorithms have a crucial role for fundamental research.

In the study of active material systems, there is significant interest in the coupling effects between elastic, electric, magnetic and thermal fields. For example, piezoelectric materials (electro-elastic coupling) have been used as ultrasonic transducers and micro-actuators; pyroelectric materials (thermal-electric coupling) have been applied in thermal imaging devices; and piezomagnetic materials (elastic-magnetic coupling) are pursued for health monitoring of civil structures (see [81], [98], [33], [96], [106], [107], [111], [59]–[67], [18], [97], [34], and the references therein).

Although natural materials rarely show full coupling between elastic, electric, magnetic, and thermal fields, some artificial materials do. In the reference [110] it is reported that the fabrication of $BaTiO_3$-$CoFe_2O_4$ composite had the magnetoelectric effect not existing in either constituent. Other examples of similar complex coupling can be found in the references [9], [8], [46], [47], [6], [69], [86], [7], [70], [71], [115], [48], [102], [87].

Here we consider the generalized thermo-electro-magneto-elasticity (GTEME) theory associated with Green–Lindsay's model. The essential feature of the generalized model under consideration is that heat propagation has a finite speed (see [103], [5]).

Thermoelasticity theories predicting a finite speed for the propagation of thermal signals have come into existence during the past sixty years. In contrast to the conventional thermoelasticity theory, these nonclassical theories involve a hyperbolic-type heat transport equation, and are motivated by experiments exhibiting the actual occurrence of wave-type heat transport (*second sound*). Several authors have formulated these theories on different grounds, and a wide variety of problems revealing characteristic features of the theories has been investigated. A detailed historical notes and extensive reviews on the literature on thermoelasticity with temperature waves of finite speed can be found in the references [85], [19], [20], [49], [55], [52], and [103].

The mathematical dynamical model of the generalized thermo-electro-magneto-elasticity theory is described by a system of second order partial differential equations with the appropriate boundary and initial conditions. The problem is to determine three components of the elastic displacement vector, the electric and magnetic scalar potential functions and the temperature distribution. Other field characteristics (e.g., mechanical stresses, electric and magnetic fields, electric displacement vector, magnetic induction vector, heat flux vector and entropy density) can be then determined by the gradient equations and the constitutive equations.

The main feature of the dynamical mathematical model under consideration is that in the corresponding second order system of partial differential equations the mechanical, thermal and electromagnetic fields are fully coupled, and the corresponding matrix differential operator generated by the dynamical equations does not belong to the well known standard classes of differential operators, such as elliptic, hyperbolic or parabolic. The case is that the second order $6 \times 6$ matrix differential operator, generated by the second order partial derivatives with respect to the spatial variables, represents a strongly elliptic part of the system, while the system contains the first and second order partial derivatives with respect to the time variable of the components of the displacement vector and the temperature function, and only the first order derivatives of the electric and magnetic potential functions with respect to the time variable. Therefore the basic matrix differential operator of dynamics possesses elliptic-hyperbolic-parabolic properties.

Moreover, additional difficulties arise in the setting of initial conditions which is caused by a special structure of the differential equations of dynamics. It turned out that the electric and magnetic potential functions should be free from initial conditions at the initial time, since they can be uniquely defined by the initial conditions associated with the components of the displacement vector and the temperature function. We willdescribe and analyze all these peculiarities in detail.

For these equations the uniqueness of solutions of some mixed initial-boundary value problems of dynamics are considered in the mathematical scientific literature. In particular, the uniqueness theorem for linear homogeneous dynamical problems with special type initial data, consisting of nine homogeneous initial conditions, is proved without making restrictions on the positive definiteness on the elastic moduli in the references [69], [4], [5]. However, as we will show in Chapter 2 of this monograph, only eight nonhomogeneous initial conditions can be prescribed arbitrarily in the GTEME model and, along with the natural boundary conditions, they form well posed initial-boundary value problems of dynamics.

The main purpose of the present monograph is detailed investigation of the existence, uniqueness, and asymptotic behaviour of solutions to the general mixed initial-boundary value, transmission and crack type problems of the GTEME theory for homogeneous and piecewise homogeneous composite bodies.

We analyze dynamical initial-boundary value problems and the corresponding boundary value problems of pseudo-oscillations obtained from the dynamical problems by the Laplace transform.

As we have mentioned above, the dynamical system of partial differential equations generate a nonstandard $6 \times 6$ matrix differential operator of second order, but the corresponding system of partial differential equations of pseudo-oscillations generates a second order strongly elliptic formally non-selfadjoint $6 \times 6$ matrix differential operator depending on a complex parameter.

First we prove uniqueness theorems of dynamical initial-boundary value problems under reasonable restrictions on material parameters and afterwards we apply the Laplace transform technique to investigate the existence of solutions. This approach reduces the dynamical problems to the corresponding elliptic problems for pseudo-oscillation equations. On the final stage, by the inverse Laplace transform the solutions of the original dynamical problems are constructed with the help of the solutions of the corresponding elliptic problems of pseudo-oscillations.

As it is well known, solutions to mixed and crack type boundary value problems and the corresponding mechanical, electrical, magnetic, and thermal characteristics usually have singularities at the so called *exceptional curves*: the crack edges and the curves where the different types of boundary conditions collide (the so-called *collision curves*). Along with the existence and uniqueness questions our main goal is a detailed theoretical investigation of regularity properties of the thermo-mechanical and electro-magnetic fields near the exceptional curves and qualitative description of their singularities. In particular, the most important question is description of the dependence of the stress singularity exponents on the material parameters.

With the help of the potential method we reduce the three-dimensional basic, mixed and crack type boundary value problems for the pseudo-oscillation equations of the thermo-electro-magneto-elasticity theory to the equivalent systems of pseudodifferential equations which live on proper parts of the boundary of the elastic body under consideration. We analyze the solvability of the resulting boundary pseudodifferential equations in the Sobolev–Slobodetskii ($W_p^s$), Bessel potential ($H_p^s$), and Besov ($B_{p,q}^s$) spaces. We show that the principal homogeneous symbol matrices of the corresponding pseudodifferential operators yield information on the existence and regularity of the solution fields of the corresponding boundary value problems. We study smoothness and asymptotic properties of solutions near the exceptional curves and establish almost best global $C^\alpha$-regularity results with some $\alpha \in (0, \frac{1}{2})$. The exponent $\alpha$ is determined with the help of the eigenvalues of special matrices which are explicitly constructed by means of the principal homogeneous symbol matrices of the corresponding pseudodifferential operators. These eigenvalues depend on the material parameters and the geometry of exceptional curves, in general, and actually they define the singularity exponents for the first order derivatives of solutions. On the basis of the asymptotic analysis, we give an efficient method for calculation of the stress singularity exponents.

Along with the dynamical problems we investigate the boundary value problems of statics of thermo-electro-magneto-elasticity theory. In the study of BVPs of statics there arise essential difficulties related to the behaviour of solutions at infinity. The case is that solutions to the exterior static BVPs for unbounded domains do not vanish at infinity, they are bounded, in general, and the function spaces for problems to be uniquely solvable should be chosen appropriately. We establish efficient structural restrictions which guarantee uniqueness of solutions to exterior BVPs of statics.

It should be mentioned that the GTEME model considered in the monograph is a rather general mathematical model of deformable solids and as particular cases it contains models of *classical elasticity, classical thermo-elasticity, thermo-electro-elasticity*, and *thermo-electro-magneto elasticity without taking into account the second sound effects*. All these models can be obtained from the GTEME model by appropriate choice of the material parameters. It should be specially mentioned that all the results obtained in the monograph for the GTEME theory remain valid for the problems of the above listed models.

The monograph is organized as follows.

In the second section, we collect the basic field equations, introduce matrix differential operators associated with the dynamical and pseudo-oscillation equations of the thermo-electro-magneto-elasticity theory, derive the corresponding Green's formulas, formulate the initial-boundary and boundary value problems for dynamical and pseudo-oscillation equations in the sense of classical and variational settings in the appropriate regular and generalized function spaces, and prove the corresponding uniqueness theorems for the problems of dynamics, pseudo-oscillations, and statics in the case of Lipschitz domains.

In the third section, the fundamental matrices for the operators of pseudo-oscillations and statics are constructed explicitly by means of the generalized Fourier transform technique, their properties near the origin and at infinity are established, the corresponding single and double layer potentials and the Newtonian volume potentials are introduced, and the general integral representation formulas of solutions are derived in the case of bounded and unbounded domains.

The fourth section is devoted to the investigation mapping and coercivity properties of the single and double layer potentials, the boundary operators generated by them, and the generalized Steklov–Poincaré type operators. Mapping properties are established in Hölder ($C^{k,\alpha}$), Sobolev–Slobodetskii ($W^s_p$), Bessel potential ($H^s_p$), and Besov ($B^s_{p,q}$) function spaces.

In the fifth section, we study existence of regular and weak solutions to the Dirichlet, Neumann, and mixed type boundary value problems of pseudo-oscillations for smooth and Lipschitz domains. We establish the almost best regularity properties of solutions to the mixed boundary value problems near the curves where the different boundary conditions collide.

In the sixth section, we analyze different type crack problems and investigate the regularity of solutions near the crack edges. An important issue in studying fracture mechanics of piezoelectric materials is the crack-face electric boundary conditions. There are two idealized crack-face boundary conditions that are extensively used in the literatures. One commonly used boundary condition is the specification that the normal components of electric displacement and magnetic induction vectors along the crack faces equal to zero. These boundary conditions ignore the permittivity in the medium interior to the crack. The other commonly used boundary condition treats the crack as being electrically permeable. In this case the appropriate transmission conditions are prescribed on the crack surface. We deal with both type problems and derive the corresponding existence and regularity results which afterwards are applied to establish the asymptotic behaviour of solutions near the crack edges.

The seventh section is devoted to the study of boundary value problems of statics. Here essential difficulties arise in the study of exterior BVPs of statics for unbounded domains. The case is that one has to consider the problem in a class of vector functions which are bounded at infinity. This complicates the proof of uniqueness and existence theorems since Green's formulas do not hold for such vector functions and analysis of null spaces of the corresponding integral operators needs special consideration. We have found efficient and natural asymptotic conditions at infinity which ensure the uniqueness of solutions in the space of bounded vector functions. Moreover, for the interior Neumann-type boundary-value problem, the complete system of linearly independent solutions of the corresponding homogeneous adjoint integral equation is constructed in polynomials and the necessary and sufficient conditions of solvability of the problem are written explicitly.

In the eight section, we investigate the boundary-transmission problems of pseudo-oscillations for piecewise homogeneous elastic bodies containing the interfacial cracks and study the smoothness properties of solutions. We investigate the asymptotic properties of solutions to the mixed transmission problems near the interfacial crack adges and characterize the so called singularity exponents for the thermo-mechanical and electro-magnetic fields.

In the Appendices A, B, C, and D we present some auxiliary results employed in the main text of the monograph. In particular, in Appendix A, we describe the structural properties of bounded solutions in unbounded domains. In Appendix B, we formulate basic results concerning Fredholm properties of pseudodifferential operators defined on manifolds with boundary which plays a crucial role in the study of mixed and crack type boundary value problems. In Appendix C, the expressions for the principal homogeneous symbols of the boundary pseudodifferential operators generated by the single and double layer potentials are written down explicitly. Appendix D, is devoted to calculation of specific integrals.

## 2. Basic Equations and Operators, Statement of Problems, and Uniqueness Theorems

### 2.1. **Field equations.**

In this subsection we collect the field equations of the generalized linear theory of thermo-electro-magneto-elasticity under Green–Lindsay's model for a general anisotropic case (see [103], [5], [43]) and introduce the corresponding matrix partial differential operators.

Throughout the monograph $u = (u_1, u_2, u_3)^\top$ denotes the displacement vector, $\sigma_{ij}$ are the components of the mechanical stress tensor, $\varepsilon_{kj} = 2^{-1}(\partial_k u_j + \partial_j u_k)$ are the compnents of the mechanical strain tensor, $E = (E_1, E_2, E_3)^\top$ and $H = (H_1, H_2, H_3)^\top$ are electric and magnetic fields, respectively, $D = (D_1, D_2, D_3)^\top$ is the electric displacement vector and $B = (B_1, B_2, B_3)^\top$ is the magnetic induction vector, $\varphi$ and $\psi$ stand for the electric and magnetic potentials and

$$E = -\operatorname{grad}\varphi\,, \quad H = -\operatorname{grad}\psi\,, \tag{2.1}$$

$\vartheta$ is the temperature change to a reference temperature $T_0$, $q = (q_1, q_2, q_3)^\top$ is the heat flux vector, and $\mathcal{S}$ is the entropy density.

We employ also the notation $\partial = \partial_x = (\partial_1, \partial_2, \partial_3)$, $\partial_j = \partial/\partial x_j$, $\partial_t = \partial/\partial t$; sometimes we use also the traditional ("overset dot") notation for the time derivatives: $\partial_t u = \dot{u}$ and $\partial_t^2 u := \partial_t\partial_t u = \ddot{u}$; the superscript $(\,\cdot\,)^\top$ denotes transposition operation. In what follows the summation over the repeated indices is meant from 1 to 3, unless stated otherwise. The over bar, applied to numbers and functions, denotes complex conjugation and the central dot denotes the scalar product of two vectors in the complex vector space $\mathbb{C}^N$, i.e., $a \cdot b \equiv (a, b) := \sum\limits_{j=1}^{N} a_j\, \overline{b_j}$ for $a, b \in \mathbb{C}^N$. Over bar, applied to a subset $\mathcal{M}$ of Euclidean space $\mathbb{R}^N$, denotes the closure of $\mathcal{M}$, i.e. $\overline{\mathcal{M}} = \mathcal{M} \cup \partial\mathcal{M}$, where $\partial\mathcal{M}$ is the boundary of $\mathcal{M}$.

The basic linear field equations of the thermo-electro-magneto-elasticity theory under Green–Lindsay's model read as follows (see [103], [5], [43]) and the references therein):

*The constitutive relations:*

$$\sigma_{rj} = \sigma_{jr} = c_{rjkl}\varepsilon_{kl} - e_{lrj}E_l - q_{lrj}H_l - \lambda_{rj}(\vartheta + \nu_0\partial_t\vartheta), \quad r, j = 1, 2, 3, \tag{2.2}$$

$$D_j = e_{jkl}\varepsilon_{kl} + \varkappa_{jl}E_l + a_{jl}H_l + p_j(\vartheta + \nu_0\partial_t\vartheta), \quad j = 1, 2, 3, \tag{2.3}$$

$$B_j = q_{jkl}\varepsilon_{kl} + a_{jl}E_l + \mu_{jl}H_l + m_j(\vartheta + \nu_0\partial_t\vartheta), \quad j = 1, 2, 3, \tag{2.4}$$

$$\varrho\,\mathcal{S} = \lambda_{kl}\varepsilon_{kl} + p_l E_l + m_l H_l + a_0 + d_0\vartheta + h_0\partial_t\vartheta. \tag{2.5}$$

*The equations of motion:*

$$\partial_j\sigma_{rj} + \varrho\,F_r = \varrho\,\partial_t^2 u_r, \quad r = 1, 2, 3. \tag{2.6}$$

*The quasi-static equations for electric and magnetic fields:*

$$\partial_j D_j = \varrho_e\,, \quad \partial_j B_j = \varrho_c\,. \tag{2.7}$$

*The linearized energy equations:*

$$\varrho\,T_0\partial_t\mathcal{S} = -\partial_j q_j + \varrho\,Q, \quad q_j = -T_0\,\eta_{jl}\partial_l\vartheta. \tag{2.8}$$

Here and in what follows we employ the following notation: $\varrho$ – the mass density, $\varrho_e$ – the electric charge density, $\varrho_c$ – the electric current density, $F = (F_1, F_2, F_3)^\top$ – the mass force density, $Q$ – the heat source intensity, $c_{rjkl}$ – the elastic constants, $e_{jkl}$ – the piezoelectric constants, $q_{jkl}$ – the piezomagnetic constants, $\varkappa_{jk}$ – the dielectric (permittivity) constants, $\mu_{jk}$ – the magnetic permeability constants,

$a_{jk}$ – the electromagnetic coupling coefficients, $p_j$, $m_j$, and $\lambda_{rj}$ – coupling coefficients connecting dissimilar fields, $\eta_{jk}$ – the heat conductivity coefficients, $T_0$ – the initial reference temperature, that is the temperature in the natural state in the absence of deformation and electromagnetic fields, $\nu_0$ and $h_0$ – two relaxation times, $a_0$ and $d_0$ – constitutive coefficients.

Note that, if the relaxation time parameters $\nu_0$ and $h_0$ equal to zero, then the generalized model under consideration coincides with the classical thermo-electro-magneto-elasticity model and the temperature distribution loses its wave nature.

The constants involved in the above equations satisfy the symmetry conditions:

$$\begin{gathered} c_{rjkl}=c_{jrkl}=c_{klrj},\ \ e_{klj}=e_{kjl},\ \ q_{klj}=q_{kjl},\\ \varkappa_{kj}=\varkappa_{jk},\ \ \lambda_{kj}=\lambda_{jk},\ \ \mu_{kj}=\mu_{jk},\ \ a_{kj}=a_{jk},\ \ \eta_{kj}=\eta_{jk},\ \ r,j,k,l=1,2,3. \end{gathered} \tag{2.9}$$

Some authors require more extended symmetry conditions for piezoelectric and piezomagnetic constants: $e_{klj}=e_{kjl}=e_{ljk}$, $q_{klj}=q_{kjl}=q_{ljk}$ (see, e.g., [68], [69], [4], [5]). However in our further analysis we will require only the symmetry properties described in (2.9).

From physical considerations it follows that (see, e.g., [95], [68], [5], [103], [43]):

$$\begin{gathered} c_{rjkl}\xi_{rj}\xi_{kl}\geq\delta_0\xi_{kl}\,\xi_{kl},\ \ \varkappa_{kj}\xi_k\xi_j\geq\delta_1|\xi|^2,\ \ \mu_{kj}\xi_k\xi_j\geq\delta_2|\xi|^2,\ \ \eta_{kj}\xi_k\xi_j\geq\delta_3|\xi|^2,\\ \text{for all }\ \xi_{kj}=\xi_{jk}\in\mathbb{R}\ \text{ and for all }\ \xi=(\xi_1,\xi_2,\xi_3)\in\mathbb{R}^3, \end{gathered} \tag{2.10}$$

$$\nu_0>0,\ \ h_0>0,\ \ d_0\nu_0-h_0>0, \tag{2.11}$$

where $\delta_0$, $\delta_1$, $\delta_2$, and $\delta_3$ are positive constants depending on material parameters.

Due to the symmetry conditions (2.9), with the help of (2.10) we easily derive

$$\begin{gathered} c_{rjkl}\zeta_{rj}\overline{\zeta_{kl}}\geq\delta_0\zeta_{kl}\overline{\zeta_{kl}},\ \ \varkappa_{kj}\zeta_k\overline{\zeta_j}\geq\delta_1|\zeta|^2,\ \ \mu_{kj}\zeta_k\overline{\zeta_j}\geq\delta_2|\zeta|^2,\ \ \eta_{kj}\zeta_k\overline{\zeta_j}\geq\delta_3|\ \zeta|^2,\\ \text{for all }\ \zeta_{kj}=\zeta_{jk}\in\mathbb{C}\ \text{ and for all }\ \zeta=(\zeta_1,\zeta_2,\zeta_3)\in\mathbb{C}^3. \end{gathered} \tag{2.12}$$

More careful analysis related to the positive definiteness of the potential energy and the thermodynamical laws insure that the following $8\times 8$ matrix

$$M=[M_{kj}]_{8\times 8}:=\begin{bmatrix} [\varkappa_{jl}]_{3\times 3} & [a_{jl}]_{3\times 3} & [p_j]_{3\times 1} & [\nu_0 p_j]_{3\times 1}\\ [a_{jl}]_{3\times 3} & [\mu_{jl}]_{3\times 3} & [m_j]_{3\times 1} & [\nu_0 m_j]_{3\times 1}\\ [p_j]_{1\times 3} & [m_j]_{1\times 3} & d_0 & h_0\\ [\nu_0 p_j]_{1\times 3} & [\nu_0 m_j]_{1\times 3} & h_0 & \nu_0 h_0 \end{bmatrix}_{8\times 8} \tag{2.13}$$

is positive definite. Note that the positive definiteness of $M$ remains valid if the parameters $p_j$ and $m_j$ in (2.13) are replaced by the opposite ones, $-p_j$ and $-m_j$ . Moreover, it follows that the matrices

$$\Lambda^{(1)}:=\begin{bmatrix} [\varkappa_{kj}]_{3\times 3} & [a_{kj}]_{3\times 3}\\ [a_{kj}]_{3\times 3} & [\mu_{kj}]_{3\times 3} \end{bmatrix}_{6\times 6},\quad \Lambda^{(2)}:=\begin{bmatrix} d_0 & h_0\\ h_0 & \nu_0 h_0 \end{bmatrix}_{2\times 2} \tag{2.14}$$

are positive definite as well, i.e.,

$$\varkappa_{kj}\zeta'_k\overline{\zeta'_j}+a_{kj}(\zeta'_k\overline{\zeta''_j}+\overline{\zeta'_k}\zeta''_j)+\mu_{kj}\zeta''_k\overline{\zeta''_j}\geq\kappa_1\big(|\zeta'|^2+|\zeta''|^2\big)\ \ \forall\,\zeta',\zeta''\in\mathbb{C}^3, \tag{2.15}$$

$$d_0|z_1|^2+h_0\big(z_1\overline{z_2}+\overline{z_1}z_2\big)+\nu_0 h_0|z_2|^2\geq\kappa_2\big(|z_1|^2+|z_2|^2\big)\ \ \forall\,z_1,z_2\in\mathbb{C}, \tag{2.16}$$

with some positive constants $\kappa_1$ and $\kappa_2$ depending on the material parameters involved in (2.14).

With the help of the symmetry conditions (2.10) we can rewrite the constitutive relations (2.2)–(2.5) as follows

$$\sigma_{rj}=c_{rjkl}\partial_l u_k+e_{lrj}\partial_l\varphi+q_{lrj}\partial_l\psi-\lambda_{rj}(\vartheta+\nu_0\partial_t\vartheta),\ \ r,j=1,2,3, \tag{2.17}$$

$$D_j=e_{jkl}\partial_l u_k-\varkappa_{jl}\partial_l\varphi-a_{jl}\partial_l\psi+p_j(\vartheta+\nu_0\partial_t\vartheta),\ \ j=1,2,3, \tag{2.18}$$

$$B_j=q_{jkl}\partial_l u_k-a_{jl}\partial_l\varphi-\mu_{jl}\partial_l\psi+m_j(\vartheta+\nu_0\partial_t\vartheta),\ \ j=1,2,3, \tag{2.19}$$

$$\mathcal{S}=\lambda_{kl}\partial_l u_k-p_l\partial_l\varphi-m_l\partial_l\psi+a_0+d_0\vartheta+h_0\partial_t\vartheta. \tag{2.20}$$

In the theory of generalized thermo-electro-magneto-elasticity the components of the three-dimensional *mechanical stress vector* acting on a surface element with a normal $n=(n_1,n_2,n_3)$ have the form

$$\sigma_{rj}n_j=c_{rjkl}n_j\partial_l u_k+e_{lrj}n_j\partial_l\varphi+q_{lrj}n_j\partial_l\psi-\lambda_{rj}n_j(\vartheta+\nu_0\partial_t\vartheta),\ \ r=1,2,3, \tag{2.21}$$

while the *normal components of the electric displacement vector, magnetic induction vector* and *heat flux vector* read as

$$D_j n_j = e_{jkl} n_j \partial_l u_k - \varkappa_{jl} n_j \partial_l \varphi - a_{jl} n_j \partial_l \psi + p_j n_j (\vartheta + \nu_0 \partial_t \vartheta), \tag{2.22}$$
$$B_j n_j = q_{jkl} n_j \partial_l u_k - a_{jl} n_j \partial_l \varphi - \mu_{jl} n_j \partial_l \psi + m_j n_j (\vartheta + \nu_0 \partial_t \vartheta), \tag{2.23}$$
$$q_j n_j = -T_0 \eta_{jl} n_j \partial_l \vartheta. \tag{2.24}$$

For convenience we introduce the following matrix differential operator

$$\mathcal{T}(\partial_x, n, \partial_t) = \big[\mathcal{T}_{pq}(\partial_x, n, \partial_t)\big]_{6\times 6}$$
$$:= \begin{bmatrix} [c_{rjkl} n_j \partial_l]_{3\times 3} & [e_{lrj} n_j \partial_l]_{3\times 1} & [q_{lrj} n_j \partial_l]_{3\times 1} & [-\lambda_{rj} n_j (1 + \nu_0 \partial_t)]_{3\times 1} \\ [-e_{jkl} n_j \partial_l]_{1\times 3} & \varkappa_{jl} n_j \partial_l & a_{jl} n_j \partial_l & -p_j n_j (1 + \nu_0 \partial_t) \\ [-q_{jkl} n_j \partial_l]_{1\times 3} & a_{jl} n_j \partial_l & \mu_{jl} n_j \partial_l & -m_j n_j (1 + \nu_0 \partial_t) \\ [0]_{1\times 3} & 0 & 0 & \eta_{jl} n_j \partial_l \end{bmatrix}_{6\times 6}. \tag{2.25}$$

Denote by $\mathcal{T}^{(0)}(\partial_x, n)$ the "main part" of the operator $\mathcal{T}(\partial_x, n, \partial_t)$ with respect to the spatial derivatives,

$$\mathcal{T}^{(0)}(\partial_x, n) = \big[\ \mathcal{T}^{(0)}_{pq}(\partial_x, n)\ \big]_{6\times 6}$$
$$:= \begin{bmatrix} [c_{rjkl} n_j \partial_l]_{3\times 3} & [e_{lrj} n_j \partial_l]_{3\times 1} & [q_{lrj} n_j \partial_l]_{3\times 1} & [0]_{3\times 1} \\ [-e_{jkl} n_j \partial_l]_{1\times 3} & \varkappa_{jl} n_j \partial_l & a_{jl} n_j \partial_l & 0 \\ [-q_{jkl} n_j \partial_l]_{1\times 3} & a_{jl} n_j \partial_l & \mu_{jl} n_j \partial_l & 0 \\ [0]_{1\times 3} & 0 & 0 & \eta_{jl} n_j \partial_l \end{bmatrix}_{6\times 6}. \tag{2.26}$$

Evidently, for a smooth six vector $U := (u, \varphi, \psi, \vartheta)^\top$ we have

$$\mathcal{T}(\partial_x, n, \partial_t) U = (\sigma_{1j} n_j, \sigma_{2j} n_j, \sigma_{3j} n_j, -D_j n_j, -B_j n_j, -T_0^{-1} q_j n_j)^\top. \tag{2.27}$$

Due to the constitutive equations, the components of the vector $\mathcal{T}U$ given by (2.27) have the following physical sense: the first three components correspond to the mechanical stress vector in the theory of generalized thermo-electro-magneto-elasticity, the forth and the fifth components correspond to the normal components of the electric displacement vector and the magnetic induction vector, respectively, with opposite sign, and finally the sixth component is $(-T_0^{-1})$ times the normal component of the heat flux vector.

Note that the following pairs are called *like fields*:

(i) $\{u = (u_1, u_2, u_3)^\top, (\sigma_{1j} n_j, \sigma_{2j} n_j, \sigma_{3j} n_j)^\top\}$ – pair of mechanical fields,

(ii) $\{\varphi, -D_j n_j\}$ – pair of electric fields,

(iii) $\{\psi, -B_j n_j\}$ – pair of magnetic fields,

(iv) $\{\vartheta, -T_0^{-1} q_j n_j\}$ – pair of thermal fields.

As we see all the thermo-mechanical and electro-magnetic characteristics can be determined by the six scalar functions: three displacement components $u_j$, $j = 1, 2, 3$, temperature function $\vartheta$, and the electric and magnetic potentials $\varphi$ and $\psi$. Therefore, all the above field relations and the corresponding boundary value problems we reformulate in terms of these six functions.

First of all, from the equations (2.6)–(2.8) with the help of the constitutive relations (2.1)–(2.5) we derive the basic linear system of dynamics of the generalized thermo-electro-magneto-elasticity theory of inhomogeneous solids, when the material parameters are functions of the spatial variables

$x = (x_1, x_2, x_3)$, but does not depend on the time variable $t$,

$$\begin{aligned}
&\partial_j\big(c_{rjkl}(x)\partial_l u_k(x,t)\big) + \partial_j\big(e_{lrj}(x)\partial_l\varphi(x,t)\big) + \partial_j\big(q_{lrj}(x)\partial_l\psi(x,t)\big)\\
&-\partial_j\Big[\lambda_{rj}(x)\big(\vartheta(x,t)+\nu_0(x)\partial_t\vartheta(x,t)\big)\Big] - \varrho(x)\partial_t^2 u_r(x,t) = -\varrho(x)F_r(x,t),\quad r=1,2,3,\\
&-\partial_j\big(e_{jkl}(x)\partial_l u_k(x,t)\big) + \partial_j\big(\varkappa_{jl}(x)\partial_l\varphi(x,t)\big) + \partial_j\big(a_{jl}(x)\partial_l\psi(x,t)\big)\\
&-\partial_j\Big[p_j(x)\big(\vartheta(x,t)+\nu_0(x)\partial_t\vartheta(x,t)\big)\Big] = -\varrho_e(x,t),\\
&-\partial_j\big(q_{jkl}(x)\partial_l u_k(x,t)\big) + \partial_j\big(a_{jl}(x)\partial_l\varphi(x,t)\big) + \partial_j\big(\mu_{jl}(x)\partial_l\psi(x,t)\big)\\
&-\partial_j(x)\Big[m_j(x)\big(\vartheta(x,t)+\nu_0(x)\partial_t\vartheta(x,t)\big)\Big] = -\varrho_c(x,t),\\
&-\lambda_{kl}(x)\partial_t\partial_l u_k(x,t) + p_l(x)\partial_l\partial_t\varphi(x,t) + m_l(x)\partial_l\partial_t\psi(x,t) + \partial_j\big(\eta_{jl}(x)\partial_l\vartheta(x,t)\big)\\
&-d_0(x)\partial_t\vartheta(x,t) - h_0(x)\partial_t^2\vartheta(x,t) = -T_0^{-1}\varrho(x)Q(x,t).
\end{aligned} \tag{2.28}$$

If the body under consideration is homogeneous, then all the material parameters are constants and the basic equations (2.28) can be rewritten as:

$$\begin{aligned}
&c_{rjkl}\partial_j\partial_l u_k(x,t) + e_{lrj}\partial_j\partial_l\varphi(x,t) + q_{lrj}\partial_j\partial_l\psi(x,t) - \lambda_{rj}\partial_j\vartheta(x,t) - \nu_0\lambda_{rj}\partial_j\partial_t\vartheta(x,t)\\
&-\varrho\partial_t^2 u_r(x,t) = -\varrho F_r(x,t),\quad r=1,2,3,\\
&-e_{jkl}\partial_j\partial_l u_k(x,t) + \varkappa_{jl}\partial_j\partial_l\varphi(x,t) + a_{jl}\partial_j\partial_l\psi(x,t) - p_j\partial_j\vartheta(x,t) - \nu_0 p_j\partial_j\partial_t\vartheta(x,t) = -\varrho_e(x,t),\\
&-q_{jkl}\partial_j\partial_l u_k(x,t) + a_{jl}\partial_j\partial_l\varphi(x,t) + \mu_{jl}\partial_j\partial_l\psi(x,t) - m_j\partial_j\vartheta(x,t) - \nu_0 m_j\partial_j\partial_t\vartheta(x,t) = -\varrho_c(x,t),\\
&-\lambda_{kl}\partial_t\partial_l u_k(x,t) + p_l\partial_l\partial_t\varphi(x,t) + m_l\partial_l\partial_t\psi(x,t) + \eta_{jl}\partial_j\partial_l\vartheta(x,t) - d_0\partial_t\vartheta(x,t)\\
&-h_0\partial_t^2\vartheta(x,t) = -T_0^{-1}\varrho Q(x,t).
\end{aligned} \tag{2.29}$$

Let us introduce the matrix differential operator generated by the left hand side expressions in equations (2.29),

$$A(\partial_x,\partial_t) = [A_{pq}(\partial_x,\partial_t)]_{6\times 6}$$
$$:= \begin{bmatrix}
[c_{rjkl}\partial_j\partial_l - \varrho\delta_{rk}\partial_t^2]_{3\times3} & [e_{lrj}\partial_j\partial_l]_{3\times1} & [q_{lrj}\partial_j\partial_l]_{3\times1} & [-\lambda_{rj}\partial_j(1+\nu_0\partial_t)]_{3\times1}\\
[-e_{jkl}\partial_j\partial_l]_{1\times3} & \varkappa_{jl}\partial_j\partial_l & a_{jl}\partial_j\partial_l & -p_j\partial_j(1+\nu_0\partial_t)\\
[-q_{jkl}\partial_j\partial_l]_{1\times3} & a_{jl}\partial_j\partial_l & \mu_{jl}\partial_j\partial_l & -m_j\partial_j(1+\nu_0\partial_t)\\
[-\lambda_{kl}\partial_l\partial_t]_{1\times3} & p_l\partial_l\partial_t & m_l\partial_l\partial_t & \eta_{jl}\partial_j\partial_l - d_0\partial_t - h_0\partial_t^2
\end{bmatrix}_{6\times6}. \tag{2.30}$$

Then the system of equations (2.29) can be rewritten in matrix form

$$A(\partial_x,\partial_t)U(x,t) = \Phi(x,t), \tag{2.31}$$

where

$$U = (u_1,u_2,u_3,u_4,u_5,u_6)^\top := (u,\varphi,\psi,\vartheta)^\top \tag{2.32}$$

is the sought for vector function and

$$\Phi = (\Phi_1,\dots,\Phi_6)^\top := (-\varrho F_1, -\varrho F_2, -\varrho F_3, -\varrho_e, -\varrho_c, -\varrho T_0^{-1}Q)^\top \tag{2.33}$$

is a given vector function.

If all the functions involved in these equations are harmonic time dependent, that is they can be represented as the product of a function of the spatial variables $(x_1, x_2, x_3)$ and the multiplier $\exp\{\tau t\}$, where $\tau = \sigma + i\omega$ is a complex parameter, we have the *pseudo-oscillation equations* of the generalized thermo-electro-magneto-elasticity theory. Note that the pseudo-oscillation equations can be obtained from the corresponding dynamical equations by the Laplace transform. If $\tau = i\omega$ is a pure imaginary number, with the so called *frequency parameter* $\omega \in \mathbb{R}$, we obtain the *steady state oscillation equations*. Finally, if $\tau = 0$, i.e., the functions involved in equations (2.29) are independent of $t$, we get the *equations of statics*:

$$A(\partial_x)U(x) = \Phi(x), \tag{2.34}$$

where

$$A(\partial_x) = [A_{pq}(\partial_x)]_{6\times 6} := A(\partial_x, 0)$$
$$= \begin{bmatrix} [c_{rjkl}\partial_j\partial_l]_{3\times 3} & [e_{lrj}\partial_j\partial_l]_{3\times 1} & [q_{lrj}\partial_j\partial_l]_{3\times 1} & [-\lambda_{rj}\partial_j]_{3\times 1} \\ [-e_{jkl}\partial_j\partial_l]_{1\times 3} & \varkappa_{jl}\partial_j\partial_l & a_{jl}\partial_j\partial_l & -p_j\partial_j \\ [-q_{jkl}\partial_j\partial_l]_{1\times 3} & a_{jl}\partial_j\partial_l & \mu_{jl}\partial_j\partial_l & -m_j\partial_j \\ [0]_{1\times 3} & 0 & 0 & \eta_{jl}\partial_j\partial_l \end{bmatrix}_{6\times 6}. \tag{2.35}$$

Recall that for a smooth function $v(t)$ which is exponentially bounded,

$$e^{-\sigma_0 t}\big[\,|v(t)| + |\partial v(t)| + |\partial_t^2 v(t)|\,\big] = \mathcal{O}(1) \ \text{ as } \ t\to+\infty, \quad \sigma_0 \geqslant 0, \tag{2.36}$$

the corresponding Laplace transform

$$\widehat{v}(\tau) \equiv L_{t\to\tau}[v(t)] := \int_0^{+\infty} e^{-\tau t} v(t)\,dt, \quad \tau = \sigma + i\omega, \ \ \sigma > \sigma_0, \tag{2.37}$$

possesses the following properties

$$L_{t\to\tau}[\partial_t v(t)] := \int_0^{+\infty} e^{-\tau t}\partial_t v(t)\,dt = -v(0) + \tau\widehat{v}(\tau), \tag{2.38}$$

$$L_{t\to\tau}[\partial_t^2 v(t)] := \int_0^{+\infty} e^{-\tau t}\partial_t^2 v(t)\,dt = -\partial_t v(0) - \tau v(0) + \tau^2\widehat{v}(\tau). \tag{2.39}$$

Assuming that all the functions involved in the dynamical equations (2.29) are exponentially bounded and applying the Laplace transform to the system (2.29), we obtain the following pseudo-oscillation equations:

$$\begin{aligned} c_{rjkl}\partial_j\partial_l\widehat{u}_k(x,\tau) - \varrho\tau^2\widehat{u}_r(x,\tau) + e_{lrj}\partial_j\partial_l\widehat{\varphi}(x,\tau) + q_{lrj}\partial_j\partial_l\widehat{\psi}(x,\tau) - (1+\nu_0\tau)\lambda_{rj}\partial_j\widehat{\vartheta}(x,\tau) \\ = -\varrho\widehat{F}_r(x,\tau) + \Psi_r^{(0)}(x,\tau), \ \ r = 1,2,3, \\ -e_{jkl}\partial_j\partial_l\widehat{u}_k(x,\tau) + \varkappa_{jl}\partial_j\partial_l\widehat{\varphi}(x,\tau) + a_{jl}\partial_j\partial_l\widehat{\psi}(x,\tau) - (1+\nu_0\tau)p_j\partial_j\widehat{\vartheta}(x,\tau) \\ = -\widehat{\varrho}_e(x,\tau) + \Psi_4^{(0)}(x,\tau), \\ -q_{jkl}\partial_j\partial_l\widehat{u}_k(x,\tau) + a_{jl}\partial_j\partial_l\widehat{\varphi}(x,\tau) + \mu_{jl}\partial_j\partial_l\widehat{\psi}(x,\tau) - (1+\nu_0\tau)m_j\partial_j\widehat{\vartheta}(x,\tau) \\ = -\widehat{\varrho}_c(x,\tau) + \Psi_5^{(0)}(x,\tau), \\ -\tau\lambda_{kl}\partial_l\widehat{u}_k(x,\tau) + \tau p_l\partial_l\widehat{\varphi}(x,\tau) + \tau m_l\partial_l\widehat{\psi}(x,\tau) + \eta_{jl}\partial_j\partial_l\widehat{\vartheta}(x,\tau) - (\tau d_0 + \tau^2 h_0)\widehat{\vartheta}(x,\tau) \\ = -T_0^{-1}\varrho\widehat{Q}(x,\tau) + \Psi_6^{(0)}(x,\tau), \end{aligned} \tag{2.40}$$

where the overset "hat" denotes the Laplace transform of the corresponding function with respect to $t$ (see (2.37)) and

$$\Psi^{(0)}(x,\tau) = \big(\Psi_1^{(0)}(x,\tau),\dots,\Psi_6^{(0)}(x,\tau)\big)^\top$$
$$:= \begin{bmatrix} -\varrho\tau u_1(x,0) - \varrho\partial_t u_1(x,0) - \nu_0\lambda_{1j}\partial_j\vartheta(x,0) \\ -\varrho\tau u_2(x,0) - \varrho\partial_t u_2(x,0) - \nu_0\lambda_{2j}\partial_j\vartheta(x,0) \\ -\varrho\tau u_3(x,0) - \varrho\partial_t u_3(x,0) - \nu_0\lambda_{3j}\partial_j\vartheta(x,0) \\ \nu_0 p_j\partial_j\vartheta(x,0) \\ \nu_0 m_j\partial_j\vartheta(x,0) \\ -\lambda_{kl}\partial_l u_k(x,0) + p_j\partial_l\varphi(x,0) + m_j\partial_l\psi(x,0) - (d_0+\tau h_0)\vartheta(x,0) - h_0\partial_t\vartheta(x,0) \end{bmatrix}. \tag{2.41}$$

Note that the relations (2.37)–(2.39) can be extended to the spaces of generalized functions (see e.g., [116]).

In matrix form these pseudo-oscillation equations can be rewritten as

$$A(\partial_x,\tau)\widehat{U}(x,\tau) = \Psi(x,\tau), \tag{2.42}$$

where

$$\widehat{U} = (\widehat{u}_1, \widehat{u}_2, \widehat{u}_3, \widehat{u}_4, \widehat{u}_5, \widehat{u}_6)^\top := (\widehat{u}, \widehat{\varphi}, \widehat{\psi}, \widehat{\vartheta})^\top \tag{2.43}$$

is the sought for complex-valued vector function,

$$\Psi(x,\tau) = \big(\Psi_1(x,\tau), \dots, \Psi_6(x,\tau)\big)^\top = \widehat{\Phi}(x,\tau) + \Psi^{(0)}(x,\tau) \tag{2.44}$$

with $\widehat{\Phi}(x,\tau)$ being the Laplace transform of the vector function $\Phi(x,t)$ defined in (2.33) and $\Psi^{(0)}(x,\tau)$ given by (2.41), and $A(\partial,\tau)$ is the pseudo-oscillation matrix differential operator generated by the left hand side expressions in equations (2.40),

$$\begin{aligned} A(\partial_x,\tau) &= [A_{pq}(\partial_x,\tau)]_{6\times 6} \\ &:= \begin{bmatrix} [c_{rjkl}\partial_j\partial_l - \varrho\tau^2\delta_{rk}]_{3\times 3} & [e_{lrj}\partial_j\partial_l]_{3\times 1} & [q_{lrj}\partial_j\partial_l]_{3\times 1} & [-(1+\nu_0\tau)\lambda_{rj}\partial_j]_{3\times 1} \\ [-e_{jkl}\partial_j\partial_l]_{1\times 3} & \varkappa_{jl}\partial_j\partial_l & a_{jl}\partial_j\partial_l & -(1+\nu_0\tau)p_j\partial_j \\ [-q_{jkl}\partial_j\partial_l]_{1\times 3} & a_{jl}\partial_j\partial_l & \mu_{jl}\partial_j\partial_l & -(1+\nu_0\tau)m_j\partial_j \\ [-\tau\lambda_{kl}\partial_l]_{1\times 3} & \tau p_l\partial_l & \tau m_l\partial_l & \eta_{jl}\partial_j\partial_l - \tau^2 h_0 - \tau d_0 \end{bmatrix}_{6\times 6} . \end{aligned} \tag{2.45}$$

From (2.35) and (2.45) we see that $A(\partial_x, 0) = A(\partial_x)$.
It is evident that the operator

$$A^{(0)}(\partial_x) := \begin{bmatrix} [c_{rjkl}\partial_j\partial_l]_{3\times 3} & [e_{lrj}\partial_j\partial_l]_{3\times 1} & [q_{lrj}\partial_j\partial_l]_{3\times 1} & [0]_{3\times 1} \\ [-e_{jkl}\partial_j\partial_l]_{1\times 3} & \varkappa_{jl}\partial_j\partial_l & a_{jl}\partial_j\partial_l & 0 \\ [-q_{jkl}\partial_j\partial_l]_{1\times 3} & a_{jl}\partial_j\partial_l & \mu_{jl}\partial_j\partial_l & 0 \\ [0]_{1\times 3} & 0 & 0 & \eta_{jl}\partial_j\partial_l \end{bmatrix}_{6\times 6} \tag{2.46}$$

is the principal part of the operators $A(\partial,\tau)$ and $A(\partial)$. Clearly, the symbol matrix $A^{(0)}(-i\xi)$, $\xi \in \mathbb{R}^3$, of the operator $A^{(0)}(\partial)$ is the principal homogeneous symbol matrix of the operator $A(\partial,\tau)$ for all $\tau \in \mathbb{C}$,

$$\begin{aligned} A^{(0)}(-i\xi) &= A^{(0)}(i\xi) = -A^{(0)}(\xi) \\ &= \begin{bmatrix} [-c_{rjkl}\xi_j\xi_l]_{3\times 3} & [-e_{lrj}\xi_j\xi_l]_{3\times 1} & [-q_{lrj}\xi_j\xi_l]_{3\times 1} & [0]_{3\times 1} \\ [e_{jkl}\xi_j\xi_l]_{1\times 3} & -\varkappa_{jl}\xi_j\xi_l & -a_{jl}\xi_j\xi_l & 0 \\ [q_{jkl}\xi_j\xi_l]_{1\times 3} & -a_{jl}\xi_j\xi_l & -\mu_{jl}\xi_j\xi_l & 0 \\ [0]_{1\times 3} & 0 & 0 & -\eta_{jl}\xi_j\xi_l \end{bmatrix}_{6\times 6} . \end{aligned} \tag{2.47}$$

From the symmetry conditions (2.9), inequalities (2.10), and positive definiteness of the matrix $\Lambda^{(1)}$ defined in (2.14) it follows that there is a positive constant $C_0$ depending only on the material parameters, such that

$$\operatorname{Re}\big(-A^{(0)}(-i\xi)\zeta\cdot\zeta\big) = \operatorname{Re}\Big(\sum_{k,j=1}^{6} A^{(0)}_{kj}(\xi)\zeta_j\overline{\zeta}_k\Big) \geq C_0|\xi|^2|\zeta|^2 \quad \text{for all } \xi \in \mathbb{R}^3 \text{ and for all } \zeta \in \mathbb{C}^6. \tag{2.48}$$

Therefore, $-A(\partial_x,\tau)$ is a non-selfadjoint strongly elliptic differential operator. We recall that the over bar denotes complex conjugation and the central dot denotes the scalar product in the respective complex-valued vector space.

By $A^*(\partial_x,\tau) := [\overline{A(-\partial_x,\tau)}]^\top = A^\top(-\partial_x,\overline{\tau})$ we denote the operator formally adjoint to $A(\partial_x,\tau)$,

$$A^*(\partial_x,\tau) = \big[A^*_{pq}(\partial_x,\tau)\big]_{6\times 6}$$
$$:= \begin{bmatrix} [c_{rjkl}\partial_j\partial_l - \varrho\overline{\tau}^2\delta_{rk}]_{3\times 3} & [-e_{lrj}\partial_j\partial_l]_{3\times 1} & [-q_{lrj}\partial_j\partial_l]_{3\times 1} & [\overline{\tau}\lambda_{kl}\partial_l]_{3\times 1} \\ [e_{jkl}\partial_j\partial_l]_{1\times 3} & \varkappa_{jl}\partial_j\partial_l & a_{jl}\partial_j\partial_l & -\overline{\tau}p_l\partial_l \\ [q_{jkl}\partial_j\partial_l]_{1\times 3} & a_{jl}\partial_j\partial_l & \mu_{jl}\partial_j\partial_l & -\overline{\tau}m_l\partial_l \\ [(1+\nu_0\overline{\tau})\lambda_{rj}\partial_j]_{1\times 3} & (1+\nu_0\overline{\tau})p_j\partial_j & (1+\nu_0\overline{\tau})m_j\partial_j & \eta_{jl}\partial_j\partial_l - \overline{\tau}^2 h_0 - \overline{\tau} d_0 \end{bmatrix}_{6\times 6}. \tag{2.49}$$

Applying the Laplace transform to the dynamical constitutive relations (2.2)–(2.4) and (2.8) we get

$$\widehat{\sigma_{rj}}(x,\tau) = c_{rjkl}\widehat{\varepsilon_{kl}}(x,\tau) + e_{lrj}\partial_l\widehat{\varphi}(x,\tau) + q_{lrj}\partial_l\widehat{\psi}(x,\tau) - (1+\nu_0\tau)\lambda_{rj}\widehat{\vartheta}(x,\tau) + \nu_0\lambda_{rj}\vartheta(x,0), \quad r,j=1,2,3, \tag{2.50}$$

$$\widehat{D_j}(x,\tau) = e_{jkl}\widehat{\varepsilon_{kl}}(x,\tau) - \varkappa_{jl}\partial_l\widehat{\varphi}(x,\tau) - a_{jl}\partial_l\widehat{\psi}(x,\tau) + (1+\nu_0\tau)p_j\widehat{\vartheta}(x,\tau) - \nu_0 p_j\vartheta(x,0), \quad j=1,2,3, \tag{2.51}$$

$$\widehat{B_j}(x,\tau) = q_{jkl}\widehat{\varepsilon_{kl}}(x,\tau) - a_{jl}\partial_l\widehat{\varphi}(x,\tau) - \mu_{jl}\partial_l\widehat{\psi}(x,\tau) + (1+\nu_0\tau)m_j\widehat{\vartheta}(x,\tau) - \nu_0 m_j\vartheta(x,0), \quad j=1,2,3, \tag{2.52}$$

$$\widehat{q_j}(x,\tau) = -T_0\eta_{jl}\partial_l\widehat{\vartheta}(x,\tau). \tag{2.53}$$

By these equalities, the Laplace transform of the dynamical stress vector $\mathcal{T}(\partial_x,n,\partial_t)U(x,t)$ defined in (2.27) can be represented as

$$L_{t\to\tau}\big[\mathcal{T}(\partial_x,n,\partial_t)U(x,t)\big] = \mathcal{T}(\partial_x,n,\tau)\widehat{U}(x,\tau) + F^{(0)}(x), \tag{2.54}$$

where

$$\mathcal{T}(\partial_x,n,\tau)\widehat{U}(x,\tau) = \big(\widehat{\sigma_{1j}}(x,\tau)n_j(x), \widehat{\sigma_{2j}}(x,\tau)n_j(x), \widehat{\sigma_{3j}}(x,\tau)n_j(x), - \widehat{D_j}(x,\tau)n_j(x),\ -\widehat{B_j}(x,\tau)n_j(x),\ -T_0^{-1}\widehat{q_j}(x,\tau)n_j(x)\big)^\top - F^{(0)}(x), \tag{2.55}$$

$$F^{(0)}(x) := \begin{bmatrix} \nu_0\lambda_{1j}n_j(x)\vartheta(x,0) \\ \nu_0\lambda_{2j}n_j(x)\vartheta(x,0) \\ \nu_0\lambda_{3j}n_j(x)\vartheta(x,0) \\ \nu_0 p_j n_j(x)\vartheta(x,0) \\ \nu_0 m_j n_j(x)\vartheta(x,0) \\ 0 \end{bmatrix} = \begin{bmatrix} \nu_0\lambda_{1j}n_j(x) \\ \nu_0\lambda_{2j}n_j(x) \\ \nu_0\lambda_{3j}n_j(x) \\ \nu_0 p_j n_j(x) \\ \nu_0 m_j n_j(x) \\ 0 \end{bmatrix} \vartheta(x,0), \tag{2.56}$$

and the pseudo-oscillation stress operator $\mathcal{T}(\partial_x,n,\tau)$ reads as (cf. (2.25))

$$\mathcal{T}(\partial_x,n,\tau) = \big[\ \mathcal{T}_{pq}(\partial_x,n,\tau)\ \big]_{6\times 6}$$
$$:= \begin{bmatrix} [c_{rjkl}n_j\partial_l]_{3\times 3} & [e_{lrj}n_j\partial_l]_{3\times 1} & [q_{lrj}n_j\partial_l]_{3\times 1} & [-(1+\nu_0\tau)\lambda_{rj}n_j]_{3\times 1} \\ [-e_{jkl}n_j\partial_l]_{1\times 3} & \varkappa_{jl}n_j\partial_l & a_{jl}n_j\partial_l & -(1+\nu_0\tau)p_j n_j \\ [-q_{jkl}n_j\partial_l]_{1\times 3} & a_{jl}n_j\partial_l & \mu_{jl}n_j\partial_l & -(1+\nu_0\tau)m_j n_j \\ [0]_{1\times 3} & 0 & 0 & \eta_{jl}n_j\partial_l \end{bmatrix}_{6\times 6}. \tag{2.57}$$

The vector $\mathcal{T}(\partial_x,n,\tau)\widehat{U}(x,\tau)$ will be referred to as a *pseudo-oscillation stress vector*.
Below, in Green's formulas there appears also the boundary operator $\mathcal{P}(\partial_x,n,\tau)$ associated with the

adjoint differential operator $A^*(\partial_x,\tau)$,

$$\mathcal{P}(\partial_x,n,\tau)=\big[\,\mathcal{P}_{pq}(\partial_x,n,\tau)\,\big]_{6\times 6}$$
$$=\begin{bmatrix} [c_{rjkl}n_j\partial_l]_{3\times 3} & [-e_{lrj}n_j\partial_l]_{3\times 1} & [-q_{lrj}n_j\partial_l]_{3\times 1} & [\overline{\tau}\lambda_{rj}n_j]_{3\times 1}\\ [e_{jkl}n_j\partial_l]_{1\times 3} & \varkappa_{jl}n_j\partial_l & a_{jl}n_j\partial_l & -\overline{\tau}p_jn_j\\ [q_{jkl}n_j\partial_l]_{1\times 3} & a_{jl}n_j\partial_l & \mu_{jl}n_j\partial_l & -\overline{\tau}m_jn_j\\ [0]_{1\times 3} & 0 & 0 & \eta_{jl}n_j\partial_l \end{bmatrix}_{6\times 6}. \tag{2.58}$$

By $\mathcal{T}^{(0)}(\partial_x,n)$ and $\mathcal{P}^{(0)}(\partial_x,n)$ we denote the principal parts of the operators $\mathcal{T}(\partial_x,n,\tau)$ and $\mathcal{P}(\partial_x,n,\tau)$, respectively,

$$\mathcal{T}^{(0)}(\partial_x,n):=\begin{bmatrix} [c_{rjkl}n_j\partial_l]_{3\times 3} & [e_{lrj}n_j\partial_l]_{3\times 1} & [q_{lrj}n_j\partial_l]_{3\times 1} & [0]_{3\times 1}\\ [-e_{jkl}n_j\partial_l]_{1\times 3} & \varkappa_{jl}n_j\partial_l & a_{jl}n_j\partial_l & 0\\ [-q_{jkl}n_j\partial_l]_{1\times 3} & a_{jl}n_j\partial_l & \mu_{jl}n_j\partial_l & 0\\ [0]_{1\times 3} & 0 & 0 & \eta_{jl}n_j\partial_l \end{bmatrix}_{6\times 6}, \tag{2.59}$$

$$\mathcal{P}^{(0)}(\partial_x,n):=\begin{bmatrix} [c_{rjkl}n_j\partial_l]_{3\times 3} & [-e_{lrj}n_j\partial_l]_{3\times 1} & [-q_{lrj}n_j\partial_l]_{3\times 1} & [0]_{3\times 1}\\ [e_{jkl}n_j\partial_l]_{1\times 3} & \varkappa_{jl}n_j\partial_l & a_{jl}n_j\partial_l & 0\\ [q_{jkl}n_j\partial_l]_{1\times 3} & a_{jl}n_j\partial_l & \mu_{jl}n_j\partial_l & 0\\ [0]_{1\times 3} & 0 & 0 & \eta_{jl}n_j\partial_l \end{bmatrix}_{6\times 6}. \tag{2.60}$$

2.2. **Initial-boundary value problems of dynamics.** Let $\Omega^+$ be a bounded 3-dimensional domain in $\mathbb{R}^3$ with a smooth boundary $S=\partial\Omega^+$, $\overline{\Omega^+}=\Omega^+\cup S$, and $\Omega^-=\mathbb{R}^3\setminus\overline{\Omega^+}$. Assume that the domain $\overline{\Omega^+}$ is occupied by an anisotropic homogeneous material with the above described generalized thermo-electro-magneto-elastic properties.

By $C^k(\overline{\Omega^\pm})$ we denote the subspace of functions from $C^k(\Omega^\pm)$ whose derivatives up to the order $k$ are continuously extendable to $S$ from $\Omega^\pm$.

By $\mathcal{D}(\Omega)$ we denote the space of infinitely differentiable test functions with compact supports in $\Omega\subset\mathbb{R}^3$.

The symbols $\{\,\cdot\,\}^+_S$ and $\{\,\cdot\,\}^-_S$ denote one-sided limits (traces) on $S$ from $\Omega^+$ and $\Omega^-$, respectively. We often drop the subscript $S$ if it does not lead to misunderstanding.

By $L_p$, $L_{p,loc}$, $L_{p,comp}$, $W^r_p$, $W^r_{p,loc}$, $W^r_{p,comp}$, $H^s_p$, and $B^s_{p,q}$ (with $r\geq 0$, $s\in\mathbb{R}$, $1<p<\infty$, $1\leq q\leq\infty$) we denote the well-known Lebesgue, Sobolev–Slobodetskii, Bessel potential, and Besov function spaces, respectively (see, e.g., [108], [74]). Recall that $H^r_2=W^r_2=B^r_{2,2}$, $H^s_2=B^s_{2,2}$, $W^t_p=B^t_{p,p}$, and $H^k_p=W^k_p$, for any $r\geq 0$, for any $s\in\mathbb{R}$, for any positive and non-integer $t$, and for any non-negative integer $k$. In our analysis we essentially employ also the spaces:

$$\widetilde{H}^s_p(\mathcal{M}):=\big\{f:\ f\in H^s_p(\mathcal{M}_0),\ \operatorname{supp}f\subset\overline{\mathcal{M}}\big\},\quad \widetilde{B}^s_{p,q}(\mathcal{M}):=\big\{f:\ f\in B^s_{p,q}(\mathcal{M}_0),\ \operatorname{supp}f\subset\overline{\mathcal{M}}\big\},$$
$$H^s_p(\mathcal{M}):=\big\{r_{\mathcal{M}}f:\ f\in H^s_p(\mathcal{M}_0)\big\},\quad B^s_{p,q}(\mathcal{M}):=\big\{r_{\mathcal{M}}f:\ f\in B^s_{p,q}(\mathcal{M}_0)\big\},$$

where $\mathcal{M}_0$ is a closed manifold without boundary and $\mathcal{M}$ is an open proper submanifold of $\mathcal{M}_0$ with nonempty smooth boundary $\partial\mathcal{M}\neq\varnothing$; $r_{\mathcal{M}}$ is the restriction operator onto $\mathcal{M}$. Below, sometimes we use also the abbreviations $H^s_2=H^s$ and $W^s_2=W^s$.

*Remark* 2.1. Let a function $f$ be defined on an open proper submanifold $\mathcal{M}$ of a closed manifold $\mathcal{M}_0$ without boundary. Let $f\in B^s_{p,q}(\mathcal{M})$ and $\widetilde{f}$ be the extension of $f$ by zero to $\mathcal{M}_0\setminus\mathcal{M}$. If the extension preserves the space, i.e., if $\widetilde{f}\in\widetilde{B}^s_{p,q}(\mathcal{M})$, then we write $f\in\widetilde{B}^s_{p,q}(\mathcal{M})$ instead of $f\in r_{\mathcal{M}}\widetilde{B}^s_{p,q}(\mathcal{M})$ when it does not lead to misunderstanding.

Further, let $J_\infty:=(0,+\infty)$, $\widetilde{J}_\infty:=[0,+\infty)$, and $J_T:=(0,T)$ for $0<T<+\infty$, $\overline{J_T}:=[0,T]$.

Now, we formulate the basic initial-boundary value problems for the dynamical equations of the generalized thermo-electro-magneto-elasticity (GTEME) theory in the classical setting.

**The Dirichlet dynamical problem $(\mathrm{D})_t^+$:** *Find a regular solution*

$$U=(u,\varphi,\psi,\vartheta)^\top\in[C^1(\overline{\Omega^+}\times\widetilde{J}_\infty)]^6\cap[C^2(\Omega^+\times J_\infty)]^6 \tag{2.61}$$

*to the dynamical equations of the GTEME theory,*

$$A(\partial_x,\partial_t)U(x,t)=\Phi(x,t),\ \ (x,t)\in\Omega^+\times J_\infty, \tag{2.62}$$

*satisfying the initial conditions:*

$$u(x,0)=u^{(0)}(x),\quad \partial_t u(x,0)=u^{(1)}(x),\ \ x\in\Omega^+, \tag{2.63}$$

$$\vartheta(x,0)=\vartheta^{(0)}(x),\quad \partial_t\vartheta(x,0)=\vartheta^{(1)}(x),\ \ x\in\Omega^+, \tag{2.64}$$

*and the Dirichlet type boundary condition,*

$$\{U(x,t)\}^+=f(x,t),\ \ (x,t)\in S\times\widetilde{J}_\infty, \tag{2.65}$$

*i.e.,*

$$\{u_r(x,t)\}^+=f_r(x,t),\ \ (x,t)\in S\times\widetilde{J}_\infty,\ \ r=1,2,3, \tag{2.66}$$

$$\{\varphi(x,t)\}^+=f_4(x,t),\ \ (x,t)\in S\times\widetilde{J}_\infty, \tag{2.67}$$

$$\{\psi(x,t)\}^+=f_5(x,t),\ \ (x,t)\in S\times\widetilde{J}_\infty, \tag{2.68}$$

$$\{\vartheta(x,t)\}^+=f_6(x,t),\ \ (x,t)\in S\times\widetilde{J}_\infty, \tag{2.69}$$

*where $\Phi=(\Phi_1,\dots,\Phi_6)^\top$, $u^{(0)}$, $u^{(1)}$, $\vartheta^{(0)}$, $\vartheta^{(1)}$, and $f=(f_1,\dots,f_6)^\top$ are given functions from appropriate smooth spaces.*

**The Neumann dynamical problem $(\mathrm{N})_t^+$:** *Find a regular solution*

$$U=(u,\varphi,\psi,\vartheta)^\top\in[C^1(\overline{\Omega^+}\times\widetilde{J}_\infty)]^6\cap[C^2(\Omega^+\times J_\infty)]^6$$

*to the dynamical equations of the GTEME theory* (2.62) *satisfying the initial conditions* (2.63), (2.64) *and the Neumann type boundary condition*

$$\{\mathcal{T}(\partial_x,n,\partial_t)U(x,t)\}^+=F(x,t),\ \ (x,t)\in S\times\widetilde{J}_\infty, \tag{2.70}$$

*i.e.*

$$\big\{[\mathcal{T}(\partial_x,n,\partial_t)U(x,t)]_r\big\}^+\equiv\{\sigma_{rj}n_j\}^+=F_r(x,t),\ \ (x,t)\in S\times\widetilde{J}_\infty,\ \ r=1,2,3, \tag{2.71}$$

$$\big\{[\mathcal{T}(\partial_x,n,\partial_t)U(x,t)]_4\big\}^+\equiv\{-D_jn_j\}^+=F_4(x,t),\ \ (x,t)\in S\times\widetilde{J}_\infty, \tag{2.72}$$

$$\big\{[\mathcal{T}(\partial_x,n,\partial_t)U(x,t)]_5\big\}^+\equiv\{-B_jn_j\}^+=F_5(x,t),\ \ (x,t)\in S\times\widetilde{J}_\infty, \tag{2.73}$$

$$\big\{[\mathcal{T}(\partial_x,n,\partial_t)U(x,t)]_6\big\}^+\equiv\{-T_0^{-1}q_jn_j\}^+=F_6(x,t),\ \ (x,t)\in S\times\widetilde{J}_\infty, \tag{2.74}$$

*where $F=(F_1,\dots,F_6)^\top$ is a given continuous vector function.*

Now, we formulate the most general mixed initial-boundary value problem. To this end, let us consider the following four dissections of the boundary surface $S$ into non-overlapping open submanifolds $S_k$, $k=1,2,\dots,8$, so that

$$\begin{gathered}S=\overline{S}_1\cup S_2=\overline{S}_3\cup S_4=\overline{S}_5\cup S_6=\overline{S}_7\cup S_8,\\ S_1\cap S_2=S_3\cap S_4=S_5\cap S_6=S_7\cap S_8=\varnothing,\quad \ell_j=\overline{S}_j\cap\overline{S}_{j+1},\ \ j=1,3,5,7.\end{gathered} \tag{2.75}$$

Further, let

$$\widetilde{\Omega}^+_\ell:=\overline{\Omega^+}\setminus\ell,\quad \ell:=\ell_1\cup\ell_3\cup\ell_5\cup\ell_7, \tag{2.76}$$

and introduce the classes of *semi-regular functions*.

**Definition 2.2.** We say that $w$ is a semi-regular function in $\widetilde{\Omega}^+_\ell$ and write $w\in\mathbf{C}(\widetilde{\Omega}^+_\ell;\alpha)$ if

(i) $w$ is continuous in $\overline{\Omega^+}$;

(ii) the first order derivatives of $w$ are continuous in $\widetilde{\Omega}^+_\ell$ and there is $\alpha\in[0,1)$, such that

$$|\partial_k w(x)|\leqslant C\sum_{j=1,3,5,7}[\mathrm{dist}(x,\ell_j)]^{-\alpha},\ \ x\in\widetilde{\Omega}^+_\ell,\ \ C=const,\ \ k=1,2,3;$$

(iii) the second order derivatives of $w$ are continuous in $\Omega^+$ and integrable over $\Omega^+$.

Evidently,

$$\mathbf{C}(\widetilde{\Omega}^+_\ell;\alpha) \subset C(\overline{\Omega^+}) \cap C^1(\widetilde{\Omega}^+_\ell) \cap C^2(\Omega^+). \tag{2.77}$$

**Definition 2.3.** We say that $w$ is a semi-regular function in $\widetilde{\Omega}^+_\ell \times J_\infty$ and write $w \in \mathbf{C}(\widetilde{\Omega}^+_\ell \times \widetilde{J}_\infty;\alpha)$ if

(i) $w$ and $\partial_t w$ are continuous in $\overline{\Omega^+} \times \widetilde{J}_\infty$;

(ii) the first order derivatives of $w$ with respect to the spatial variables are continuous in $\widetilde{\Omega}^+_\ell \times \widetilde{J}_\infty$ and there is $\alpha \in [0,1)$, such that for any $T \in \widetilde{J}_\infty$

$$|\partial_k w(x,t)| \leqslant C \sum_{j=1,3,5,7} [\mathrm{dist}(x,\ell_j)]^{-\alpha}, \quad (x,t) \in \widetilde{\Omega}^+_\ell \times \overline{J}_T, \quad C = C(T) = const, \quad k = 1,2,3;$$

(iii) the second order derivatives of $w$ are continuous in $\Omega^+ \times J_\infty$ and integrable over $\Omega^+ \times J_T$ for any $T \in \widetilde{J}_\infty$.

Evidently,

$$\mathbf{C}(\widetilde{\Omega}^+_\ell \times \widetilde{J}_\infty;\alpha) \subset C(\overline{\Omega^+} \times \widetilde{J}_\infty) \cap C^1(\widetilde{\Omega}^+_\ell \times \widetilde{J}_\infty) \cap C^2(\Omega^+ \times J_\infty) \tag{2.78}$$

and if $w \in \mathbf{C}(\widetilde{\Omega}^+_\ell \times \widetilde{J}_\infty;\alpha)$, then for any fixed $t \in J_\infty$ the function $w(\,\cdot\,,t)$ is semi-regular in $\widetilde{\Omega}^+_\ell$ as a function of spatial variable $x$, i.e., $w(\,\cdot\,,t) \in \mathbf{C}(\widetilde{\Omega}^+_\ell;\alpha)$.

**Mixed type dynamical problem $(\mathrm{M})^+_t$:** *Find a semi-regular solution*

$$U = (u,\varphi,\psi,\vartheta)^\top \in [\mathbf{C}(\widetilde{\Omega}^+_\ell \times \widetilde{J}_\infty;\alpha)]^6$$

*to the dynamical equations of the GTEME theory* (2.62) *satisfying the initial conditions* (2.63), (2.64) *and the mixed type boundary conditions:*

$$\{u_r(x,t)\}^+ = f^*_r(x,t), \quad (x,t) \in S_1 \times \widetilde{J}_\infty, \quad r = 1,2,3, \tag{2.79}$$

$$\big\{[\mathcal{T}(\partial_x,n,\partial_t)U(x,t)]_r\big\}^+ \equiv \{\sigma_{rj}n_j\}^+ = F^*_r(x,t), \quad (x,t) \in S_2 \times \widetilde{J}_\infty, \quad r = 1,2,3, \tag{2.80}$$

$$\{\varphi(x,t)\}^+ = f^*_4(x,t), \quad (x,t) \in S_3 \times \widetilde{J}_\infty, \tag{2.81}$$

$$\big\{[\mathcal{T}(\partial_x,n,\partial_t)U(x,t)]_4\big\}^+ \equiv \{-D_j n_j\}^+ = F^*_4(x,t), \quad (x,t) \in S_4 \times \widetilde{J}_\infty, \tag{2.82}$$

$$\{\psi(x,t)\}^+ = f^*_5(x,t), \quad (x,t) \in S_5 \times \widetilde{J}_\infty, \tag{2.83}$$

$$\big\{[\mathcal{T}(\partial_x,n,\partial_t)U(x,t)]_5\big\}^+ \equiv \{-B_j n_j\}^+ = F^*_5(x,t), \quad (x,t) \in S_6 \times \widetilde{J}_\infty, \tag{2.84}$$

$$\{\vartheta(x,t)\}^+ = f^*_6(x,t), \quad (x,t) \in S_7 \times \widetilde{J}_\infty, \tag{2.85}$$

$$\big\{[\mathcal{T}(\partial_x,n,\partial_t)U(x,t)]_6\big\}^+ \equiv \{-T_0^{-1}q_j n_j\}^+ = F^*_6(x,t), \quad (x,t) \in S_8 \times \widetilde{J}_\infty, \tag{2.86}$$

*where $f^*_k$ and $F^*_k$, $k = 1,2,\dots,6$, are given functions from appropriate smooth spaces.*

*Remark* 2.4. In the case of a special particular dissection of the boundary $S$ when

$$S_1 = S_3 = S_5 = S_7 := S_D, \quad S_2 = S_4 = S_6 = S_8 := S_N, \quad \ell = \overline{S}_D \cap \overline{S}_N, \tag{2.87}$$

we refer the mixed problem as **Basic mixed dynamical problem $(\mathbf{M})^+_t$** associated with the dissection $S = \overline{S}_D \cup \overline{S}_N$.

Now, let an elastic solid occupying the domain $\Omega^+$ (respectively, $\Omega^-$) contain an interior crack. We identify the crack surface as a two-dimensional, two-sided smooth manifold $\Sigma \subset \Omega^\pm$ with the crack edge $\ell_c := \partial\Sigma$. We assume that $\Sigma$ is a proper submanifold of a closed surface $S_0$ surrounding a domain $\overline{\Omega}_0$ which is a proper subdomain of $\Omega^+$ (respectively, $\Omega^-$). We choose the direction of the unit normal vector to the fictitious surface $S_0$ such that it is outward with respect to the domain $\Omega_0$. This agreement defines uniquely the direction of the normal vector to the crack surface $\Sigma$. The symbols $\{\,\cdot\,\}^+_\Sigma$ and $\{\,\cdot\,\}^-_\Sigma$ denote the one-sided limits on $\Sigma$ from $\Omega_0$ and $\Omega^+ \setminus \overline{\Omega}_0$, respectively.

Further, let $\Omega^+_\Sigma := \Omega^+ \setminus \overline{\Sigma}$ and $\widetilde{\Omega}^+_\Sigma := \overline{\Omega^+} \setminus \overline{\Sigma}$ with $\overline{\Sigma} = \Sigma \cup \ell_c$.

**Definition 2.5.** We say that $w$ is a semi-regular function in $\widetilde{\Omega}^+_\Sigma$ and write $w \in \mathbf{C}(\widetilde{\Omega}^+_\Sigma;\alpha)$ if

(i) $w$ is continuous in $\widetilde{\Omega}^+_\Sigma$ and one-sided continuously extendable to $\overline{\Sigma}$ from $\Omega_0$ and from $\Omega^+ \setminus \overline{\Omega}_0$, i.e., $w$ is continuous in $\widetilde{\Omega}^+_\Sigma$, $\overline{\Omega^+} \setminus \Omega_0$, and $\overline{\Omega}_0$;

(ii) the first order derivatives of $w$ are continuous in $\widetilde{\Omega}^+_\Sigma$ and one-sided continuously extendable to $\Sigma$ from $\Omega_0$ and from $\Omega^+ \setminus \overline{\Omega}_0$, and there is $\alpha \in [0,1)$, such that at the crack edge $\ell_c = \partial\Sigma$ the following estimates hold

$$|\partial_k w(x)| \leqslant C[\mathrm{dist}(x,\ell_c)]^{-\alpha}, \ \ x \in \widetilde{\Omega}^+_\Sigma, \ \ C = const, \ \ k = 1,2,3;$$

(iii) the second order derivatives of $w$ are continuous in $\Omega^+_\Sigma$ and integrable over $\Omega^+_\Sigma$.

Evidently, formally we can write

$$\mathbf{C}(\widetilde{\Omega}^+_\Sigma;\alpha) \subset C(\overline{\Omega_0}) \cap C(\overline{\Omega^+} \setminus \Omega_0) \cap C^1(\widetilde{\Omega}^+_\Sigma) \cap C^2(\Omega^+_\Sigma), \tag{2.88}$$

which is understood in the following sense: if $w \in \mathbf{C}(\widetilde{\Omega}^+_\Sigma;\alpha)$, then

$$r_{\overline{\Omega}_0} w \in C(\overline{\Omega}_0), \quad r_{\overline{\Omega^+} \setminus \Omega_0} w \in C(\overline{\Omega^+} \setminus \Omega_0), \ \ w \in C^1(\widetilde{\Omega}^+_\Sigma), \ \ w \in C^2(\Omega^+_\Sigma).$$

**Definition 2.6.** We say that $w$ is a semi-regular function in $\widetilde{\Omega}^+_\Sigma \times J_\infty$ and write $w \in \mathbf{C}(\widetilde{\Omega}^+_\Sigma \times \widetilde{J}_\infty;\alpha)$ if

(i) $w$ and $\partial_t w$ are continuous in $\widetilde{\Omega}^+_\Sigma \times \widetilde{J}_\infty$ and one-sided continuously extendable to $\overline{\Sigma}$ from $\Omega_0$ and from $\Omega^+ \setminus \overline{\Omega}_0$ for all $t \in \widetilde{J}_\infty$;

(ii) the first order derivatives of $w$ with respect to the spatial variables are continuous in $\widetilde{\Omega}^+_\Sigma \times \widetilde{J}_\infty$ and one-sided continuously extendable to $\Sigma$ from $\Omega_0$ and from $\Omega^+ \setminus \overline{\Omega}_0$ for all $t \in \widetilde{J}_\infty$, and there is $\alpha \in [0,1)$, such that at the crack edge $\ell_c = \partial\Sigma$ the following estimates hold for any $T \in \widetilde{J}_\infty$

$$|\partial_k w(x,t)| \leqslant C[\mathrm{dist}(x,\ell_c)]^{-\alpha}, \ \ (x,t) \in \widetilde{\Omega}^+_\Sigma \times \overline{J}_T, \ \ C = C(T) = const, \ \ k = 1,2,3;$$

(iii) the second order derivatives of $w$ are continuous in $\Omega^+_\Sigma \times J_\infty$ and integrable over $\Omega^+_\Sigma \times J_T$ for any $T \in \widetilde{J}_\infty$.

Evidently, we can formally write (in the same sense as (2.88))

$$\mathbf{C}(\widetilde{\Omega}^+_\Sigma \times \widetilde{J}_\infty;\alpha) \subset C(\overline{\Omega_0} \times \widetilde{J}_\infty) \cap C\big((\overline{\Omega^+} \setminus \Omega_0) \times \widetilde{J}_\infty\big) \cap C^1(\widetilde{\Omega}^+_\Sigma \times \widetilde{J}_\infty) \cap C^2(\Omega^+_\Sigma \times J_\infty). \tag{2.89}$$

As usual, we assume that the crack faces are mechanically traction free, i.e., the traces of the components of the mechanical stress vector equal to zero on $\Sigma$, i.e., $\{\sigma_{rj}n_j\}^\pm = 0$, $r = 1,2,3$.

Depending on the physical properties of the crack gap, one can consider different conditions on the crack faces for the electric, magnetic and thermal fields (the same applies to the transmission and interfacial crack problems where the physical thermo-mechanical and electro-magnetic properties of the interface thin layer modeled as a two dimensional surface between adjacent contacting parts of a composite body should be taken into account). In particular,

(1) if the crack gap is a dielectric medium, then the traces of the normal component of the electric displacement vector $\{D_jn_j\}^\pm$ should be zero on $\Sigma$;

(2) if the crack gap is a conductor, then the electric potential function and the normal component of the electric displacement vector should satisfy the electrically permeable boundary conditions on the crack surface $\Sigma$, i.e., $\{\varphi\}^+ = \{\varphi\}^-$ and $\{D_jn_j\}^+ = \{D_jn_j\}^-$ on $\Sigma$;

(3) if the crack gap is not magnetically permeable, then the traces of the normal component of the magnetic induction vector $\{B_jn_j\}^\pm$ should be zero on $\Sigma$;

(4) if the crack gap is magnetically permeable, then the magnetic potential function and the normal component of the magnetic induction vector should be continuous across the crack surface $\Sigma$, i.e., $\{\psi\}^+ = \{\psi\}^-$ and $\{B_jn_j\}^+ = \{B_jn_j\}^-$ on $\Sigma$;

(5) if the crack gap is thermally insulated, then the traces of the normal heat flux function $\{q_jn_j\}^\pm$ should be zero on the crack surface $\Sigma$;

(6) if the crack gap is not thermally insulated, then the temperature and the normal heat flux functions should be continuous on the crack surface $\Sigma$, i.e., $\{\vartheta\}^+ = \{\vartheta\}^-$ and $\{q_j n_j\}^+ = \{q_j n_j\}^-$ on $\Sigma$.

The applicability and effect of the crack-face electrical boundary conditions in piezoelectric fracture are investigated in many papers and by treating flaws in a medium as notches with a finite width, the results from different electrical boundary condition assumptions on the crack faces are compared. It is found that the electrically impermeable boundary is a reasonable one for engineering problems. Unless the flaw interior is filled with conductive media, the permeable crack assumption may not be directly applied to the fracture of piezoelectric materials in engineering applications (see, e.g. [112], [113], [114], [94], [32], [72], and the references therein).

As model cases we consider the following three type of the crack problems.

**Crack type dynamical problems:** *Find a semi-regular vector function*

$$U = (u, \varphi, \psi, \vartheta)^\top \in \big[\mathbf{C}(\widetilde{\Omega}^+_\Sigma \times \widetilde{J}_\infty; \alpha)\big]^6$$

*satisfying*

(i) *the dynamical equations of the GTEME theory* (2.62) *in* $\Omega^+_\Sigma \times J_\infty$*;*

(ii) *the initial conditions* (2.63)*,* (2.64) *in* $\Omega^+_\Sigma$*;*

(iii) *the Dirichlet or Neumann type boundary conditions on* $S \times \widetilde{J}_\infty$*;*

(iv) *one of the following type boundary and/or transmission conditions on the crack surface* $\Sigma$*:*

(a) *screen type conditions:*

$$\{U(x,t)\}^+ = f^{(+)}(x,t), \ \ (x,t) \in \Sigma \times \widetilde{J}_\infty, \tag{2.90}$$

$$\{U(x,t)\}^- = f^{(-)}(x,t), \ \ (x,t) \in \Sigma \times \widetilde{J}_\infty, \tag{2.91}$$

*or*

(b) *crack type conditions:*

$$\{\mathcal{T}(\partial_x, n, \partial_t)U(x,t)\}^+ = F^{(+)}(x,t), \ \ (x,t) \in \Sigma \times \widetilde{J}_\infty, \tag{2.92}$$

$$\{\mathcal{T}(\partial_x, n, \partial_t)U(x,t)\}^- = F^{(-)}(x,t), \ \ (x,t) \in \Sigma \times \widetilde{J}_\infty, \tag{2.93}$$

*or*

(c) *mixed crack type conditions:*

$$\big\{[\mathcal{T}(\partial_x, n, \partial_t)U(x,t)]_r\big\}^\pm = F^{(\pm)}_r(x,t), \ \ (x,t)\in\Sigma\times\widetilde{J}_\infty, \ r=1,2,3, \tag{2.94}$$

$$\{\varphi(x,t)\}^+ - \{\varphi(x,t)\}^- = f^{**}_4(x,t), \ \ (x,t) \in \Sigma \times \widetilde{J}_\infty, \tag{2.95}$$

$$\big\{[\mathcal{T}(\partial_x, n, \partial_t)U(x,t)]_4\big\}^+ - \big\{[\mathcal{T}(\partial_x, n, \partial_t)U(x,t)]_4\big\}^- = F^{**}_4(x,t), \ \ (x,t) \in \Sigma \times \widetilde{J}_\infty, \tag{2.96}$$

$$\{\psi(x,t)\}^+ - \{\psi(x,t)\}^- = f^{**}_5(x,t), \ \ (x,t) \in \Sigma \times \widetilde{J}_\infty, \tag{2.97}$$

$$\big\{[\mathcal{T}(\partial_x, n, \partial_t)U(x,t)]_5\big\}^+ - \big\{[\mathcal{T}(\partial_x, n, \partial_t)U(x,t)]_5\big\}^- = F^{**}_5(x,t), \ \ (x,t) \in \Sigma \times \widetilde{J}_\infty, \tag{2.98}$$

$$\{\vartheta(x,t)\}^+ - \{\vartheta(x,t)\}^- = f^{**}_6(x,t), \ \ (x,t) \in \Sigma \times \widetilde{J}_\infty, \tag{2.99}$$

$$\big\{[\mathcal{T}(\partial_x, n, \partial_t)U(x,t)]_6\big\}^+ - \big\{[\mathcal{T}(\partial_x, n, \partial_t)U(x,t)]_6\big\}^- = F^{**}_6(x,t), \ \ (x,t) \in \Sigma \times \widetilde{J}_\infty, \tag{2.100}$$

*where* $f^{(\pm)} = (f^{(\pm)}_1, \dots, f^{(\pm)}_6)^\top$*,* $F^{(\pm)} = (F^{(\pm)}_1, \dots, F^{(\pm)}_6)^\top$*,* $f^{**}_k$*, and* $F^{**}_k$*,* $k = 4, 5, 6$*, are given functions from the appropriate smooth spaces.*

Evidently, we can formulate the semi-regular setting of crack type dynamical problem with mixed conditions on the exterior boundary of the body $S \times \widetilde{J}_\infty$ as well, but in this case we have to require that for some open neighbourhood $\mathcal{U}_\Sigma \times \widetilde{J}_\infty \subset \Omega^+_\Sigma \times \widetilde{J}_\infty$ of the crack surface $\Sigma$ the sought for vector belongs to the semi-regular class of functions $[\mathbf{C}(\widetilde{\mathcal{U}}_\Sigma \times \widetilde{J}_\infty; \alpha)]^6$ and, at the same time, for some open one-sided interior neighbourhood $\mathcal{V}^+_S \times \widetilde{J}_\infty \subset \Omega^+_\Sigma \times \widetilde{J}_\infty$ of the exterior boundary surface $S$ it belongs to the semi-regular class of functions $[\mathbf{C}(\widetilde{\mathcal{V}}^+_{S,\ell} \times \widetilde{J}_\infty; \alpha)]^6$, where the curve $\ell$ is defined by the dissection of the surface $S$ associated with the mixed boundary conditions and $\widetilde{\mathcal{V}}^+_{S,\ell} = \overline{\mathcal{V}^+_S} \setminus \ell$. Without loss of generality one can assume that $\overline{\mathcal{U}_\Sigma} \cap S = \varnothing$ and $\overline{\mathcal{V}^+_S} \cap \overline{\Sigma} = \varnothing$.

Crack type dynamical problem with the Dirichlet type conditions (2.90), (2.91) on the crack faces will be referred to as Problem (B-CR-D)$^+_t$, with the Neumann type conditions (2.92), (2.93) – as Problem (B-CR-N)$^+_t$, and with the mixed type Neumann-Transmission conditions (2.94)–(2.100) – as Problem (B-CR-NT)$^+_t$. Here B stands for D or for N or for M and shows which kind of boundary conditions (Dirichlet, Neumann or Mixed type boundary conditions) are prescribed on the exterior surface $S$.

*The initial-boundary value problems of dynamics for an exterior unbounded domain* $\Omega^-$ can be formulated quite similarly. The only difference in the formal setting is that the one-sided limiting values of the corresponding functions on the boundary surface should be taken from the domain $\Omega^-$, i.e., the interior traces $\{\,\cdot\,\}^+$ should be replaced by the exterior ones $\{\,\cdot\,\}^-$. We denote these problems by (D)$^-_t$, (N)$^-_t$, (M)$^-_t$, (B-CR-D)$^-_t$, (B-CR-N)$^-_t$, and (B-CR-NT)$^-_t$ with B$\in$ {D, N, M}.

Evidently, the known functions involved in the formulation of dynamical problems should possess appropriate smoothness properties and, in addition, they should satisfy some compatibility conditions. These aspects will be discussed and specified later when we start investigation of uniqueness and existence questions.

Note that, some authors formulate the dynamical initial-boundary value problems with different initial conditions. In particular, in the reference [5] which deals with the uniqueness of solutions to the dynamical problems, instead of conditions (2.64) for the temperature function, the initial conditions for the electric and magnetic potentials $\varphi(x,0)$ and $\psi(x,0)$ and for the entropy function $\mathcal{S}(x,0)$ are given along with the initial condition for the displacement vector (2.63). As a result, in [5] there are given 9 independent initial conditions, while in our formulation and further analysis we have only 8 independent initial conditions.

2.2.1. *Essential remarks concerning the initial data of electro-magnetic potentials.* **Remarks concerning the Dirichlet problem.** Special attention should be paid to the fact that in the formulation of the dynamical problems stated above we have not initial conditions for the electric and magnetic potentials $\varphi$ and $\psi$. The case is that for $x \in \Omega^+$ the values of the functions $\varphi(x,0)$ and $\psi(x,0)$ at the initial moment $t = 0$ can be found by the data of the corresponding dynamical problems if the corresponding necessary smoothness conditions are satisfied. Indeed, the fourth and fifth equations of the system (2.62) for $t = 0$ read as (see system (2.29) and the initial conditions (2.63), (2.64))

$$\begin{aligned} &\varkappa_{jl}\partial_j\partial_l\varphi(x,0) + a_{jl}\partial_j\partial_l\psi(x,0) = Y_4(x),\\ &a_{jl}\partial_j\partial_l\varphi(x,0) + \mu_{jl}\partial_j\partial_l\psi(x,0) = Y_5(x), \end{aligned} \quad x \in \Omega^+, \tag{2.101}$$

where $Y_4(x)$ and $Y_5(x)$ are known functions, defined by the relations

$$\begin{aligned} Y_4(x) &:= -\varrho_e(x,0) + e_{jkl}\partial_j\partial_l u^{(0)}_k(x) + p_j\partial_j\vartheta^{(0)}(x) + \nu_0 p_j\partial_j\vartheta^{(1)}(x),\\ Y_5(x) &:= -\varrho_c(x,0) + q_{jkl}\partial_j\partial_l u^{(0)}_k(x) + m_j\partial_j\vartheta^{(0)}(x) + \nu_0 m_j\partial_j\vartheta^{(1)}(x) \end{aligned} \tag{2.102}$$

(see below Remark 2.18 concerning the weak formulation of these equations).

Further, from the Dirichlet boundary conditions (2.67), (2.68) for $t = 0$ we have

$$\begin{aligned} \{\varphi(x,0)\}^+ &= f_4(x,0),\\ \{\psi(x,0)\}^+ &= f_5(x,0), \end{aligned} \quad x \in S. \tag{2.103}$$

Note that, if $\nu_0 = 0$, i.e., in the case of the classical thermo-electro-magneto-elasticity theory, the function $\vartheta^{(1)}$ is not involved in the right hand side expressions in (2.102).

Now, let

$$L(\partial_x) := \begin{bmatrix} \varkappa_{jl}\partial_j\partial_l & a_{jl}\partial_j\partial_l\\ a_{jl}\partial_j\partial_l & \mu_{jl}\partial_j\partial_l \end{bmatrix}_{2\times 2}, \quad \Upsilon(\partial_x, n) := \begin{bmatrix} \varkappa_{jl}n_j\partial_l & a_{jl}n_j\partial_l\\ a_{jl}n_j\partial_l & \mu_{jl}n_j\partial_l \end{bmatrix}_{2\times 2}. \tag{2.104}$$

Note that, for vector functions $X = (X_1, X_2)^\top$ and $\widetilde{X} = (\widetilde{X}_1, \widetilde{X}_2)^\top$ possessing appropriate smoothness properties, we have the following Green's formula

$$\int_{\Omega^+} L(\partial_x)X(x)\cdot\widetilde{X}(x)\,dx = -\int_{\Omega^+} \Lambda^{(1)}\nabla X(x)\cdot\nabla\widetilde{X}(x)\,dx + \int_S \big\{\Upsilon(\partial_x, n)X(x)\big\}^+\cdot\big\{\widetilde{X}(x)\big\}^+\,dS, \tag{2.105}$$

where the $6\times 6$ matrix $\Lambda^{(1)}$ is given by (2.14), $\nabla X(x) := (\nabla X_1(x)), \nabla X_2(x))^\top$ and $\nabla\widetilde{X}(x) := (\nabla\widetilde{X}_1(x)), \nabla\widetilde{X}_2(x))^\top$ are six dimensional vectors.

This formula can be extended to arbitrary vector functions $X, \widetilde{X} \in [H_2^1(\Omega^+)]^2$, where in addition $L(\partial_x)X \in [L_2(\Omega^+)]^2$. Note that, for such $X$ the generalized trace on $S$ of the co-normal derivative $\{\Upsilon(\partial_x, n)X\}^+$ is well defined by (2.105) and belongs to the space $[H_2^{-\frac{1}{2}}(S)]^2$,

$$\big\langle\{\Upsilon(\partial_x, n)X\}^+, \{\widetilde{X}\}^+\big\rangle_S := \int_{\Omega^+} \Lambda^{(1)}\nabla X(x)\cdot\nabla\widetilde{X}(x)\,dx + \int_{\Omega^+} L(\partial_x)X(x)\cdot\widetilde{X}(x)\,dx, \tag{2.106}$$

where $\langle\,\cdot\,,\,\cdot\,\rangle_S$ denotes the duality relation between the mutually adjoint function spaces $[H_2^{-\frac{1}{2}}(S)]^2$ and $[H_2^{\frac{1}{2}}(S)]^2$.

Due to positive definiteness of the matrix $\Lambda^{(1)}$ and the Friedrichs inequality, the bilinear form

$$B(X, \widetilde{X}) := \int_{\Omega^+} \Lambda^{(1)}\nabla X(x)\cdot\nabla\widetilde{X}(x)\,dx, \quad X, \widetilde{X} \in [\widetilde{H}_2^1(\Omega^+)]^2 \times [\widetilde{H}_2^1(\Omega^+)]^2 \tag{2.107}$$

is strongly coercive in the space $[\widetilde{H}_2^1(\Omega^+)]^2 \times [\widetilde{H}_2^1(\Omega^+)]^2$, i.e.,

$$B(X, X) \geqslant \kappa_1\|\nabla X\|^2_{[L_2(\Omega^+)]^2} \geqslant \kappa_1^*\|X\|^2_{[H_2^1(\Omega^+)]^2}, \quad \kappa_1^* = const > 0, \tag{2.108}$$

for all $X = (X_1, X_2)^\top \in [\widetilde{H}_2^1(\Omega^+)]^2$ with a positive constant $\kappa_1$ involved in (2.15).

In view of the coercivity property (2.108), for a Lipschitz domain $\Omega^+$ and

$$\begin{gathered} \varrho_e(\,\cdot\,,0), \varrho_c(\,\cdot\,,0) \in H_2^{-1}(\Omega^+), \quad u_k^{(0)} \in H_2^1(\Omega^+), \quad \vartheta^{(0)}, \vartheta^{(1)} \in L_2(\Omega^+), \\ f_4(\,\cdot\,,0), f_5(\,\cdot\,,0) \in H_2^{\frac{1}{2}}(S), \end{gathered} \tag{2.109}$$

the nonhomogeneous Dirichlet boundary value problem (2.101), (2.103) possesses a unique weak solution $\varphi(\,\cdot\,,0), \psi(\,\cdot\,,0) \in H_2^1(\Omega^+)$, where equations (2.101) are understood in the distributional sense, i.e.,

$$\begin{aligned} B(X, \widetilde{X}) = \langle\varrho_e, \widetilde{X}_1\rangle_{\Omega^+} + \langle\varrho_c, \widetilde{X}_2\rangle_{\Omega^+} + \int_{\Omega^+} \Big\{ \big[e_{jkl}\partial_l u_k^{(0)}(x) + p_j\vartheta^{(0)}(x) + \nu_0 p_j\vartheta^{(1)}(x)\big]\partial_j\widetilde{X}_1(x) \\ + \big[q_{jkl}\partial_l u_k^{(0)}(x) + m_j\vartheta^{(0)}(x) + \nu_0 m_j\vartheta^{(1)}(x)\big]\partial_j\widetilde{X}_2(x)\Big\}\,dx \ \text{ for all } \ \widetilde{X} = (\widetilde{X}_1, \widetilde{X}_2)^\top \in [\mathcal{D}(\Omega^+)]^2, \end{aligned}$$

while the Dirichlet boundary conditions (2.103) are understood in the usual trace sense.

Indeed, we can decompose the nonhomogeneous Dirichlet boundary value problem (2.101), (2.103) into two Dirichlet boundary value problems:

$$\begin{gathered} L(\partial_x)X^{(1)} = Y^{(1)} \ \text{ in } \ \Omega^+, \\ \{X^{(1)}\}^+ = 0 \ \text{ on } \ S, \end{gathered} \tag{2.110}$$

with

$$X^{(1)} := (\varphi^{(1)}, \psi^{(1)})^\top \in [\widetilde{H}_2^1(\Omega^+)]^2, \quad Y^{(1)} := (Y_4, Y_5)^\top \in [H_2^{-1}(\Omega^+)]^2, \tag{2.111}$$

and

$$\begin{gathered} L(\partial_x)X^{(2)} = 0 \ \text{ in } \ \Omega^+, \\ \{X^{(2)}\}^+ = g \ \text{ on } \ S, \end{gathered} \tag{2.112}$$

with

$$X^{(2)} := (\varphi^{(2)}, \psi^{(2)})^\top \in [H_2^1(\Omega^+)]^2, \quad g = \big(f_4(\,\cdot\,,0), f_5(\,\cdot\,,0)\big)^\top \in [H_2^{\frac{1}{2}}(S)]^2. \tag{2.113}$$

In view of (2.106) and (2.107), the weak formulation of the first problem (2.110) reads as

$$B(X^{(1)}, \widetilde{X}) = -\langle Y^{(1)}, \widetilde{X}\rangle_{\Omega^+} \ \text{ for all } \ \widetilde{X} = (\widetilde{X}_1, \widetilde{X}_2)^\top \in [\widetilde{H}_2^1(\Omega^+)]^2, \tag{2.114}$$

and is uniquely solvable in $[\widetilde{H}_2^1(\Omega^+)]^2$ by the Lax–Milgram theorem due to the coercivity property (2.108) and since for arbitrary $\widetilde{X} = (\widetilde{X}_1, \widetilde{X}_2)^\top \in [\widetilde{H}_2^1(\Omega^+)]^2$ the duality

$$- \langle Y^{(1)}, \widetilde{X}\rangle_{\Omega^+} = \langle \varrho_e, \widetilde{X}_1\rangle_{\Omega^+} + \langle \varrho_c, \widetilde{X}_2\rangle_{\Omega^+} + \int_{\Omega^+} \Big\{ \big[e_{jkl}\partial_l u_k^{(0)}(x) + p_j\vartheta^{(0)}(x) + \nu_0 p_j \vartheta^{(1)}(x)\big]\partial_j \widetilde{X}_1(x) + \big[q_{jkl}\partial_l u_k^{(0)}(x) + m_j\vartheta^{(0)}(x) + \nu_0 m_j \vartheta^{(1)}(x)\big]\partial_j \widetilde{X}_2(x)\Big\}\, dx$$

defines a bounded linear functional in $[\widetilde{H}_2^1(\Omega^+)]^2$ under conditions (2.109).

The second problem (2.112) is also uniquely solvable in $[H_2^1(\Omega^+)]^2$ which can be proved by standard potential method (for details see, e.g. [25], [75, Ch. 7]; see also [51, Ch. 9], [27, Ch. 11]). A unique weak solution of the problem (2.112) is representable in the form

$$X^{(2)} = \mathbf{V}(\mathbb{H}^{-1}g) \in [H_2^1(\Omega^+)]^2,$$

where $\mathbf{V}$ is a single layer potential operator constructed by the fundamental matrix $\Gamma^{(L)} = [\Gamma_{kj}^{(L)}]_{2\times 2}$ of the operator $L(\partial)$,

$$\mathbf{V}(f)(x) = \int_{\partial\Omega^+} \Gamma^{(L)}(x-y) f(y)\, dS, \quad f = (f_1, f_2)^\top,$$

possessing the mapping properties

$$\mathbf{V} : [H_2^{-\frac{1}{2}}(S)]^2 \to [H_2^1(\Omega^+)]^2 \ \text{ for Lipschitz } \ S = \partial\Omega^+,$$
$$\mathbf{V} : [C^{r,\kappa'}(S)]^2 \to [C^{r+1,\kappa'}(\overline{\Omega^+})]^2, \ \ S \in C^{m,\kappa}, \ \ m \in \mathbb{N}, \ \ 0 \leqslant r \leq m-1, \ \ 0 < \kappa' < \kappa \leqslant 1,$$

while $\mathbb{H}$ is an integral operator generated by the trace on $S = \partial\Omega^+$ of the single layer potential, $\mathbb{H}(f) = \{\mathbf{V}(f)\}_S^+$. Note that, due to the positive definiteness of the matrix $\Lambda^{(1)}$ given by (2.14), the operator $\mathbb{H}$ is coercive on $[H_2^{-\frac{1}{2}}(S)]^2$, i.e., there is a positive constant $C_0 = const > 0$ such that for all $f \in [H_2^{-\frac{1}{2}}(S)]^2$

$$\langle \mathbb{H} f, f\rangle_S \geqslant C_0 \|f\|_{[H_2^{-\frac{1}{2}}(S)]^2},$$

implying the invertibility of the operator

$$\mathbb{H} : [H_2^{-\frac{1}{2}}(S)]^2 \to [H_2^{\frac{1}{2}}(S)]^2.$$

Now, it is evident that the vector function

$$X = \big(\varphi(\,\cdot\,,0), \psi(\,\cdot\,,0)\big)^\top := X^{(1)} + X^{(2)} \in [H_2^1(\Omega^+)]^2$$

is a unique weak solution of the nonhomogeneous Dirichlet problem (2.101), (2.103) for a Lipschitz domain $\Omega^+$.

*Remark* 2.7. It is evident that in the case of the homogeneous dynamical Dirichlet problem $(\mathrm{D})_t^+$ the vector function $(\varphi(x,0), \psi(x,0))^\top$ solves the homogeneous Dirichlet boundary value problem (2.101)–(2.103) with $Y_4 = Y_5 = 0$ and $f_4 = f_5 = 0$, and, consequently, $\varphi(\,\cdot\,,0) = \psi(\,\cdot\,,0) = 0$ in $\Omega^+$.

Due to the classical interior and boundary regularity results, if the surface $S$ and the functions (2.109) are smoother, then the functions $\varphi(\,\cdot\,,0)$ and $\psi(\,\cdot\,,0)$ are smoother as well (for details see, e.g. [75, Ch. 4, Theorem 4.18]). In particular, if for some integer $r \geqslant 0$ the following conditions hold

$$\begin{gathered} S \in C^{r+1,1}, \ \ \varrho_e(\,\cdot\,,0), \varrho_c(\,\cdot\,,0) \in H_2^r(\Omega^+), \ \ u_k^{(0)} \in H_2^{r+2}(\Omega^+), \\ \vartheta^{(0)}, \vartheta^{(1)} \in H_2^{r+1}(\Omega^+), \ \ f_4(\,\cdot\,,0), f_5(\,\cdot\,,0) \in H_2^{r+\frac{3}{2}}(S), \end{gathered} \tag{2.115}$$

then $\varphi(\,\cdot\,,0), \psi(\,\cdot\,,0) \in H_2^{r+2}(\Omega^+)$. Further, if for some integer $r \geqslant 0$ and $0 < \kappa' < \kappa \leqslant 1$ the following conditions hold (cf. [45], [78], [57], [3])

$$\begin{gathered} S \in C^{r+1,\kappa}, \ \ \varrho_e(\,\cdot\,,0), \varrho_c(\,\cdot\,,0) \in C^{r,\kappa'}(\overline{\Omega^+}), \ \ u_k^{(0)} \in C^{r+2,\kappa'}(\overline{\Omega^+}), \\ \vartheta^{(0)}, \vartheta^{(1)} \in C^{r+1,\kappa'}(\overline{\Omega^+}), \ \ f_4(\,\cdot\,,0), f_5(\,\cdot\,,0) \in C^{r,\kappa'}(S), \end{gathered} \tag{2.116}$$

then $\varphi(\,\cdot\,,0), \ \psi(\,\cdot\,,0) \in C^{r,\kappa'}(\overline{\Omega^+})$.

**Remarks concerning the Neumann problem.** Now, let us consider the dynamical Neumann initial-boundary value problem $(\mathrm{N})_t^+$. In this case, if the problem $(\mathrm{N})_t^+$ possesses a solution and the corresponding smoothness conditions are satisfied, the function $\varphi(\,\cdot\,,0)$ and $\psi(\,\cdot\,,0)$ solve again the equations (2.101), (2.102) and satisfy the Neumann type boundary conditions (see (2.72), (2.73), and (2.25))

$$\begin{aligned}&\big\{\varkappa_{jl}n_j\partial_l\varphi(x,0)+a_{jl}n_j\partial_l\psi(x,0)\big\}^+=G_4(x),\\&\big\{a_{jl}n_j\partial_l\varphi(x,0)+\mu_{jl}n_j\partial_l\psi(x,0)\big\}^+=G_5(x),\end{aligned}\quad x\in S, \tag{2.117}$$

with

$$\begin{aligned}G_4(x)&:=F_4(x,0)+\Big\{e_{jkl}n_j\partial_l u_k^{(0)}(x)+n_jp_j\big(\vartheta^{(0)}(x)+\nu_0\vartheta^{(1)}(x)\big)\Big\}^+,\ \ x\in S,\\G_5(x)&:=F_5(x,0)+\Big\{q_{jkl}n_j\partial_l u_k^{(0)}(x)+n_jm_j\big(\vartheta^{(0)}(x)+\nu_0\vartheta^{(1)}(x)\big)\Big\}^+,\ \ x\in S.\end{aligned} \tag{2.118}$$

For sufficiently smooth data involved in (2.118) and (2.102), the corresponding solution vector $X=(\varphi(\,\cdot\,,0),\psi(\,\cdot\,,0))^\top$ to the Neumann type problem (2.101), (2.117) will be also regular and for an arbitrary smooth vector function $\widetilde{X}=(\widetilde{X}_1,\widetilde{X}_2)^\top$, in view of (2.105) the following relation

$$\int_{\Omega^+}\Lambda^{(1)}\nabla X(x)\cdot\nabla\widetilde{X}(x)\,dx=\int_S G(x)\cdot\big\{\widetilde{X}(x)\big\}^+\,dS-\int_{\Omega^+}Y^{(1)}(x)\cdot\widetilde{X}(x)\,dx \tag{2.119}$$

holds, where $G=(G_4,G_5)^\top$ is given by (2.118) and $Y^{(1)}=(Y_4,Y_5)^\top$ with $Y_4$ and $Y_5$ being defined in (2.102).

With the help of the equalities (2.118) and (2.102), and applying the integration by parts formula, the right hand side of (2.119) can be rewritten as

$$\begin{aligned}\mathcal{G}(\widetilde{X})&:=\int_S G(x)\cdot\big\{\widetilde{X}(x)\big\}^+\,dS-\int_{\Omega^+}Y^{(1)}(x)\cdot\widetilde{X}(x)\,dx\\&=\int_S\Big[F_4(x,0)\big\{\widetilde{X}_1(x)\big\}^++F_5(x,0)\big\{\widetilde{X}_2(x)\big\}^+\Big]\,dS\\&\quad+\int_{\Omega^+}\Big\{\varrho_e(x,0)\widetilde{X}_1(x)+\big[e_{jkl}\partial_l u_k^{(0)}(x)+p_j\big(\vartheta^{(0)}(x)+\nu_0\vartheta^{(1)}(x)\big)\big]\partial_j\widetilde{X}_1(x)\\&\quad+\varrho_c(x,0)\widetilde{X}_2(x)+\big[q_{jkl}\partial_l u_k^{(0)}(x)+m_j\big(\vartheta^{(0)}(x)+\nu_0\vartheta^{(1)}(x)\big)\big]\partial_j\widetilde{X}_2(x)\Big\}\,dx.\end{aligned} \tag{2.120}$$

Finally, employing the notations (2.107) and (2.120) we can rewrite (2.119) as

$$B(X,\widetilde{X})=\mathcal{G}(\widetilde{X}). \tag{2.121}$$

This equation can be extended to vector functions $X,\widetilde{X}\in\big[H_2^1(\Omega^+)\big]^2$, if

$$\begin{aligned}&\varrho_e(\,\cdot\,,0),\ \varrho_c(\,\cdot\,,0)\in\widetilde{H}_2^{-1}(\Omega^+),\ \ u_k^{(0)}\in H_2^1(\Omega^+),\ \ k=1,2,3,\\&\vartheta^{(0)},\vartheta^{(1)}\in L_2(\Omega^+),\ \ F_4(\,\cdot\,,0),F_5(\,\cdot\,,0)\in H_2^{-\frac12}(S).\end{aligned} \tag{2.122}$$

In this case

$$\begin{aligned}\mathcal{G}(\widetilde{X})&:=\big\langle F_4(x,0),\{\widetilde{X}_1(x)\}^+\big\rangle_S+\big\langle F_5(x,0),\{\widetilde{X}_2(x)\}^+\big\rangle_S\\&\quad+\big\langle\varrho_e(x,0),\widetilde{X}_1(x)\big\rangle_{\Omega^+}+\big\langle\varrho_c(x,0),\widetilde{X}_2(x)\big\rangle_{\Omega^+}\\&\quad+\int_{\Omega^+}\Big\{\big[e_{jkl}\partial_l u_k^{(0)}(x)+p_j\big(\vartheta^{(0)}(x)+\nu_0p_j\vartheta^{(1)}(x)\big)\big]\partial_j\widetilde{X}_1(x)\\&\quad+\big[q_{jkl}\partial_l u_k^{(0)}(x)+m_j\big(\vartheta^{(0)}(x)+\nu_0m_j\vartheta^{(1)}(x)\big)\big]\partial_j\widetilde{X}_2(x)\Big\}\,dx,\end{aligned} \tag{2.123}$$

where the dualities on $S$ and $\Omega^+$ are well defined due to (2.122).

Note that, under conditions (2.122) the linear functional $\mathcal{G}$ given by formula (2.123) is bounded on $[H_2^1(\Omega^+)]^2$ and due to positive definiteness of the matrix $\Lambda^{(1)}$ given by (2.14) we have

$$B(X,X)\geqslant\kappa_1\big(\|\nabla X_1\|^2_{L_2(\Omega^+)}+\|\nabla X_2\|^2_{L_2(\Omega^+)}\big) \tag{2.124}$$

for arbitrary $X=(X_1,X_2)^\top\in[H_2^1(\Omega^+)]^2$ with $\kappa_1$ involved in (2.15).

The weak formulation of the nonhomogeneous Neumann type boundary value problem (2.101) and (2.117) then reads as follows: *Find* $X = (\varphi(\,\cdot\,,0), \psi(\,\cdot\,,0))^\top \in [H^1_2(\Omega^+)]^2$ *such that the following equation*

$$B(X, \widetilde{X}) \equiv \int_{\Omega^+} \Lambda^{(1)} \nabla X(x) \cdot \nabla \widetilde{X}(x)\, dx = \mathcal{G}(\widetilde{X}) \tag{2.125}$$

*holds for arbitrary* $\widetilde{X} \in [H^1_2(\Omega^+)]^2$*, where* $\mathcal{G}(\widetilde{X})$ *is given by* (2.123) *and* $\Omega^+$ *is assumed to be a Lipschitz domain.*

It is easy to show that in the space of regular vector functions, $\varphi(\,\cdot\,,0) = c_4$ and $\psi(\,\cdot\,,0) = c_5$ with $c_4$ and $c_5$ being arbitrary constants, represent the general solution to the homogeneous boundary value problem (2.101), (2.117) with $Y_4 = Y_5 = 0$ and $G_4 = G_5 = 0$.

The same is valid for the homogeneous variational equation (2.125) with $\mathcal{G} = 0$: the vector $X = \big(c_4, c_5\big)^\top$ represents a general weak solution to the equation $B(X, \widetilde{X}) = 0$ for arbitrary $\widetilde{X} \in [H^1_2(\Omega^+)]^2$.

It is well known that the necessary and sufficient condition for the nonhomogeneous problem (2.101), (2.117) with sufficiently smooth data to be solvable in the space of regular vector functions is the following relation (see, e.g., [78])

$$\int_S G(x)\, dS = \int_{\Omega^+} Y^{(1)}(x)\, dx. \tag{2.126}$$

In view of (2.102) and (2.118) it then follows that equality (2.126) is equivalent to the conditions

$$\int_S F_4(x,0)\, dS + \int_{\Omega^+} \varrho_e(x,0)\, dx = 0, \quad \int_S F_5(x,0)\, dS + \int_{\Omega^+} \varrho_c(x,0)\, dx = 0. \tag{2.127}$$

Similarly, in the case of weak formulation of the Neumann problem under assumptions (2.122), it can be shown that the necessary and sufficient conditions for solvability of equation (2.125) read as (cf. [74], [51], [75], [93])

$$\mathcal{G}(E^{(1)}) \equiv \langle F_4(x,0), 1\rangle_S + \langle \varrho_e(x,0), 1\rangle_{\Omega^+} = 0, \quad E^{(1)} = (1,0)^\top, \tag{2.128}$$

$$\mathcal{G}(E^{(2)}) \equiv \langle F_5(x,0), 1\rangle_S + \langle \varrho_c(x,0), 1\rangle_{\Omega^+} = 0, \quad E^{(2)} = (0,1)^\top, \tag{2.129}$$

where $\mathcal{G}$ is defined in (2.123).

As a byproduct, from these results it follows that the relations (2.128), (2.129) are the necessary conditions for solvability of the dynamical Neumann problem $(\mathrm{N})^+_t$ as well.

*Remark* 2.8. Note that in the case of the weak formulation (2.125) of the Neumann problem (2.101), (2.102) and (2.117), the sought for vector function $X = (\varphi(\,\cdot\,,0), \psi(\,\cdot\,,0))^\top$ belongs to the function space $[H^1(\Omega^+)]^2$ and is a distributional solution of the differential equation

$$L(\partial_x)X = Y^{(1)} \ \text{ in } \ \Omega^+,$$

where $L(\partial_x)$ is defined in (2.104) and $Y^{(1)} := (Y_4, Y_5)^\top$ with $Y_4$ and $Y_5$ given by (2.102). Due to the conditions (2.122) we can assume that $Y^{(1)} \in [\widetilde{H}^{-1}(\Omega^+)]^2$. But in this case, the trace of the conormal derivative $\{\Upsilon(\partial_x, n)X\}^+$ (see (2.104)) can not be defined uniquely by a reasonable definition, say by Green's formula (2.106), where the second integral over $\Omega^+$ in the right hand side is understood as the duality $\langle L(\partial_x)X, \widetilde{X}\rangle_{\Omega^+}$ between the mutually adjoint spaces $[\widetilde{H}^{-1}(\Omega^+)]^2$ and $[H^{-1}(\Omega^+)]^2$. The case is that here we need extension of the distribution $L(\partial_x)X$ from the open domain $\Omega^+$ to $\Omega^-$ by zero which is not unique and in general is defined modulo a nontrivial distribution from $[\widetilde{H}^{-1}(\Omega^+)]^2$ with support in $\partial\Omega^+$. Note that the space $[\widetilde{H}^{-1}(\Omega^+)]^2$ contains nontrivial distributions from $[H^{-1}(\mathbb{R}^3)]^2$ with support in $\partial\Omega^+$ (for details see [75, Ch. 4], [76]).

Therefore in the framework of the weak formulation of the Neumann problem we can speak about the Neumann condition (2.117) if the vector $L(\partial_x)X = Y^{(1)} \in [H^s(\Omega^+)]^2$ as a distribution admits a unique continuation from $\Omega^+$ to $\Omega^-$ by zero. This is the case, e.g., when $L(\partial_x)X = Y^{(1)} \in [L_2(\Omega^+)]^2$, and then the generalised trace of the conormal derivative $\{\Upsilon(\partial_x, n)X\}^+$ can be defined by Green's formula (2.106) with $Y^{(1)}$ for $L(\partial_x)X$. Evidently, in this case we need higher regularity for the initial data and the right hand side functions $\varrho_e(\,\cdot\,,0)$, $\varrho_c(\,\cdot\,,0)$, and we require that

$$\begin{gathered} \varrho_e(\,\cdot\,,0), \varrho_c(\,\cdot\,,0) \in L_2(\Omega^+), \ \ u_k^{(0)} \in H^2_2(\Omega^+), \ \ k = 1,2,3, \\ \vartheta^{(0)}, \vartheta^{(1)} \in H^1_2(\Omega^+), \ \ F_4(\,\cdot\,,0), F_5(\,\cdot\,,0) \in H_2^{-\frac{1}{2}}(S), \end{gathered} \tag{2.130}$$

implying (see (2.102), (2.117), and (2.118))

$$Y^{(1)} \in [L_2(\Omega^+)]^2, \ \ G_4(\,\cdot\,,0), G_5(\,\cdot\,,0) \in H_2^{-\frac{1}{2}}(S), \tag{2.131}$$

which in turn imply that $X = (\varphi(\,\cdot\,,0), \psi(\,\cdot\,,0))^\top \in [H^1(\Omega^+)]^2$ with $L(\partial_x)X = Y^{(1)} \in [L_2(\Omega^+)]^2$. Therefore we conclude that the conormal derivative $\{\Upsilon(\partial_x, n)X\}^+$ is correctly defined by Green's formula (2.106) and, consequently, the Neumann boundary conditions (2.117) are understood in the generalized functional trace sense (cf., e.g., [75, Ch. 4]).

If instead of inclusions (2.130), there hold

$$\begin{gathered} \varrho_e(\,\cdot\,,0), \varrho_c(\,\cdot\,,0) \in L_2(\Omega^+), \ \ u_k^{(0)} \in H_2^2(\Omega^+), \ \ k = 1,2,3, \\ \vartheta^{(0)}, \vartheta^{(1)} \in H_2^1(\Omega^+), \ \ F_4(\,\cdot\,,0), F_5(\,\cdot\,,0) \in H_2^{\frac{1}{2}}(S), \end{gathered} \tag{2.132}$$

then

$$Y^{(1)} \in [L_2(\Omega^+)]^2, \ \ G_4(\,\cdot\,,0), G_5(\,\cdot\,,0) \in H_2^{\frac{1}{2}}(S), \tag{2.133}$$

which in turn imply that $X = (\varphi(\,\cdot\,,0), \psi(\,\cdot\,,0))^\top \in [H^2(\Omega^+)]^2$ and, consequently, the Neumann boundary conditions (2.117) are satisfied in the usual trace sense (cf., e.g., [75, Ch. 4, Theorem 4.18]).

*Remark* 2.9. From the above results it follows that if the nonhomogeneous dynamical Neumann problem $(\mathrm{N})_t^+$ is solvable and the conditions (2.128), (2.129), and (2.132) hold, then the functions $\varphi(x,0)$ and $\psi(x,0)$ belong to the space $H^2(\Omega^+)$ and they are defined modulo constant summands by the data of the problem $(\mathrm{N})_t^+$. In particular, it is evident that in the case of the homogeneous dynamical Neumann problem $(\mathrm{N})_t^+$, all the above required conditions are satisfied, and the functions $\varphi(x,0)$ and $\psi(x,0)$ could be only arbitrary constants and do not vanish, in general.

**Remarks concerning the mixed problem.** It can analogously be shown that in the case of the mixed type dynamical problem $(\mathrm{M})_t^+$, as in the case of the dynamical Dirichlet problem, the functions $\varphi(\,\cdot\,,0), \psi(\,\cdot\,,0) \in H_2^1(\Omega^+)$ are defined uniquely by the data of the problem. In particular, this implies that for the homogeneous dynamical mixed problem $(\mathrm{M})_t^+$ they vanish, $\varphi(\,\cdot\,,0) = \psi(\,\cdot\,,0) = 0$ in $\Omega^+$.

Indeed, for illustration let us consider the semi-regular setting of the *basic mixed dynamical problem associated with the dissection* $S = \overline{S}_D \cup \overline{S}_N$ (see Remark 2.4). In this case the vector $X(x,0) = (\varphi(\,\cdot\,,0), \psi(\,\cdot\,,0))^\top$ solves the system of differential equations (2.101) in $\Omega^+$ and satisfies the Dirichlet type condition on $S_D$ and the Neumann type condition on $S_N$:

$$L(\partial_x)X = Y^{(1)} \ \text{ in } \ \Omega^+, \tag{2.134}$$

$$\{X\}^+ = g^* \ \text{ on } \ S_D, \tag{2.135}$$

$$\{\Upsilon(\partial_x, n)X\}^+ = G^* \ \text{ on } \ S_N, \tag{2.136}$$

where $Y^{(1)} = (Y_4, Y_5)$ with $Y_4$ and $Y_5$ defined in (2.102), the boundary operator $\Upsilon(\partial, n)$ is defined in (2.104), and in accordance with boundary conditions (2.81)–(2.84): $g^* = (f_4^*(\,\cdot\,,0), f_5^*(\,\cdot\,,0))^\top$ and $G^* = (G_4^*(\,\cdot\,,0), G_5^*(\,\cdot\,,0))^\top$ where $G_4^*$ and $G_5^*$ are given by (2.118) on $S_N$ with $F_4^*$ and $F_5^*$ for $F_4$ and $F_5$, respectively.

We decompose this problem into two boundary value problems:

$$L(\partial_x)X^{(1)} = 0 \ \text{ in } \ \Omega^+, \tag{2.137}$$

$$\{X^{(1)}\}^+ = g^* \ \text{ on } \ S_D, \tag{2.138}$$

$$\{\Upsilon(\partial_x, n)X^{(1)}\}^+ = 0 \ \text{ on } \ S_N, \tag{2.139}$$

and

$$L(\partial_x)X^{(2)} = Y^{(1)} \ \text{ in } \ \Omega^+, \tag{2.140}$$

$$\{X^{(2)}\}^+ = 0 \ \text{ on } \ S_D, \tag{2.141}$$

$$\{\Upsilon(\partial_x, n)X^{(2)}\}^+ = G^* \ \text{ on } \ S_N, \tag{2.142}$$

Using the potential method it can be shown that the mixed boundary value problem (2.137)–(2.139) possesses a unique solution (see, e.g. [25], [75, Ch. 7])

$$X^{(1)} = (\varphi^{(1)}, \psi^{(1)})^\top \in [H_2^1(\Omega^+)]^2 \tag{2.143}$$

if $\Omega^+$ is Lipschitz domain, $S_D$ and $S_N$ are Lipschitz manifolds with Lipschitz boundary, and

$$g^* = \big(f_4^*(\,\cdot\,,0), f_5^*(\,\cdot\,,0)\big)^\top \in [H^{\frac{1}{2}}(S_D)]^2. \tag{2.144}$$

On the other hand, applying the same arguments as in the case of the Neumann problem, the second mixed problem (2.140)–(2.142) under conditions

$$\begin{gathered} \varrho_e(\,\cdot\,,0), \varrho_c(\,\cdot\,,0) \in \widetilde{H}_2^{-1}(\Omega^+), \ \ u_k^{(0)} \in H_2^1(\Omega^+), \ \ \vartheta^{(0)}, \vartheta^{(1)} \in L_2(\Omega^+), \\ F_4^*(\,\cdot\,,0), F_5^*(\,\cdot\,,0) \in [H_2^{-\frac{1}{2}}(S_N)]^2, \end{gathered} \tag{2.145}$$

can be reformulated as a variational equation (see (2.119), (2.120))

$$B(X^{(2)}, \widetilde{X}^{(2)}) \equiv \int_{\Omega^+} \Lambda^{(1)} \nabla X^{(2)}(x) \cdot \nabla \widetilde{X}^{(2)}(x)\, dx = \mathcal{G}_2(\widetilde{X}^{(2)}) \tag{2.146}$$

for the unknown vector function

$$X^{(2)} = (\varphi^{(2)}, \psi^{(2)})^\top \in [H_2^1(\Omega^+, S_D)]^2, \tag{2.147}$$

where $[H_2^1(\Omega^+, S_D)]^2$ is a subspace of $[H_2^1(\Omega^+)]^2$,

$$[H_2^1(\Omega^+, S_D)]^2 = \big\{X \in [H_2^1(\Omega^+)]^2 : \ r_{S_D} X = 0\big\}, \tag{2.148}$$

endowed with the norm of the space $[H_2^1(\Omega^+)]^2$, $\widetilde{X}^{(2)} = (\widetilde{X}_1^{(2)}, \widetilde{X}_2^{(2)})^\top \in [H_2^1(\Omega^+, S_D)]^2$, and $\mathcal{G}_2$ is a well defined bounded linear functional on $[H_2^1(\Omega^+, S_D)]^2$ (cf. (2.123))

$$\begin{aligned} \mathcal{G}_2(\widetilde{X}^{(2)}) := {}& \big\langle F_4^*(x,0), \{\widetilde{X}_1^{(2)}(x)\}^+ \big\rangle_{S_N} + \big\langle F_5^*(x,0), \{\widetilde{X}_2^{(2)}(x)\}^+ \big\rangle_{S_N} \\ & + \big\langle \varrho_e(x,0), \widetilde{X}_1^{(2)}(x) \big\rangle_{\Omega^+} + \big\langle \varrho_c(x,0), \widetilde{X}_2^{(2)}(x) \big\rangle_{\Omega^+} \\ & + \int_{\Omega^+} \Big\{ \big[ e_{jkl} \partial_l u_k^{(0)}(x) + p_j \big(\vartheta^{(0)}(x) + \nu_0 \vartheta^{(1)}(x)\big)\big] \partial_j \widetilde{X}_1^{(2)}(x) \\ & \qquad + \big[ q_{jkl} \partial_l u_k^{(0)}(x) + m_j \big(\vartheta^{(0)}(x) + \nu_0 \vartheta^{(1)}(x)\big)\big] \partial_j \widetilde{X}_2^{(2)}(x) \Big\}\, dx. \end{aligned} \tag{2.149}$$

Evidently, the bilinear form $B(X^{(2)}, \widetilde{X}^{(2)})$ is bounded and strongly coercive in $[H_2^1(\Omega^+, S_D)]^2$ (see, e.g., [93, Section 1.1.8, Theorem 1.10]), i.e., there is a positive constant $\kappa$ such that

$$B(X^{(2)}, X^{(2)}) \geqslant \kappa \|X^{(2)}\|^2_{H_2^1(\Omega^+)} \ \text{ for all } \ X^{(2)} \in [H_2^1(\Omega^+, S_D)]^2. \tag{2.150}$$

Consequently, equation (2.146) is uniquely solvable in the space $[H_2^1(\Omega^+, S_D)]^2$ by Lax-Milgram theorem.

Therefore, we conclude that if the conditions (2.144) and (2.145) hold, then the mixed problem (2.134)–(2.136) possesses a unique weak solution $X$ which can be represented as

$$X = \big(\varphi(\,\cdot\,,0), \psi(\,\cdot\,,0)\big)^\top = X^{(1)} + X^{(2)} \in [H_2^1(\Omega^+)]^2.$$

*Remark* 2.10. Here we have to take into consideration the arguments and reasonings presented in Remark 2.8 concerning existence of the trace of conormal derivative $\{\Upsilon(\partial_x, n)X\}^+$ on $S_N$. In particular, if the conditions

$$\begin{gathered} \varrho_e(\,\cdot\,,0), \varrho_c(\,\cdot\,,0) \in L_2(\Omega^+), \ \ u_k^{(0)} \in H_2^2(\Omega^+), \ \ k=1,2,3, \ \ \vartheta^{(0)}, \vartheta^{(1)} \in H_2^1(\Omega^+), \\ f_4^*(\,\cdot\,,0), f_5^*(\,\cdot\,,0) \in H_2^{\frac{1}{2}}(S_D), \ \ F_4^*(\,\cdot\,,0), F_5^*(\,\cdot\,,0) \in H_2^{-\frac{1}{2}}(S_N), \end{gathered} \tag{2.151}$$

hold, then (see (2.102), (2.117), and (2.118))

$$Y^{(1)} \in [L_2(\Omega^+)]^2, \ \ g^* \in H_2^{\frac{1}{2}}(S_D), \ \ G^* \in H_2^{-\frac{1}{2}}(S_N) \tag{2.152}$$

which in turn imply that $X = (\varphi(\,\cdot\,,0), \psi(\,\cdot\,,0))^\top \in [H^1(\Omega^+)]^2$ with $L(\partial_x)X = Y^{(1)} \in [L_2(\Omega^+)]^2$. Therefore we conclude that the conormal derivative $\{\Upsilon(\partial_x, n)X\}^+$ is correctly defined by Green's formula (2.106) and, consequently, the Neumann boundary conditions (2.136), (2.139), and (2.142) are understood in the generalized functional trace sense.

Unfortunately, for mixed and crack type problems we can not improve regularity of solutions up to $H^2(\Omega^+)$, in general, by increasing smoothness of data of the problems (for details see Section 5).

By the word for word arguments we can show that in the case of the crack type dynamical problems the initial values of the electric and magnetic potentials, $\varphi(x,0)$ and $\psi(x,0)$ are defined uniquely if the boundary conditions of the dynamical problems contain the Dirichlet conditions for the electric and magnetic potentials, $\varphi(x,t)$ and $\psi(x,t)$, otherwise they are defined modulo constant summands. In particular, for the homogeneous dynamical crack problems the functions $\varphi(x,0)$ and $\psi(x,0)$ either vanish or they are arbitrary constants in $\Omega_\Sigma^+$.

From the above results it follows that under appropriate assumptions, the initial data of the electric and magnetic potentials, $\varphi(x,0)$ and $\psi(x,0)$, are defined either uniquely or modulo constant summand by the data of the dynamical problems formulated in Subsection 2.2. In particular, in the case of the homogeneous dynamical problems, the functions $\varphi(x,0)$ and $\psi(x,0)$ equal to zero if the Dirichlet data for the electro-magnetic potentials are prescribed on some non-empty part of the boundary, otherwise they are arbitrary constants.

*Remark* 2.11. It is important to note here that for the pseudo-oscillation elliptic problems (see Subsection 2.3), obtained from the corresponding dynamical problems with the help of the Laplace transform, all the data are defined uniquely:

(i) for the Dirichlet type boundary data it is evident due to the Laplace transform definition, see (2.37);

(ii) for the Neumann type boundary data it follows from the fact that in view of formula (2.54) the pseudo-oscillation stress vector $\{\mathcal{T}(\partial_x,n,\tau)\widehat{U}(x,\tau)\}^+$ is uniquely defined by the Laplace transform of the given dynamical stress vector $L_{t\to\tau}[\{\mathcal{T}(\partial_x,n,\partial_t)U(x,t)\}^+]$ and the vector function $F^{(0)}$ defined by (2.56).

(iii) for the right hand sides of the pseudo-oscillation equations (2.40) (see also (2.42), (2.44)) it follows from the fact that the vector function $\Psi^{(0)}(x,\tau)$ given by (2.41), is defined uniquely since the functions $\varphi(x,0)$ and $\psi(x,0)$ are defined at most modulo constant summands, as it has been shown above.

2.3. **Boundary value problems for pseudo-oscillation equations.** Applying the Laplace transform and formulas (2.37)–(2.39), the above formulated dynamical problems can be reduced to the corresponding elliptic problems containing a complex parameter $\tau$, assuming that all the data involved in the formulation of the dynamical problems and the sought for solutions are exponentially bounded with respect to the variable $t$ (see (2.36)).

Here we preserve the notation for the operators introduced in the previous subsections and formulate the boundary value problems for the pseudo-oscillation equations of the GTEME theory. The operators $A(\partial,\tau)$ and $\mathcal{T}(\partial,n,\tau)$ involved in the formulations below are determined by the relations (2.45) and (2.57), respectively. In our analysis we always assume that

$$\tau=\sigma+i\omega,\ \ \sigma>\sigma_0\geqslant 0,\ \ \omega\in\mathbb{R}, \tag{2.153}$$

if not otherwise stated.

We recall that over posed "hat" denotes the Laplace transform of the corresponding function defined by (2.37).

**The Dirichlet pseudo-oscillation problem $(\mathrm{D})_\tau^+$:** *Find a regular complex-valued solution vector*

$$\widehat{U}=(\widehat{u},\widehat{\varphi},\widehat{\psi},\widehat{\vartheta})^\top\in[C^1(\overline{\Omega^+})]^6\cap[C^2(\Omega^+)]^6 \tag{2.154}$$

*to the pseudo-oscillation equations of the GTEME theory,*

$$A(\partial_x,\tau)\widehat{U}(x)=\Psi(x),\ \ x\in\Omega^+, \tag{2.155}$$

*satisfying the Dirichlet type boundary condition*

$$\{\widehat{U}(x)\}^+=\widehat{f}(x),\quad x\in S, \tag{2.156}$$

*i.e.*

$$\{\widehat{u}_r(x)\}^+=\widehat{f}_r(x),\ \ x\in S,\ \ r=1,2,3, \tag{2.157}$$

$$\{\widehat{\varphi}(x)\}^+=\widehat{f}_4(x),\ \ x\in S, \tag{2.158}$$

$$\{\widehat{\psi}(x)\}^+ = \widehat{f}_5(x), \ \ x \in S, \tag{2.159}$$

$$\{\widehat{\vartheta}(x)\}^+ = \widehat{f}_6(x), \ \ x \in S, \tag{2.160}$$

*where* $\Psi = (\Psi_1, \dots, \Psi_6)^\top$ *defined by* (2.44) *and* (2.41)*, and* $\widehat{f} = (\widehat{f}_1, \dots, \widehat{f}_6)^\top$ *are given smooth functions from the appropriate spaces.*

**The Neumann pseudo-oscillation problem $(\mathrm{N})^+_\tau$:** *Find a regular complex-valued solution vector*

$$\widehat{U} = (\widehat{u}, \widehat{\varphi}, \widehat{\psi}, \widehat{\vartheta})^\top \in [C^1(\overline{\Omega^+})]^6 \cap [C^2(\Omega^+)]^6$$

*to the pseudo-oscillation equations of the GTEME theory* (2.155) *satisfying the Neumann type boundary condition*

$$\{\mathcal{T}(\partial_x, n, \tau)\widehat{U}(x)\}^+ = \widehat{F}(x) - F^{(0)}(x), \quad x \in S, \tag{2.161}$$

*i.e.*

$$\big\{[\mathcal{T}(\partial_x, n, \tau)\widehat{U}(x)]_r\big\}^+ \equiv \{\widehat{\sigma_{rj}} n_j\}^+ - F^{(0)}_r(x) = \widehat{F}_r(x) - F^{(0)}_r(x), \ \ x \in S, \ r = 1,2,3, \tag{2.162}$$

$$\big\{[\mathcal{T}(\partial_x, n, \tau)\widehat{U}(x)]_4\big\}^+ \equiv \{-\widehat{D}_j n_j\}^+ - F^{(0)}_4(x) = \widehat{F}_4(x) - F^{(0)}_4(x), \ \ x \in S, \tag{2.163}$$

$$\big\{[\mathcal{T}(\partial_x, n, \tau)\widehat{U}(x)]_5\big\}^+ \equiv \{-\widehat{B}_j n_j\}^+ - F^{(0)}_5(x) = \widehat{F}_5(x) - F^{(0)}_5(x), \ \ x \in S, \tag{2.164}$$

$$\big\{[\mathcal{T}(\partial_x, n, \tau)\widehat{U}(x)]_6\big\}^+ \equiv \{-T_0^{-1}\widehat{q}_j n_j\}^+ = \widehat{F}_6(x), \ \ x \in S, \tag{2.165}$$

*where* $\widehat{F} - F^{(0)}$ *is a given continuous vector function with* $F^{(0)} = (F^{(0)}_1, \dots, F^{(0)}_6)^\top$ *defined in* (2.56).

**Mixed type pseudo-oscillation problem $(\mathrm{M})^+_\tau$:** *Find a semi-regular complex-valued solution vector*

$$\widehat{U} = (\widehat{u}, \widehat{\varphi}, \widehat{\psi}, \widehat{\vartheta})^\top \in [\mathbf{C}(\widetilde{\Omega}^+_\ell; \alpha)]^6$$

*to the pseudo-oscillation equations of the GTEME theory* (2.155) *satisfying the mixed type boundary conditions:*

$$\{\widehat{u}_r(x)\}^+ = \widehat{f}^*_r(x), \ \ x \in S_1, \ \ r = 1,2,3, \tag{2.166}$$

$$\big\{[\mathcal{T}(\partial_x, n, \tau)\widehat{U}(x)]_r\big\}^+ \equiv \{\widehat{\sigma_{rj}} n_j\}^+ - F^{(0)}_r(x) = \widehat{F}^*_r(x) - F^{(0)}_r(x), \ \ x \in S_2, \ \ r = 1,2,3, \tag{2.167}$$

$$\{\widehat{\varphi}(x)\}^+ = \widehat{f}^*_4(x), \ \ x \in S_3, \tag{2.168}$$

$$\big\{[\mathcal{T}(\partial_x, n, \tau)\widehat{U}(x)]_4\big\}^+ \equiv \{-\widehat{D}_j n_j\}^+ - F^{(0)}_4(x) = \widehat{F}^*_4(x) - F^{(0)}_4(x), \ \ x \in S_4, \tag{2.169}$$

$$\{\widehat{\psi}(x)\}^+ = \widehat{f}^*_5(x), \ \ x \in S_5, \tag{2.170}$$

$$\big\{[\mathcal{T}(\partial_x, n, \tau)\widehat{U}(x)]_5\big\}^+ \equiv \{-\widehat{B}_j n_j\}^+ - F^{(0)}_5(x) = \widehat{F}^*_5(x) - F^{(0)}_5(x), \ \ x \in S_6, \tag{2.171}$$

$$\{\widehat{\vartheta}(x)\}^+ = \widehat{f}^*_6(x), \ \ x \in S_7, \tag{2.172}$$

$$\big\{[\mathcal{T}(\partial_x, n, \tau)\widehat{U}(x)]_6\big\}^+ \equiv \{-T_0^{-1}\widehat{q}_j n_j\}^+ = \widehat{F}^*_6(x), \ \ x \in S_8, \tag{2.173}$$

*where* $\widehat{f}^*_k$ *and* $\widehat{F}^*_k - F^{(0)}_k$, $k = 1,2,\dots,6$*, are given functions form the appropriate spaces.*

**Crack type pseudo-oscillation problems:** *Find a semi-regular complex-valued vector function*

$$\widehat{U} = (\widehat{u}, \widehat{\varphi}, \widehat{\psi}, \widehat{\vartheta})^\top \in [\mathbf{C}(\widetilde{\Omega}^+_\Sigma; \alpha)]^6$$

*satisfying*

(i) *the pseudo-oscillation equations of the GTEME theory* (2.155) *in* $\Omega^+_\Sigma$*;*

(ii) *the Dirichlet or Neumann type boundary conditions on* $S$*;*

(iii) *one of the following type boundary and/or transmission conditions on the crack surface* $\Sigma$*:*

(a) *screen type conditions:*

$$\{\widehat{U}(x)\}^+ = \widehat{f}^{(+)}(x), \ \ x \in \Sigma, \tag{2.174}$$

$$\{\widehat{U}(x)\}^- = \widehat{f}^{(-)}(x), \ \ x \in \Sigma, \tag{2.175}$$

*or*

(b) *crack type conditions:*

$$\big\{\mathcal{T}(\partial_x,n,\tau)\widehat{U}(x)\big\}^+ = \widehat{F}^{(+)}(x) - \{F^{(0)}(x)\}^+, \ \ x\in\Sigma, \tag{2.176}$$
$$\big\{\mathcal{T}(\partial_x,n,\tau)U(x)\big\}^- = \widehat{F}^{(-)}(x) - \{F^{(0)}(x)\}^-, \ \ x\in\Sigma, \tag{2.177}$$

*or*

(c) *mixed crack type conditions:*

$$\big\{[\mathcal{T}(\partial_x,n,\tau)\widehat{U}(x)]_r\big\}^{\pm} = \widehat{F}_r^{(\pm)}(x) - \{F_r^{(0)}(x)\}^{\pm}, \ \ x\in\Sigma, \ \ r=1,2,3, \tag{2.178}$$
$$\{\widehat{\varphi}(x)\}^+ - \{\widehat{\varphi}(x)\}^- = \widehat{f}_4^{**}(x), \ \ x\in\Sigma, \tag{2.179}$$
$$\big\{[\mathcal{T}(\partial_x,n,\tau)\widehat{U}(x)]_4\big\}^+ - \big\{[\mathcal{T}(\partial_x,n,\tau)\widehat{U}(x)]_4\big\}^- = \widehat{F}_4^{**}(x) - \big[\{F_4^{(0)}(x)\}^+ - \{F_4^{(0)}(x)\}^-\big], \ \ x\in\Sigma, \tag{2.180}$$
$$\{\widehat{\psi}(x)\}^+ - \{\widehat{\psi}(x)\}^- = \widehat{f}_5^{**}(x), \ \ x\in\Sigma, \tag{2.181}$$
$$\big\{[\mathcal{T}(\partial_x,n,\tau)\widehat{U}(x)]_5\big\}^+ - \big\{[\mathcal{T}(\partial_x,n,\tau)\widehat{U}(x)]_5\big\}^- = \widehat{F}_5^{**}(x) - [\{F_5^{(0)}(x)\}^+ - \{F_5^{(0)}(x)\}^-\big], \ \ x\in\Sigma, \tag{2.182}$$
$$\{\widehat{\vartheta}(x)\}^+ - \{\widehat{\vartheta}(x)\}^- = \widehat{f}_6^{**}(x), \ \ x\in\Sigma, \tag{2.183}$$
$$\big\{[\mathcal{T}(\partial_x,n,\tau)\widehat{U}(x)]_6\big\}^+ - \big\{[\mathcal{T}(\partial_x,n,\tau)\widehat{U}(x)]_6\big\}^- = \widehat{F}_6^{**}(x), \ \ x\in\Sigma, \tag{2.184}$$

*where* $\widehat{f}^{(\pm)} = (\widehat{f}_1^{(\pm)},\dots,\widehat{f}_6^{(\pm)})^\top$, $\widehat{F}^{(\pm)} = (\widehat{F}_1^{(\pm)},\dots,\widehat{F}_6^{(\pm)})^\top$, $F^{(0)} = (F_1^{(0)},\dots,F_6^{(0)})^\top$, $\widehat{f}_k^{**}$, *and* $\widehat{F}_k^{**}$, $k=4,5,6$, *are given functions form the appropriate spaces.*

Evidently, as in the dynamical case, here we can formulate the semi-regular setting of crack type pseudo-oscillation problem with mixed conditions on the exterior boundary of the body $S$ as well, but in this case we have to require that in some open neighbourhood $\mathcal{U}_\Sigma\subset\Omega_\Sigma$ of the crack surface $\Sigma$ the sought for vector belongs to the semi-regular class of functions $[\mathbf{C}(\widetilde{\mathcal{U}}_\Sigma;\alpha)]^6$ and at the same time for some open one-sided interior neighbourhood $\mathcal{V}_S^+\subset\Omega_\Sigma$ of the exterior boundary surface $S$ it belongs to the semi-regular class of functions $[\mathbf{C}(\widetilde{\mathcal{V}}^+_{S,\ell};\alpha)]^6$, where $\ell$ is defined by the dissection of the surface $S$ associated with the mixed boundary conditions and $\widetilde{\mathcal{V}}^+_{S,\ell} := \mathcal{V}_S^+\setminus\ell$. Without loss of generality one can assume that $\overline{\mathcal{U}_\Sigma}\cap S=\varnothing$ and $\overline{\mathcal{V}_S^+}\cap\overline{\Sigma}=\varnothing$.

It is evident that the right hand side of differential equation (2.155) as well as the boundary data involved in the above formulated pseudo-oscillation problems, associated to the corresponding dynamical problems via the Laplace transform, depend also on the complex parameter $\tau$.

As in the dynamical case, here the crack type pseudo-oscillation problem with the Dirichlet type conditions (2.174), (2.175) on the crack faces, will be referred to as Problem $(\text{B-CR-D})^+_\tau$, with the Neumann type conditions (2.176), (2.177) – as Problem $(\text{B-CR-N})^+_\tau$, and with the mixed type Neumann-Transmission conditions (2.178)–(2.184) – as Problem $(\text{B-CR-NT})^+_\tau$. Again, here B stands for D or for N or for M and shows which kind of boundary conditions (Dirichlet, Neumann or Mixed type boundary conditions) are prescribed on the surface $S$.

*The boundary value problems of pseudo-oscillations for an exterior unbounded domain* $\Omega^-$ can be formulated quite similarly. The only difference in the formal setting is that solution vectors have to satisfy certain conditions at infinity, which will be specified later (see (2.207)), and the one-sided limiting values of the corresponding functions on the boundary surface should be taken from the domain $\Omega^-$, i.e., the interior traces $\{\,\cdot\,\}^+$ should be replaced by the exterior ones $\{\,\cdot\,\}^-$. We denote these problems by $(\text{D})^-_\tau$, $(\text{N})^-_\tau$, $(\text{M})^-_\tau$, $(\text{B-CR-D})^-_\tau$, $(\text{B-CR-N})^-_\tau$, and $(\text{B-CR-NT})^-_\tau$ with $\text{B}\in\{\text{D},\text{N},\text{M}\}$.

*Remark* 2.12. In order to simplify the notation, in what follows, treating the general boundary value problems of pseudo-oscillations we do not use over posed "hat" and employ the simple notation $U(x,\tau) = (u(x,\tau),\varphi(x,\tau),\psi(x,\tau),\vartheta(x,\tau))^\top$ for the sought vector assuming that all the data of the problem depend upon the complex parameter $\tau$ as well.

2.4. **Green's formulas for the dynamical model.** Let $U=(u,\varphi,\psi,\vartheta)^\top\in[C^2(\overline{\Omega^+}\times\widetilde{J}_\infty)]^6$ and $U'=(u',\varphi',\psi',\vartheta')^\top\in[C^1(\overline{\Omega^+}\times\widetilde{J}_\infty)]^6$. In view of the relations (2.1)–(2.8), (2.17)–(2.25), and (2.30), by the Gauss divergence theorem we derive the following formulas:

$$\begin{aligned}
&\int_{\Omega^+}\big[A(\partial_x,\partial_t)U(x,t)\big]_r u_r'(x,t)\,dx=\int_{\Omega^+}\big[\partial_j\sigma_{rj}(x,t)-\varrho\partial_t^2u_r(x,t)\big]u_r'(x,t)\,dx\\
&=\int_{\partial\Omega^+}\big\{\sigma_{rj}(x,t)n_j(x)\big\}^+\{u_r'(x,t)\}^+\,dS-\int_{\Omega^+}\big[\sigma_{rj}(x,t)\partial_ju_r'(x,t)+\varrho\partial_t^2u_r(x,t)u_r'(x,t)\big]\,dx\\
&=\int_{\partial\Omega^+}\big\{\big[\mathcal{T}(\partial_x,n,\partial_t)U(x,t)\big]_r\big\}^+\{u_r'(x,t)\}^+\,dS-\int_{\Omega^+}\sigma_{rj}(x,t)\partial_ju_r'(x,t)\,dx\\
&\qquad-\int_{\Omega^+}\varrho\partial_t^2u_r(x,t)u_r'(x,t)\,dx,
\end{aligned}\tag{2.185}$$

$$\begin{aligned}
&\int_{\Omega^+}\big[A(\partial_x,\partial_t)U(x,t)\big]_4\varphi'(x,t)\,dx=-\int_{\Omega^+}\partial_jD_j(x,t)\varphi'(x,t)\,dx\\
&=-\int_{\partial\Omega^+}\{D_j(x,t)n_j(x)\}^+\{\varphi'(x,t)\}^+\,dS+\int_{\Omega^+}D_j(x,t)\partial_j\varphi'(x,t)\,dx\\
&=\int_{\partial\Omega^+}\big\{\big[\mathcal{T}(\partial_x,n,\partial_t)U(x,t)\big]_4\big\}^+\{\varphi'(x,t)\}^+\,dS+\int_{\Omega^+}D_j(x,t)\partial_j\varphi'(x,t)\,dx,
\end{aligned}\tag{2.186}$$

$$\begin{aligned}
&\int_{\Omega^+}[A(\partial_x,\partial_t)U(x,t)]_5\psi'(x,t)\,dx=-\int_{\Omega^+}\partial_jB_j(x,t)\psi'(x,t)\,dx\\
&=-\int_{\partial\Omega^+}\{B_j(x,t)n_j(x)\}^+\{\psi'(x,t)\}^+\,dS+\int_{\Omega^+}B_j(x,t)\partial_j\psi'(x,t)\,dx\\
&=\int_{\partial\Omega^+}\big\{\big[\mathcal{T}(\partial_x,n,\partial_t)U(x,t)\big]_5\big\}^+\{\psi'(x,t)\}^+\,dS+\int_{\Omega^+}B_j(x,t)\partial_j\psi'(x,t)\,dx,
\end{aligned}\tag{2.187}$$

$$\begin{aligned}
&\int_{\Omega^+}[A(\partial_x,\partial_t)U(x,t)]_6\vartheta'(x,t)\,dx=-\int_{\Omega^+}\big[\varrho\partial_t\mathcal{S}(x,t)+T_0^{-1}\partial_jq_j(x,t)\big]\vartheta'(x,t)\,dx\\
&=-\int_{\partial\Omega^+}\big\{T_0^{-1}q_j(x,t)n_j(x)\big\}^+\{\vartheta'(x,t)\}^+\,dS-\int_{\Omega^+}\varrho\partial_t\mathcal{S}(x,t)\vartheta'(x,t)\,dx\\
&+\int_{\Omega^+}T_0^{-1}q_j(x,t)]\partial_j\vartheta'(x,t)\,dx=\int_{\partial\Omega^+}\big\{\big[\mathcal{T}(\partial_x,n,\partial_t)U(x,t)\big]_6\big\}^+\{\vartheta'(x,t)\}^+\,dS\\
&\qquad-\int_{\Omega^+}\varrho\partial_t\mathcal{S}(x,t)\vartheta'(x,t)\,dx-\int_{\Omega^+}\eta_{jl}\partial_l\vartheta(x,t)\partial_j\vartheta'(x,t)\,dx.
\end{aligned}\tag{2.188}$$

Combining these formulas and keeping in mind that $U$ and $U'$ are real valued vector functions, we get:

$$\begin{aligned}
&\int_{\Omega^+}\big[A(\partial_x,\partial_t)U(x,t)\big]\cdot U'(x,t)\,dx=\int_{\partial\Omega^+}\big\{\mathcal{T}(\partial_x,n,\partial_t)U(x,t)\big\}^+\cdot\{U'(x,t)\}^+\,dS\\
&\quad-\int_{\Omega^+}\Big[\sigma_{rj}(x,t)\varepsilon_{rj}'(x,t)+\varrho\partial_t^2u_r(x,t)u_r'(x,t)-D_j(x,t)\partial_j\varphi'(x,t)-B_j(x,t)\partial_j\psi'(x,t)\\
&\qquad+\eta_{jl}\partial_l\vartheta(x,t)\partial_j\vartheta'(x,t)+\varrho\partial_t\mathcal{S}(x,t)\vartheta'(x,t)\Big]\,dx,
\end{aligned}\tag{2.189}$$

where $\sigma_{rj}$, $D_j$, $B_j$, and $\mathcal{S}$ are defined in (2.2)–(2.5).

Note that the above Green's formulas by standard limiting procedure can be extended to Lipschitz domains and to vector functions from the Sobolev–Slobodetskii spaces, such that the following inclusions are continuous in the sense of appropriate function spaces with respect to $t\in\widetilde{J}_\infty$,

$$\begin{gathered}
U(\,\cdot\,,t)\in[W_p^1(\Omega^+)]^6,\qquad U'(\,\cdot\,,t)\in[W_{p'}^1(\Omega^+)]^6,\quad [A(\partial_x,\partial_t)U(\,\cdot\,,t)]_r\in L_p(\Omega^+),\\
\partial_t^2u_k(\,\cdot\,,t),\ \ \partial_t\partial_ju_k(\,\cdot\,,t),\ \ \partial_t\partial_j\varphi(\,\cdot\,,t),\ \partial_t\partial_j\psi(\,\cdot\,,t),\ \ \partial_t^2\vartheta(\,\cdot\,,t)\in L_p(\Omega^+),\\
r=1,2,\dots,6,\ \ k,j=1,2,3,\ \ 1<p<\infty,\ \ \frac1p+\frac1{p'}=1.
\end{gathered}\tag{2.190}$$

With the help of Green's formula (2.189) we can correctly determine a *generalized trace vector* $\{\mathcal{T}(\partial_x,n,\partial_t)U(\,\cdot\,,t)\}^+\in[B^{-1/p}_{p,p}(\partial\Omega^+)]^6$ for a function $U(\,\cdot\,,t)\in[W^1_p(\Omega^+)]^6$ provided the conditions (2.190) are satisfied (cf. [93], [89], [12]),

$$\begin{aligned}\Big\langle\big\{\mathcal{T}(\partial_x,n,\partial_t)U(x,t)\big\}^+,\{U'(x,t)\}^+\Big\rangle_{\partial\Omega^+}:=\int_{\Omega^+}[A(\partial_x,\partial_t)U(x,t)]\cdot U'(x,t)\,dx\\ +\int_{\Omega^+}\Big[\sigma_{rj}(x,t)\varepsilon'_{rj}(x,t)+\varrho\partial_t^2u_r(x,t)u'_r(x,t)-D_j(x,t)\partial_j\varphi'(x,t)-B_j(x,t)\partial_j\psi'(x,t)\\ +\eta_{jl}\partial_l\vartheta(x,t)\partial_j\vartheta'(x,t)+\varrho\partial_t\mathcal{S}(x,t)\vartheta'(x,t)\Big]\,dx.\qquad(2.191)\end{aligned}$$

Here the symbol $\langle\,\cdot\,,\,\cdot\,\rangle_{\partial\Omega^+}$ denotes the duality between the real valued Besov spaces $[B^{-1/p}_{p,p}(\partial\Omega^+)]^6$ and $[B^{1/p}_{p',p'}(\partial\Omega^+)]^6$ which extends the usual $L_2$ inner product for real valued vector functions,

$$\langle f,g\rangle_{\partial\Omega^+}=\int_{\partial\Omega^+}\sum_{j=1}^6 f_j(x)g_j(x)\,dS\ \text{ for }\ f,g\in[L_2(\partial\Omega^+)]^6.\qquad(2.192)$$

Let us return to identity (2.189) written for smooth functions and in addition assume that

$$u'_r(x,T)=0,\ \ r=1,2,3,\quad \vartheta'(x,T)=0\ \text{ in }\ \Omega^+,\qquad(2.193)$$

where $T$ is some positive number. Integrating (2.189) over the interval $(0,T)$ with respect to $t$, using the integration by parts formula for the terms involving $\varrho\partial_t^2u_r(x,t)$ and $\partial_t\mathcal{S}(x,t)$, and the equalities (2.193) and (2.5) we get

$$\begin{aligned}\int_0^T\int_{\Omega^+}\big[A(\partial_x,\partial_t)U(x,t)\big]\cdot U'(x,t)\,dx\,dt=\int_0^T\int_{\partial\Omega^+}\big\{\mathcal{T}(\partial_x,n,\partial_t)U(x,t)\big\}^+\cdot\{U'(x,t)\}^+\,dS\,dt\\ -\int_0^T\int_{\Omega^+}\Big[\sigma_{rj}(x,t)\varepsilon'_{rj}(x,t)-\varrho\partial_tu_r(x,t)\partial_tu'_r(x,t)-D_j(x,t)\partial_j\varphi'(x,t)-B_j(x,t)\partial_j\psi'(x,t)\\ +\eta_{jl}\partial_l\vartheta(x,t)\partial_j\vartheta'(x,t)-[\varrho\mathcal{S}(x,t)-a_0]\partial_t\vartheta'(x,t)\Big]\,dx\,dt\\ +\int_{\Omega^+}\big\{\varrho\partial_tu_r(x,0)u'_r(x,0)+[\varrho\mathcal{S}(x,0)-a_0]\vartheta'(x,0)\big\}\,dx.\qquad(2.194)\end{aligned}$$

Here we have taken into consideration that $\partial_t\mathcal{S}(x,t)=\partial_t[\mathcal{S}(x,t)-a_0]$, where $a_0$ is the constant involved in (2.5).

With the help of the constitutive relations (2.2)–(2.5) we can rewrite (2.194) in the form that will be very useful in our further analysis

$$\begin{aligned}\int_0^T\int_{\Omega^+}\Big\{\Big[c_{rjkl}\varepsilon_{kl}(x,t)+e_{lrj}\partial_l\varphi(x,t)+q_{lrj}\partial_l\psi(x,t)-\lambda_{rj}\big(\vartheta(x,t)+\nu_0\partial_t\vartheta(x,t)\big)\Big]\varepsilon'_{rj}(x,t)\\ +\Big[-e_{jkl}\varepsilon_{kl}(x,t)+\varkappa_{jl}\partial_l\varphi(x,t)+a_{jl}\partial_l\psi(x,t)-p_j\big(\vartheta(x,t)+\nu_0\partial_t\vartheta(x,t)\big)\Big]\partial_j\varphi'(x,t)\\ +\Big[-q_{jkl}\varepsilon_{kl}(x,t)+a_{jl}\partial_l\varphi(x,t)+\mu_{jl}\partial_l\psi(x,t)-m_j\big(\vartheta(x,t)+\nu_0\partial_t\vartheta(x,t)\big)\Big]\partial_j\psi'(x,t)\\ +\Big[-\lambda_{kl}\varepsilon_{kl}(x,t)+p_l\partial_l\varphi(x,t)+m_l\partial_l\psi(x,t)-d_0\vartheta(x,t)-h_0\partial_t\vartheta(x,t)\Big]\partial_t\vartheta'(x,t)\\ +\eta_{jl}\partial_l\vartheta(x,t)\partial_j\vartheta'(x,t)-\varrho\partial_tu_r(x,t)\partial_tu'_r(x,t)\Big\}\,dx\,dt\\ =-\int_0^T\int_{\Omega^+}\big[A(\partial_x,\partial_t)U(x,t)\big]\cdot U'(x,t)\,dx\,dt+\int_0^T\int_{\partial\Omega^+}\big\{\mathcal{T}(\partial_x,n,\partial_t)U(x,t)\big\}^+\cdot\{U'(x,t)\}^+\,dS\,dt\\ +\int_{\Omega^+}\Big\{\Big[\lambda_{kl}\varepsilon_{kl}(x,0)-p_l\partial_l\varphi(x,0)-m_l\partial_l\psi(x,0)+d_0\vartheta(x,0)+h_0\partial_t\vartheta(x,0)\Big]\vartheta'(x,0)\\ +\varrho\partial_tu_r(x,0)u'_r(x,0)\Big\}\,dx.\qquad(2.195)\end{aligned}$$

This formula can also be extended to generalized function spaces. In particular, (2.195) holds if

$$
\begin{gathered}
U=(u_1,u_2,u_3,\varphi,\psi,\vartheta)^\top\in\big[W_p^1(\Omega^+\times J_T)\big]^6,\;\; A(\partial_x,\partial_t)U(\,\cdot\,,t)\in\big[L_p(\Omega^+\times J_T)\big]^6,\\
\partial_t^2u_k(\,\cdot\,,t),\;\;\partial_t\partial_ju_k(\,\cdot\,,t),\;\;\partial_t\partial_j\varphi(\,\cdot\,,t),\;\partial_t\partial_j\psi(\,\cdot\,,t),\;\;\partial_t^2\vartheta(\,\cdot\,,t)\in L_p(\Omega^+),\\
U'=(u_1',u_2',u_3',\varphi',\psi',\vartheta')^\top\in\big[W_{p'}^1(\Omega^+\times J_T)\big]^6,\\
u_k'(x,T)=0,\;\vartheta'(x,T)=0\;\text{ in }\;\Omega^+,\\
k,l=1,2,3,\;\;1<p<\infty,\;\;\frac1p+\frac1{p'}=1.
\end{gathered}
\tag{2.196}
$$

Note that in view of the inclusions in the second row of (2.196) we have

$$
\partial_lu_k(\,\cdot\,,0),\partial_tu_k(\,\cdot\,,0),\partial_l\varphi(\,\cdot\,,0),\partial_l\psi(\,\cdot\,,0),\vartheta(\,\cdot\,,0),\partial_t\vartheta(\,\cdot\,,0)\in L_p(\Omega^+),\;\;k=1,2,3.
$$

Evidently, in this case, the integral involving the trace of the generalized stress vector should be understood as the appropriate duality relation, and under conditions (2.196) we have

$$
\begin{aligned}
&\Big\langle\big\{\mathcal T(\partial_x,n,\partial_t)U(x,t)\big\}^+,\{U'(x,t)\}^+\Big\rangle_{\partial\Omega^+\times J_T}\\
&:=\int_0^T\int_{\Omega^+}\bigg\{\Big[c_{rjkl}\varepsilon_{kl}(x,t)+e_{lrj}\partial_l\varphi(x,t)+q_{lrj}\partial_l\psi(x,t)-\lambda_{rj}\big(\vartheta(x,t)+\nu_0\partial_t\vartheta(x,t)\big)\Big]\varepsilon_{rj}'(x,t)\\
&\quad+\Big[-e_{jkl}\varepsilon_{kl}(x,t)+\varkappa_{jl}\partial_l\varphi(x,t)+a_{jl}\partial_l\psi(x,t)-p_j\big(\vartheta(x,t)+\nu_0\partial_t\vartheta(x,t)\big)\Big]\partial_j\varphi'(x,t)\\
&\quad+\Big[-q_{jkl}\varepsilon_{kl}(x,t)+a_{jl}\partial_l\varphi(x,t)+\mu_{jl}\partial_l\psi(x,t)-m_j\big(\vartheta(x,t)+\nu_0\partial_t\vartheta(x,t)\big)\Big]\partial_j\psi'(x,t)\\
&\quad+\Big[-\lambda_{kl}\varepsilon_{kl}(x,t)+p_l\partial_l\varphi(x,t)+m_l\partial_l\psi(x,t)-d_0\vartheta(x,t)-h_0\partial_t\vartheta(x,t)\Big]\partial_t\vartheta'(x,t)\\
&\quad+\eta_{jl}\partial_l\vartheta(x,t)\partial_j\vartheta'(x,t)-\varrho\partial_tu_r(x,t)\partial_tu_r'(x,t)\bigg\}\,dx\,dt\\
&\quad+\int_0^T\int_{\Omega^+}\big[A(\partial_x,\partial_t)U(x,t)\big]\cdot U'(x,t)\,dx\,dt-\int_{\Omega^+}\Big\{\varrho\partial_tu_r(x,0)u_r'(x,0)\\
&\quad+\Big[\lambda_{kl}\varepsilon_{kl}(x,0)-p_l\partial_l\varphi(x,0)-m_l\partial_l\psi(x,0)+d_0\vartheta(x,0)+h_0\partial_t\vartheta(x,0)]\vartheta'(x,0)\Big\}\,dx.
\end{aligned}
\tag{2.197}
$$

This formula correctly defines the generalized trace of the stress vector on the boundary $\partial\Omega^+\times J_T$ (cf. (2.191))

$$
\big\{\mathcal T(\partial_x,n,\partial_t)U(x,t)\big\}^+\in\big[B_{p,p}^{-1/p}(\partial\Omega^+\times J_T)\big]^6.
$$

*Remark* 2.13. Note that formula (2.195) remains valid for vector functions $U\in\big[H_p^1(\Omega^+\times J_T)\big]^6$ and $U'\in\big[\mathcal D(\Omega^+\times J_T)\big]^6$, but the first summand in the right hand side should be replaced by the well defined duality $\langle -A(\partial_x,\partial_t)U,U'\rangle_{\Omega^+\times J_T}$ between the mutually adjoint spaces $[H_p^{-1}(\Omega^+\times J_T)]^6$ and $[\widetilde H_{p'}(\Omega^+\times J_T)]^6$, while the second and third summands equal to zero since the traces of $U'$ on $\partial\Omega^+\times J_T$ and $\Omega^+\times\{0\}$ vanish.

2.5. **Green's formulas for the pseudo-oscillation model.** For arbitrary vector functions

$$
U=(u_1,u_2,u_3,\varphi,\psi,\vartheta)^\top\in[C^2(\overline{\Omega^+})]^6\;\text{ and }\;U'=(u_1',u_2',u_3',\varphi',\psi',\vartheta')^\top\in[C^2(\overline{\Omega^+})]^6
$$

we can derive the following Green's identities with the help of the Gauss divergence theorem:

$$
\int_{\Omega^+}\big[A(\partial_x,\tau)U\cdot U'+\mathcal E_\tau(U,\overline{U'})\big]\,dx=\int_{\partial\Omega^+}\{\mathcal T(\partial_x,n,\tau)U\}^+\cdot\{U'\}^+\,dS,
\tag{2.198}
$$

$$
\int_{\Omega^+}\big[U\cdot A^*(\partial_x,\tau)U'+\mathcal E_\tau(U,\overline{U'})\big]\,dx=\int_{\partial\Omega^+}\{U\}^+\cdot\{\mathcal P(\partial_x,n,\tau)U'\}^+\,dS,
\tag{2.199}
$$

$$
\begin{aligned}
&\int_{\Omega^+}\big[A(\partial_x,\tau)U\cdot U'-U\cdot A^*(\partial_x,\tau)U'\big]\,dx\\
&\qquad=\int_{\partial\Omega^+}\Big[\{\mathcal T(\partial_x,n,\tau)U\}^+\cdot\{U'\}^+-\{U\}^+\cdot\{\mathcal P(\partial_x,n,\tau)U'\}^+\Big]\,dS,
\end{aligned}
\tag{2.200}
$$

where the operators $A(\partial_x,\tau)$, $\mathcal{T}(\partial_x,n,\tau)$, $A^*(\partial_x,\tau)$ and $\mathcal{P}(\partial_x,n,\tau)$ are given in (2.45), (2.57), (2.49), and (2.58), respectively,

$$\begin{aligned}\mathcal{E}_\tau(U,\overline{U'}) := c_{rjkl}\partial_l u_k\overline{\partial_j u_r'} + \varrho\tau^2 u_r\overline{u_r'} + e_{lrj}(\partial_l\varphi\overline{\partial_j u_r'} - \partial_j u_r\overline{\partial_l\varphi'})\\ + q_{lrj}(\partial_l\psi\overline{\partial_j u_r'} - \partial_j u_r\overline{\partial_l\psi'}) + \varkappa_{jl}\partial_l\varphi\overline{\partial_j\varphi'} + a_{jl}(\partial_l\varphi\overline{\partial_j\psi'} + \partial_j\psi\overline{\partial_l\varphi'})\\ + \mu_{jl}\partial_l\psi\overline{\partial_j\psi'} + \lambda_{kj}\big[\tau\overline{\vartheta'}\partial_j u_k - (1+\nu_0\tau)\vartheta\overline{\partial_j u_k'}\,\big] - p_l\big[\tau\overline{\vartheta'}\partial_l\varphi + (1+\nu_0\tau)\vartheta\overline{\partial_l\varphi'}\,\big]\\ - m_l\big[\tau\overline{\vartheta'}\partial_l\psi + (1+\nu_0\tau)\vartheta\overline{\partial_l\psi'}\,\big] + \eta_{jl}\partial_l\vartheta\overline{\partial_j\vartheta'} + \tau(h_0\tau + d_0)\vartheta\overline{\vartheta'}. \quad (2.201)\end{aligned}$$

Note that, the above Green's formula (2.198) by standard limiting procedure can be generalized to Lipschitz domains and to vector functions $U\in[W^1_p(\Omega^+)]^6$ and $U'\in[W^1_{p'}(\Omega^+)]^6$ with

$$A(\partial_x,\tau)U\in[L_p(\Omega^+)]^6,\ \ 1<p<\infty,\ \ \frac{1}{p}+\frac{1}{p'}=1.$$

With the help of Green's formula (2.198) we can correctly determine a *generalized trace of the stress vector* $\{\mathcal{T}(\partial_x,n,\tau)U\}^+\in[B^{-1/p}_{p,p}(\partial\Omega^+)]^6$ for a function $U\in[W^1_p(\Omega^+)]^6$ with $A(\partial_x,\tau)U\in[L_p(\Omega^+)]^6$ by the following relation (cf. [93], [75], [89], [12])

$$\big\langle\big\{\mathcal{T}(\partial_x,n,\tau)U\big\}^+,\{U'\}^+\big\rangle_{\partial\Omega^+} := \int_{\Omega^+}\big[A(\partial_x,\tau)U\cdot U' + \mathcal{E}_\tau(U,\overline{U'})\big]\,dx, \quad (2.202)$$

where $U'\in[W^1_{p'}(\Omega^+)]^6$ is an arbitrary vector function. Here the symbol $\langle\,\cdot\,,\,\cdot\,\rangle_{\partial\Omega^+}$ denotes the duality between the function spaces $[B^{-1/p}_{p,p}(\partial\Omega^+)]^6$ and $[B^{1/p}_{p',p'}(\partial\Omega^+)]^6$ which extends the usual $L_2$ inner product for complex-valued vector functions,

$$\langle f,g\rangle_{\partial\Omega^+} = \int_{\partial\Omega^+}\sum_{j=1}^6 f_j(x)\overline{g_j(x)}\,dS \ \text{ for } \ f,g\in[L_2(\partial\Omega^+)]^6. \quad (2.203)$$

We have the following evident estimate

$$\begin{aligned}\big\|\big\{\mathcal{T}(\partial_x,n,\tau)U\big\}^+\big\|_{[B^{-1/p}_{p,p}(\partial\Omega^+)]^6}\\ \le C_0\Big\{\|A(\partial_x,\tau)U\|_{[L_p(\Omega^+)]^6} + (1+|\tau|)\|U\|_{[W^1_p(\Omega^+)]^6} + |\tau|^2\big(\|u\|_{[L_p(\Omega^+)]^3} + \|\vartheta\|_{L_p(\Omega^+)}\big)\Big\}, \quad (2.204)\end{aligned}$$

where a positive constants $C_0$ does not depend on $U$ and $\tau$; in general $C_0$ depends on the material parameters and on the geometrical characteristics of the domain $\Omega^+$.

Let us introduce a sesquilinear form on $[H^1_2(\Omega^+)]^6\times[H^1_2(\Omega^+)]^6$

$$\mathcal{B}(U,V) := \int_{\Omega^+}\mathcal{E}_\tau(U,\overline{V})\,dx, \quad (2.205)$$

where $\mathcal{E}_\tau(U,\overline{V})$ is defined by (2.201). With the help of the relations (2.10) and (2.201), positive definiteness of the matrix (2.14), and the well known Korn's inequality we deduce the following estimate

$$\operatorname{Re}\mathcal{B}(U,U)\ge c_1\|U\|^2_{[H^1_2(\Omega^+)]^6} - c_2\|U\|^2_{[H^0_2(\Omega^+)]^6} \ \text{ for all } U\in[H^1_2(\Omega^+)]^6 \quad (2.206)$$

with some positive constants $c_1$ and $c_2$ depending on the material parameters and on the complex parameter $\tau$ (cf. [38], [89]), which shows that the sesquilinear form defined in (2.205) is coercive.

From Green's identities (2.198)–(2.200) by standard limiting procedure one can derive similar formulas for the exterior domain $\Omega^-$ provided the regular vector functions $U$ and $U'$ belong to the space $[C^2(\overline{\Omega^-})]^6$ and satisfy certain decay conditions at infinity. In particular, let $A(\partial_x,\tau)U$ be compactly supported and the following asymptotic conditions hold at infinity as $|x|\to\infty$

$$\begin{aligned}u_k(x)=\mathcal{O}(|x|^{-2}),\quad \partial_j u_k(x)=\mathcal{O}(|x|^{-2}),\quad \varphi(x)=\mathcal{O}(|x|^{-1}),\quad \partial_j\varphi(x)=\mathcal{O}(|x|^{-2}),\\ \psi(x)=\mathcal{O}(|x|^{-1}),\ \ \partial_j\psi(x)=\mathcal{O}(|x|^{-2}),\ \ \vartheta(x)=\mathcal{O}(|x|^{-2}),\ \ \partial_j\vartheta(x)=\mathcal{O}(|x|^{-2}),\ \ k,j,l=1,2,3.\end{aligned} \quad (2.207)$$

**Definition 2.14.** We say that a vector functions $U=(u_1,u_2,u_3,\varphi,\psi,\vartheta)^\top\in[W^1_{p,loc}(\Omega^-)]^6$ with some $p\in(1,+\infty)$ belongs to the class $\mathbf{Z}_\tau(\Omega^-)$ if the components of $U$ satisfy the decay conditions (2.207) at infinity.

As we shall see below the columns of the fundamental matrix of the operator $A(\partial_x,\tau)$ with $\tau=\sigma+i\omega$, $\sigma>\sigma_0$, possesses the decay properties (2.207) (see Section 3).

Assume that $A^*(\partial_x,\tau)U'$ is compactly supported as well and $U'$ satisfies the decay conditions of type (2.207). Then for regular vector functions the following Green formulas hold true for the exterior unbounded domain $\Omega^-$:

$$\int_{\Omega^-}\big[A(\partial_x,\tau)U\cdot U'+\mathcal{E}_\tau(U,\overline{U'})\big]\,dx=-\int_{\partial\Omega^-}\big\{\mathcal{T}(\partial_x,n,\tau)U\big\}^-\cdot\{U'\}^-\,dS, \tag{2.208}$$

$$\int_{\Omega^-}\big[U\cdot A^*(\partial_x,\tau)U'+\mathcal{E}_\tau(U,\overline{U'})\big]\,dx=-\int_{\partial\Omega^-}\{U\}^-\cdot\big\{\mathcal{P}(\partial_x,n,\tau)U'\big\}^-\,dS, \tag{2.209}$$

$$\begin{aligned}&\int_{\Omega^-}\big[A(\partial_x,\tau)U\cdot U'-U\cdot A^*(\partial_x,\tau)U'\big]\,dx\\&\qquad=-\int_{\partial\Omega^-}\Big[\big\{\mathcal{T}(\partial_x,n,\tau)U\big\}^-\cdot\{U'\}^- -\{U\}^-\cdot\big\{\mathcal{P}(\partial_x,n,\tau)U'\big\}^-\Big]\,dS,\end{aligned} \tag{2.210}$$

where $\mathcal{E}_\tau$ is defined by (2.201). We recall that the direction of the unit normal vector to $S=\partial\Omega^-$ is outward with respect to the domain $\Omega^+$.

As in the case of bounded domains, by standard limiting procedure Green's formula (2.208), can be extended to vector functions $U\in[W^1_{p,loc}(\Omega^-)]^6$ and $U'\in[W^1_{p',loc}(\Omega^-)]^6$ with $1/p+1/p'=1$ satisfying the decay conditions at infinity (2.207) and possessing the property $A(\partial_x,\tau)U\in[L_{2,comp}(\Omega^-)]^6$,

$$\int_{\Omega^-}\big[A(\partial_x,\tau)U\cdot U'+\mathcal{E}_\tau(U,\overline{U'})\big]\,dx=-\big\langle\{\mathcal{T}(\partial_x,n,\tau)U\}^-,\{U'\}^-\big\rangle_S, \tag{2.211}$$

where the duality brackets in the right hand side between the function spaces $[B^{-1/p}_{p,p}(S)]^6$ and $[B^{1/p}_{p',p'}(S)]^6$ correctly defines the *generalized trace of the stress vector* $\{\mathcal{T}(\partial_x,n,\tau)U\}^-\in[B^{-1/p}_{p,p}(S)]^6$ on the boundary surface $S=\partial\Omega^-$.

Note that, since the operator $A(\partial_x,\tau)$ is strongly elliptic and $A(\partial_x,\tau)U$ has a compact support, then actually $U$ is an analytic vector function of the real variables $(x_1,x_2,x_3)$ in a vicinity of infinity (in the domain $\Omega^-\setminus\operatorname{supp}A(\partial_x,\tau)U$) and the conditions (2.207) can be understood in the usual classical pointwise sense. Therefore, the improper integral over $\Omega^-$ in formula (2.211) is convergent and well defined.

*Remark* 2.15. The above Green's formula (2.195), (2.198), and (2.211) remains valid for semi-regular vector functions (see Definitions 2.2–2.3, 2.5–2.6). Indeed, e.g., for the class $\mathbf{C}(\widetilde{\Omega}^+_\ell;\alpha)$, we have to consider a domain $\Omega^+_\varepsilon:=\Omega^+\setminus\mathcal{U}_\ell(\varepsilon)$, where $\mathcal{U}_\ell(\varepsilon)\subset\Omega^+$ is a tubular neighbourhood of the curve $\ell$ with Lebesgue measure equivalent to $\varepsilon^2$ (say, a cylindrical neighbourhood of circular cross section of radius $\varepsilon$) and write Green's formula of type (2.195) for the domain $\Omega^+_\varepsilon$. Under the conditions stated in the definition of the class of semi-regular functions $\mathbf{C}(\widetilde{\Omega}^+_\ell;\alpha)$, it can be shown that the limits of the integrals involved in Green's identity written for the domain $\Omega^+_\varepsilon$ exist as $\varepsilon$ tends to zero.

Similarly, for semi-regular functions from the class $\widetilde{\Omega}^+_\Sigma$ the corresponding Green's first formula reads as

$$\begin{aligned}\int_{\Omega^+_\Sigma}\big[A(\partial_x,\tau)U\cdot U'+\mathcal{E}_\tau(U,\overline{U'})\big]\,dx&=\int_S\{\mathcal{T}(\partial_x,n,\tau)U\}^+\cdot\{U'\}^+\,dS\\&+\int_\Sigma\Big[\big\{\mathcal{T}(\partial_x,n,\tau)U\big\}^+\cdot\{U'\}^+-\big\{\mathcal{T}(\partial_x,n,\tau)U\big\}^-\cdot\{U'\}^-\Big]\,dS.\end{aligned} \tag{2.212}$$

2.6. **Weak formulation of the problems.** As we have mentioned above in Introduction, solutions to the mixed and crack type boundary value problems and the corresponding thermo-mechanical and electro-magnetic characteristics usually have singularities at the so called exceptional curves: the crack edges and the curves where the different types of boundary conditions collide (the collision curves). In general, these types of problems do not possess regular solutions in a neighbourhood of the exceptional curves even for infinitely smooth data. Therefore we have to look for solutions either in classes of semi-regular vector functions or in properly chosen Sobolev–Slobodetskii, Bessel potential, and Besov spaces. To this end, we need the appropriate weak reformulations of the above stated classical settings of the problems (see Subsections 2.2 and 2.3).

2.6.1. *Weak solutions of the dynamical problems.* In the case of weak statements of the dynamical problems for the domain $\Omega^+$, we look for a vector function (cf., e.g., [73], [36])

$$U(\,\cdot\,,t) = \big(u(\,\cdot\,,t), \varphi(\,\cdot\,,t), \psi(\,\cdot\,,t), \vartheta(\,\cdot\,,t)\big)^\top \in [H_2^1(\Omega^+)]^6, \tag{2.213}$$

which is two times continuously differentiable with respect to $t$ in $\widetilde{J}_\infty$ and satisfies:

(i) the differential equation (2.62) in the weak sense which means that for an arbitrary test vector function

$$U' = \big(u', \varphi', \psi', \vartheta'\big)^\top \in [\mathcal{D}(\Omega^+)]^6 \tag{2.214}$$

the following equation holds true for all $t \in \widetilde{J}_\infty$ (see Green's formula (2.189))

$$\int_{\Omega^+} \Big[\sigma_{rj}(x,t)\varepsilon'_{rj}(x) + \varrho\partial_t^2 u_r(x,t)u'_r(x) - D_j(x,t)\partial_j\varphi'(x) - B_j(x,t)\partial_j\psi'(x) \\ + \eta_{jl}(x,t)\partial_l\vartheta(x,t)\partial_j\vartheta'(x) + \varrho\partial_t\mathcal{S}(x,t)\vartheta'(x)\Big]\,dx = -\int_{\Omega^+}\Phi(x,t)\cdot U'(x)\,dx, \tag{2.215}$$

where $\varepsilon'_{rj} = 2^{-1}(\partial_r u'_j + \partial_j u'_r)$;

(ii) initial conditions (2.63), (2.64);

(iii) one of the following boundary conditions:

- the Dirichlet type boundary condition (2.65) in the usual trace sense,

or

- the Neumann type boundary condition (2.70) in the generalized trace sense defined by Green's formula (2.191),

or

- the mixed type boundary conditions (2.79)–(2.86), where the Dirichlet type conditions are understood in the usual trace sense, while the Neumann type conditions are understood in the generalized trace sense defined by Green's formula (2.191).

The weak dynamical crack type problems for the domain $\Omega_\Sigma^+$ are formulated similarly. In this case, the boundary conditions (2.90)–(2.100) on the crack faces are understood in the usual trace sense for the Dirichlet data and in the sense of generalized trace for the Neumann data.

We require that the data involved in the weak formulations of initial-boundary value problems satisfy the following conditions for arbitrary $t \in \widetilde{J}_\infty$:

$$\Phi(\,\cdot\,,t) \in [L_2(\Omega^+)]^6, \quad u_r^{(0)}, \varphi^{(0)}, \psi^{(0)}, \vartheta^{(0)}, u^{(1)}, \vartheta^{(1)} \in H_2^1(\Omega^+), \tag{2.216}$$

$$f_r(\,\cdot\,,t) \in H_2^{\frac12}(S), \quad F_r(\,\cdot\,,t) \in H_2^{-\frac12}(S), \ \ r = 1,2,\dots,6, \tag{2.217}$$

$$f_k^*(\,\cdot\,,t) \in H_2^{\frac12}(S_1), \quad F_k^*(\,\cdot\,,t) \in H_2^{-\frac12}(S_2), \ \ k = 1,2,3, \tag{2.218}$$

$$f_4^*(\,\cdot\,,t) \in H_2^{\frac12}(S_3), \quad F_4^*(\,\cdot\,,t) \in H_2^{-\frac12}(S_4), \tag{2.219}$$

$$f_5^*(\,\cdot\,,t) \in H_2^{\frac12}(S_5), \quad F_5^*(\,\cdot\,,t) \in H_2^{-\frac12}(S_6), \tag{2.220}$$

$$f_6^*(\,\cdot\,,t) \in H_2^{\frac12}(S_7), \quad F_6^*(\,\cdot\,,t) \in H_2^{-\frac12}(S_8), \tag{2.221}$$

$$f_r^{(\pm)}(\,\cdot\,,t) \in H_2^{\frac12}(\Sigma), \quad F_r^{(\pm)}(\,\cdot\,,t) \in H_2^{-\frac12}(\Sigma)], \quad r = 1,2,\dots,6, \tag{2.222}$$

$$f_j^{**}(\,\cdot\,,t) \in H_2^{\frac12}(\Sigma), \quad F_j^{**}(\,\cdot\,,t) \in H_2^{-\frac12}(\Sigma), \ \ j = 4,5,6. \tag{2.223}$$

Moreover, we assume that these functions satisfy certain natural compatibility conditions.

In the case of the exterior dynamical problems for the unbounded domain $\Omega^-$, we assume that the right hand side vector function $\Phi$ has a compact support in $x$ for arbitrary $t \in \widetilde{J}_\infty$ and

$$\Phi(\,\cdot\,,t) \in [L_{2,comp}(\Omega^-)]^6. \tag{2.224}$$

Motivated by Green's formulas (2.194), (2.195), now we formulate an alternative, more general definition of a weak solution for the dynamical equation (2.62) (cf. e.g. [58]).

**Definition 2.16.** A vector function
$$U=(u,\varphi,\psi,\vartheta))^\top\in[H_2^1(\Omega^+\times J_T)]^6,\ \ J_T=(0,T),\ \ T>0, \tag{2.225}$$
is said to be **a weak solution** to the differential equation (2.62) if the following relation
$$\int_0^T\int_{\Omega^+}\Big[\sigma_{rj}(x,t)\widetilde{\varepsilon}_{rj}(x,t)-\varrho\partial_t u_r(x,t)\partial_t\widetilde{u}_r(x,t)-D_j(x,t)\partial_j\widetilde{\varphi}(x,t)-B_j(x,t)\partial_j\widetilde{\psi}(x,t)$$
$$+\eta_{jl}\partial_l\vartheta(x,t)\partial_j\widetilde{\vartheta}(x,t)-[\varrho\mathcal{S}(x,t)7-a_0]\partial_t\widetilde{\vartheta}(x,t)\Big]\,dx\,dt=-\int_0^T\int_{\Omega^+}\Phi(x,t)\cdot\widetilde{U}(x,t)\,dx\,dt, \tag{2.226}$$
or what is the same
$$\int_0^T\int_{\Omega^+}\bigg\{\Big[c_{rjkl}\varepsilon_{kl}(x,t)+e_{lrj}\partial_l\varphi(x,t)+q_{lrj}\partial_l\psi(x,t)-\lambda_{rj}\big(\vartheta(x,t)+\nu_0\partial_t\vartheta(x,t)\big)\Big]\widetilde{\varepsilon}_{rj}(x,t)$$
$$+\Big[-e_{jkl}\varepsilon_{kl}(x,t)+\varkappa_{jl}\partial_l\varphi(x,t)+a_{jl}\partial_l\psi(x,t)-p_j\big(\vartheta(x,t)+\nu_0\partial_t\vartheta(x,t)\big)\Big]\partial_j\widetilde{\varphi}(x,t)$$
$$+\Big[-q_{jkl}\varepsilon_{kl}(x,t)+a_{jl}\partial_l\varphi(x,t)+\mu_{jl}\partial_l\psi(x,t)-m_j\big(\vartheta(x,t)+\nu_0\partial_t\vartheta(x,t)\big)\Big]\partial_j\widetilde{\psi}(x,t)$$
$$+\Big[-\lambda_{kl}\varepsilon_{kl}(x,t)+p_l\partial_l\varphi(x,t)+m_l\partial_l\psi(x,t)-d_0\vartheta(x,t)-h_0\partial_t\vartheta(x,t)\Big]\partial_t\widetilde{\vartheta}(x,t)$$
$$+\eta_{jl}\partial_l\vartheta(x,t)\partial_j\widetilde{\vartheta}(x,t)-\varrho\partial_t u_r(x,t)\partial_t\widetilde{u}_r(x,t)\bigg\}\,dx\,dt$$
$$=-\int_0^T\int_{\Omega^+}\Phi(x,t)\cdot\widetilde{U}(x,t)\,dx\,dt \tag{2.227}$$
holds for an arbitrary test vector function $\widetilde{U}=(\widetilde{u},\widetilde{\varphi},\widetilde{\psi},\widetilde{\vartheta})^\top\in[\mathcal{D}(\Omega^+\times J_T)]^6$.

*Remark* 2.17. Evidently, a weak solution $U$ to the differential equation (2.213) in the sense of the relation (2.215) is also a weak solution in the sense of the relation (2.227), but the reverse assertion is not valid, in general. Clearly, the reverse assertion holds true if the weak solution $U$ to the functional equation (2.227) is in addition two times continuously differentiable with respect to $t\in\widetilde{J}_T$.

*Remark* 2.18. It should be noted that for the electric and magnetic potentials, $\varphi$ and $\psi$, both definitions lead to the same relations if $u'=0$ and $\vartheta'=0$ in (2.215), and $\widetilde{u}=0$ and $\widetilde{\vartheta}=0$ in (2.227). These relations correspond to the weak formulation of the forth and fifth equations of the basic system (2.29).

Further, based on Green's formula (2.197), we can formulate the weak setting of nonhomogeneous dynamical initial-boundary value problems that takes into consideration the corresponding nonhomogeneous initial and boundary conditions.

For illustration below we consider the weak setting of the following basic mixed type problem associated with the dissection $\partial\Omega^+=\overline{S}_D\cup\overline{S}_N$ (see Remark 2.4): Find a vector function
$$U=(u,\varphi,\psi,\vartheta)^\top\in[\mathbf{C}(\widetilde{\Omega}^+_\ell\times\widetilde{J}_\infty;\alpha)]^6 \tag{2.228}$$
satisfying the dynamical equation of the GTEME theory
$$A(\partial_x,\partial_t)U(x,t)=\Phi(x,t),\ \ (x,t)\in\Omega^+\times J_T, \tag{2.229}$$
the initial conditions
$$u(x,0)=u^{(0)}(x),\quad \partial_t u(x,0)=u^{(1)}(x),\ \ x\in\Omega^+, \tag{2.230}$$
$$\vartheta(x,0)=\vartheta^{(0)}(x),\quad \partial_t\vartheta(x,0)=\vartheta^{(1)}(x),\ \ x\in\Omega^+, \tag{2.231}$$
and the Dirichlet and Neumann type boundary condition on $S_D$ and $S_N$, respectively,
$$\{U(x,t)\}^+=0,\ \ (x,t)\in S_D\times J_T, \tag{2.232}$$
$$\big\{\mathcal{T}(\partial_x,n,\partial_t)U(x,t)\big\}^+=F^*(x,t),\ \ (x,t)\in S_N\times J_T, \tag{2.233}$$
i.e.,
$$\{u_r(x,t)\}^+=0,\ \ (x,t)\in S_D\times J_T,\ \ r=1,2,3, \tag{2.234}$$
$$\{\varphi(x,t)\}^+=0,\ \ (x,t)\in S_D\times J_T, \tag{2.235}$$

$$\{\psi(x,t)\}^+ = 0, \quad (x,t) \in S_D \times J_T, \tag{2.236}$$

$$\{\vartheta(x,t)\}^+ = 0, \quad (x,t) \in S_D \times J_T, \tag{2.237}$$

$$\big\{[\mathcal{T}(\partial_x, n, \partial_t)U(x,t)]_r\big\}^+ \equiv \{\sigma_{rj}n_j\}^+ = F_r^*(x,t), \quad (x,t) \in S_N \times J_T, \quad r = 1,2,3, \tag{2.238}$$

$$\big\{[\mathcal{T}(\partial_x, n, \partial_t)U(x,t)]_4\big\}^+ \equiv \{-D_j n_j\}^+ = F_4^*(x,t), \quad (x,t) \in S_N \times J_T, \tag{2.239}$$

$$\big\{[\mathcal{T}(\partial_x, n, \partial_t)U(x,t)]_5\big\}^+ \equiv \{-B_j n_j\}^+ = F_5^*(x,t), \quad (x,t) \in S_N \times J_T, \tag{2.240}$$

$$\big\{[\mathcal{T}(\partial_x, n, \partial_t)U(x,t)]_6\big\}^+ \equiv \{-T_0^{-1} q_j n_j\}^+ = F_6^*(x,t), \quad (x,t) \in S_N \times J_T, \tag{2.241}$$

where $S_D$ and $S_N$ are nonempty submanifolds of the surface $S$, $S_D \cap S_N = \varnothing$, $S = \overline{S_D} \cup \overline{S_N}$, $\Phi = (\Phi_1, \dots, \Phi_6)^\top$, $F^* = (F_1^*, \dots, F_6^*)^\top$, $u^{(0)} = (u_1^{(0)}, u_2^{(0)}, u_3^{(0)})^\top$, $u^{(1)} = (u_1^{(1)}, u_2^{(1)}, u_3^{(1)})^\top$, $\vartheta^{(0)}$, and $\vartheta^{(1)}$ are given smooth vector functions.

Note that, the general nonhomogeneous Dirichlet condition $\{U(x,t)\}^+ = f^*(x,t)$ on $S_D \times J_T$ can be reduced to the homogeneous condition (2.232) by standard approach. Therefore the homogeneous condition (2.232) does not restrict generality of the problem under consideration.

Let us introduce a subspace of $[H_2^1(\Omega^+ \times J_T)]^6$ consisting of vector functions satisfying the Dirichlet homogeneous condition (2.232) on $S_D \times J_T$,

$$[H_2^1(\Omega^+ \times J_T; S_D)]^6 = [W_2^1(\Omega^+ \times J_T; S_D)]^6 := \Big\{ U \in [H_2^1(\Omega^+ \times J_T)]^6 : \ \{U(x,t)\}^+ = 0 \text{ on } S_D \times J_T \Big\}, \tag{2.242}$$

and also set

$$[H_2^1(\Omega^+; S_D)]^2 = \big\{ X \in [H_2^1(\Omega^+)]^2 : \ r_{S_D}\{X\}^+ = 0 \big\}. \tag{2.243}$$

These spaces are endowed with the norms of the spaces $[H_2^1(\Omega^+ \times J_T)]^6$ and $[H_2^1(\Omega^+)]^2$, respectively.

With the help of Green's formula (2.197) and the arguments presented in Subsection 2.2.1, the mixed initial-boundary value problem (2.228)–(2.241) can be reformulated in the following weak sense.

**Weak setting of the basic mixed dynamical problem $(\mathbf{WM})_t$:** *Find a vector function*

$$U = (u, \varphi, \psi, \vartheta)^\top \in [H_2^1(\Omega^+ \times J_T; S_D)]^6 \tag{2.244}$$

*satisfying the initial conditions*

$$u_k(x,0) = u_k^{(0)}(x), \quad x \in \Omega^+, \quad k = 1,2,3, \tag{2.245}$$

$$\vartheta(x,0) = \vartheta^{(0)}(x), \quad x \in \Omega^+, \tag{2.246}$$

$$\lim_{t\to 0} \int_{\Omega^+} \partial_t \vartheta(x,t) h(x)\,dx = \int_{\Omega^+} \vartheta^{(1)}(x) h(x)\,dx \ \ \textit{for all} \ \ h \in L_2(\Omega^+), \tag{2.247}$$

*and the functional equation*

$$\begin{aligned}
&\int_0^T \int_{\Omega^+} \bigg\{ \Big[ c_{rjkl}\varepsilon_{kl}(x,t) + e_{lrj}\partial_l\varphi(x,t) + q_{lrj}\partial_l\psi(x,t) - \lambda_{rj}\big(\vartheta(x,t) + \nu_0\partial_t\vartheta(x,t)\big)\Big] \widetilde{\varepsilon}_{rj}(x,t) \\
&\quad + \Big[ - e_{jkl}\varepsilon_{kl}(x,t) + \varkappa_{jl}\partial_l\varphi(x,t) + a_{jl}\partial_l\psi(x,t) - p_j\big(\vartheta(x,t) + \nu_0\partial_t\vartheta(x,t)\big)\Big] \partial_j\widetilde{\varphi}(x,t) \\
&\quad + \Big[ - q_{jkl}\varepsilon_{kl}(x,t) + a_{jl}\partial_l\varphi(x,t) + \mu_{jl}\partial_l\psi(x,t) - m_j\big(\vartheta(x,t) + \nu_0\partial_t\vartheta(x,t)\big)\Big] \partial_j\widetilde{\psi}(x,t) \\
&\quad + \Big[ - \lambda_{kl}\varepsilon_{kl}(x,t) + p_l\partial_l\varphi(x,t) + m_l\partial_l\psi(x,t) - d_0\vartheta(x,t) - h_0\partial_t\vartheta(x,t)\Big] \partial_t\widetilde{\vartheta}(x,t) \\
&\quad + \eta_{jl}\partial_l\vartheta(x,t)\partial_j\widetilde{\vartheta}(x,t) - \varrho\partial_t u_r(x,t)\partial_t\widetilde{u}_r(x,t) \bigg\}\,dx\,dt \\
&= \big\langle F^*(x,t), \{\widetilde{U}(x,t)\}^+ \big\rangle_{S_N \times J_T} - \int_0^T \int_{\Omega^+} \Phi(x,t)\cdot\widetilde{U}(x,t)\,dx\,dt + \int_{\Omega^+} \Big\{ \varrho u_r^{(1)}(x)\widetilde{u}_r(x,0) \\
&\quad + \Big[ \lambda_{kl}\varepsilon_{kl}^{(0)}(x) - p_l\partial_l\varphi^{(0)}(x) - m_l\partial_l\psi^{(0)}(x) + d_0\vartheta^{(0)}(x) + h_0\,\vartheta^{(1)}(x)\Big] \widetilde{\vartheta}(x,0) \Big\}\,dx,
\end{aligned} \tag{2.248}$$

*where*

$$\widetilde{U} = (\widetilde{u}_1, \widetilde{u}_2, \widetilde{u}_3, \widetilde{\varphi}, \widetilde{\psi}, \widetilde{\vartheta})^\top \in [H_2^1(\Omega^+ \times J_T)]^6 \tag{2.249}$$

*is an arbitrary vector function such that*

$$\{\widetilde{U}(x,t)\}^+ = 0 \ \ for \ \ (x,t) \in S_D \times J_T, \tag{2.250}$$

$$\widetilde{u}_k(x,T) = 0, \ \ \widetilde{\vartheta}(x,T) = 0 \ \ for \ \ x \in \Omega^+, \ \ k = 1,2,3, \tag{2.251}$$

*and*

$$\widetilde{\varepsilon}_{kl}(x,t) := \frac{1}{2}\big(\partial_k \widetilde{u}_l(x,t) + \partial_l \widetilde{u}_k(x,t)\big) \in L_2(\Omega^+ \times J_T).$$

*Here we assume that* $\Phi$ *(see* (2.33)*),* $F^*$, $u_k^{(0)}$, $u_k^{(1)}$, $\vartheta^{(0)}$, *and* $\vartheta^{(1)}$ *are given functions satisfying the inclusions*

$$\Phi = (\Phi_1, \dots, \Phi_6)^\top := (-\varrho F_1, -\varrho F_2, -\varrho F_3, -\varrho_e, -\varrho_c, -\varrho T_0^{-1} Q)^\top \in [L_2(\Omega^+ \times J_T)]^6, \tag{2.252}$$

$$F^* = (F_1^*, F_2^*, \dots, F_6^*)^\top \in [H_2^{-\frac{1}{2}}(S_N \times J_T)]^6, \tag{2.253}$$

$$u_k^{(1)}, \vartheta^{(0)}, \vartheta^{(1)} \in L_2(\Omega^+), \quad u_k^{(0)} \in H_2^1(\Omega^+), \ \ k = 1,2,3, \tag{2.254}$$

$$\varepsilon_{kl}^{(0)}(x) := \frac{1}{2}\big(\partial_k u_l^{(0)}(x) + \partial_l u_k^{(0)}(x)\big) \in L_2(\Omega^+), \tag{2.255}$$

*and the vector function*

$$X^{(0)} := \big(\varphi^{(0)}, \psi^{(0)}\big)^\top \in [H_2^1(\Omega^+; S_D)]^2 \tag{2.256}$$

*is defined by the variational equation (see* (2.146)*)*

$$B(X^{(0)}, \widetilde{X}) \equiv \int_{\Omega^+} \Lambda^{(1)} \nabla X^{(0)}(x) \cdot \nabla \widetilde{X}(x)\, dx = \mathcal{G}_2(\widetilde{X}) \ \ for \ all \ \ \widetilde{X} \in [H_2^1(\Omega^+; S_D)]^2, \tag{2.257}$$

*where* $\Lambda^{(1)}$ *is a positive definite matrix given by* (2.14) *and* $\mathcal{G}_2$ *is a well defined bounded linear functional on* $[H_2^1(\Omega^+; S_D)]^2$ *introduced in Subsection* (2.2.1) *(see* (2.149)*),*

$$\begin{aligned} \mathcal{G}_2(\widetilde{X}) := \big\langle F_4^*(x,0), \{\widetilde{X}_1(x)\}^+\big\rangle_{S_N} + \big\langle F_5^*(x,0), \{\widetilde{X}_2(x)\}^+\big\rangle_{S_N} + \int_{\Omega^+} \big[\varrho_e(x,0)\widetilde{X}_1(x) + \varrho_c(x,0)\widetilde{X}_2(x)\big]\, dx \\ + \int_{\Omega^+} \Big\{\big[e_{jkl}\partial_l u_k^{(0)}(x) + p_j\big(\vartheta^{(0)}(x) + \nu_0 \vartheta^{(1)}(x)\big)\big]\partial_j \widetilde{X}_1(x) \\ + \big[q_{jkl}\partial_l u_k^{(0)}(x) + m_j\big(\vartheta^{(0)}(x) + \nu_0 \vartheta^{(1)}(x)\big)\big]\partial_j \widetilde{X}_2(x)\Big\}\, dx. \end{aligned} \tag{2.258}$$

*Due to the conditions* (2.250) *and* (2.253)*, and the inclusion* $\widetilde{X} \in [H_2^1(\Omega^+; S_D)]^2$, *the duality relations on* $S_N \times J_T$ *and* $S_N$ *in the right hand sides of* (2.248) *and* (2.258) *are well defined.*

*Evidently, the natural compatibility equalities related to the conditions* (2.245)*,* (2.246) *and inclusion* (2.244) *should be fulfilled, in particular,*

$$r_{s_D}\{u^{(0)}\}_S^+ = 0, \ \ r_{s_D}\{\vartheta^{(0)}\}_S^+ = 0. \tag{2.259}$$

The weak formulation (2.248) covers the Dirichlet and Neumann type problems as particular cases when either $S_N$ or $S_D$ is an empty set. We denote them by the symbols, $(\mathrm{WD})_t$ and $(\mathrm{WN})_t$.

In what follows by *a weak solutions of a dynamical problem* under consideration we always understand a weak solution in the sense of the relation (2.248) if not otherwise stated.

*Remark* 2.19. It is evident that any weak solution $U$ of the problem $(\mathrm{WM})_t$ is a distributional solution to the equation

$$A(\partial_x, \partial_t)U(x,t) = \Phi(x,t) \ \text{ in } \ \Omega^+ \times J_T. \tag{2.260}$$

This easily follows from (2.248) with $\widetilde{U} \in [\mathcal{D}(\Omega^+ \times J_T)]^6$.

By standard arguments it can be shown that if the functions $\Phi$, $u^{(0)}$, $u^{(1)}$, $\vartheta^{(0)}$, $\vartheta^{(1)}$, $\varphi^{(0)}$, $\psi^{(0)}$, $F^*$, and the weak solution $U$ possesses higher regularity property, then all the conditions of the mixed initial-boundary value problem (2.229)–(2.241) are satisfied pointwise and moreover, $\varphi(x,0) = \varphi^{(0)}(x)$ and $\psi(x,0) = \psi^{(0)}(x)$. It follows from the above weak formulation of the problem (WSBMDP) and Green's formula (2.197).

Indeed, let us assume that the functions $\Phi$, $u^{(0)}$, $u^{(1)}$, $\vartheta^{(0)}$, $\vartheta^{(1)}$, $\varphi^{(0)}$, $\psi^{(0)}$, and $F^*$ are sufficiently smooth, and the weak solution $U$ of the problem $(\mathrm{WM})_t$ possesses the semi-regularity property. Then equation (2.260) is satisfied pointwise and Green's formula (2.197) holds true for $U$ and $U' = \widetilde{U}$ satisfying the conditions (2.249), (2.250), and (2.251). Therefore we have

$$\begin{aligned}
&\int_0^T \int_{\Omega^+} \Big\{ \Big[ c_{rjkl}\varepsilon_{kl}(x,t) + e_{lrj}\partial_l\varphi(x,t) + q_{lrj}\partial_l\psi(x,t) - \lambda_{rj}\big(\vartheta(x,t) + \nu_0\partial_t\vartheta(x,t)\big)\Big]\widetilde{\varepsilon}_{rj}(x,t) \\
&\qquad + \Big[ - e_{jkl}\varepsilon_{kl}(x,t) + \varkappa_{jl}\partial_l\varphi(x,t) + a_{jl}\partial_l\psi(x,t) - p_j\big(\vartheta(x,t) + \nu_0\partial_t\vartheta(x,t)\big)\Big]\partial_j\widetilde{\varphi}(x,t) \\
&\qquad + \Big[ - q_{jkl}\varepsilon_{kl}(x,t) + a_{jl}\partial_l\varphi(x,t) + \mu_{jl}\partial_l\psi(x,t) - m_j\big(\vartheta(x,t) + \nu_0\partial_t\vartheta(x,t)\big)\Big]\partial_j\widetilde{\psi}(x,t) \\
&\qquad + \Big[ - \lambda_{kl}\varepsilon_{kl}(x,t) + p_l\partial_l\varphi(x,t) + m_l\partial_l\psi(x,t) - d_0\vartheta(x,t) - h_0\partial_t\vartheta(x,t)\Big]\partial_t\widetilde{\vartheta}(x,t) \\
&\qquad\qquad + \eta_{jl}\partial_l\vartheta(x,t)\partial_j\widetilde{\vartheta}(x,t) - \varrho\partial_t u_r(x,t)\partial_t\widetilde{u}_r(x,t)\Big\}\,dx\,dt \\
&= \int_0^T \int_{S_N} \{\mathcal{T}(\partial_x,n,\partial_t)U(x,t)\}^+ \cdot \{\widetilde{U}(x,t)\}^+\,dS\,dt - \int_0^T \int_{\Omega^+} [\Phi(x,t)U(x,t)]\cdot\widetilde{U}(x,t)\,dx\,dt \\
&\qquad + \int_{\Omega^+} \Big\{ \varrho\partial_t u_r(x,0)\widetilde{u}_r(x,0) \\
&\qquad + \Big[\lambda_{kl}\varepsilon^{(0)}_{kl}(x) - p_l\partial_l\varphi(x,0) - m_l\partial_l\psi(x,0) + d_0\vartheta^{(0)}(x) + h_0\vartheta^{(1)}(x)\Big]\widetilde{\vartheta}(x,0)\Big\}\,dx. \qquad (2.261)
\end{aligned}$$

Here we have taken into consideration that in view of the semi-regularity property of $U$ we have $\varepsilon_{kl}(x,0) = \varepsilon^{(0)}_{kl}(x)$, $\vartheta(x,0) = \vartheta^{(0)}(x)$, and $\partial_t\vartheta(x,0) = \vartheta^{(1)}(x)$ due to (2.245), (2.246), and (2.247). Comparing equalities (2.248) and (2.261) we deduce

$$\begin{aligned}
&\int_0^T\int_{S_N}\Big[\{\mathcal{T}(\partial_x,n,\partial_t)U(x,t)\}^+ - F^*(x,t)\Big]\cdot\{\widetilde{U}(x,t)\}^+\,dS\,dt + \int_{\Omega^+}\varrho\big[\partial_t u_r(x,0) - u^{(1)}_r(x)\big]\widetilde{u}_r(x,0)\,dx \\
&\qquad - \int_{\Omega^+}\Big\{p_l\big[\partial_l\varphi(x,0) - \partial_l\varphi^{(0)}(x)\big] + m_l\big[\partial_l\psi(x,0) - \partial_l\psi^{(0)}(x)\big]\Big\}\widetilde{\vartheta}(x,0)\,dx = 0. \qquad (2.262)
\end{aligned}$$

By taking an arbitrary vector function $\widetilde{U}(x,t)$ possessing in addition the property $\widetilde{U}(x,0) = 0$, we conclude from (2.262)

$$\big\{\mathcal{T}(\partial_x,n,\partial_t)U(x,t)\big\}^+ = F^*(x,t) \ \text{ on } \ \partial\Omega^+\times J_T. \qquad (2.263)$$

Now, if in (2.262) we take $\widetilde{U}(x,t)$ with arbitrary $\widetilde{u}_r(x,0)$ and $\widetilde{\varphi}(x,0) = \widetilde{\psi}(x,0) = \widetilde{\vartheta}(x,0) = 0$ we find

$$\partial_t u_r(x,0) = u^{(1)}_r(x) \ \text{ in } \ \Omega^+. \qquad (2.264)$$

Then (2.262) leads to the relation

$$\int_{\Omega^+}\Big\{p_l\big[\partial_l\varphi(x,0) - \partial_l\varphi^{(0)}(x)\big] + m_l\big[\partial_l\psi(x,0) - \partial_l\psi^{(0)}(x)\big]\Big\}\widetilde{\vartheta}(x,0)\,dx = 0. \qquad (2.265)$$

Now, we show that $\varphi(x,0) = \varphi^{(0)}(x)$ and $\psi(x,0) = \psi^{(0)}(x)$ for $x\in\Omega^+$. To this end let us take in (2.248) an arbitrary vector function $\widetilde{U}$ with $\widetilde{u} = 0$ and $\widetilde{\vartheta} = 0$ and take into consideration(2.252),

$$\begin{aligned}
&\int_0^T \int_{\Omega^+} \Big\{ \Big[ - e_{jkl}\varepsilon_{kl}(x,t) + \varkappa_{jl}\partial_l\varphi(x,t) + a_{jl}\partial_l\psi(x,t) - p_j\big(\vartheta(x,t) + \nu_0\partial_t\vartheta(x,t)\big)\Big]\partial_j\widetilde{\varphi}(x,t) \\
&\quad + \Big[ - q_{jkl}\varepsilon_{kl}(x,t) + a_{jl}\partial_l\varphi(x,t) + \mu_{jl}\partial_l\psi(x,t) - m_j\big(\vartheta(x,t) + \nu_0\partial_t\vartheta(x,t)\big)\Big]\partial_j\widetilde{\psi}(x,t)\Big\}\,dx\,dt \\
&\qquad = \int_0^T \int_{S_N} \big[F^*_4(x,t)\{\widetilde{\varphi}(x,t)\}^+ + F^*_5(x,t)\{\widetilde{\psi}(x,t)\}^+\big]\,dS\,dt \\
&\qquad\qquad + \int_0^T \int_{\Omega^+} \big[\varrho_e(x,t)\widetilde{\varphi}(x,t) + \varrho_c(x,t)\widetilde{\psi}(x,t)\big]\,dx\,dt. \qquad (2.266)
\end{aligned}$$

Taking here $\widetilde{U}_4(x,t) = \widetilde{\varphi}(x,t) = \widetilde{X}_2(x)\widetilde{\chi}(t)$ and $\widetilde{U}_5(x,t) = \widetilde{\psi}(x,t) = \widetilde{X}_2(x)\widetilde{\chi}(t)$, where $\widetilde{X}_1, \widetilde{X}_2 \in H^1_2(\Omega^+;S_D)$ and $\widetilde{\chi}$ is an absolutely continuous function in $\overline{J_T}$ with $\widetilde{\chi}(T) = 0$, and keeping in mind that $U$ is a semi-regular vector, from (2.266) we deduce

$$\int_{\Omega^+}\Big\{\Big[-e_{jkl}\varepsilon_{kl}(x,t)+\varkappa_{jl}\partial_l\varphi(x,t)+a_{jl}\partial_l\psi(x,t)-p_j\big(\vartheta(x,t)+\nu_0\partial_t\vartheta(x,t)\big)\Big]\partial_j\widetilde{X}_1(x)$$
$$+\Big[-q_{jkl}\varepsilon_{kl}(x,t)+a_{jl}\partial_l\varphi(x,t)+\mu_{jl}\partial_l\psi(x,t)-m_j\big(\vartheta(x,t)+\nu_0\partial_t\vartheta(x,t)\big)\Big]\partial_j\widetilde{X}_2(x)\Big\}\,dx$$
$$=\int_{S_N}\big[F_4^*(x,t)\{\widetilde{X}_1(x)\}^+ + F_5^*(x,t)\{\widetilde{X}_2(x)\}^+\big]\,dS$$
$$+\int_{\Omega^+}\big[\varrho_e(x,t)\widetilde{X}_1(x)+\varrho_c(x,t)\widetilde{X}_2(x)\big]\,dx,\ \ t\in\overline{J_T}. \tag{2.267}$$

Passing to the limit in (2.267) as $t\to 0$ and taking into consideration (2.245)–(2.247) we arrive at the relation

$$\int_{\Omega^+}\Big\{\big[\varkappa_{jl}\partial_l\varphi(x,0)+a_{jl}\partial_l\psi(x,0)\big]\partial_j\widetilde{X}_1(x)+\big[a_{jl}\partial_l\varphi(x,0)+\mu_{jl}\partial_l\psi(x,0)\big]\partial_j\widetilde{X}_2(x)\Big\}\,dx$$
$$=\int_{S_N}\big[F_4^*(x,0)\{\widetilde{X}_1(x)\}^+ + F_5^*(x,0)\{\widetilde{X}_2(x)\}^+\big]\,dS+\int_{\Omega^+}\big[\varrho_e(x,0)\widetilde{X}_1(x)+\varrho_c(x,0)\widetilde{X}_2(x)\big]\,dx$$
$$+\int_{\Omega^+}\Big\{\big[e_{jkl}\varepsilon_{kl}^{(0)}(x)+p_j\big(\vartheta^{(0)}(x)+\nu_0\vartheta^{(1)}(x)\big)\big]\partial_j\widetilde{X}_1(x)$$
$$+\big[q_{jkl}\varepsilon_{kl}^{(0)}(x)+m_j\big(\vartheta^{(0)}(x)+\nu_0\vartheta^{(1)}(x)\big)\big]\partial_j\widetilde{X}_2(x)\Big\}\,dx, \tag{2.268}$$

which holds true for arbitrary $\widetilde{X}_1,\widetilde{X}_2\in H_2^1(\Omega^+;S_D)$.

By comparing the relations (2.268) and (2.256)–(2.258) we conclude that the semi-regular vector $(\varphi(x,0),\psi(x,0))^\top\in[H_2^1(\Omega^+;S_D)]^2$ is a solution to the uniquely solvable weak problem (2.256)–(2.258). Therefore, $\varphi(x,0)=\varphi^{(0)}(x)$ and $\psi(x,0)=\psi^{(0)}(x)$ for $x\in\Omega^+$. Consequently, equality (2.265) holds true.

*Remark* 2.20. Some additional structural restrictions on the problem data associated with the higher order compatibility conditions will be treated in detail later on when we investigate existence and regularity of solutions.

2.6.2. *Weak formulation of the pseudo-oscillation problems.* In the case of weak formulation of the pseudo-oscillation problems for the domain $\Omega^+$, we look for a complex-valued vector function

$$U=(u,\varphi,\psi,\vartheta)^\top\in[W_p^1(\Omega^+)]^6,\ \ 1<p<\infty, \tag{2.269}$$

which satisfies:

(i) the differential equation $A(\partial_x,\tau)U(x)=\Psi(x)$ (see (2.155)) in the weak sense which means that for arbitrary vector function $U'\in[\mathcal{D}(\Omega^+)]^6$ the following functional equation holds true (see Green's formula (2.198))
$$\int_{\Omega^+}\mathcal{E}_\tau(U,\overline{U'})\,dx=-\int_{\Omega^+}\Psi(x)\cdot U'(x)\,dx \tag{2.270}$$
with $\mathcal{E}_\tau(U,\overline{U'})$ defined in (2.201);

(ii) one of the boundary conditions
- the Dirichlet type boundary condition (2.161) in the usual trace sense,

or
- the Neumann type boundary condition (2.156) in the generalized trace sense defined by Green's formula (2.202),

or
- the mixed type boundary conditions (2.166)–(2.173), where the Dirichlet type conditions are understood in the usual trace sense, while the Neumann type conditions are understood in the generalized trace sense defined by Green's formula (2.202).

The weak statements of the crack type pseudo-oscillation problems for the domain $\Omega_\Sigma^+$ are formulated similarly; in this case the boundary conditions (2.174)–(2.184) on the crack faces are understood in the usual trace sense for the Dirichlet data and in the sense of generalized trace for the Neumann data.

Note that, if $U \in [W^1_p(\Omega)]^6$ is a weak solution to the homogeneous differential equation (2.155) in an open domain $\Omega \subset \mathbb{R}^3$, i.e., $\Psi = 0$ in $\Omega$, then actually we have the inclusion $U \in [C^\infty(\Omega)]^6$ due to the strong ellipticity of the differential operator $A(\partial, \tau)$. In fact, in this case $U$ is a complex-valued analytic vector function of the spatial real variables $(x_1, x_2, x_3)$ in the domain $\Omega$.

*Remark* 2.21. The data involved in the weak formulations of the boundary value problems of pseudo-oscillations should satisfy natural smoothness conditions, in particular, the right hand side of the differential equation $\Psi \in [L_p(\Omega^+)]^6$, the Dirichlet and Neumann data belong to the spaces $B^{1-\frac{1}{p}}_{p,p}$ and $B^{-\frac{1}{p}}_{p,p}$, respectively, on the corresponding parts of the boundary.

In the case of the exterior pseudo-oscillation problems for an unbounded domain $\Omega^-$ we assume that the right hand side vector function $\Psi$ has a compact support, $\Phi \in [L_{p,comp}(\Omega^-)]^6$.

Some additional restrictions associated with the asymptotic behaviour of solutions with respect to the complex parameter $\tau$ will be treated later on when we analyse sufficient conditions for existence of the inverse Laplace transform.

2.7. **Uniqueness theorems.** Here we will prove the uniqueness theorems for the general dynamical and pseudo-oscillation problems. Note that the uniqueness theorem for classical solutions to the homogeneous mixed initial-boundary value problem with special type initial data, consisting of nine homogeneous initial conditions, is proved without making restrictions on the positive definiteness on the elastic moduli in the references [69], [4], [5], but the additional symmetry assumption is assumed for the the piezoelectric and the piezomagnetic constants: $e_{jkl} = e_{ljk} = e_{lkj}$ and $q_{jkl} = q_{ljk} = q_{lkj}$ (cf. (2.9)).

However, as we have shown in Subsection 2.2.1, only eight nonhomogeneous initial conditions can be prescribed arbitrarily in the GTEME model and they along with the natural boundary conditions form well posed initial-boundary value problems of dynamics.

We will apply here a different technique which allows to prove uniqueness of weak solutions and to avoid the additional symmetry requirements but instead we need the positive definiteness of both the elasticity tensor and the matrix $M$ defined by (2.13).

2.7.1. *Uniqueness theorems for regular solutions of the dynamical problems.*

**Theorem 2.22.** *Let the conditions* (2.9), (2.10) *hold and the matrix $M$ defined by* (2.13) *be positive definite. In the class of regular vector functions $[C^3(\Omega^+ \times J_\infty)]^6 \cap [C^2(\overline{\Omega^+} \times \widetilde{J}_\infty)]^6$, the homogeneous Dirichlet and mixed type initial-boundary value problems $(D)^+_t$ and $(M)^+_t$ possess only the trivial solution, provided the subsurfaces $S_3$ and $S_5$ in the setting of mixed problem are not empty, while the general solution of the Neumann type initial-boundary value problem* $(\mathrm{N})^+_t$ *reads as*

$$U(x,t) = (0,0,0,C_1(t),C_2(t),0)^\top \tag{2.271}$$

*with arbitrary differentiable functions $C_1$ and $C_2$.*

*Proof.* Let $U = (u, \varphi, \psi, \vartheta)^\top \in [C^3(\Omega^+ \times J_\infty)]^6 \cap [C^2(\overline{\Omega^+} \times \widetilde{J}_\infty)]^6$ be a solution vector to the homogeneous initial-boundary value problem $(\mathrm{D})^+_t$, or $(\mathrm{N})^+_t$, or $(\mathrm{M})^+_t$. Thus the vector $U$ solves the homogeneous system of dynamical differential equations $A(\partial_x, \partial_t)U(x,t) = 0$ in $(x,t) \in \Omega^+ \times J_\infty$, i.e.,

$$\begin{aligned} c_{rjkl}\partial_j\partial_l u_k + e_{lrj}\partial_j\partial_l\varphi + q_{lrj}\partial_j\partial_l\psi - \lambda_{rj}\partial_j\vartheta - \nu_0\lambda_{rj}\partial_j\dot{\vartheta} - \varrho\ddot{u}_r = 0, \quad r = 1,2,3,\\ -e_{jkl}\partial_j\partial_l u_k + \varkappa_{jl}\partial_j\partial_l\varphi + a_{jl}\partial_j\partial_l\psi - p_j\partial_j\vartheta - \nu_0 p_j\partial_j\dot{\vartheta} = 0,\\ -q_{jkl}\partial_j\partial_l u_k + a_{jl}\partial_j\partial_l\varphi + \mu_{jl}\partial_j\partial_l\psi - m_j\partial_j\vartheta - \nu_0 m_j\partial_j\dot{\vartheta} = 0,\\ -\lambda_{kl}\partial_l\dot{u}_k + p_l\partial_l\dot{\varphi} + m_l\partial_l\dot{\psi} + \eta_{jl}\partial_j\partial_l\vartheta - d_0\dot{\vartheta} - h_0\ddot{\vartheta} = 0, \end{aligned} \tag{2.272}$$

where all the unknown functions $u_k$, $\varphi$, $\psi$, and $\vartheta$ depend on the variables $x \in \Omega^+$ and $t \in J_\infty$, and the overset dots denote the time derivatives.

By multiplying the first three equations in (2.272) by $-\dot{u}_r$, $r = 1,2,3$, integrating over $\Omega^+$, applying the Gauss divergence theorem and taking into account the homogenous boundary conditions and the symmetry properties of the material parameters (2.9), we get (cf. (2.185) with $u'_r = \dot u_r$)

$$\int_{\Omega^+}\Big[c_{rjkl}\varepsilon_{kl}\dot\varepsilon_{rj} + e_{lrj}\partial_l\varphi\dot\varepsilon_{rj} + q_{lrj}\partial_l\psi\dot\varepsilon_{rj} - \lambda_{rj}(\vartheta+\nu_0\dot\vartheta)\dot\varepsilon_{rj} + \frac{\varrho}{2}\,\partial_t(\dot u)^2\Big]\,dx = 0, \tag{2.273}$$

where $\varepsilon_{kl} = 2^{-1}(\partial_k u_l + \partial_l u_k)$ and $(\dot u)^2 = (\partial_t u_1)^2 + (\partial_t u_2)^2 + (\partial_t u_3)^2$.

Further, differentiate the fourth and the fifth equations in (2.272) with respect to $t$, multiply by $-\varphi$ and $-\psi$, respectively, integrate over $\Omega^+$, again apply the Gauss divergence theorem, the homogeneous boundary conditions and the symmetry properties of the material parameters (2.9) to obtain (cf. (2.186) and (2.187) with $\varphi' = -\varphi$ and $\psi' = -\psi$, respectively)

$$\int_{\Omega^+}\Big[-e_{jkl}\partial_j\varphi\dot\varepsilon_{kl} + \varkappa_{jl}\partial_j\varphi\partial_l\dot\varphi + a_{jl}\partial_j\varphi\partial_l\dot\psi - (\dot\vartheta+\nu_0\ddot\vartheta)p_j\partial_j\varphi\Big]\,dx = 0, \tag{2.274}$$

$$\int_{\Omega^+}\Big[-q_{jkl}\partial_j\psi\dot\varepsilon_{kl} + a_{jl}\partial_j\psi\partial_l\dot\varphi + \mu_{jl}\partial_j\psi\partial_l\dot\psi - (\dot\vartheta+\nu_0\ddot\vartheta)m_j\partial_j\psi\Big]\,dx = 0. \tag{2.275}$$

Quite similarly, by multiplying the sixth equation in (2.272) by $-(\vartheta+\nu_0\dot\vartheta)$, integrating over $\Omega^+$ and applying the same manipulations as in the derivation of the previous relations, we get (cf. (2.188) with $\vartheta' = -(\vartheta+\nu_0\dot\vartheta)$)

$$\int_{\Omega^+}\Big[\lambda_{kl}\dot\varepsilon_{kl}(\vartheta+\nu_0\dot\vartheta) - (\vartheta+\nu_0\dot\vartheta)p_l\partial_l\dot\varphi - (\vartheta+\nu_0\dot\vartheta)m_l\partial_l\dot\psi$$
$$+ \eta_{jl}\partial_l\vartheta(\partial_j\vartheta+\nu_0\partial_j\dot\vartheta) + (d_0\dot\vartheta + h_0\ddot\vartheta)(\vartheta+\nu_0\dot\vartheta)\Big]\,dx = 0. \tag{2.276}$$

In the derivation of the relations (2.273)–(2.276), the surface integrals vanish for all initial-boundary value problems stated in the theorem due to the homogeneous boundary conditions.

Using the following evident relations

$$c_{rjkl}\varepsilon_{kl}\dot\varepsilon_{rj} = 2^{-1}\partial_t\big(c_{rjkl}\varepsilon_{kl}\varepsilon_{rj}\big),\quad \varkappa_{jl}\partial_j\varphi\partial_l\dot\varphi = 2^{-1}\partial_t\big(\varkappa_{jl}\partial_j\varphi\partial_l\varphi\big),$$
$$\mu_{jl}\partial_j\psi\partial_l\dot\psi = 2^{-1}\partial_t\big(\mu_{jl}\partial_j\psi\partial_l\psi\big),\quad \eta_{jl}\partial_j\vartheta\partial_l\dot\vartheta = 2^{-1}\partial_t\big(\eta_{jl}\partial_j\vartheta\partial_l\vartheta\big),$$
$$a_{jl}\big(\partial_j\varphi\partial_l\dot\psi + \partial_l\dot\varphi\partial_j\psi\big) = \partial_t\big(a_{jl}\partial_j\varphi\partial_l\psi\big),$$

the sum of equalities (2.273)–(2.276) can be written as

$$\int_{\Omega^+}\Big[\frac12\,\partial_t\big(c_{rjkl}\varepsilon_{kl}\varepsilon_{rj}\big) + \frac{\varrho}{2}\partial_t(\dot u)^2 + \frac12\,\partial_t\big[\varkappa_{jl}\partial_j\varphi\partial_l\varphi + 2a_{jl}\partial_j\varphi\partial_l\psi + \mu_{jl}\partial_j\psi\partial_l\psi\big]$$
$$-\partial_t\big[(\vartheta+\nu_0\dot\vartheta)(p_j\partial_j\varphi + m_j\partial_j\psi)\big] + \frac{\nu_0}{2}\,\partial_t(\eta_{jl}\partial_j\vartheta\partial_l\vartheta) + \frac{d_0}{2}\,\partial_t(\vartheta^2) + \frac{h_0\nu_0}{2}\partial_t(\dot\vartheta)^2$$
$$+ h_0\vartheta\ddot\vartheta + \eta_{jl}\partial_j\vartheta\partial_l\vartheta + d_0\nu_0(\dot\vartheta)^2\Big]\,dx = 0. \tag{2.277}$$

In our analysis below, we take into consideration that due to the results obtained in Sub-subsection 2.2.1, the functions $\varphi(x,0)$ and $\psi(x,0)$ either vanish in $\Omega^+$ or they are constants. Therefore, we have $\partial_j\varphi(x,0) = 0$ and $\partial_j\psi(x,0) = 0$ for $x\in\Omega^+$.

Now, multiply equation (2.277) by 2, integrate over the interval $(0,t)$ and take into account the corresponding homogeneous initial conditions to obtain

$$\int_{\Omega^+}\Big[c_{rjkl}\varepsilon_{kl}\varepsilon_{rj} + \varrho(\dot u)^2 + \varkappa_{jl}\partial_j\varphi\partial_l\varphi + 2a_{jl}\partial_j\varphi\partial_l\psi + \mu_{jl}\partial_j\psi\partial_l\psi$$
$$-2(\vartheta+\nu_0\dot\vartheta)(p_j\partial_j\varphi + m_j\partial_j\psi) + \nu_0\eta_{jl}\partial_j\vartheta\partial_l\vartheta + d_0\vartheta^2 + h_0\nu_0(\dot\vartheta)^2\Big]\,dx$$
$$+\int_{\Omega^+}\int_0^t\big[2h_0\vartheta\ddot\vartheta + 2\eta_{jl}\partial_j\vartheta\partial_l\vartheta + 2d_0\nu_0(\dot\vartheta)^2\big]dt'\,dx = 0. \tag{2.278}$$

With the help of the equality

$$\int_0^t \vartheta(x,t')\ddot\vartheta(x,t')\,dt' = \vartheta(x,t)\dot\vartheta(x,t) - \int_0^t \big[\dot\vartheta(x,t')\big]^2\,dt',$$

from (2.278) we finally arrive at the relation

$$\int_{\Omega^+}\Big[c_{rjkl}\varepsilon_{kl}\varepsilon_{rj}+\varrho(\dot u)^2+\nu_0\eta_{jl}\partial_j\vartheta\partial_l\vartheta+\varkappa_{jl}\partial_j\varphi\partial_l\varphi+2a_{jl}\partial_j\varphi\partial_l\psi+\mu_{jl}\partial_j\psi\partial_l\psi$$
$$-2(\vartheta+\nu_0\dot\vartheta)(p_j\partial_j\varphi+m_j\partial_j\psi)+d_0\vartheta^2+h_0\nu_0(\dot\vartheta)^2+2h_0\vartheta\dot\vartheta\Big]\,dx$$
$$+\int_{\Omega^+}\int_0^t\big[2\eta_{jl}\partial_j\vartheta\partial_l\vartheta+2(d_0\nu_0-h_0)(\dot\vartheta)^2\big]dt'\,dx=0. \quad (2.279)$$

The first, the second, and the third summands in the first integral, as well as the summands in the second integral are non-negative in accordance with the inequalities in (2.10) and (2.11). Further, we show that the sum of the last seven terms in the first integrand containing the functions $\partial_j\varphi$, $\partial_j\psi$, $\vartheta$, and $\dot\vartheta$ is also non-negative. Indeed let us set

$$\zeta_j:=\partial_j\varphi,\quad \zeta_{j+3}:=\partial_j\psi,\quad \zeta_7:=-\vartheta,\quad \zeta_8:=-\dot\vartheta,\ \ j=1,2,3, \quad (2.280)$$

and introduce the vector

$$\Theta:=(\zeta_1,\zeta_2,\dots,\zeta_8)^\top. \quad (2.281)$$

Keeping in mind the structure of the matrix $M=[M_{pq}]_{8\times8}$ defined by (2.13), it can easily be checked that (summation over repeated indices is meant from 1 to 3)

$$\varkappa_{jl}\partial_j\varphi\partial_l\varphi+2a_{jl}\partial_j\varphi\partial_l\psi+\mu_{jl}\partial_j\psi\partial_l\psi-2(\vartheta+\nu_0\dot\vartheta)(p_j\partial_j\varphi+m_j\partial_j\psi)+d_0\vartheta^2+h_0\nu_0(\dot\vartheta)^2+2h_0\vartheta\dot\vartheta$$
$$=\big[\varkappa_{jl}\partial_l\varphi+a_{jl}\partial_l\psi+p_j(-\vartheta)+\nu_0p_j(-\dot\vartheta)\big]\partial_j\varphi+\big[a_{jl}\partial_l\varphi+\mu_{jl}\partial_l\psi+m_j(-\vartheta)+\nu_0m_j(-\dot\vartheta)\big]\partial_j\psi$$
$$+\big[p_l\partial_l\varphi+m_l\partial_l\psi+d_0(-\vartheta)+h_0(-\dot\vartheta)\big](-\vartheta)+\big[\nu_0p_l\partial_l\varphi+\nu_0m_l\partial_l\psi+h_0(-\vartheta)+\nu_0h_0(-\dot\vartheta)\big](-\dot\vartheta)$$
$$=\big[\varkappa_{jl}\zeta_l+a_{jl}\zeta_{l+3}+p_j\zeta_7+\nu_0p_j\zeta_8\big]\zeta_j+\big[a_{jl}\zeta_l+\mu_{jl}\zeta_{l+3}+m_j\zeta_7+\nu_0m_j\zeta_8\big]\zeta_{j+3}$$
$$+\big[p_l\zeta_l+m_l\zeta_{l+3}+d_0\zeta_7+h_0\zeta_8\big]\zeta_7+\big[\nu_0p_l\zeta_l+\nu_0m_l\zeta_{l+3}+h_0\zeta_7+\nu_0h_0\zeta_8\big]\zeta_8$$
$$=\sum_{p,q=1}^{8}M_{pq}\zeta_q\zeta_p=M\Theta\cdot\Theta\ge C_0|\Theta|^2 \quad (2.282)$$

with some positive constant $C_0$ due to positive definiteness of the matrix $M$. Therefore from (2.279) in view of (2.280)–(2.282) it follows that

$$2\varepsilon_{rj}(x,t)=\partial_ru_j+\partial_ju_r=0,\ \ \dot u_r(x,t)=0,\ \ \vartheta(x,t)=0, \quad (2.283)$$
$$\partial_j\varphi(x,t)=0,\ \ \partial_j\psi(x,t)=0,\ \ r,j=1,2,3,\ \ (x,t)\in\Omega^+\times J_\infty, \quad (2.284)$$

implying that $u$ is independent of $t$, while $\varphi$ and $\psi$ are independent of $x$. As it is well known (see, e.g., [57]), the general solution to the first group of equations in (2.283) is a rigid displacement vector which reads as

$$\chi(x)=a\times x+b, \quad (2.285)$$

where $a=(a_1,a_2,a_3)$ and $b=(b_1,b_2,b_3)$ are arbitrary real constant vectors and the symbol "$\times$" denotes the cross product. Therefore from (2.283), (2.284) we find

$$u(x,t)=\chi(x)=a\times x+b,\quad \vartheta(x,t)=0,\ \ \varphi(x,t)=C_1(t),\ \ \psi(x,t)=C_2(t), \quad (2.286)$$

where $a=(a_1,a_2,a_3)$ and $b=(b_1,b_2,b_3)$ are arbitrary constant vectors, while $C_1$ and $C_2$ are arbitrary differentiable functions of $t$. In view of the homogeneous initial conditions we finally conclude

$$u(x,t)=0,\ \ \vartheta(x,t)=0,\ \ \varphi(x,t)=C_1(t),\ \ \psi(x,t)=C_2(t) \quad (2.287)$$

with arbitrary differentiable functions $C_1$ and $C_2$.

In the case of the dynamical Dirichlet and mixed type problems (with non-empty subsurfaces $S_3$ and $S_5$) it is evident that $C_1=C_2=0$, while in the case of the dynamical Neumann problem the functions $C_1$ and $C_2$ remain arbitrary differentiable functions. This completes the proof of the theorem. □

2.7.2. *Uniqueness theorems for weak solutions of the dynamical problems.*

**Theorem 2.23.** *Let the matrix $M$ given by* (2.13) *be positive definite. The homogeneous Dirichlet and mixed type initial-boundary value problems $(WD)_t^+$ and $(WM)_t^+$ possess only the trivial weak solutions in the spaces $[W_2^1(\Omega^+\times J_T;S)]^6$ and $[W_2^1(\Omega^+\times J_T;S_D)]^6$, respectively, while the general weak solution of the Neumann type problem $(WN)_t^+$ in the space $[W_2^1(\Omega^+\times J_T)]^6$ reads as*

$$U(x,t)=\big(0,0,0,C_1(t),C_2(t),0\big)^\top,\ \ (x,t)\in\Omega^+\times J_T, \tag{2.288}$$

*where $C_1(t)$ and $C_2(t)$ are arbitrary absolutely continuous functions on $[0,T]$.*

*Proof.* Let $U=(u,\varphi,\psi,\vartheta)^\top\in[W_2^1(\Omega^+\times J_T)]^6$ be a weak solution to the homogeneous initial-boundary value problem $(\mathrm{WD})_t^+$, or $(\mathrm{WN})_t^+$, or $(\mathrm{WM})_t^+$ in the sense of the definition (2.244)–(2.259), where we have to substitute

$$\begin{gathered}\Phi(x,t)=0,\quad F^*(x,t)=0,\quad u_r(x,0)=u_r^{(0)}(x)=0,\quad \partial_t u_r(x,0)=u_r^{(1)}(x)=0,\\ \vartheta(x,0)=\vartheta^{(0)}(x)=0,\quad \partial_t\vartheta(x,0)=\vartheta^{(1)}(x)=0,\\ \varphi(x,0)=\varphi^{(0)}(x)=c_4,\quad \psi(x,0)=\psi^{(0)}(x,0)=c_5,\end{gathered} \tag{2.289}$$

where $c_4$ and $c_5$ are arbitrary constants in the case of the Neumann problem and they equal to zero for the Dirichlet and mixed type problems. Therefore $U$ satisfies the homogeneous functional equation

$$\begin{aligned}\int_0^T\int_{\Omega^+}\bigg\{&\Big[c_{rjkl}\varepsilon_{kl}(x,t)+e_{lrj}\partial_l\varphi(x,t)+q_{lrj}\partial_l\psi(x,t)-\lambda_{rj}\big(\vartheta(x,t)+\nu_0\partial_t\vartheta(x,t)\big)\Big]\widetilde\varepsilon_{rj}(x,t)\\ &+\Big[-e_{jkl}\varepsilon_{kl}(x,t)+\varkappa_{jl}\partial_l\varphi(x,t)+a_{jl}\partial_l\psi(x,t)-p_j\big(\vartheta(x,t)+\nu_0\partial_t\vartheta(x,t)\big)\Big]\partial_j\widetilde\varphi(x,t)\\ &+\Big[-q_{jkl}\varepsilon_{kl}(x,t)+a_{jl}\partial_l\varphi(x,t)+\mu_{jl}\partial_l\psi(x,t)-m_j\big(\vartheta(x,t)+\nu_0\partial_t\vartheta(x,t)\big)\Big]\partial_j\widetilde\psi(x,t)\\ &+\big[-\lambda_{kl}\varepsilon_{kl}(x,t)+p_l\partial_l\varphi(x,t)+m_l\partial_l\psi(x,t)-d_0\vartheta(x,t)-h_0\partial_t\vartheta(x,t)\Big]\partial_t\widetilde\vartheta(x,t)\\ &+\eta_{jl}\partial_l\vartheta(x,t)\partial_j\widetilde\vartheta(x,t)-\varrho\partial_t u_r(x,t)\partial_t\widetilde u_r(x,t)\bigg\}\,dx\,dt=0\end{aligned} \tag{2.290}$$

for arbitrary vector function

$$\widetilde U=(\widetilde u_1,\widetilde u_2,\widetilde u_3,\widetilde\varphi,\widetilde\psi,\widetilde\vartheta)^\top\in[W_2^1(\Omega^+\times J_T)]^6, \tag{2.291}$$

such that

$$\{\widetilde U(x,t)\}^+=0\ \text{ for }\ (x,t)\in S_D\times J_T, \tag{2.292}$$

$$\widetilde u_k(x,T)=0,\ \ \widetilde\vartheta(x,T)=0\ \text{ for }\ x\in\Omega^+,\ \ k=1,2,3. \tag{2.293}$$

Recall that $S_D=S$ in the case of the Dirichlet problem, $S_D=\varnothing$ in the case of the Neumann problem, and $S_D$ is a proper part of $S$ in the case of the mixed problem.

Let us define the following functions

$$\widetilde u_r(x,t):=-\int_t^T u_r(x,t_1)\,dt_1,\ \ r=1,2,3, \tag{2.294}$$

$$\widetilde\vartheta(x,t):=-\int_t^T\Big[\int_0^{t_1}\vartheta(x,t_2)\,dt_2+\nu_0\vartheta(x,t_1)\Big]\,dt_1, \tag{2.295}$$

$$\widetilde\varphi(x,t):=-\int_0^t\varphi(x,t_1)\,dt_1, \tag{2.296}$$

$$\widetilde\psi(x,t):=-\int_0^t\psi(x,t_1)\,dt_1,\ \ (x,t)\in\Omega^+\times(0,T]. \tag{2.297}$$

It is evident that the vector function $\widetilde U=(\widetilde u_1,\widetilde u_2,\widetilde u_3,\widetilde\varphi,\widetilde\psi,\widetilde\vartheta)^\top$ defined by the equalities (2.294)–(2.297) belongs to the space $[W_2^1(\Omega^+\times J_T)]^6$ and the following relations hold

$$\begin{gathered}\partial_t\widetilde u_r(x,t)=u_r(x,t),\ \ \widetilde u(x,T)=0,\ \ \widetilde\varepsilon_{rj}(x,T)=0,\\ \partial_j\partial_t\widetilde u_r(x,t)=\partial_j u(x,t),\ \ r,j=1,2,3,\end{gathered} \tag{2.298}$$

$$\begin{gathered}\partial_t\widetilde{\vartheta}(x,t)=\int_0^t\vartheta(x,t_2)\,dt_2+\nu_0\vartheta(x,t),\ \ \partial_j\partial_t\widetilde{\vartheta}(x,t)=\int_0^t\partial_j\vartheta(x,t_2)\,dt_2+\nu_0\partial_j\vartheta(x,t),\\ \partial_t\partial_t\widetilde{\vartheta}(x,t)=\vartheta(x,t)+\nu_0\partial_t\vartheta(x,t),\ \ \widetilde{\vartheta}(x,T)=0,\ \ \partial_t\widetilde{\vartheta}(x,0)=0,\ \ \partial_j\widetilde{\vartheta}(x,T)=0,\end{gathered} \tag{2.299}$$

$$\partial_t\widetilde{\varphi}(x,t)=-\varphi(x,t),\ \ \partial_j\partial_t\widetilde{\varphi}(x,t)=-\partial_j\varphi(x,t),\ \ \widetilde{\varphi}(x,0)=0,\ \ \partial_j\widetilde{\varphi}(x,0)=0, \tag{2.300}$$

$$\partial_t\widetilde{\psi}(x,t)=-\psi(x,t),\ \ \partial_j\partial_t\widetilde{\psi}(x,t)=-\partial_j\psi(x,t),\ \ \widetilde{\psi}(x,0)=0,\ \ \partial_j\widetilde{\psi}(x,0)=0, \tag{2.301}$$

implying in particular that

$$\partial_j\partial_t\widetilde{u}_r,\partial_t\partial_t\widetilde{u}_r,\partial_j\partial_t\widetilde{\varphi},\partial_t\partial_t\widetilde{\varphi},\partial_j\partial_t\widetilde{\psi},\partial_t\partial_t\widetilde{\psi},\partial_j\partial_t\widetilde{\vartheta},\partial_t\partial_t\widetilde{\vartheta}\in L_2(\Omega^+\times J_T),\ \ r,j=1,2,3. \tag{2.302}$$

Denote by $\mathcal{I}(U)$ the expression in the left hand side of (2.290) with $\widetilde{U}$ defined by the equalities (2.294)–(2.297) and represent $\mathcal{I}(U)$ as the sum,

$$\mathcal{I}(U)=\mathcal{I}_1+\mathcal{I}_2+\mathcal{I}_3+\mathcal{I}_4+\mathcal{I}_5+\mathcal{I}_6+\mathcal{I}_7=0, \tag{2.303}$$

where

$$\mathcal{I}_1:=-\int_0^T\int_{\Omega^+}\varrho\partial_t u_r(x,t)\partial_t\widetilde{u}_r(x,t)\,dx\,dt, \tag{2.304}$$

$$\mathcal{I}_2:=\int_0^T\int_{\Omega^+}c_{rjkl}\varepsilon_{kl}(x,t)\widetilde{\varepsilon}_{rj}(x,t)\,dx\,dt, \tag{2.305}$$

$$\mathcal{I}_3:=\int_0^T\int_{\Omega^+}e_{lrj}\big[\partial_l\varphi(x,t)\widetilde{\varepsilon}_{rj}(x,t)-\partial_l\widetilde{\varphi}(x,t)\varepsilon_{rj}(x,t)\big]\,dx\,dt, \tag{2.306}$$

$$\mathcal{I}_4:=\int_0^T\int_{\Omega^+}q_{lrj}\big[\partial_l\psi(x,t)\widetilde{\varepsilon}_{rj}(x,t)-\partial_l\widetilde{\psi}(x,t)\varepsilon_{rj}(x,t)\big]\,dx\,dt, \tag{2.307}$$

$$\begin{aligned}\mathcal{I}_5:=\int_0^T\int_{\Omega^+}\Big\{&\varkappa_{jl}\partial_l\varphi(x,t)\partial_j\widetilde{\varphi}(x,t)+\mu_{jl}\partial_l\psi(x,t)\partial_j\widetilde{\psi}(x,t)\\ &+a_{jl}\big[\partial_l\psi(x,t)\partial_j\widetilde{\varphi}(x,t)+\partial_l\widetilde{\psi}(x,t)\partial_j\varphi(x,t)\big]\Big\}\,dx\,dt,\end{aligned} \tag{2.308}$$

$$\mathcal{I}_6:=-\int_0^T\int_{\Omega^+}\lambda_{rj}\Big\{\big[\vartheta(x,t)+\nu_0\partial_t\vartheta(x,t)\big]\widetilde{\varepsilon}_{rj}(x,t)+\varepsilon_{rj}(x,t)\partial_t\widetilde{\vartheta}(x,t)\Big\}\,dx\,dt, \tag{2.309}$$

$$\begin{aligned}\mathcal{I}_7:=\int_0^T\int_{\Omega^+}\Big\{&-\big[\vartheta(x,t)+\nu_0\partial_t\vartheta(x,t)\big]\,\big[p_j\partial_j\widetilde{\varphi}(x,t)+m_j\partial_j\widetilde{\psi}(x,t)\big]\\ &+\big[p_l\partial_l\varphi(x,t)+m_l\partial_l\psi(x,t)-d_0\vartheta(x,t)-h_0\partial_t\vartheta(x,t)\big]\partial_t\widetilde{\vartheta}(x,t)\\ &+\eta_{jl}\partial_l\vartheta(x,t)\partial_j\widetilde{\vartheta}(x,t)\Big\}\,dx\,dt.\end{aligned} \tag{2.310}$$

Applying the homogeneous initial conditions (2.289), the relations (2.293) and (2.294)–(2.302), and using the integration by parts formula from (2.304)–(2.310) we find

$$\mathcal{I}_1=-\frac{\varrho}{2}\int_0^T\int_{\Omega^+}\partial_t\big[\partial_t\widetilde{u}_r(x,t)\partial_t\widetilde{u}_r(x,t)\big]\,dx\,dt=-\frac{\varrho}{2}\int_{\Omega^+}[u(x,T)]^2\,dx, \tag{2.311}$$

$$\mathcal{I}_2=\frac{1}{2}\int_0^T\int_{\Omega^+}c_{rjkl}\partial_t\big[\widetilde{\varepsilon}_{kl}(x,t)\;\widetilde{\varepsilon}_{rj}(x,t)\big]\,dx\,dt=-\frac{1}{2}\int_{\Omega^+}c_{rjkl}\widetilde{\varepsilon}_{kl}(x,0)\;\widetilde{\varepsilon}_{rj}(x,0)\,dx, \tag{2.312}$$

$$\mathcal{I}_3=-\int_0^T\int_{\Omega^+}e_{lrj}\partial_t\big[\partial_l\widetilde{\varphi}(x,t)\widetilde{\varepsilon}_{rj}(x,t)\big]\,dx\,dt=0, \tag{2.313}$$

$$\mathcal{I}_4=\int_0^T\int_{\Omega^+}q_{lrj}\partial_t\big[\partial_l\widetilde{\psi}(x,t)\widetilde{\varepsilon}_{rj}(x,t)\big]\,dx\,dt=0, \tag{2.314}$$

$$\begin{aligned}\mathcal{I}_5&=-\frac{1}{2}\int_0^T\int_{\Omega^+}\partial_t\Big[\varkappa_{jl}\partial_l\widetilde{\varphi}(x,t)\partial_j\widetilde{\varphi}(x,t)+2a_{jl}\partial_l\widetilde{\psi}(x,t)\partial_j\widetilde{\varphi}(x,t)+\mu_{jl}\partial_l\widetilde{\psi}(x,t)\partial_j\widetilde{\psi}(x,t)\Big]\,dx\,dt\\ &=-\frac{1}{2}\int_{\Omega^+}\Big[\varkappa_{jl}\partial_l\widetilde{\varphi}(x,T)\partial_j\widetilde{\varphi}(x,T)+2a_{jl}\partial_l\widetilde{\psi}(x,T)\partial_j\widetilde{\varphi}(x,T)+\mu_{jl}\partial_l\widetilde{\psi}(x,T)\partial_j\widetilde{\psi}(x,T)\Big]\,dx,\end{aligned} \tag{2.315}$$

$$\mathcal{I}_6 = -\int_0^T \int_{\Omega^+} \lambda_{rj}\big[\vartheta(x,t) + \nu_0\partial_t\vartheta(x,t)\big]\widetilde{\varepsilon}_{rj}(x,t)\,dx\,dt - \int_{\Omega^+} \lambda_{rj}\big[\widetilde{\varepsilon}_{rj}(x,t)\partial_t\widetilde{\vartheta}(x,t)\big]_0^T\,dx$$
$$+ \int_0^T \int_{\Omega^+} \lambda_{rj}\widetilde{\varepsilon}_{rj}(x,t)\partial_t\partial_t\widetilde{\vartheta}(x,t)\,dx\,dt = 0. \tag{2.316}$$

To transform the integral $\mathcal{I}_7$ let us introduce the notation

$$\theta(x,t) := \int_0^t \vartheta(x,t_1)\,dt_1. \tag{2.317}$$

Using the relations

$$\partial_t\widetilde{\vartheta}(x,t) = \theta(x,t) + \nu_0\partial_t\theta(x,t), \quad \theta(x,0) = \partial_j\theta(x,0) = 0, \quad \partial_t\theta(x,t) = \vartheta(x,t), \tag{2.318}$$

we get from (2.310)

$$\mathcal{I}_7 = \int_0^T \int_{\Omega^+} \Big\{ -\partial_t\partial_t\widetilde{\vartheta}(x,t)\big[p_j\partial_j\widetilde{\varphi}(x,t) + m_j\partial_j\widetilde{\psi}(x,t)\big] - \big[p_j\partial_t\partial_j\widetilde{\varphi}(x,t) + m_j\partial_t\partial_j\widetilde{\psi}(x,t)\big]\partial_t\widetilde{\vartheta}(x,t)$$
$$- \big[d_0\vartheta(x,t) + h_0\partial_t\vartheta(x,t)\big]\partial_t\widetilde{\vartheta}(x,t) + \eta_{jl}\partial_l\vartheta(x,t)\partial_j\widetilde{\vartheta}(x,t)\Big\}\,dx\,dt$$
$$= -\int_{\Omega^+} \big[\theta(x,T) + \nu_0\partial_t\theta(x,T)\big]\,\big[p_j\partial_j\widetilde{\varphi}(x,T) + m_j\partial_j\widetilde{\psi}(x,T)\big]\,dx$$
$$+ \int_0^T \int_{\Omega^+} \Big\{\eta_{jl}\partial_l\vartheta(x,t)\partial_j\widetilde{\vartheta}(x,t) - \big[d_0\vartheta(x,t) + h_0\partial_t\vartheta(x,t)\big]\partial_t\widetilde{\vartheta}(x,t)\Big\}\,dx\,dt. \tag{2.319}$$

By simple manipulations we arrive at the following chain of equalities

$$\int_0^T \int_{\Omega^+} \eta_{jl}\partial_l\vartheta(x,t)\partial_j\widetilde{\vartheta}(x,t)\,dx\,dt = \int_0^T \int_{\Omega^+} \big\{\eta_{jl}\partial_t\partial_l\theta(x,t)\partial_j\widetilde{\vartheta}(x,t)\big\}\,dx\,dt$$
$$= \int_{\Omega^+} \eta_{jl}\big[\partial_l\theta(x,t)\partial_j\widetilde{\vartheta}(x,t)\big]_0^T\,dx - \int_0^T \int_{\Omega^+} \eta_{jl}\partial_l\theta(x,t)\partial_t\partial_j\widetilde{\vartheta}(x,t)\,dx\,dt$$
$$= -\int_0^T \int_{\Omega^+} \eta_{jl}\partial_l\theta(x,t)\big[\partial_j\theta(x,t) + \nu_0\partial_t\partial_j\theta(x,t)\big]\,dx\,dt$$
$$= -\int_0^T \int_{\Omega^+} \Big\{\eta_{jl}\partial_l\theta(x,t)\partial_j\theta(x,t) + \frac{\nu_0}{2}\,\partial_t\big[\eta_{jl}\partial_j\theta(x,t)\partial_l\theta(x,t)\big]\Big\}\,dx\,dt$$
$$= -\int_0^T \int_{\Omega^+} \eta_{jl}\partial_l\theta(x,t)\partial_j\theta(x,t)\,dx\,dt - \int_{\Omega^+} \frac{\nu_0}{2}\,\eta_{jl}\partial_j\theta(x,T)\partial_l\theta(x,T)\,dx, \tag{2.320}$$

$$-\int_0^T \int_{\Omega^+} \big[d_0\vartheta(x,t) + h_0\partial_t\vartheta(x,t)\big]\partial_t\widetilde{\vartheta}(x,t)\,dx\,dt$$
$$= -\int_0^T \int_{\Omega^+} \Big\{\frac{d_0}{2}\,\partial_t\big[\theta(x,t)\big]^2 + d_0\nu_0[\partial_t\theta(x,t)]^2 + h_0\theta(x,t)\partial_t\partial_t\theta(x,t) + \frac{\nu_0 h_0}{2}\,\partial_t\big[\partial_t\theta(x,t)\big]^2\Big\}\,dx\,dt$$
$$= -\int_{\Omega^+} \Big[\frac{d_0}{2}\,[\theta(x,T)]^2 + \frac{\nu_0 h_0}{2}\,[\partial_t\theta(x,T)]^2 + h_0\theta(x,T)\partial_t\theta(x,T)\Big]\,dx$$
$$-\int_0^T \int_{\Omega^+} (d_0\nu_0 - h_0)[\partial_t\theta(x,t)]^2\,dx\,dt. \tag{2.321}$$

Now substitute (2.320), (2.321) into (2.319) to obtain

$$\mathcal{I}_7 = -\int_{\Omega^+} \Big\{\big[\theta(x,T) + \nu_0\partial_t\theta(x,T)\big]\,\big[p_j\partial_j\widetilde{\varphi}(x,T) + m_j\partial_j\widetilde{\psi}(x,T)\big] + \frac{d_0}{2}\,[\theta(x,T)]^2$$
$$+ \frac{\nu_0 h_0}{2}\,\big[\partial_t\theta(x,T)^2 + h_0\theta(x,T)\partial_t\theta(x,T)\big] + \frac{\nu_0}{2}\,\eta_{jl}\partial_j\theta(x,T)\partial_l\theta(x,T)\Big\}\,dx$$
$$-\int_0^T \int_{\Omega^+} \Big[\eta_{jl}\partial_l\theta(x,t)\partial_j\theta(x,t) + (d_0\nu_0 - h_0)\big[\partial_t\theta(x,t)\big]^2\Big]\,dx\,dt. \tag{2.322}$$

Thus we have found that

$$\mathcal{I}(U)=\mathcal{I}_1+\mathcal{I}_2+\mathcal{I}_5+\mathcal{I}_7=0. \tag{2.323}$$

From (2.311) and (2.312) it follows that

$$\mathcal{I}_1\leqslant 0,\ \ \mathcal{I}_2\leqslant 0. \tag{2.324}$$

Moreover, with the help of the arguments applied in the proof of Theorem 2.22 we get

$$\begin{aligned}\mathcal{I}_5+\mathcal{I}_7=&-\frac{1}{2}\int_{\Omega^+}\Big\{\varkappa_{jl}\partial_l\widetilde{\varphi}(x,T)\partial_j\widetilde{\varphi}(x,T)+2a_{jl}\partial_l\widetilde{\psi}(x,T)\partial_j\widetilde{\varphi}(x,T)+\mu_{jl}\partial_l\widetilde{\psi}(x,T)\partial_j\widetilde{\psi}(x,T)\\
&+2\big[\theta(x,T)+\nu_0\partial_t\theta(x,T)\big]\,\big[p_j\partial_j\widetilde{\varphi}(x,T)+m_j\partial_j\widetilde{\psi}(x,T)\big]+d_0[\theta(x,T)]^2\\
&+\nu_0h_0[\partial_t\theta(x,T)]^2+2h_0\theta(x,T)\partial_t\theta(x,T)+\nu_0\eta_{jl}\partial_j\theta(x,T)\partial_l\theta(x,T)\Big\}\,dx\\
&-\int_0^T\int_{\Omega^+}\Big[\eta_{jl}\partial_l\theta(x,t)\partial_j\theta(x,t)+(d_0\nu_0-h_0)\big[\partial_t\theta(x,t)\big]^2\Big]\,dx\,dt\\
=&-\frac{1}{2}\int_{\Omega^+}\Big\{\sum_{p,q=1}^{8}M_{pq}\zeta_p(x,T)\zeta_q(x,T)+\nu_0\eta_{jl}\partial_j\theta(x,T)\partial_l\theta(x,T)\Big\}\,dx\\
&-\int_0^T\int_{\Omega^+}\Big[\eta_{jl}\partial_l\theta(x,t)\partial_j\theta(x,t)+(d_0\nu_0-h_0)[\partial_t\theta(x,t)]^2\Big]\,dx\,dt,\end{aligned} \tag{2.325}$$

where $M=[M_{pq}]_{8\times 8}$ is the positive definite matrix defined by (2.13) and

$$\zeta_j:=\partial_j\widetilde{\varphi}(x,T),\ \ \zeta_{j+3}:=\partial_j\widetilde{\psi}(x,T),\ \ \zeta_7:=-\theta(x,T),\ \ \zeta_8:=-\partial_t\theta(x,T),\ \ j=1,2,3. \tag{2.326}$$

Evidently,

$$\sum_{p,q=1}^{8}M_{pq}\zeta_q\zeta_p\geqslant C_0(\zeta_1^2+\zeta_2^2+\cdots+\zeta_8^2) \tag{2.327}$$

with some positive constant $C_0$.

Due to the properties of the material parameters described in (2.10) and (2.11) we deduce that each summand in (2.325) is nonnegative, which implies

$$\mathcal{I}_5+\mathcal{I}_7\leqslant 0. \tag{2.328}$$

Finally, in accordance with (2.311), (2.312), and (2.325), from (2.323) we get

$$\begin{aligned}\mathcal{I}(U)=-\frac{1}{2}\int_{\Omega^+}\Big\{&\varrho[u(x,T)]^2+c_{rjkl}\widetilde{\varepsilon}_{kl}(x,0)\widetilde{\varepsilon}_{rj}(x,0)+\sum_{p,q=1}^{8}M_{pq}\zeta_p(x,T)\zeta_q(x,T)\\
&+\nu_0\eta_{jl}\partial_j\theta(x,T)\partial_l\theta(x,T)\Big\}\,dx-\int_0^T\int_{\Omega^+}\Big[\eta_{jl}\partial_l\theta(x,t)\partial_j\theta(x,t)+(d_0\nu_0-h_0)[\partial_t\theta(x,t)]^2\Big]\,dx\,dt=0.\end{aligned} \tag{2.329}$$

Taking into account that each summand of the integrands in (2.329) is nonnegative, we conclude that

$$u_r(x,T)=0,\ \ 2\widetilde{\varepsilon}_{rj}(x,0)=\partial_r\widetilde{u}_j(x,0)+\partial_j\widetilde{u}_r(x,0)=0, \tag{2.330}$$

$$\partial_j\widetilde{\varphi}(x,T)=0,\ \ \partial_j\widetilde{\psi}(x,T)=0, \tag{2.331}$$

$$\partial_j\theta(x,t)=0,\ \ \partial_t\theta(x,t)=0,\ \ r,j=1,2,3,\ \ t\in(0,T),\ \ x\in\Omega^+. \tag{2.332}$$

From (2.332) and (2.317) we find that

$$\vartheta(x,t)=0\ \text{ for }\ (x,t)\in\Omega^+\times J_T. \tag{2.333}$$

In view of (2.333), now from (2.290) it follows that $U=(u,\varphi,\psi,0)^\top$ satisfies the functional equation

$$\begin{aligned}\int_0^T\int_{\Omega^+}\Big\{&\big[c_{rjkl}\varepsilon_{kl}(x,t)+e_{lrj}\partial_l\varphi(x,t)+q_{lrj}\partial_l\psi(x,t)\big]\widetilde{\varepsilon}_{rj}(x,t)\\
&+\big[-e_{jkl}\varepsilon_{kl}(x,t)+\varkappa_{jl}\partial_l\varphi(x,t)+a_{jl}\partial_l\psi(x,t)\big]\partial_j\widetilde{\varphi}(x,t)\\
&\big[-q_{jkl}\varepsilon_{kl}(x,t)+a_{jl}\partial_l\varphi(x,t)+\mu_{jl}\partial_l\psi(x,t)\big]\partial_j\widetilde{\psi}(x,t)-\varrho\partial_tu_r(x,t)\partial_t\widetilde{u}_r(x,t)\Big\}\,dx\,dt=0\end{aligned} \tag{2.334}$$

for arbitrary vector function $\widetilde{U}=(\widetilde{u}_1,\widetilde{u}_2,\widetilde{u}_3,\widetilde{\varphi},\widetilde{\psi},0)^\top\in[W_2^1(\Omega^+\times J_T)]^6$ such that

$$\{\widetilde{U}(x,t)\}^+=0 \ \text{ for } \ (x,t)\in S_D\times J_T, \quad \widetilde{u}_r(x,T)=0 \ \text{ for } \ x\in\Omega^+, \quad r=1,2,3. \tag{2.335}$$

Let $a\in(0,T]$ be an arbitrary number and for $(x,t)\in\Omega^+\times(0,T]$ define the functions

$$\widetilde{u}_r(x,t):=\begin{cases}-\displaystyle\int_t^a u_r(x,t_1)\,dt_1 & \text{for } \ t\leqslant a,\\ 0 & \text{for } \ a<t\leqslant T,\end{cases} \qquad r=1,2,3, \tag{2.336}$$

$$\widetilde{\varphi}(x,t):=\begin{cases}-\displaystyle\int_t^a \varphi(x,t_1)\,dt_1 & \text{for } \ t\leqslant a,\\ 0 & \text{for } \ a<t\leqslant T,\end{cases} \tag{2.337}$$

$$\widetilde{\psi}(x,t):=\begin{cases}-\displaystyle\int_t^a \psi(x,t_1)\,dt_1 & \text{for } \ t\leqslant a,\\ 0 & \text{for } \ a<t\leqslant T.\end{cases} \tag{2.338}$$

It is evident that the vector function $\widetilde{U}=(\widetilde{u}_1,\widetilde{u}_2,\widetilde{u}_3,\widetilde{\varphi},\widetilde{\psi},0)^\top$ defined by the equalities (2.336)–(2.338) belongs to the space $[W_2^1(\Omega^+\times J_T)]^6$, vanishes for $(x,t)\in\Omega^+\times(a,T]$, and the following relations hold for $(x,t)\in\Omega^+\times(0,a]$,

$$\partial_t\widetilde{u}_r(x,t)=u_r(x,t), \quad \partial_j\partial_t\widetilde{u}_r(x,t)=\partial_j u_r(x,t), \quad \widetilde{u}_r(x,a)=\partial_j\widetilde{u}_r(x,a)=0, \tag{2.339}$$

$$\partial_t\widetilde{\varphi}(x,t)=\varphi(x,t), \quad \partial_j\partial_t\widetilde{\varphi}(x,t)=\partial_j\varphi(x,t), \quad \widetilde{\varphi}(x,a)=0, \tag{2.340}$$

$$\partial_t\widetilde{\psi}(x,t)=\psi(x,t), \quad \partial_j\partial_t\widetilde{\psi}(x,t)=\partial_j\psi(x,t), \quad \widetilde{\psi}(x,a)=0, \ \ r,j=1,2,3. \tag{2.341}$$

These relations imply that

$$\partial_j\partial_t\widetilde{u}_r,\partial_t\partial_t\widetilde{u}_r,\partial_j\partial_t\widetilde{\varphi},\partial_j\partial_t\widetilde{\psi}\in L_2(\Omega^+\times J_a), \ \ J_a=(0,a) \ \ r,j=1,2,3. \tag{2.342}$$

Applying the homogeneous initial conditions (2.289) and relations (2.336)–(2.342), with the help of the integration by parts formula we can rewrite the functional equation (2.334) in the following form

$$\mathcal{I}_1(a)+\mathcal{I}_2(a)+\mathcal{I}_3(a)+\mathcal{I}_4(a)+\mathcal{I}_5(a)=0, \tag{2.343}$$

where

$$\begin{aligned}\mathcal{I}_1(a)&:=-\int_0^T\int_{\Omega^+}\varrho\partial_t u_r(x,t)\partial_t\widetilde{u}_r(x,t)\,dx\,dt=-\frac{\varrho}{2}\int_0^a\int_{\Omega^+}\partial_t\big[\partial_t\widetilde{u}_r(x,t)\partial_t\widetilde{u}_r(x,t)\big]\,dx\,dt\\&=-\frac{\varrho}{2}\int_{\Omega^+}\big[u_r(x,t)u_r(x,t)\big]_0^a\,dx\,dt=-\frac{\varrho}{2}\int_{\Omega^+}[u(x,a)]^2\,dx,\end{aligned} \tag{2.344}$$

$$\begin{aligned}\mathcal{I}_2(a)&:=\int_0^T\int_{\Omega^+}c_{rjkl}\varepsilon_{kl}(x,t)\widetilde{\varepsilon}_{rj}(x,t)\,dx\,dt=\frac{1}{2}\int_0^a\int_{\Omega^+}c_{rjkl}\partial_t\big[\widetilde{\varepsilon}_{kl}(x,t)\widetilde{\varepsilon}_{rj}(x,t)\big]\,dx\,dt\\&=-\frac{1}{2}\int_{\Omega^+}c_{rjkl}\widetilde{\varepsilon}_{kl}(x,0)\widetilde{\varepsilon}_{rj}(x,0)\,dx,\end{aligned} \tag{2.345}$$

$$\begin{aligned}\mathcal{I}_3(a)&:=\int_0^T\int_{\Omega^+}e_{lrj}\big[\partial_l\varphi(x,t)\widetilde{\varepsilon}_{rj}(x,t)-\partial_l\widetilde{\varphi}(x,t)\varepsilon_{rj}(x,t)\big]\,dx\,dt\\&=\int_0^a\int_{\Omega^+}e_{lrj}\big[\partial_t\partial_l\widetilde{\varphi}(x,t)\widetilde{\varepsilon}_{rj}(x,t)-\partial_l\widetilde{\varphi}(x,t)\partial_t\widetilde{\varepsilon}_{rj}(x,t)\big]\,dx\,dt,\end{aligned} \tag{2.346}$$

$$\begin{aligned}\mathcal{I}_4(a)&:=\int_0^T\int_{\Omega^+}q_{lrj}\big[\partial_l\psi(x,t)\widetilde{\varepsilon}_{rj}(x,t)-\partial_l\widetilde{\psi}(x,t)\varepsilon_{rj}(x,t)\big]\,dx\,dt\\&=\int_0^a\int_{\Omega^+}q_{lrj}\big[\partial_t\partial_l\widetilde{\psi}(x,t)\widetilde{\varepsilon}_{rj}(x,t)-\partial_l\widetilde{\psi}(x,t)\partial_t\widetilde{\varepsilon}_{rj}(x,t)\big]\,dx\,dt,\end{aligned} \tag{2.347}$$

$$\begin{aligned}\mathcal{I}_5(a):=\int_0^T\int_{\Omega^+}\Big\{&\varkappa_{jl}\partial_l\varphi(x,t)\partial_j\widetilde{\varphi}(x,t)+\mu_{jl}\partial_l\psi(x,t)\partial_j\widetilde{\psi}(x,t)\\&+a_{jl}\Big[\partial_l\psi(x,t)\partial_j\widetilde{\varphi}(x,t)+\partial_l\widetilde{\psi}(x,t)\partial_j\varphi(x,t)\Big]\Big\}\,dx\,dt\end{aligned}$$

$$= \frac{1}{2}\int_0^a \int_{\Omega^+} \partial_t \Big[ \varkappa_{jl}\partial_l\widetilde{\varphi}(x,t)\partial_j\widetilde{\varphi}(x,t) + 2a_{jl}\partial_l\widetilde{\psi}(x,t)\partial_j\widetilde{\varphi}(x,t) + \mu_{jl}\partial_l\widetilde{\psi}(x,t)\partial_j\widetilde{\psi}(x,t) \Big]\, dx\, dt$$

$$= -\frac{1}{2}\int_{\Omega^+} \Big[ \varkappa_{jl}\partial_l\widetilde{\varphi}(x,0)\partial_j\widetilde{\varphi}(x,0) + 2a_{jl}\partial_l\widetilde{\psi}(x,0)\partial_j\widetilde{\varphi}(x,0) + \mu_{jl}\partial_l\widetilde{\psi}(x,0)\partial_j\widetilde{\psi}(x,0) \Big]\, dx. \quad (2.348)$$

Note that, the vector function $\widetilde{U}^* = (\widetilde{u}_1, \widetilde{u}_2, \widetilde{u}_3, -\widetilde{\varphi}, -\widetilde{\psi}, 0)^\top$, with the components given by (2.336)–(2.338), belongs to the space $[W_2^1(\Omega^+ \times J_T)]^6$ and satisfies the conditions (2.335). Therefore, in the functional equation (2.334) instead of $\widetilde{U} = (\widetilde{u}_1, \widetilde{u}_2, \widetilde{u}_3, \widetilde{\varphi}, \widetilde{\psi}, 0)^\top \in [W_2^1(\Omega^+ \times J_T)]^6$ we can substitute the vector function $\widetilde{U}^*$. As a result we obtain

$$\mathcal{I}_1(a) + \mathcal{I}_2(a) - \mathcal{I}_3(a) - \mathcal{I}_4(a) + \mathcal{I}_5(a) = 0 \quad (2.349)$$

with $\mathcal{I}_k(a)$ defined in (2.344)–(2.348). Now, from (2.343) and (2.349) it follows that

$$\mathcal{I}_1(a) + \mathcal{I}_2(a) + \mathcal{I}_5(a) = 0. \quad (2.350)$$

Due to the inequalities (2.10), (2.15), and since $\varrho > 0$, from (2.350) we get

$$u_j(x,a) = 0, \;\; \widetilde{\varepsilon}_{rj}(x,0) = 0, \;\; \partial_j\widetilde{\varphi}(x,0) = \partial_j\widetilde{\psi}(x,0) = 0, \;\; x \in \Omega^+, \;\; r,j = 1,2,3, \quad (2.351)$$

Since $a$ is an arbitrary number from $(0,T]$, in view of (2.336)–(2.338) we find that

$$u_j(x,t) = 0, \;\; \partial_j\varphi(x,t) = \partial_j\psi(x,t) = 0, \;\; (x,t) \in \Omega^+ \times J_T, \;\; j = 1,2,3, \quad (2.352)$$

whence we finally conclude

$$u_j(x,t) = 0, \;\; \varphi(x,t) = C_1(t), \;\; \psi(x,t) = C_2(t), \;\; (x,t) \in \Omega^+ \times J_T, \;\; j = 1,2,3, \quad (2.353)$$

where $C_1$ and $C_2$ are arbitrary absolutely continuous functions of $t \in [0,T]$ due to the inclusion $C_1, C_2 \in H^1(\overline{J_T})$.

Therefore, if $S_D \neq \varnothing$, form (2.353) we conclude that

$$u_j(x,t) = 0, \;\; \varphi(x,t) = 0, \;\; \psi(x,t) = 0, \;\; (x,t) \in \Omega^+ \times J_T, \;\; j = 1,2,3, \quad (2.354)$$

due to the homogeneous Dirichlet conditions on $S_D \times J_T$.

If $S_D = \varnothing$, i.e., in the case of the Neumann type problem, $C_1$ and $C_2$ remain arbitrary absolutely continuous functions of $t \in [0,T]$. This completes the proof. □

From Theorem 2.23 and Remark 2.19 the following assertion follows which generalizes Theorem 2.22.

**Corollary 2.24.** *Let the matrix $M$ given by (2.13) be positive definite. Then*

(i) *The homogeneous Dirichlet type initial-boundary value problem $(\mathrm{D})_t^+$ possesses only the trivial solutions in the class of regular vector-functions;*

(ii) *The homogeneous mixed type initial-boundary value problem $(\mathrm{M})_t^+$ possesses only the trivial solutions in the class of semi-regular vector functions;*

(iii) *The general solution of the homogeneous Neumann type problem $(\mathrm{N})_t^+$ in the space of regular vector-functions reads as follows*

$$U(x,t) = (0,0,0,C_1(t),C_2(t),0)^\top, \;\; (x,t) \in \Omega^+ \times J_T, \quad (2.355)$$

*where $C_1(t)$ and $C_2(t)$ are arbitrary regular functions of the class $C^1[0,T] \cap C^2(0,T)$.*

2.7.3. *Uniqueness theorems for the pseudo-oscillation problems.* We start with the following uniqueness results for weak solutions of the pseudo-oscillation problems in the case of $p = 2$.

**Theorem 2.25.** *Let $S$ be Lipschitz surface and $\tau = \sigma + i\omega$ with $\sigma > \sigma_0 \geqslant 0$ and $\omega \in \mathbb{R}$.*

(i) *The basic boundary value problems $(\mathrm{D})_\tau^+$ and $(\mathrm{M})_\tau^+$ have at most one solution in the space $[W_2^1(\Omega^+)]^6$ provided the subsurfaces $S_3$ and $S_5$ in the setting of mixed problem are not empty.*

(ii) *Solutions to the Neumann type boundary value problem $(\mathrm{N})_\tau^+$ in the space $[W_2^1(\Omega^+)]^6$ are defined modulo a vector of type $U^{(\mathcal{N})} = (0,0,0,b_1,b_2,0)^\top$, where $b_1$ and $b_2$ are arbitrary constants.*

(iii) *Crack type boundary value problems $(\text{D-CR-D})_\tau^+$, $(\text{N-CR-D})_\tau^+$, $(\text{M-CR-D})_\tau^+$, $(\text{D-CR-N})_\tau^+$, $(\text{M-CR-N})_\tau^+$, $(\text{D-CR-NT})_\tau^+$, and $(\text{M-CR-NT})_\tau^+$ have at most one solution in the space $[W_2^1(\Omega_\Sigma^+)]^6$.*

(iv) *Solutions to the crack type boundary value problems* (N-CR-N)$_\tau^+$ *and* (N-CR-NT)$_\tau^+$ *in the space* $[W_2^1(\Omega_\Sigma^+)]^6$ *are defined modulo a vector of type* $U^{(\mathcal{N})} = (0,0,0,b_1,b_2,0)^\top$*, where* $b_1$ *and* $b_2$ *are arbitrary constants.*

*Proof.* Due to the linearity of the boundary value problems in question it suffices to consider the corresponding homogeneous problems.

First we demonstrate the proof for the problems stated in the items (i) and (ii) of the theorem. Let $U = (u,\varphi,\psi,\vartheta)^\top \in \big[W_2^1(\Omega^+)\big]^6$ be a solution to the homogeneous problem $(\mathrm{D})_\tau^+$ or $(\mathrm{N})_\tau^+$ or $(\mathrm{M})_\tau^+$. For arbitrary $U' = (u',\varphi',\psi',\vartheta')^\top \in [W_2^1(\Omega^+)]^6$ from Green's formula (2.202) then we have

$$\int_{\Omega^+} \mathcal{E}_\tau(U,\overline{U'})\,dx = \big\langle \big\{\mathcal{T}(\partial_x,n,\tau)U\big\}^+, \{U'\}^+\big\rangle_{\partial\Omega^+}, \tag{2.356}$$

where $\mathcal{E}_\tau(U,\overline{U'})$ is given by (2.201).

If in (2.356) we substitute the vectors

$$(u_1,u_2,u_3,0,0,0)^\top,\ \ (0,0,0,\varphi,0,0)^\top,\ \ (0,0,0,0,\psi,0)^\top,\ \ (0,0,0,0,0,(1+\nu_0\tau)[\overline{\tau}]^{-1}\vartheta)^\top$$

for the vector $U'$ successively and take into consideration the homogeneous boundary conditions, we get

$$\int_{\Omega^+} \Big[c_{rjkl}\partial_l u_k\overline{\partial_j u_r} + \varrho\tau^2 u_r\overline{u_r} + e_{lrj}\partial_l\varphi\overline{\partial_j u_r} + q_{lrj}\partial_l\psi\overline{\partial_j u_r} - (1+\nu_0\tau)\lambda_{kj}\vartheta\overline{\partial_j u_k}\Big]\,dx = 0, \tag{2.357}$$

$$\int_{\Omega^+} \Big[-e_{lrj}\partial_j u_r\overline{\partial_l\varphi} + \varkappa_{jl}\partial_l\varphi\overline{\partial_j\varphi} + a_{jl}\partial_j\psi\overline{\partial_l\varphi} - (1+\nu_0\tau)p_l\vartheta\overline{\partial_l\varphi}\Big]\,dx = 0, \tag{2.358}$$

$$\int_{\Omega^+} \Big[-q_{lrj}\partial_j u_r\overline{\partial_l\psi} + a_{jl}\partial_l\varphi\overline{\partial_j\psi} + \mu_{jl}\partial_l\psi\overline{\partial_j\psi} - (1+\nu_0\tau)m_l\vartheta\overline{\partial_l\psi}\Big]\,dx = 0, \tag{2.359}$$

$$\int_{\Omega^+} \Big\{(1+\nu_0\overline{\tau})\Big[\lambda_{kj}\overline{\vartheta}\partial_j u_k - p_l\overline{\vartheta}\partial_l\varphi - m_l\overline{\vartheta}\partial_l\psi + (h_0\tau+d_0)|\vartheta|^2\Big] + \frac{1+\nu_0\overline{\tau}}{\tau}\eta_{jl}\partial_l\vartheta\overline{\partial_j\vartheta}\Big\}\,dx = 0. \tag{2.360}$$

Add to equation (2.357) the complex conjugate of equations (2.358)–(2.360) and use the symmetry properties (2.9) to obtain

$$\begin{aligned}\int_{\Omega^+} \Big\{& c_{rjkl}\partial_l u_k\overline{\partial_j u_r} + \varrho\tau^2|u|^2 + \varkappa_{jl}\partial_l\varphi\overline{\partial_j\varphi} + a_{jl}(\partial_l\psi\overline{\partial_j\varphi} + \partial_j\varphi\overline{\partial_l\psi}) + \mu_{jl}\partial_l\psi\overline{\partial_j\psi} - 2\,\mathrm{Re}\,\big[p_l(1+\nu_0\tau)\vartheta\overline{\partial_l\varphi}\,\big]\\ &- 2\,\mathrm{Re}\,\big[m_l(1+\nu_0\tau)\vartheta\overline{\partial_l\psi}\,\big] + (1+\nu_0\tau)(h_0\overline{\tau}+d_0)|\vartheta|^2 + \frac{1+\nu_0\tau}{\overline{\tau}}\,\eta_{jl}\partial_l\vartheta\overline{\partial_j\vartheta}\Big\}\,dx = 0.\end{aligned} \tag{2.361}$$

Due to the relations (2.12) and the positive definiteness of the matrix $\Lambda^{(1)}$ defined in (2.14), we find that

$$\begin{gathered} c_{ijlk}\partial_i u_j\overline{\partial_l u_k} \ge 0, \quad \eta_{jl}\partial_l\vartheta\overline{\partial_j\vartheta} \ge 0,\\ \Big[\varkappa_{jl}\partial_l\varphi\overline{\partial_j\varphi} + a_{jl}(\partial_l\psi\overline{\partial_j\varphi} + \partial_j\varphi\overline{\partial_l\psi}) + \mu_{jl}\partial_l\psi\overline{\partial_j\psi}\Big] \ge \lambda_0\big(|\nabla\varphi|^2 + |\nabla\psi|^2\big), \end{gathered} \tag{2.362}$$

where $\lambda_0$ is a positive constant. Using the equalities

$$\begin{gathered} \tau^2 = \sigma^2 - \omega^2 + 2i\sigma\omega, \quad \frac{1+\nu_0\tau}{\overline{\tau}} = \frac{\sigma + \nu_0(\sigma^2-\omega^2)}{|\tau|^2} + i\frac{\omega(1+2\sigma\nu_0)}{|\tau|^2}\,,\\ (1+\nu_0\tau)(h_0\overline{\tau}+d_0) = d_0 + \nu_0 h_0|\tau|^2 + (h_0+\nu_0 d_0)\sigma + i\omega(\nu_0 d_0 - h_0), \end{gathered} \tag{2.363}$$

and separating the imaginary part of (2.361), we deduce

$$\omega\int_{\Omega^+} \Big\{2\varrho\sigma|u|^2 + (\nu_0 d_0 - h_0)|\vartheta|^2 + \frac{1+2\sigma\nu_0}{|\tau|^2}\,\eta_{jl}\partial_l\vartheta\overline{\partial_j\vartheta}\Big\}\,dx = 0. \tag{2.364}$$

By the inequalities (2.11) and since $\sigma > \sigma_0 \geqslant 0$, we conclude $u = 0$ and $\vartheta = 0$ in $\Omega^+$ for $\omega \ne 0$. From (2.361) we then have

$$\int_{\Omega^+} \Big[\varkappa_{jl}\partial_l\varphi\overline{\partial_j\varphi} + a_{jl}(\partial_l\psi\overline{\partial_j\varphi} + \partial_j\varphi\overline{\partial_l\psi}) + \mu_{jl}\partial_l\psi\overline{\partial_j\psi}\Big]\,dx = 0. \tag{2.365}$$

Whence, in view of the last inequality in (2.362), we get $\partial_l\varphi = 0$, $\partial_l\psi = 0$, $l = 1, 2, 3$, in $\Omega^+$. Thus, if $\omega \neq 0$,

$$u = 0, \;\; \varphi = b_1 = const, \;\; \psi = b_2 = const, \;\; \vartheta = 0 \;\text{ in }\; \Omega^+. \tag{2.366}$$

If $\omega = 0$, then $\tau = \sigma > 0$ and (2.361) can be rewritten in the form

$$\int_{\Omega^+}\Big\{c_{rjkl}\partial_l u_k\overline{\partial_j u_r} + \varrho\sigma^2|u|^2 + \frac{1+\nu_0\sigma}{\sigma}\,\eta_{jl}\partial_l\vartheta\overline{\partial_j\vartheta}\Big\}\,dx \\
+ \int_{\Omega^+}\Big\{\varkappa_{jl}\partial_l\varphi\overline{\partial_j\varphi} + a_{jl}(\partial_l\psi\overline{\partial_j\varphi} + \partial_j\varphi\overline{\partial_l\psi}) + \mu_{jl}\partial_l\psi\overline{\partial_j\psi} - 2p_l(1+\nu_0\sigma)\,\mathrm{Re}\,\big[\vartheta\overline{\partial_l\varphi}\,\big] \\
- 2m_l(1+\nu_0\sigma)\,\mathrm{Re}\,\big[\vartheta\overline{\partial_l\psi}\,\big] + (1+\nu_0\sigma)(h_0\sigma + d_0)|\vartheta|^2\Big\}\,dx = 0. \tag{2.367}$$

The integrand in the first integral is nonnegative. Let us show that the integrand in the second integral is also nonnegative. To this end, as in the proof of Theorem 2.22, we set

$$\zeta_j := \partial_j\varphi, \;\; \zeta_{j+3} := \partial_j\psi, \;\; \zeta_7 := -\vartheta, \;\; \zeta_8 := -\sigma\vartheta, \;\; j = 1, 2, 3, \tag{2.368}$$

and introduce the vector

$$\Theta := (\zeta_1, \zeta_2, \dots, \zeta_8)^\top. \tag{2.369}$$

It can easily be checked that (summation over repeated indices is meant from 1 to 3)

$$\varkappa_{jl}\partial_l\varphi\overline{\partial_j\varphi} + a_{jl}(\partial_l\psi\overline{\partial_j\varphi} + \partial_j\varphi\overline{\partial_l\psi}) + \mu_{jl}\partial_l\psi\overline{\partial_j\psi} - 2p_l(1+\nu_0\sigma)\,\mathrm{Re}\,\big[\vartheta\overline{\partial_l\varphi}\,\big] \\
-2m_l(1+\nu_0\sigma)\,\mathrm{Re}\,\big[\vartheta\overline{\partial_l\psi}\,\big] + (1+\nu_0\sigma)(h_0\sigma + d_0)|\vartheta|^2 \\
= \big[\varkappa_{jl}\partial_l\varphi + a_{jl}\partial_l\psi + p_j(-\vartheta) + \nu_0 p_j(-\sigma\vartheta)\big]\overline{\partial_j\varphi} + \big[a_{jl}\partial_l\varphi + \mu_{jl}\partial_l\psi + m_j(-\vartheta) + \nu_0 m_j(-\sigma\vartheta)\big]\overline{\partial_j\psi} \\
+\big[p_l\partial_l\varphi + m_l\partial_l\psi + d_0(-\vartheta) + h_0(-\sigma\vartheta)\big](-\overline{\vartheta}) \\
+\big[\nu_0 p_l\partial_l\varphi + \nu_0 m_l\partial_l\psi + h_0(-\vartheta) + \nu_0 h_0(-\sigma\vartheta)\big](-\sigma\overline{\vartheta}) + \sigma(d_0\nu_0 - h_0)|\vartheta|^2 \\
= \big[\varkappa_{jl}\zeta_l + a_{jl}\zeta_{l+3} + p_j\zeta_7 + \nu_0 p_j\zeta_8\big]\overline{\zeta}_j + \big[a_{jl}\zeta_l + \mu_{jl}\zeta_{l+3} + m_j\zeta_7 + \nu_0 m_j\zeta_8\big]\overline{\zeta}_{j+3} \\
+\big[p_l\zeta_l + m_l\zeta_{l+3} + d_0\zeta_7 + h_0\zeta_8\big]\overline{\zeta}_7 + \big[\nu_0 p_l\zeta_l + \nu_0 m_l\zeta_{l+3} + h_0\zeta_7 + \nu_0 h_0\zeta_8\big]\overline{\zeta}_8 + \sigma(d_0\nu_0 - h_0)|\vartheta|^2 \\
= \sum_{p,q=1}^{8} M_{pq}\zeta_q\overline{\zeta}_p + \sigma(d_0\nu_0 - h_0)|\vartheta|^2 = M\Theta\cdot\Theta + \sigma(d_0\nu_0 - h_0)|\vartheta|^2 \geq C_0|\Theta|^2 \tag{2.370}$$

with some positive constant $C_0$, due to the positive definiteness of the matrix $M$ defined by (2.13) and the inequality $\sigma(d_0\nu_0 - h_0) > 0$.

Therefore, from (2.367) we see that the relations (2.366) hold for $\omega = 0$ as well.

Thus the equalities (2.366) hold for arbitrary $\tau = \sigma + i\omega$ with $\sigma > \sigma_0 \geqslant 0$ and $\omega \in \mathbb{R}$, whence the items (i) and (ii) of the theorem follow immediately, since the homogeneous Dirichlet conditions for $\varphi$ and $\psi$ imply $b_1 = b_2 = 0$, while a vector $U^{(\mathcal{N})} = (0, 0, 0, b_1, b_2, 0)^\top$, where $b_1$ and $b_2$ are arbitrary complex constants, solves the homogeneous Neumann BVP $(\mathrm{N})^+_\tau$.

To prove the remaining items of the theorem we have to add together two Green's formulas of type (2.356) for the domains $\Omega\setminus\overline{\Omega}_0$ and $\Omega_0$, where the auxiliary domain $\Omega_0 \subset \Omega^+$ is introduced in the beginning of Subsection 2.2. We recall that the crack surface $\Sigma$ is a proper part of the boundary $S_0 = \partial\Omega_0 \subset \Omega^+$ and any solution to the homogeneous differential equation $A(\partial, \tau)U = 0$ of the class $[W^1_2(\Omega^+_\Sigma)]^6$ and its derivatives are continuous across the surface $S_0\setminus\overline{\Sigma}$. If $U$ is a solution to one of the homogeneous crack type BVPs listed in items (iii) and (iv), by the same arguments as above, we arrive at the relation (cf. (2.212))

$$\int_{\Omega^+_\Sigma}\Big\{c_{rjkl}\partial_l u_k\overline{\partial_j u_r} + \varrho\tau^2|u|^2 + \varkappa_{jl}\partial_l\varphi\overline{\partial_j\varphi} + a_{jl}(\partial_l\psi\overline{\partial_j\varphi} + \partial_j\varphi\overline{\partial_l\psi}) + \mu_{jl}\partial_l\psi\overline{\partial_j\psi} - 2\,\mathrm{Re}\,\big[p_l(1+\nu_0\tau)\vartheta\overline{\partial_l\varphi}\,\big] \\
- 2\,\mathrm{Re}\,\big[m_l(1+\nu_0\tau)\vartheta\overline{\partial_l\psi}\,\big] + (1+\nu_0\tau)(h_0\overline{\tau} + d_0)|\vartheta|^2 + \frac{1+\nu_0\tau}{\overline{\tau}}\,\eta_{jl}\partial_l\vartheta\overline{\partial_j\vartheta}\Big\}\,dx = 0. \tag{2.371}$$

The surface integrals vanish due to the homogeneous boundary and crack type conditions and the above mentioned continuity of solutions and its derivatives across the auxiliary surface $S_0\setminus\overline{\Sigma}$. Therefore, the proof of items (iii) and (iv) can be verbatim performed. □

For the exterior BVPs of pseudo-oscillations we have the following uniqueness results.

**Theorem 2.26.** *Let $S$ be Lipschitz surface and $\tau = \sigma + i\omega$ with $\sigma > \sigma_0 \geqslant 0$ and $\omega \in \mathbb{R}$. The exterior basic boundary value problems* $(\mathrm{D})^-_\tau$, $(\mathrm{N})^-_\tau$ *and* $(\mathrm{M})^-_\tau$, *and the crack type boundary value problems* $(\text{D-CR-D})^-_\tau$, $(\text{N-CR-D})^-_\tau$, $(\text{M-CR-D})^-_\tau$, $(\text{D-CR-N})^-_\tau$, $(\text{M-CR-N})^-_\tau$, $(\text{D-CR-NT})^-_\tau$, $(\text{M-CR-NT})^-_\tau$, $(\text{N-CR-N})^-_\tau$ *and* $(\text{N-CR-NT})^-_\tau$ *have at most one solution in the space* $[W^1_{2,loc}(\Omega^-)]^6$ *and* $[W^1_{2,loc}(\Omega^-_\Sigma)]^6$, *respectively, satisfying the decay conditions* (2.207) *at infinity.*

*Proof.* With the help of Green's formula (2.211) and the decay conditions (2.207) by the word for word arguments applied in the proof of Theorem 2.25 we can show that the homogeneous basic and crack type exterior BVPs possess only the trivial solution. □

*Remark* 2.27. From Theorem 2.25 it follows that in the case of the homogenous problems $(\mathrm{N})^+_\tau$, $(\text{N-CR-N})^+_\tau$, and $(\text{N-CR-NT})^+_\tau$ the functions $\varphi$ and $\psi$ are arbitrary constants $b_1$ and $b_2$ which do not depend on the variable $x$ but may depend on the parameter $\tau$:

$$\varphi(x,\tau) = b_1(\tau), \quad \psi(x,\tau) = b_2(\tau). \tag{2.372}$$

If we assume that these boundary value problems are associated with the corresponding dynamical problems via the Laplace transform, then $\varphi(x,\tau) = b_1(\tau)$ and $\psi(x,\tau) = b_2(\tau)$, must be Laplace transforms of exponentially bounded regular generalized functions. Therefore they have to satisfy some natural restrictions. In particular, $b_1(\tau)$ and $b_2(\tau)$ must be analytic in the half-plane $\operatorname{Re}\tau > \sigma_0$ with respect to $\tau$ and the inverse Laplace transforms

$$v_j(t) := \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} b_j(\tau)e^{\tau t}\,d\tau, \ \ j=1,2, \ \ \sigma > \sigma_0, \tag{2.373}$$

must define exponentially bounded regular generalized functions. Some sufficient conditions for $b_j$ having the above properties can be found in [116].

2.7.4. *Uniqueness theorems for interior static problems.* The setting of the BVPs of statics coincides with the above formulated pseudo-oscillation BVPs with $\tau = 0$ (see Subsection 2.3). We denote these problems by the same symbols as in the pseudo-oscillation case but without subscript $\tau$: $(\mathrm{D})^\pm$, $(\mathrm{N})^\pm$, $(\mathrm{M})^\pm$, $(\text{B-CR-D})^+$, $(\text{B-CR-N})^+$, and $(\text{B-CR-NT})^+$ with $\mathrm{B} \in \{\mathrm{D},\mathrm{N},\mathrm{M}\}$. Recall that the differential operator of statics $A(\partial)$ coincides with the operator $A(\partial,0)$ (see (2.35) and (2.45)). Moreover, note that in the static case the differential equation for the temperature function and the corresponding boundary conditions are then decoupled and we obtain separated BVPs for $\vartheta$, since

$$[A(\partial_x)U]_6 = \eta_{jl}\partial_j\partial_l\vartheta \ \text{ and } \ \big\{\mathcal{T}(\partial_x,n)U\big\}_6 = \big\{\eta_{jl}n_j\partial_l\vartheta\big\},$$

where (see (2.57))

$$\begin{aligned}\mathcal{T}(\partial_x,n) &:= \mathcal{T}(\partial_x,n,0) = \big[\mathcal{T}_{pq}(\partial_x,n,0)\big]_{6\times 6}\\ &:= \begin{bmatrix} [c_{rjkl}n_j\partial_l]_{3\times3} & [e_{lrj}n_j\partial_l]_{3\times1} & [q_{lrj}n_j\partial_l]_{3\times1} & [-\lambda_{rj}n_j]_{3\times1}\\ [-e_{jkl}n_j\partial_l]_{1\times3} & \varkappa_{jl}n_j\partial_l & a_{jl}n_j\partial_l & -p_jn_j\\ [-q_{jkl}n_j\partial_l]_{1\times3} & a_{jl}n_j\partial_l & \mu_{jl}n_j\partial_l & -m_jn_j\\ [0]_{1\times3} & 0 & 0 & \eta_{jl}n_j\partial_l \end{bmatrix}_{6\times6}.\end{aligned} \tag{2.374}$$

Note also that in the static problems, without loss of generality, we can assume that all unknowns and given data are real functions, since the coefficients of the differential operators in $\Omega^\pm$ and the boundary operators on $\partial\Omega^\pm$ are real valued quantities. For the interior static BVPs we have the following uniqueness results.

**Theorem 2.28.** *Let $S$ be a Lipschitz surface.*

(i) *The homogeneous boundary value problems of statics* $(\mathrm{D})^+$ *and* $(\mathrm{M})^+$ *have only the trivial solution in the space* $[W^1_2(\Omega^+)]^6$.

(ii) *The homogeneous crack type boundary value problems of statics* $(\text{D-CR-D})^+$, $(\text{N-CR-D})^+$, $(\text{M-CR-D})^+$, $(\text{D-CR-N})^+$, $(\text{M-CR-N})^+$, $(\text{D-CR-NT})^+$, *and* $(\text{M-CR-NT})^+$ *have anly the trivial solution in the space* $[W^1_2(\Omega^+_\Sigma)]^6$.

*Proof.* Let $U = (u, \varphi, \psi, \vartheta)^\top$ be a solution to the homogeneous mixed boundary value problem $(\mathrm{M})^+$. Then $\vartheta$ solves the following decoupled mixed BVP

$$\eta_{jl}\partial_j\partial_l\vartheta = 0 \ \text{ in } \ \Omega^+, \tag{2.375}$$

$$\{\vartheta\}^+ = 0 \ \text{ on } \ S_7, \tag{2.376}$$

$$\{\eta_{jl}n_j\partial_l\vartheta\}^+ = 0 \ \text{ on } \ S_8. \tag{2.377}$$

By Green's formula

$$\int_{\Omega^+} \eta_{jl}\partial_l\vartheta\partial_j\vartheta\,dx = \big\langle\{\eta_{jl}n_j\partial_l\vartheta\}^+, \{\vartheta\}^+\big\rangle_S \tag{2.378}$$

and with the help of the homogeneous boundary conditions we derive $\vartheta = \mathit{const}$ in $\Omega^+$, since the right hand side duality expression in (2.378) vanishes and the matrix $[\eta_{jl}]_{3\times 3}$ is positive definite. Consequently, $\vartheta = 0$ in $\Omega^+$ due to the homogeneous Dirichlet condition (2.376). Therefore, the five dimensional vector $V = (u, \varphi, \psi)^\top$, constructed by the first five components of the solution vector $U$, solves the following homogeneous mixed BVP (see the formulation of BVP $(\mathrm{M})^+$, formulas (2.30) and (2.25) and take into account that $\vartheta = 0$)

$$\begin{aligned}
\widetilde{A}^{(0)}(\partial_x)V(x) &= 0, \ \ x \in \Omega^+,\\
\{u_r(x)\}^+ &= 0, \ \ \ x \in S_1, \ \ r = 1,2,3,\\
\big\{[T(\partial_x, n)V(x)]_r\big\}^+ &= 0, \ \ x \in S_2, \ \ r = 1,2,3,\\
\{\varphi(x)\}^+ &= 0, \ \ x \in S_3,\\
\big\{[T(\partial_x, n)V(x)]_4\big\}^+ &= 0, \ \ x \in S_4,\\
\{\psi(x)\}^+ &= 0, \ \ x \in S_5,\\
\big\{[T(\partial_x, n)V(x)]_5\big\}^+ &= 0, \ \ x \in S_6,
\end{aligned} \tag{2.379}$$

where $\widetilde{A}^{(0)}(\partial_x)$ is the $5 \times 5$ differential operator of statics of the electro-magneto-elasticity theory without taking into account thermal effects

$$\widetilde{A}^{(0)}(\partial_x) = \big[\widetilde{A}^{(0)}_{pq}(\partial_x)\big]_{5\times 5} := \begin{bmatrix} [c_{rjkl}\partial_j\partial_l]_{3\times 3} & [e_{lrj}\partial_j\partial_l]_{3\times 1} & [q_{lrj}\partial_j\partial_l]_{3\times 1}\\ [-e_{jkl}\partial_j\partial_l]_{1\times 3} & \varkappa_{jl}\partial_j\partial_l & a_{jl}\partial_j\partial_l\\ [-q_{jkl}\partial_j\partial_l]_{1\times 3} & a_{jl}\partial_j\partial_l & \mu_{jl}\partial_j\partial_l \end{bmatrix}_{5\times 5}, \tag{2.380}$$

and $T(\partial, n)$ is the corresponding $5 \times 5$ generalized stress operator (cf. (2.25), (2.26) and (2.58))

$$T(\partial_x, n) = \big[T_{pq}(\partial_x, n)\big]_{5\times 5} = \begin{bmatrix} [c_{rjkl}n_j\partial_l]_{3\times 3} & [e_{lrj}n_j\partial_l]_{3\times 1} & [q_{lrj}n_j\partial_l]_{3\times 1}\\ [-e_{jkl}n_j\partial_l]_{1\times 3} & \varkappa_{jl}n_j\partial_l & a_{jl}n_j\partial_l\\ [-q_{jkl}n_j\partial_l]_{1\times 3} & a_{jl}n_j\partial_l & \mu_{jl}n_j\partial_l \end{bmatrix}_{5\times 5} \tag{2.381}$$

In this case, Green's identity for arbitrary vectors $V = (u, \varphi, \psi)^\top, V' = (u', \varphi', \psi')^\top \in [W^1_2(\Omega^+)]^5$ with $\widetilde{A}^{(0)}(\partial_x)V \in L_2(\Omega^+)$ reads as

$$\int_{\Omega^+} \big[\widetilde{A}^{(0)}(\partial_x)V \cdot V' + \widetilde{\mathcal{E}}(V, V')\big]\,dx = \big\langle\{T(\partial_x, n)V\}^+, \{V'\}^+\big\rangle_{\partial\Omega^+}, \tag{2.382}$$

where

$$\begin{aligned}
\widetilde{\mathcal{E}}(V, V') = c_{rjkl}\partial_l u_k\partial_j u'_r &+ e_{lrj}(\partial_l\varphi\partial_j u'_r - \partial_j u_r\partial_l\varphi') + q_{lrj}(\partial_l\psi\partial_j u'_r - \partial_j u_r\partial_l\psi')\\
&+ \varkappa_{jl}\partial_l\varphi\partial_j\varphi' + a_{jl}(\partial_l\varphi\partial_j\psi' + \partial_j\psi\partial_l\varphi') + \mu_{jl}\partial_l\psi\partial_j\psi'.
\end{aligned} \tag{2.383}$$

Write the above Green's formula for a solution $V$ of the problem (2.379) and $V' = V$ to obtain

$$\int_{\Omega^+} \widetilde{\mathcal{E}}(V, V)\,dx = 0, \tag{2.384}$$

where

$$\widetilde{\mathcal{E}}(V, V) := c_{rjkl}\partial_l u_k\partial_j u_r + \varkappa_{jl}\partial_l\varphi\partial_j\varphi + 2a_{jl}\partial_l\varphi\partial_j\psi + \mu_{jl}\partial_l\psi\partial_j\psi. \tag{2.385}$$

Due to the inequalities (2.10) and positive definiteness of the matrix $\Lambda^{(1)}$ defined in (2.14) we conclude that $\partial_j\varphi = 0$ and $\partial_j\psi = 0$ in $\Omega^+$ for $j = 1, 2, 3$, and

$$c_{rjkl}\partial_l u_k \partial_j u_r = 0 \ \text{ implying } \ \partial_l u_k + \partial_k u_l = 0 \ \text{ in } \ \Omega^+, \ \ k, l = 1, 2, 3. \tag{2.386}$$

Therefore from (2.385) it follows that

$$u(x) = \chi(x) = a \times x + b, \ \ \varphi = b_4, \ \ \psi = b_5, \tag{2.387}$$

where $\chi(x)$ is a rigid displacement vector with $a = (a_1, a_2, a_3)^\top$ and $b = (b_1, b_2, b_3)^\top$ being arbitrary real constant vectors and $b_4$ and $b_5$ being arbitrary real constants. Now, the homogeneous Dirichlet condition in (2.379) implies $u = 0$, $\varphi = 0$, and $\psi = 0$ in $\Omega^+$, which proves the uniqueness theorem for the homogenous problem $(\mathrm{M})^+$.

It is evident that the proof for the problem $(\mathrm{D})^+$ is word-for-word.

The uniqueness results for the homogeneous crack type problems, $(\text{D-CR-D})^+$, $(\text{N-CR-D})^+$, $(\text{M-CR-D})^+$, $(\text{D-CR-N})^+$, $(\text{M-CR-N})^+$, $(\text{D-CR-NT})^+$, and $(\text{M-CR-NT})^+$, follow from the identities

$$\int_{\Omega^+_\Sigma} \eta_{jl}\partial_l\vartheta\partial_j\vartheta\,dx = 0, \qquad \int_{\Omega^+_\Sigma} \widetilde{\mathcal{E}}(V, V)\,dx = 0,$$

which can be obtained with the help of the same arguments applied in the proof of Theorem 2.25; here $U = (u, \varphi, \psi, \vartheta)^\top$ is a solution vector to one of the above listed homogeneous crack type static problems, $V = (u, \varphi, \psi)^\top$, and $\widetilde{\mathcal{E}}(V, V)$ is defined by (2.385). Therefore the proof can be verbatim performed. □

Further, we analyze the homogenous Neumann type boundary value problem $(\mathrm{N})^+$. Let a vector $U = (u, \varphi, \psi, \vartheta)^\top$ solve the homogenous problem $(\mathrm{N})^+$. In this case the temperature function $\vartheta$ solves the following decoupled problem

$$\eta_{jl}\partial_j\partial_l\vartheta = 0 \ \text{ in } \ \Omega^+, \tag{2.388}$$

$$\big\{\eta_{jl}n_j\partial_l\vartheta\big\}^+ = 0 \ \text{ on } \ S = \partial\Omega^+. \tag{2.389}$$

Whence, by (2.378), we get $\vartheta = b_6 = const$ in $\Omega^+$. Therefore, the vector $V = (u, \varphi, \psi)^\top$ solves then the nonhomogeneous BVP (see the formulation of BVP $(\mathrm{N})^+$, formulas (2.30) and (2.25), and take into account that $\vartheta = b_6 = const$ in $\Omega^+$)

$$\widetilde{A}^{(0)}(\partial_x)V = 0 \ \text{ in } \ \Omega^+, \tag{2.390}$$

$$\{T(\partial_x, n)V\}^+ = b_6 G^* \ \text{ on } \ S, \tag{2.391}$$

where $\widetilde{A}^{(0)}(\partial_x)$ and $T(\partial_x, n)$ are defined by (2.380) and (2.381), and $G^*$ is a special type given five dimensional vector function

$$G^* = (\lambda_{1j}n_j, \lambda_{2j}n_j, \lambda_{3j}n_j, p_j n_j, m_j n_j)^\top. \tag{2.392}$$

Due to Green's formula (2.382) we easily derive that a solution to the BVP (2.390), (2.391) is defined modulo the summand

$$\widetilde{V} = (\chi(x), b_4, b_5)^\top, \tag{2.393}$$

where $\chi(x)$ is an arbitrary rigid displacement vector function given by (2.285), and $b_4$ and $b_5$ are arbitrary real constants. This follows from the fact that the vector (2.393) is a general solution of the equation $\widetilde{\mathcal{E}}(\widetilde{V}, \widetilde{V}) = 0$ in $\Omega^+$ and $T(\partial_x, n)\widetilde{V} = 0$ everywhere for arbitrary unit vector $n$. Therefore, an arbitrary solution to the homogeneous Neumann type BVP (2.390), (2.391) can be represented as

$$V = \widetilde{V} + b_6 V^*, \tag{2.394}$$

where $\widetilde{V}$ is given by (2.393) and $V^* = (u^*, \varphi^*, \psi^*)^\top$ is a particular solution to the BVP

$$\widetilde{A}^{(0)}(\partial_x)V^* = 0 \ \text{ in } \ \Omega^+, \tag{2.395}$$

$$\{T(\partial_x, n)V^*\}^+ = G^* \ \text{ on } \ S, \tag{2.396}$$

with $G^*$ defined by (2.392).

Now, we show that the vector $V^*$ can be constructed explicitly in terms of linear functions for arbitrary domain $\Omega^+$. To this end, let us look for a solution to the problem (2.395), (2.396) in the form

$$V^* = (u^*, \varphi^*, \psi^*)^\top, \quad u_k^* = \widetilde{b}_{kq}^* x_q, \quad k = 1,2,3, \quad \varphi^* = \widetilde{c}_q^* x_q, \quad \psi^* = \widetilde{d}_q^* x_q, \tag{2.397}$$

where $\widetilde{b}_{kq}^* = \widetilde{b}_{qk}^*$, $\widetilde{c}_q^*$ and $\widetilde{d}_q^*$, $k, q = 1,2,3$, are 12 unknown real coefficients. Evidently, the vector $V^*$ solves the differential equation (2.395) and in view of (2.381) the boundary condition (2.396) leads to the simultaneous equations

$$\begin{aligned} &c_{rjkl} n_j \widetilde{b}_{kl}^* + e_{lrj} n_j \widetilde{c}_l^* + q_{lrj} n_j \widetilde{d}_l^* = \lambda_{rj} n_j, \quad r = 1,2,3, \\ &-e_{jkl} n_j \widetilde{b}_{kl}^* + \varkappa_{jl} n_j \widetilde{c}_l^* + a_{jl} n_j \widetilde{d}_l^* = p_j n_j, \\ &-q_{jkl} n_j \widetilde{b}_{kl}^* + a_{jl} n_j \widetilde{c}_l^* + \mu_{jl} n_j \widetilde{d}_l^* = m_j n_j. \end{aligned} \tag{2.398}$$

Note that, the normal components $n_j$ are linearly independent functions due to arbitrariness of the domain $\Omega^+$. Therefore we equate the expressions at the components $n_j$ and obtain 12 linear equations for the 12 unknown coefficients

$$\begin{aligned} &c_{rjkl} \widetilde{b}_{kl}^* + e_{lrj} \widetilde{c}_l^* + q_{lrj} \widetilde{d}_l^* = \lambda_{rj}, \quad r, j = 1,2,3, \\ &-e_{jkl} \widetilde{b}_{kl}^* + \varkappa_{jl} \widetilde{c}_l^* + a_{jl} \widetilde{d}_l^* = p_j, \quad j = 1,2,3, \\ &-q_{jkl} \widetilde{b}_{kl}^* + a_{jl} \widetilde{c}_l^* + \mu_{jl} \widetilde{d}_l^* = m_j, \quad j = 1,2,3. \end{aligned} \tag{2.399}$$

Due to the first inequality in (2.10) and positive definiteness of the matrix $\Lambda^{(1)}$ defined in (2.14), and since $\widetilde{b}_{kq}^* = \widetilde{b}_{qk}^*$, it follows that the homogeneous version of the system (2.399) possesses only the trivial solution, i.e., the determinant of the system is different from zero. Therefore, the nonhomogeneous system (2.399) is uniquely solvable and we can define the twelve unknown coefficients $\widetilde{b}_{kq}^* = \widetilde{b}_{qk}^*$, $\widetilde{c}_q^*$ and $\widetilde{d}_q^*$, $k, q = 1,2,3$. It is evident that then the boundary conditions (2.398) are satisfied and, consequently, the vector $V^*$ solves the BVP (2.395), (2.396) for arbitrary domain $\Omega^+$.

Thus, we have constructed the general solution of the homogeneous Neumann problem $(\mathrm{N})^+$ of statics explicitly $U = (V, b_6)^\top = (\widetilde{V}, 0)^\top + b_6(V^*, 1)^\top$, where $V$ is defined by (2.394), and $\widetilde{V}$ and $V^*$ are given by (2.393) and (2.397).

It is easy to check that the same vector is a general solution to the homogeneous crack type problems $(\mathrm{N\text{-}CR\text{-}N})^+$ and $(\mathrm{N\text{-}CR\text{-}NT})^+$ for arbitrary domain $\Omega_\Sigma^+$ with arbitrary crack surface $\Sigma$.

Thus, we have the following uniqueness theorem.

**Theorem 2.29.** *A general solution to the homogeneous Neumann type boundary value problem of statics* $(\mathrm{N})^+$ *and to the homogeneous crack type boundary value problems of statics* $(\mathrm{N\text{-}CR\text{-}N})^+$ *and* $(\mathrm{N\text{-}CR\text{-}NT})^+$ *in the space* $[W_2^1(\Omega^+)]^6$ *and* $[W_2^1(\Omega_\Sigma^+)]^6$, *respectively, reads as*

$$U = (\widetilde{V}, 0)^\top + b_6(V^*, 1)^\top,$$

*where* $\widetilde{V} = (a \times x + b, b_4, b_5)^\top$ *with* $a = (a_1, a_2, a_3)^\top$ *and* $b = (b_1, b_2, b_3)^\top$ *and* $V^*$ *is given by* (2.397) *with coefficients* $\widetilde{b}_{kq}^* = \widetilde{b}_{qk}^*$, $\widetilde{c}_q^*$, $\widetilde{d}_q^*$, $k, q = 1,2,3$, *defined by the uniquely solvable system* (2.399), *and where* $a_1$, $a_2$, $a_3$, *and* $b_1, \dots, b_6$ *are arbitrary real constants.*

Uniqueness theorems for exterior BVPs of statics will be considered later since it needs a quite different approach based on the properties of the corresponding fundamental matrix of the operator $A(\partial)$.

2.8. **Auxiliary boundary value problems for the adjoint operator** $A^*(\partial_x, \tau)$**.** In our further analysis we need also uniqueness theorems for some auxiliary BVPs for the operator $A^*(\partial_x, \tau)$ adjoint to $A(\partial_x, \tau)$. In particular, in the study of properties of boundary operators generated by the layer potentials we will use the uniqueness theorems for the following homogeneous Dirichlet and Neumann type BVPs.

**Dirichlet problem** $(\mathrm{D}_0^*)_\tau^\pm$**:** *Find a solution vector* $U = (u_1, \dots, u_6)^\top \in [W_2^1(\Omega^+)]^6$ *(respectively,* $U = (u_1, \dots, u_6)^\top \in [W_{2,loc}^1(\Omega^-)]^6$*) to the equation*

$$A^*(\partial_x, \tau) U = 0 \ \text{ in } \ \Omega^\pm \tag{2.400}$$

*satisfying the Dirichlet type boundary condition*

$$\{U\}^{\pm} = 0 \ \ on \ \ S. \tag{2.401}$$

**Neumann problem** $(\mathrm{N}_0^*)_\tau^\pm$**:** *Find a solution vector* $U = (u_1, \dots, u_6)^\top \in [W_2^1(\Omega^+)]^6$ (*respectively,* $U = (u_1, \dots, u_6)^\top \in [W_{2,loc}^1(\Omega^-)]^6$) *to equation* (2.211) *satisfying the Neumann type boundary condition*

$$\big\{\mathcal{P}(\partial_x, n, \tau)U\big\}^{\pm} = 0 \ \ on \ \ S, \tag{2.402}$$

*where the operator* $\mathcal{P}(\partial_x, n, \tau)$ *is defined by* (2.58). *In the case of the exterior BVPs we assume that solutions satisfy the decay conditions* (2.207) *at infinity.*

We have the following uniqueness results for these auxiliary problems.

**Theorem 2.30.** *Let* $\tau = \sigma + i\omega$ *with* $\sigma > \sigma_0 \geqslant 0$ *and* $\omega \in \mathbb{R}$.

(i) *The homogeneous boundary value problems* $(\mathrm{D}_0^*)_\tau^\pm$ *and* $(\mathrm{N}_0^*)_\tau^-$ *have only the trivial solution.*

(ii) *A general solution to the homogeneous Neumann type boundary value problem* $(\mathrm{N}_0^*)_\tau^+$ *reads as* $U = b_1 U^{(1)} + b_2 U^{(1)}$, *where* $b_1$ *and* $b_2$ *are arbitrary constants,* $U^{(1)} = (0,0,0,1,0,0)^\top$ *and* $U^{(2)} = (0,0,0,0,1,0)^\top$.

*Proof.* The proof is quite similar to the proofs of Theorems 2.25 and 2.26 and follows from Green's formulas (2.199) and (2.209). □

## 3. Fundamental Matrices

3.1. **Fundamental matrix of the operator** $A^{(0)}(\partial_x)$**.** We start with the construction of a fundamental matrix of the operator $A^{(0)}(\partial_x)$ given by (2.46) by the Fourier transform technique. Let $\mathcal{F}_{x\to\xi}$ and $\mathcal{F}^{-1}_{\xi\to x}$ denote the direct and inverse generalized Fourier transform in the space of tempered distributions (Schwartz space $\mathcal{S}'(\mathbb{R}^3)$) which for regular summable functions $f$ and $g$ read as follows

$$\mathcal{F}_{x\to\xi}[f] = \int_{\mathbb{R}^3} f(x) e^{ix\cdot\xi}\, dx, \quad \mathcal{F}^{-1}_{\xi\to x}[g] = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} g(\xi) e^{-ix\cdot\xi}\, d\xi, \tag{3.1}$$

where $x = (x_1, x_2, x_3)$ and $\xi = (\xi_1, \xi_2, \xi_3)$. Note that for an arbitrary multi–index $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ and $f \in \mathcal{S}'(\mathbb{R}^3)$

$$\mathcal{F}[\partial^\alpha f] = (-i\xi)^\alpha \mathcal{F}[f], \quad \mathcal{F}^{-1}[\xi^\alpha g] = (i\partial)^\alpha \mathcal{F}^{-1}[g], \tag{3.2}$$

where $\xi^\alpha = \xi_1^{\alpha_1}\xi_2^{\alpha_2}\xi_3^{\alpha_3}$ and $\partial^\alpha = \partial_1^{\alpha_1}\partial_2^{\alpha_2}\partial_3^{\alpha_3}$.

Denote by $\Gamma^{(0)}(x) = [\Gamma^{(0)}_{kj}(x)]_{6\times 6}$ the fundamental matrix of the operator $A^{(0)}(\partial)$,

$$A^{(0)}(\partial_x)\Gamma^{(0)}(x) = \delta(x) I_6. \tag{3.3}$$

Here $\delta(\,\cdot\,)$ is Dirac's delta distribution and $I_k$ stands for the unit $k\times k$ matrix. By standard arguments we can show that (cf., e.g., [14])

$$\begin{aligned}\Gamma^{(0)}(x) &= \mathcal{F}^{-1}_{\xi\to x}\big[\{A^{(0)}(-i\xi)\}^{-1}\big] = \frac{1}{8\pi^3} \lim_{R\to\infty} \int_{|\xi|<R} \{A^{(0)}(-i\xi)\}^{-1} e^{-ix\cdot\xi}\, d\xi \\ &= \frac{1}{8\pi^2|x|} \int_0^{2\pi} \big\{A^{(0)}(-iE(\widetilde{x})\eta)\big\}^{-1} d\phi, \ \ \eta = (\cos\phi, \sin\phi, 0)^\top, \ \ \widetilde{x} = \frac{x}{|x|},\end{aligned} \tag{3.4}$$

where $E(\widetilde{x})$ is an orthogonal matrix with properties $E^\top(\widetilde{x})x^\top = (0,0,|x|)^\top$ and $\det E(\widetilde{x}) = 1$,

$$\{A^{(0)}(-i\xi)\}^{-1} = \frac{1}{\det A^{(0)}(-i\xi)}\, A^{(0c)}(-i\xi)$$

is the inverse to the symbol matrix $A^{(0)}(-i\xi)$ given by (2.47), while the matrix $A^{(0c)}(-i\xi) = [A^{(0c)}_{kj}(-i\xi)]_{6\times 6}$ is the corresponding matrix of cofactors.

Note that, the entries of the matrix $\Gamma^{(0)}(x)$ are homogeneous even functions of order $-1$ and

$$\Gamma^{(0)}(x) = \begin{bmatrix} \widetilde{\Gamma}^{(0)}(x) & [0]_{5\times 1} \\ [0]_{1\times 5} & \Gamma^{(0)}_{66}(x) \end{bmatrix}_{6\times 6}, \quad \Gamma^{(0)}(-x) = \Gamma^{(0)}(x), \tag{3.5}$$

where $\widetilde{\Gamma}^{(0)}(x) = [\Gamma^{(0)}_{kj}(x)]_{5\times 5}$ is a fundamental matrix of the operator $\widetilde{A}^{(0)}(\partial)$ defined by (2.380) and $\Gamma^{(0)}_{66}(x)$ is a fundamental solution of the operator $A^{(0)}_{66}(\partial) = \eta_{jl}\partial_j\partial_l$ which reads as (see, e.g., [78, Ch. 1, § 8])

$$\Gamma^{(0)}_{66}(x) = -\frac{\alpha_0}{4\pi(Dx\cdot x)^{1/2}} = -\frac{\alpha_0}{4\pi[d_{kj}x_kx_j]^{1/2}}, \quad \alpha_0 = (\det D)^{1/2}, \tag{3.6}$$

where $D = [d_{kj}]_{3\times 3}$ is the inverse to the positive definite matrix $[\eta_{kj}]_{3\times 3}$.

Now, we derive an alternative representation of the fundamental matrix $\Gamma^{(0)}(x)$, which is very useful and convenient, in particular, for calculation of the principal homogeneous symbol matrices of the boundary integral operators generated by the layer potentials.

With the help of the Cauchy integral theorem for analytic functions, we can represent the matrix $\Gamma^{(0)}(x)$ in the form

$$\Gamma^{(0)}(x) = \mathcal{F}^{-1}_{\xi\to x}\big[\{A^{(0)}(-i\xi)\}^{-1}\big] = \mathcal{F}^{-1}_{\xi'\to x'}\big[\mathcal{F}^{-1}_{\xi_3\to x_3}\{A^{(0)}(-i\xi)\}^{-1}\big] = \mathcal{F}^{-1}_{\xi'\to x'}[\Psi(\xi',x_3)], \tag{3.7}$$

where $\xi' = (\xi_1,\xi_2)$, $x' = (x_1',x_2')$ and

$$\begin{aligned}\Psi(\xi',x_3) &= \frac{1}{2\pi}\int_{\mathbb{R}^1}\big\{A^{(0)}(-i\xi)\big\}^{-1}e^{-ix_3\xi_3}\,d\xi_3\\ &= \begin{cases}\dfrac{1}{2\pi}\displaystyle\int_{\ell^+}\big\{A^{(0)}(-i\xi)\big\}^{-1}e^{-ix_3\xi_3}\,d\xi_3 & \text{for } x_3\le 0,\\[2ex] \dfrac{1}{2\pi}\displaystyle\int_{\ell^-}\big\{A^{(0)}(-i\xi)\big\}^{-1}e^{-ix_3\xi_3}\,d\xi_3 & \text{for } x_3\ge 0.\end{cases}\end{aligned} \tag{3.8}$$

Here $\ell^+$ (respectively, $\ell^-$) is a closed simple curve of positive counterclockwise orientation (respectively, negative clockwise orientation) in the upper (respectively, lower) complex half-plane $\operatorname{Re}\xi_3 > 0$ (respectively, $\operatorname{Re}\xi_3 < 0$) enclosing all the roots with respect to $\xi_3$ of the equation $\det A^{(0)}(-i\xi) = 0$ with positive (respectively, negative) imaginary parts. Clearly, (3.8) does not depend on the shape of $\ell^+$ (respectively, $\ell^-$). It can easily be shown that the entries of the matrix (3.8) for $x_3 = 0$ are even, homogeneous real-valued functions in $\xi'$ of order $-1$. Moreover, from (3.8) and the inequality (2.48) it follows that there is a positive constant $c$ depending only on the material parameters, such that

$$\operatorname{Re}\big[-\Psi(\xi',0)\zeta\cdot\zeta\big] \ge c|\xi'|^{-1}|\zeta|^2 \ \text{ for all } \ \xi'\in\mathbb{R}^2\setminus\{0\} \ \text{ and for all } \ \zeta\in\mathbb{C}^6. \tag{3.9}$$

3.2. **Fundamental matrix of the operator $A(\partial_x,\tau)$.** Now, let us construct a fundamental matrix $\Gamma(x,\tau) = [\Gamma_{kj}(x,\tau)]_{6\times 6}$ of the operator $A(\partial_x,\tau)$ given by (2.30),

$$A(\partial_x,\tau)\Gamma(x,\tau) = \delta(x)I_6.$$

Applying the Fourier transform we get

$$A(-i\xi,\tau)\mathcal{F}_{x\to\xi}[\Gamma(x,\tau)] = I_6, \tag{3.10}$$

where

$$\begin{aligned}A(-i\xi,\tau) &= [A_{pq}(-i\xi,\tau)]_{6\times 6}\\ &= -\begin{bmatrix}[c_{rjkl}\xi_j\xi_l + \varrho\tau^2\delta_{rk}]_{3\times 3} & [e_{lrj}\xi_j\xi_l]_{3\times 1} & [q_{lrj}\xi_j\xi_l]_{3\times 1} & [-i(1+\nu_0\tau)\lambda_{rj}\xi_j]_{3\times 1}\\ [-e_{jkl}\xi_j\xi_l]_{1\times 3} & \varkappa_{jl}\xi_j\xi_l & a_{jl}\xi_j\xi_l & -i(1+\nu_0\tau)p_j\xi_j\\ [-q_{jkl}\xi_j\xi_l]_{1\times 3} & a_{jl}\xi_j\xi_l & \mu_{jl}\xi_j\xi_l & -i(1+\nu_0\tau)m_j\xi_j\\ [-i\tau\lambda_{kl}\xi_l]_{1\times 3} & i\tau p_l\xi_l & i\tau m_l\xi_l & \eta_{jl}\xi_j\xi_l + \tau^2h_0 + \tau d_0\end{bmatrix}_{6\times 6}.\end{aligned} \tag{3.11}$$

From (3.10) we find

$$\Gamma(x,\tau) = \mathcal{F}^{-1}_{\xi\to x}\big[A^{-1}(-i\xi,\tau)\big], \tag{3.12}$$

where $A^{-1}(-i\xi,\tau)$ is the matrix inverse to $A(-i\xi,\tau)$ defined in (3.11).

Now, we establish some properties of the matrix $A(-i\xi,\tau)$ needed in our further analysis.

**Lemma 3.1.** *Let $\tau = \sigma + i\omega$ with $\sigma > 0$ and $\omega\in\mathbb{R}$. Then*

$$\det A(-i\xi,\tau) \ne 0 \ \textit{ for all } \ \xi\in\mathbb{R}^3\setminus\{0\}. \tag{3.13}$$

*Proof.* It suffices to show that for all $\xi\in\mathbb{R}^3\setminus\{0\}$ the homogeneous system of linear algebraic equations $-A(-i\xi,\tau)\zeta=0$ for unknowns $\zeta=(\zeta_1,\dots,\zeta_6)^\top\in\mathbb{C}^6$, i.e. the simultaneous equations

$$\begin{gathered}c_{rjkl}\xi_j\xi_l\zeta_k+\varrho\tau^2\zeta_r+e_{lrj}\xi_j\xi_l\zeta_4+q_{lrj}\xi_j\xi_l\zeta_5-i(1+\nu_0\tau)\lambda_{rj}\xi_j\zeta_6=0,\quad r=1,2,3,\\ -e_{jkl}\xi_j\xi_l\zeta_k+\varkappa_{jl}\xi_j\xi_l\zeta_4+a_{jl}\xi_j\xi_l\zeta_5-i(1+\nu_0\tau)p_j\xi_j\zeta_6=0,\\ -q_{jkl}\xi_j\xi_l\zeta_k+a_{jl}\xi_j\xi_l\zeta_4+\mu_{jl}\xi_j\xi_l\zeta_5-i(1+\nu_0\tau)m_j\xi_j\zeta_6=0,\\ -i\tau\lambda_{kl}\xi_l\zeta_k+i\tau p_l\xi_l\zeta_4+i\tau m_l\xi_l\zeta_5+(\eta_{jl}\xi_j\xi_l+\tau^2h_0+\tau d_0)\zeta_6=0,\end{gathered}\tag{3.14}$$

possess only the trivial solution.

Multiply the first three equations by $\overline{\zeta}_r$, $r=1,2,3$, the complex conjugate of the fourth equation by $\zeta_4$, the complex conjugate of the fifth equation by $\zeta_5$, the complex conjugate of the sixth equation by $\frac{(1+\nu_0\tau)}{\overline{\tau}}\zeta_6$, and add together to obtain

$$\begin{aligned}c_{rjkl}(\xi_l\zeta_k)(\xi_j\overline{\zeta}_r)+\varrho\tau^2\zeta_r\overline{\zeta}_r+\varkappa_{jl}(\xi_j\zeta_4)(\xi_l\overline{\zeta}_4)+\mu_{jl}(\xi_j\zeta_5)(\xi_l\overline{\zeta}_5)&\\ +a_{jl}\big[(\xi_j\zeta_4)(\xi_l\overline{\zeta}_5)+(\xi_j\overline{\zeta}_4)(\xi_l\zeta_5)\big]+2p_l\xi_l\mathrm{Re}\big[(1+\nu_0\overline{\tau})(i\zeta_4)\overline{\zeta_6}\,\big]&\\ +2m_l\xi_l\mathrm{Re}\big[(1+\nu_0\overline{\tau})(i\zeta_5)\overline{\zeta_6}\,\big]+\frac{(1+\nu_0\tau)}{\overline{\tau}}\,\eta_{jl}\xi_j\xi_l|\zeta_6|^2+(\overline{\tau}h_0+d_0)(1+\nu_0\tau)|\zeta_6|^2=0,&\end{aligned}\tag{3.15}$$

In view of the inequalities (2.12) and (2.15), we have

$$\begin{gathered}c_{rjkl}(\xi_l\zeta_k)(\xi_j\overline{\zeta}_r)=\frac14\,c_{rjkl}(\xi_l\zeta_k+\xi_k\zeta_l)(\xi_j\overline{\zeta}_r+\xi_r\overline{\zeta}_j)\geq\frac{\delta_0}{4}\sum_{k,j=1}^3|\xi_l\zeta_k+\xi_k\zeta_l|^2,\\ \varkappa_{jl}(\xi_j\zeta_4)(\xi_l\overline{\zeta}_4)+\mu_{jl}(\xi_j\zeta_5)(\xi_l\overline{\zeta}_5)+a_{jl}\big[(\xi_j\zeta_4)(\xi_l\overline{\zeta}_5)+(\xi_j\overline{\zeta}_4)(\xi_l\zeta_5)\big]\geq\kappa_1\big(|\xi|^2|\zeta_4|^2+|\xi|^2|\zeta_5|^2\big),\\ \eta_{jl}\xi_j\xi_l\geq\delta_3|\xi|^2.\end{gathered}$$

Therefore, by separating the imaginary part of (3.15) we find

$$\omega\bigg\{2\varrho\sigma\big(|\zeta_1|^2+|\zeta_2|^2+|\zeta_3|^2\big)+\Big[\frac{1+2\nu_0\sigma}{|\tau|^2}\,\eta_{jl}\xi_j\xi_l+(\nu_0d_0-h_0)\Big]|\zeta_6|^2\bigg\}=0.$$

Since $\varrho>0$, $\sigma>0$, and $\nu_0d_0-h_0>0$ (see (2.11)) we get $\zeta_1=\zeta_2=\zeta_3=\zeta_6=0$ if $\omega\neq0$. The relation (3.15) then takes the form

$$\varkappa_{jl}(\xi_j\zeta_4)(\xi_l\overline{\zeta}_4)+\mu_{jl}(\xi_j\zeta_5)(\xi_l\overline{\zeta}_5)+a_{jl}\big[(\xi_j\zeta_4)(\xi_l\overline{\zeta}_5)+(\xi_j\overline{\zeta}_4)(\xi_l\zeta_5)\big]=0$$

and, by positive definiteness of the matrix $\Lambda^{(1)}$ defined in (2.14), we conclude $\zeta_4=\zeta_5=0$ since $\xi\in\mathbb{R}^3\setminus\{0\}$. Consequently, system (3.14) possesses only the trivial solution for $\omega\neq0$.

Now, if $\omega=0$, then $\tau=\sigma>0$ and (3.15) can be rewritten as

$$\begin{aligned}c_{rjkl}(\xi_l\zeta_k)(\xi_j\overline{\zeta}_r)+\varrho\sigma^2\zeta_r\overline{\zeta}_r+\frac{(1+\nu_0\sigma)}{\sigma}\,\eta_{jl}\xi_j\xi_l|\zeta_6|^2+\varkappa_{jl}(\xi_j\zeta_4)(\xi_l\overline{\zeta}_4)&\\ +\mu_{jl}(\xi_j\zeta_5)(\xi_l\overline{\zeta}_5)+a_{jl}\big[(\xi_j\zeta_4)(\xi_l\overline{\zeta}_5)+(\xi_j\overline{\zeta}_4)(\xi_l\zeta_5)\big]&\\ +2p_l\xi_l\mathrm{Re}\big[(1+\nu_0\sigma)(i\zeta_4)\overline{\zeta_6}\,\big]+2m_l\xi_l\mathrm{Re}\big[(1+\nu_0\sigma)(i\zeta_5)\overline{\zeta_6}\,\big]+(\sigma h_0+d_0)(1+\nu_0\sigma)|\zeta_6|^2=0,&\end{aligned}\tag{3.16}$$

Evidently, the sum of the first three summands is nonnegative. Let us show that the sum of the remaining terms is also nonnegative. To this end, let us set

$$\eta_j:=\xi_j\zeta_4,\ \ \eta_{j+3}:=\xi_j\zeta_5,\ \ \eta_7:=-\zeta_6,\ \ \eta_8:=-\sigma\zeta_6,\ \ j=1,2,3,\tag{3.17}$$

and introduce the vector

$$Q:=(\eta_1,\eta_2,\dots,\eta_8)^\top.\tag{3.18}$$

It can be easily checked that (summation over repeated indices is meant from 1 to 3)

$$\begin{aligned}&\varkappa_{jl}(\xi_j\zeta_4)(\xi_l\overline{\zeta}_4)+\mu_{jl}(\xi_j\zeta_5)(\xi_l\overline{\zeta}_5)+a_{jl}\big[(\xi_j\zeta_4)(\xi_l\overline{\zeta}_5)+(\xi_j\overline{\zeta}_4)(\xi_l\zeta_5)\big]+2p_l\xi_l\mathrm{Re}\big[(1+\nu_0\sigma)(i\zeta_4)\overline{\zeta}_6\big]\\ &\qquad+2m_l\xi_l\mathrm{Re}\big[(1+\nu_0\sigma)(i\zeta_5)\overline{\zeta}_6\big]+(\sigma h_0+d_0)(1+\nu_0\sigma)|\zeta_6|^2\\ &=\big[\varkappa_{jl}\eta_l+a_{jl}\eta_{l+3}+p_j\eta_7+\nu_0p_j\eta_8\big]\overline{\eta}_j+\big[a_{jl}\eta_l+\mu_{jl}\eta_{l+3}+m_j\eta_7+\nu_0m_j\eta_8\big]\overline{\eta}_{j+3}\\ &\quad+\big[p_l\eta_l+m_l\eta_{l+3}+d_0\eta_7+h_0\eta_8\big]\overline{\eta}_7+\big[\nu_0p_l\eta_l+\nu_0m_l\eta_{l+3}+h_0\eta_7+\nu_0h_0\eta_8\big]\overline{\eta}_8+\sigma(d_0\nu_0-h_0)|\zeta_6|^2\end{aligned}$$

$$= \sum_{p,q=1}^{8} M_{pq}\eta_q\overline{\eta}_p + \sigma(d_0\nu_0 - h_0)|\zeta_6|^2 = MQ\cdot Q + \sigma(d_0\nu_0 - h_0)|\zeta_6|^2 \geq C_0|Q|^2$$

with some positive constant $C_0$ due to the positive definiteness of the matrix $M$ defined in (2.13) and the inequality $\nu_0 d_0 - h_0 > 0$ (see (2.11)). Therefore from (3.16) it follows that $\zeta_j = 0$, $j = 1,\dots,6$, which completes the proof. □

Further we analyze properties of the inverse matrix $A^{-1}(-i\xi,\tau)$. First of all let us note that the determinant $\det A(-i\xi,\tau)$ is representable as (see (3.11))

$$\det A(-i\xi,\tau) = P_{12}(\xi) + P_{10}(\xi,\tau) + P_8(\xi,\tau) + P_6(\xi,\tau) + P_4(\xi,\tau), \tag{3.19}$$

where $P_k$ are homogeneous polynomials in $\xi$ of order $k$. In particular,

$$P_{12}(\xi) = \det A^{(0)}(-i\xi) = \det[-A^{(0)}(-i\xi)] = \det \mathbf{A}^{(0)}(\xi), \tag{3.20}$$

where $A^{(0)}(-i\xi)$ is given by (2.47) and

$$\mathbf{A}^{(0)}(\xi) := -A^{(0)}(-i\xi) = \begin{bmatrix} [c_{rjkl}\xi_j\xi_l]_{3\times3} & [e_{lrj}\xi_j\xi_l]_{3\times1} & [q_{lrj}\xi_j\xi_l]_{3\times1} & [0]_{3\times1} \\ [-e_{jkl}\xi_j\xi_l]_{1\times3} & \varkappa_{jl}\xi_j\xi_l & a_{jl}\xi_j\xi_l & 0 \\ [-q_{jkl}\xi_j\xi_l]_{1\times3} & a_{jl}\xi_j\xi_l & \mu_{jl}\xi_j\xi_l & 0 \\ [0]_{1\times3} & 0 & 0 & \eta_{jl}\xi_j\xi_l \end{bmatrix}_{6\times6}. \tag{3.21}$$

In view of (2.48) it can be shown that

$$P_{12}(\xi) = \det \mathbf{A}^{(0)}(\xi) > 0 \ \text{ for all } \ \xi \in \mathbb{R}^3 \setminus \{0\}. \tag{3.22}$$

Indeed, denote by $\lambda_j$, $j = 1,\dots,6$, the eigenvalues of the matrix (3.21). Since $\lambda_j$ are roots of the equation $\det[\mathbf{A}^{(0)}(\xi) - \lambda I_6] = 0$, we have

$$\det \mathbf{A}^{(0)}(\xi) = \lambda_1\lambda_2\cdots\lambda_6. \tag{3.23}$$

Without loss of generality we can assume that the eigenvectors associated with the real eigenvalues are real-valued vectors. Therefore it is easy to see that the real eigenvalues of the matrix (3.21) are positive due to (2.48). On the other side, since the entries of the matrix $\mathbf{A}^{(0)}(\xi)$ are real valued, it is evident that if $\lambda_j$ is a complex eigenvalue of the matrix then its complex conjugate $\overline{\lambda}_j$ is also an eigenvalue of the same matrix. Thus the product of the eigenvalues $\lambda_1\lambda_2\cdots\lambda_6$ is positive which proves the desired inequality (3.22).

Consequently, for the homogeneous function $P_{12}(\xi)$ we have the following two-sided estimate

$$a_0^*|\xi|^{12} \leq P_{12}(\xi) \leq a_0^{**}|\xi|^{12} \ \text{ for all } \ \xi \in \mathbb{R}^3 \tag{3.24}$$

with

$$a_0^* = \min_{|\xi|=1} \det \mathbf{A}^{(0)}(\xi) > 0, \quad a_0^{**} = \max_{|\xi|=1} \det \mathbf{A}^{(0)}(\xi) > 0. \tag{3.25}$$

Further, the fourth order homogeneous polynomial $P_4(\xi,\tau)$ in (3.19) reads as

$$P_4(\xi,\tau) = \varrho^3\tau^6 B(\xi,\tau), \tag{3.26}$$

where

$$B(\xi,\tau) = \det \begin{bmatrix} \varkappa_{jl}\xi_j\xi_l & a_{jl}\xi_j\xi_l & -i(1+\nu_0\tau)p_j\xi_j \\ a_{jl}\xi_j\xi_l & \mu_{jl}\xi_j\xi_l & -i(1+\nu_0\tau)m_j\xi_j \\ i\tau p_l\xi_l & i\tau m_l\xi_l & \tau^2 h_0 + \tau d_0 \end{bmatrix} = \tau \det M^{(1)}(\xi) + \frac{\tau^2}{\nu_0} \det M^{(2)}(\xi) \tag{3.27}$$

with

$$M^{(1)}(\xi) = \begin{bmatrix} \varkappa_{jl}\xi_j\xi_l & a_{jl}\xi_j\xi_l & p_j\xi_j \\ a_{jl}\xi_j\xi_l & \mu_{jl}\xi_j\xi_l & m_j\xi_j \\ p_j\xi_j & m_j\xi_j & d_0 \end{bmatrix}, \tag{3.28}$$

$$M^{(2)}(\xi) = \begin{bmatrix} \varkappa_{jl}\xi_j\xi_l & a_{jl}\xi_j\xi_l & \nu_0 p_j\xi_j \\ a_{jl}\xi_j\xi_l & \mu_{jl}\xi_j\xi_l & \nu_0 m_j\xi_j \\ \nu_0 p_j\xi_j & \nu_0 m_j\xi_j & \nu_0 h_0 \end{bmatrix}. \tag{3.29}$$

Due to positive definiteness of the matrix (2.13), the matrices $M^{(j)}(\xi)$, $j=1,2$, are positive definite as well and for all $\xi\in\mathbb{R}^3\setminus\{0\}$ and for all $\eta\in\mathbb{R}^3$ we have

$$M^{(j)}(\xi)\eta\cdot\eta\geq a_j^*\big[(\eta_1^2+\eta_2^2)|\xi|^2+\eta_3^2\big],\ \ a_j^*=const>0,\ \ j=1,2. \tag{3.30}$$

In particular, $\det M^{(j)}(\xi)>0$ for all $\xi\in\mathbb{R}^3\setminus\{0\}$. Denote

$$b_j^*:=\min_{|\xi|=1}\det M^{(j)}(\xi)>0,\ \ j=1,2. \tag{3.31}$$

Keeping in mind that $\nu_0>0$, $\sigma>0$, and $\det M^{(j)}(\xi)>0$ for all $\xi\in\mathbb{R}^3\setminus\{0\}$, from (3.26)–(3.31) we then derive

$$\begin{aligned}|P_4(\xi,\tau)|&=\varrho^3|\tau|^7\Big|\det M^{(1)}(\xi)+\frac{\sigma}{\nu_0}\det M^{(2)}(\xi)+\frac{i\omega}{\nu_0}\det M^{(2)}(\xi)\Big|\\&\geq\frac{\varrho^3|\tau|^7}{\sqrt{2}}\Big\{\det M^{(1)}(\xi)+\frac{\sigma}{\nu_0}\det M^{(2)}(\xi)+\frac{|\omega|}{\nu_0}\det M^{(2)}(\xi)\Big\}\\&\geq\frac{\varrho^3|\tau|^7}{\sqrt{2}\,\nu_0}\big(b_1^*\nu_0+|\tau|b_2^*\big)|\xi|^4\ \text{ for all }\ \xi\in\mathbb{R}^3.\end{aligned} \tag{3.32}$$

Evidently, we have also the following estimate

$$|P_4(\xi,\tau)|\leq\frac{2\varrho^3|\tau|^7}{\nu_0}\big(b_1^{**}\nu_0+|\tau|b_2^{**}\big)|\xi|^4\ \text{ for all }\ \xi\in\mathbb{R}^3 \tag{3.33}$$

with

$$b_j^{**}:=\max_{|\xi|=1}\det M^{(j)}(\xi)>0,\ \ j=1,2. \tag{3.34}$$

Now, the relation (3.19) and estimates (3.24), (3.32), and (3.33) lead to the following assertion.

**Lemma 3.2.** *Let $\tau=\sigma+i\omega$ with $\sigma>0$ and $\omega\in\mathbb{R}$. There hold the following asymptotic relations*

$$\det A(-i\xi,\tau)=|\xi|^{12}\big[a(\widetilde{\xi})+\mathcal{O}(|\xi|^{-2})\big]\ \ as\ \ |\xi|\to\infty, \tag{3.35}$$

$$\det A(-i\xi,\tau)=|\xi|^{4}\big[b(\widetilde{\xi})+\mathcal{O}(|\xi|^{2})\big]\ \ as\ \ |\xi|\to 0, \tag{3.36}$$

*where $\widetilde{\xi}=\xi/|\xi|$ and*

$$a_0^*\leq|a(\widetilde{\xi})|\leq a_0^{**},\quad \frac{\varrho^3|\tau|^7}{\sqrt{2}\nu_0}\big(b_1^*\nu_0+|\tau|b_2^*\big)\leq|b(\widetilde{\xi})|\leq\frac{2\varrho^3|\tau|^7}{\nu_0}\big(b_1^{**}\nu_0+|\tau|b_2^{**}\big), \tag{3.37}$$

*with $a_0^*$, $a_0^{**}$, $b_j^*$, and $b_j^{**}$ given by* (3.25), (3.31), *and* (3.34), *respectively.*

Further, we investigate behaviour of the inverse matrix $A^{-1}(-i\xi,\tau)$ at infinity and near the origin. By (3.11) and the representation formula

$$A^{-1}(-i\xi,\tau)=\frac{1}{\det A(-i\xi,\tau)}A^{(c)}(-i\xi,\tau),$$

where $A^{(c)}(-i\xi,\tau)=[A_{kj}^{(c)}(-i\xi,\tau)]_{6\times6}$ is the corresponding matrix of cofactors, we derive the following asymptotic relation for sufficiently large $|\xi|$ (as $|\xi|\to\infty$),

$$A^{(c)}(-i\xi,\tau)=\begin{bmatrix}[\mathcal{O}(|\xi|^{10})]_{5\times5} & [\mathcal{O}(|\xi|^{9})]_{5\times1}\\ [\mathcal{O}(|\xi|^{9})]_{1\times5} & \mathcal{O}(|\xi|^{10})\end{bmatrix}_{6\times6}, \tag{3.38}$$

implying

$$A^{-1}(-i\xi,\tau)=\begin{bmatrix}[\mathcal{O}(|\xi|^{-2})]_{5\times5} & [\mathcal{O}(|\xi|^{-3})]_{5\times1}\\ [\mathcal{O}(|\xi|^{-3})]_{1\times5} & \mathcal{O}(|\xi|^{-2})\end{bmatrix}_{6\times6}, \tag{3.39}$$

and for sufficiently small $|\xi|$ (as $|\xi| \to 0$),

$$A^{(c)}(-i\xi,\tau) = \begin{bmatrix} \mathcal{O}(|\xi|^4) & \mathcal{O}(|\xi|^6) & \mathcal{O}(|\xi|^6) & \mathcal{O}(|\xi|^4) & \mathcal{O}(|\xi|^4) & \mathcal{O}(|\xi|^5) \\ \mathcal{O}(|\xi|^6) & \mathcal{O}(|\xi|^4) & \mathcal{O}(|\xi|^6) & \mathcal{O}(|\xi|^4) & \mathcal{O}(|\xi|^4) & \mathcal{O}(|\xi|^5) \\ \mathcal{O}(|\xi|^6) & \mathcal{O}(|\xi|^6) & \mathcal{O}(|\xi|^4) & \mathcal{O}(|\xi|^4) & \mathcal{O}(|\xi|^4) & \mathcal{O}(|\xi|^5) \\ \mathcal{O}(|\xi|^4) & \mathcal{O}(|\xi|^4) & \mathcal{O}(|\xi|^4) & \mathcal{O}(|\xi|^2) & \mathcal{O}(|\xi|^2) & \mathcal{O}(|\xi|^3) \\ \mathcal{O}(|\xi|^4) & \mathcal{O}(|\xi|^4) & \mathcal{O}(|\xi|^4) & \mathcal{O}(|\xi|^2) & \mathcal{O}(|\xi|^2) & \mathcal{O}(|\xi|^3) \\ \mathcal{O}(|\xi|^5) & \mathcal{O}(|\xi|^5) & \mathcal{O}(|\xi|^5) & \mathcal{O}(|\xi|^3) & \mathcal{O}(|\xi|^3) & \mathcal{O}(|\xi|^4) \end{bmatrix}_{6\times 6}, \tag{3.40}$$

implying

$$A^{-1}(-i\xi,\tau) = \begin{bmatrix} \mathcal{O}(1) & \mathcal{O}(|\xi|^2) & \mathcal{O}(|\xi|^2) & \mathcal{O}(1) & \mathcal{O}(1) & \mathcal{O}(|\xi|) \\ \mathcal{O}(|\xi|^2) & \mathcal{O}(1) & \mathcal{O}(|\xi|^2) & \mathcal{O}(1) & \mathcal{O}(1) & \mathcal{O}(|\xi|) \\ \mathcal{O}(|\xi|^2) & \mathcal{O}(|\xi|^2) & \mathcal{O}(1) & \mathcal{O}(1) & \mathcal{O}(1) & \mathcal{O}(|\xi|) \\ \mathcal{O}(1) & \mathcal{O}(1) & \mathcal{O}(1) & \mathcal{O}(|\xi|^{-2}) & \mathcal{O}(|\xi|^{-2}) & \mathcal{O}(|\xi|^{-1}) \\ \mathcal{O}(1) & \mathcal{O}(1) & \mathcal{O}(1) & \mathcal{O}(|\xi|^{-2}) & \mathcal{O}(|\xi|^{-2}) & \mathcal{O}(|\xi|^{-1}) \\ \mathcal{O}(|\xi|) & \mathcal{O}(|\xi|) & \mathcal{O}(|\xi|) & \mathcal{O}(|\xi|^{-1}) & \mathcal{O}(|\xi|^{-1}) & \mathcal{O}(1) \end{bmatrix}_{6\times 6}. \tag{3.41}$$

Since the entries of the matrix $A^{-1}(-i\xi,\tau)$ are rational functions in $\xi$ it follows that the dominant terms in asymptotic expansions at infinity and at the origin are homogenous functions of order mentioned in the relations (3.39) and (3.41). Furthermore, in (3.39) the entries $A^{-1}_{6j}(-i\xi,\tau)$ and $A^{-1}_{j6}(-i\xi,\tau)$, $j = 1,\dots,5$, with the asymptotic $\mathcal{O}(|\xi|^{-3})$ have dominant terms of type $|\xi|^{-3}\chi(\xi)$, where $\chi(\xi)$ is an odd homogeneous function of order 0. Therefore

$$\int_{|\xi|=1} \chi(\xi)\, dS = 0,$$

and, consequently, the generalized inverse Fourier transform of the function $|\xi|^{-3}\chi(\xi)$, considered in the Cauchy Principal Value sense, is a homogeneous function of order 0 (see, e.g., [77, Ch. 10, § 1], [35, Ch. 1, § 2], [80, Ch. 2, § 6]).

Let $h$ be an infinitely differentiable function with compact support,

$$h \in C^\infty(\mathbb{R}^3), \quad h(\xi) = \begin{cases} 1 & \text{for } |\xi| < 1, \\ 0 & \text{for } |\xi| > 2. \end{cases}$$

Then we can represent the fundamental matrix $\Gamma(x,\tau)$ in the form

$$\Gamma(x,\tau) = \mathcal{F}^{-1}_{\xi\to x}[A^{-1}(-i\xi,\tau)] = \Gamma^{(1)}(x,\tau) + \Gamma^{(2)}(x,\tau), \tag{3.42}$$

where

$$\Gamma^{(1)}(x,\tau) := \mathcal{F}^{-1}_{\xi\to x}\big[h(\xi)A^{-1}(-i\xi,\tau)\big], \quad \Gamma^{(2)}(x,\tau) := \mathcal{F}^{-1}_{\xi\to x}\big[(1-h(\xi))A^{-1}(-i\xi,\tau)\big].$$

Applying properties (3.2) of the generalized Fourier transform we derive that the entries of the matrix $\Gamma^{(2)}(x,\tau)$ decay at infinity (as $|x| \to \infty$) faster than any positive power of $|x|^{-1}$, while at the origin (as $|x| \to 0$) the singularity is defined by the asymptotic behaviour (3.39) and with the help of the Fourier transform of homogeneous functions we find (see, e.g., [35, Ch. 1, § 2], [80, Ch. 2, § 6])

$$\Gamma^{(2)}(x,\tau) = \begin{bmatrix} [\mathcal{O}(|x|^{-1})]_{5\times 5} & [\mathcal{O}(1)]_{5\times 1} \\ [\mathcal{O}(1)]_{1\times 5} & \mathcal{O}(|x|^{-1}) \end{bmatrix}_{6\times 6}. \tag{3.43}$$

Here the dominant parts of the entries of the block matrices are homogeneous functions of the corresponding order.

On the other hand, we easily establish that the entries of the matrix $\Gamma^{(1)}(x,\tau)$ are infinitely differentiable functions in $\mathbb{R}^3$ and due to formula (3.41) they have the following asymptotic behaviour at infinity (as $|x| \to \infty$)

$$\Gamma^{(1)}(x,\tau) = \begin{bmatrix} \mathcal{O}(|x|^{-3}) & \mathcal{O}(|x|^{-5}) & \mathcal{O}(|x|^{-5}) & \mathcal{O}(|x|^{-3}) & \mathcal{O}(|x|^{-3}) & \mathcal{O}(|x|^{-4}) \\ \mathcal{O}(|x|^{-5}) & \mathcal{O}(|x|^{-3}) & \mathcal{O}(|x|^{-5}) & \mathcal{O}(|x|^{-3}) & \mathcal{O}(|x|^{-3}) & \mathcal{O}(|x|^{-4}) \\ \mathcal{O}(|x|^{-5}) & \mathcal{O}(|x|^{-5}) & \mathcal{O}(|x|^{-3}) & \mathcal{O}(|x|^{-3}) & \mathcal{O}(|x|^{-3}) & \mathcal{O}(|x|^{-4}) \\ \mathcal{O}(|x|^{-3}) & \mathcal{O}(|x|^{-3}) & \mathcal{O}(|x|^{-3}) & \mathcal{O}(|x|^{-1}) & \mathcal{O}(|x|^{-1}) & \mathcal{O}(|x|^{-2}) \\ \mathcal{O}(|x|^{-3}) & \mathcal{O}(|x|^{-3}) & \mathcal{O}(|x|^{-3}) & \mathcal{O}(|x|^{-1}) & \mathcal{O}(|x|^{-1}) & \mathcal{O}(|x|^{-2}) \\ \mathcal{O}(|x|^{-4}) & \mathcal{O}(|x|^{-4}) & \mathcal{O}(|x|^{-4}) & \mathcal{O}(|x|^{-2}) & \mathcal{O}(|x|^{-2}) & \mathcal{O}(|x|^{-3}) \end{bmatrix}_{6\times 6} \tag{3.44}$$

As above, here the dominant parts of the entries of the matrix $\Gamma^{(1)}(x,\tau)$ are homogeneous functions of the corresponding order. Therefore, the asymptotic formulas (3.43) and (3.44) can be differentiated any times with respect to the variables $x_j$, $j = 1,2,3$, to obtain similar asymptotic formulas for derivatives $\partial^\alpha \Gamma^{(2)}(x,\tau)$ and $\partial^\alpha \Gamma^{(1)}(x,\tau)$ with arbitrary multi-index $\alpha = (\alpha_1,\alpha_2,\alpha_3)$.

Finally, we arrive at the following asymptotic relations for the fundamental matrix

$$\begin{gathered} \Gamma(x,\tau) = \mathcal{O}(\Gamma^{(2)}(x,\tau)) \ \text{ as } \ |x| \to 0, \quad \Gamma(x,\tau) = \mathcal{O}(\Gamma^{(1)}(x,\tau)) \ \text{ as } \ |x| \to \infty, \\ \partial^\alpha \Gamma(x,\tau) = \mathcal{O}(\partial^\alpha \Gamma^{(2)}(x,\tau)) \ \text{ as } \ |x| \to 0, \quad \partial^\alpha \Gamma(x,\tau) = \mathcal{O}(\partial^\alpha \Gamma^{(1)}(x,\tau)) \ \text{ as } \ |x| \to \infty, \end{gathered} \tag{3.45}$$

where $\alpha = (\alpha_1,\alpha_2,\alpha_3)$ is an arbitrary multi-index.

Note that, the fundamental matrices $\Gamma^{(0)}(x)$ and $\Gamma(x,\tau)$ have essentially different properties at infinity in view of the relations (3.5) and (3.45). To describe the relationship between them in a neighbourhood of the origin we prove the following assertion.

**Lemma 3.3.** *Let ether $\tau = \sigma + i\omega$ with $\sigma > 0$ and $\omega \in \mathbb{R}$ or $\tau = 0$. For an arbitrary multi-index $\alpha = (\alpha_1,\alpha_2,\alpha_3)$ and for sufficiently small $|x|$ there hold the estimates*

$$\begin{gathered} \Gamma_{kj}(x,\tau) - \Gamma^{(0)}_{kj}(x) = \mathcal{O}(1), \quad \partial^\alpha \Gamma_{kj}(x,\tau) - \partial^\alpha \Gamma^{(0)}_{kj}(x) = \mathcal{O}(|x|^{-|\alpha|}), \\ k,j = 1,\dots,6, \ |\alpha| = |\alpha_1| + |\alpha_2| + |\alpha_3|. \end{gathered} \tag{3.46}$$

*Proof.* For arbitrary $P \neq 0$ and $P + Q \neq 0$, one can easily verify the identity

$$\frac{1}{P+Q} = \frac{1}{P} + \sum_{k=1}^{N} \frac{(-1)^k Q^k}{P^{k+1}} + \frac{(-1)^{N+1} Q^{N+1}}{P^{N+1}(P+Q)}. \tag{3.47}$$

Taking into consideration (3.19), (3.20), (3.24), Lemma 3.1, and applying (3.47) with $P = P_{12}(\xi)$ and $Q = P_{10}(\xi,\tau) + P_8(\xi,\tau) + P_6(\xi,\tau) + P_4(\xi,\tau)$, we find

$$A^{-1}(-i\xi,\tau) = \frac{1}{\det A(-i\xi,\tau)} A^{(c)}(-i\xi,\tau) = \frac{1}{P_{12}(\xi)} A^{(c)}(-i\xi,\tau) + \mathcal{O}(|\xi|^{-4}) \ \text{ as } \ |\xi| \to \infty. \tag{3.48}$$

It can be checked that for sufficiently large $|\xi|$

$$A^{(c)}(-i\xi,\tau) - A^{(c,0)}(-i\xi) = [\mathcal{O}(|\xi|^9)]_{6\times 6},$$

where $A^{(c,0)}(-i\xi)$ is the matrix of cofactors of $A^{(0)}(-i\xi)$ and the dominant parts of the entries of the right hand side matrix are homogeneous polynomials of order 9. Therefore, in view of (3.20) we get

$$A^{-1}(-i\xi,\tau) = \{A^{(0)}(-i\xi)\}^{-1} + |\xi|^{-3}[\chi_{kj}(\xi)]_{6\times 6} + [\mathcal{O}(|\xi|^{-4})]_{6\times 6} \ \text{ as } \ |\xi| \to \infty,$$

where $\chi_{kj}(\xi)$ are odd homogeneous functions of order 0. Whence the relations (3.46) follow from properties of the generalized Fourier transform of homogeneous functions. □

*Remark* 3.4. Note that, the matrix $\Gamma^*(x,\tau) := [\overline{\Gamma(-x,\tau)}]^\top = \Gamma^\top(-x,\overline{\tau})$ represents a fundamental solution to the formally adjoint differential operator $A^*(\partial_x,\tau) \equiv [A(-\partial_x,\overline{\tau})]^\top$,

$$A^*(\partial_x,\tau)\Gamma^\top(-x,\overline{\tau}) = I_6\delta(x). \tag{3.49}$$

This follows from the relations (3.11), (3.12), and

$$\big[\,\overline{\Gamma(-x,\tau)}\,\big]^\top = \mathcal{F}^{-1}_{\xi\to x}\big[\{A^\top(i\xi,\overline{\tau})\}^{-1}\big] = \mathcal{F}^{-1}_{\xi\to x}\big[\{A^*(-i\xi,\tau)\}^{-1}\big]. \tag{3.50}$$

3.3. **Fundamental matrix of the operator** $A(\partial,0)$**.** If $\tau=0$, than it is evident that $\det A(-i\xi,0)=\det A^{(0)}(-i\xi)$ due to (3.11) and (2.47), and by the same approach as above we get the following expression for the fundamental matrix of the operator of statics $A(\partial,0)$

$$\Gamma(x,0)=\mathcal{F}^{-1}_{\xi\to x}\big[A^{-1}(-i\xi,0)\big] \tag{3.51}$$

where

$$A^{-1}(-i\xi,0)=\frac{1}{\det A(-i\xi,0)}\,A^{(c)}(-i\xi,0). \tag{3.52}$$

Note that, $\det A(-i\xi,0)$ is a homogeneous polynomial in $\xi=(\xi_1,\xi_2,\xi_3)$ of order 12. Moreover, the cofactors $A^{(c)}_{kj}(-i\xi,0)$ are homogeneous polynomials in $\xi$ as well, namely,

$$\operatorname{ord}A^{(c)}_{kj}(-i\xi,0)=10,\ \ k,j=1,\dots,5,\quad \operatorname{ord}A^{(c)}_{66}(-i\xi,0)=10,$$
$$\operatorname{ord}A^{(c)}_{j6}(-i\xi,0)=9,\quad A^{(c)}_{6j}(-i\xi,0)=0,\ \ j=1,\dots,5.$$

Therefore, the functions

$$K_j(\xi):=\frac{A^{(c)}_{j6}(-i\xi,0)}{\det A(-i\xi,0)},\ \ j=1,\dots,5, \tag{3.53}$$

are odd homogeneous rational functions of order $-3$ in $\xi$ and, consequently,

$$\int_{|\xi|=1}K_j(\xi)\,dS=0,\ \ j=1,\dots,5. \tag{3.54}$$

Then it follows that the inverse Fourier transform of the function $K_j(\xi)$, considered in the Principal Value sense, is a homogeneous function of order 0 (see, e.g., [35, Ch. 1, § 2], [80, Ch. 2, § 6]) and

$$\int_{|x|=1}\mathcal{F}^{-1}_{\xi\to x}[K_j(\xi)]\,dS_x=0,\ \ j=1,\dots,5, \tag{3.55}$$

i.e.,

$$\int_{|x|=1}\Gamma_{j6}(x,0)\,dS=0,\ \ j=1,\dots,5, \tag{3.56}$$

Therefore, the entries of the fundamental matrix $\Gamma(x,0)$ are homogeneous functions in $x$ and

$$\Gamma(x,0)=\begin{bmatrix}[\mathcal{O}(|x|^{-1})]_{5\times5} & [\mathcal{O}(1)]_{5\times1}\\ [0]_{1\times5} & \mathcal{O}(|x|^{-1})\end{bmatrix}_{6\times6}. \tag{3.57}$$

Moreover, since $A^{(c)}_{kj}(-i\xi,0)=A^{(c,0)}_{kj}(-i\xi)$ for $k,j=1,\dots,5$ and $A^{(c)}_{66}(-i\xi,0)=A^{(c,0)}_{66}(-i\xi)$, we conclude that (see (3.4)–(3.6))

$$\Gamma_{kj}(x,0)=\Gamma^{(0)}_{kj}(x),\ \ k,j=1,\dots,5,\quad \Gamma_{66}(x,0)=\Gamma^{(0)}_{66}(x).$$

As we see from formulas (3.57), (3.44), and (3.45) the entries of the fundamental matrices $\Gamma(x,0)$ and $\Gamma(x,\tau)$ with $\operatorname{Re}\tau=\sigma>0$ have essentially different properties at infinity.

3.4. **Integral representation formulae of solutions.** For simplicity, in this subsection we assume (if not otherwise stated) that

$$\begin{aligned}&S=\partial\Omega^{\pm}\in C^{m,\kappa}\quad\text{with integer } m\geq1 \text{ and } 0<\kappa\leq1,\\ &\tau=\sigma+i\omega\ \ \text{with}\ \ \sigma>0,\ \ \omega\in\mathbb{R}.\end{aligned} \tag{3.58}$$

Let us introduce the generalized single and double layer potentials, and the Newton type volume potential

$$V_S(g)(x)=V(g)(x)=\int_S\Gamma(x-y,\tau)g(y)\,dS_y,\ \ x\in\mathbb{R}^3\setminus S, \tag{3.59}$$

$$W_S(g)(x)=W(g)(x)=\int_S\big[\mathcal{P}(\partial_y,n(y),\overline{\tau})\Gamma^\top(x-y,\tau)\big]^\top g(y)\,dS_y,\ \ x\in\mathbb{R}^3\setminus S, \tag{3.60}$$

$$N_{\Omega^\pm}(h)(x)=\int_{\Omega^\pm}\Gamma(x-y,\tau)h(y)dy,\ \ x\in\mathbb{R}^3, \tag{3.61}$$

where $\mathcal{P}(\partial,n,\tau)$ is the boundary differential operator defined by (2.58), $\Gamma(\,\cdot\,,\tau)$ is the fundamental matrix of the operator $A(\partial,\tau)$, $g=(g_1,\dots,g_6)^\top$ is a density vector function defined on $S$, while a density vector function $h=(h_1,\dots,h_6)^\top$ is defined on $\Omega^\pm$. We assume that in the case of unbounded domain $\Omega^-$ the support of the vector function $h$ is a compact domain.

Due to the equality

$$\sum_{j=1}^{6} A_{kj}(\partial_x,\tau)\big([\mathcal{P}(\partial_y,n(y),\overline{\tau})\Gamma^\top(x-y,\tau)]^\top\big)_{jp}=\sum_{j,q=1}^{6} A_{kj}(\partial_x,\tau)\mathcal{P}_{pq}(\partial_y,n(y),\overline{\tau})\Gamma_{jq}(x-y,\tau)$$
$$=\sum_{j,q=1}^{6}\mathcal{P}_{pq}(\partial_y,n(y),\overline{\tau})A_{kj}(\partial_x,\tau)\Gamma_{jq}(x-y,\tau)=0,\ \ x\neq y,\ \ k,p=1,\dots,6,$$

it can easily be checked that the potentials defined by (3.59) and (3.60) are $C^\infty$–smooth in $\mathbb{R}^3\setminus S$ and solve the homogeneous equation $A(\partial,\tau)U(x)=0$ in $\mathbb{R}^3\setminus S$ for an arbitrary $L_p$-summable vector function $g$. By standard arguments it can be shown that the volume potential solves the nonhomogeneous equation (cf., e.g., [57, Ch. 5, § 10])

$$A(\partial_x,\tau)N_{\Omega^\pm}(h)=h\ \text{ in }\ \Omega^\pm\ \text{ for }\ h\in[C^{0,\kappa}(\Omega^\pm)]^6.$$

This formula holds also true almost everywhere in $\Omega^\pm$ for $h\in[L_p(\Omega^\pm)]^6$, provided that in the case of unbounded domain $\Omega^-$ the support of the vector function $h$ is a compact domain (cf. [57]).

With the help of Green's formulas (2.200) and (2.210) we can prove the following assertions (cf., e.g., [75, Ch. 6, Theorem 6.10], [92, Ch. 1, Lemma 2.1; Ch. 2, Lemma 8.2]).

**Theorem 3.5.** *Let $S=\partial\Omega^+$ be $C^{1,\kappa}$-smooth with $0<\kappa\le 1$ and let $U$ be a regular vector of the class $[C^2(\overline{\Omega^+})]^6$. Then there holds the integral representation formula*

$$W(\{U\}^+)(x)-V(\{\mathcal{T}U\}^+)(x)+N_{\Omega^+}(A(\partial,\tau)U)(x)=\begin{cases}U(x) & \text{for } x\in\Omega^+,\\ 0 & \text{for } x\in\Omega^-.\end{cases}\tag{3.62}$$

*This formula can be extended to Lipschitz domains and to vector functions satisfying the conditions $U\in[W^1_p(\Omega^+)]^6$ and $A(\partial,\tau)U\in[L_p(\Omega^+)]^6$ with $1<p<\infty$.*

*Proof.* For the smooth case it easily follows from Green's formula (2.200) with the domain of integration $\Omega^+\setminus B(x,\varepsilon)$, where $x\in\Omega^+$ is treated as a fixed parameter and $B(x,\varepsilon)\subset\Omega^+$ is a ball centered at $x$ and radius $\varepsilon$. One needs to take the $j$-th column of the fundamental matrix $\Gamma^*(y-x,\tau)=[\overline{\Gamma(x-y,\tau)}]^\top$ for $U'(y)$ and pass to the limit as $\varepsilon\to 0$, taking into account that

$$\lim_{\varepsilon\to 0}\int_{\Sigma(x,\varepsilon)}\mathcal{P}(\partial_y,n(y),\overline{\tau})\Gamma^\top(x-y,\tau)\,dS=I_6,\tag{3.63}$$

where $\Sigma(x,\varepsilon)=\partial B(x,\varepsilon)$ and $n(y)=(y-x)/\varepsilon$ is the exterior normal vector to the sphere $\Sigma(x,\varepsilon)$ (see Appendix D).

The second part of the theorem can be shown by standard limiting procedure. □

Similar representation formula holds in the exterior domain $\Omega^-$.

**Theorem 3.6.** *Let $S=\partial\Omega^-$ be $C^{1,\kappa}$-smooth with $0<\kappa\le 1$ and let $U$ be a regular vector of the class $[C^2(\overline{\Omega^-})]^6$ satisfying the decay conditions at infinity* (2.207). *Then there holds the integral representation formula*

$$-W(\{U\}^-)(x)+V(\{\mathcal{T}U\}^-)(x)+N_{\Omega^-}(A(\partial,\tau)U)(x)=\begin{cases}0 & \text{for } x\in\Omega^+,\\ U(x) & \text{for } x\in\Omega^-.\end{cases}\tag{3.64}$$

*This integral representation formula can be extended to Lipschitz domains and to vector functions $U\in[W^1_{p,loc}(\Omega^-)]^6\cap\mathbf{Z}_\tau(\Omega^-)$ with $A(\partial,\tau)U\in[L_{p,comp}(\Omega^-)]^6$ with $1<p<\infty$.*

*Proof.* The proof for regular case immediately follows from Theorem 3.5. Indeed, one needs to write the integral representation formula (3.62) for bounded domain $\Omega^-\cap B(0,R)$, send then $R$ to $+\infty$ and take into consideration that the surface integrals over $\Sigma(0,R)$ tend to zero due to the conditions (2.207) and the decay properties of the fundamental matrix at infinity.

The second part of the theorem again can be shown by standard limiting procedure. □

**Corollary 3.7.** *Let $S = \partial\Omega^{\pm}$ be $C^{1,\kappa}$-smooth with $0 < \kappa \le 1$ and $U \in [C^2(\overline{\Omega^{\pm}})]^6$ be a solution to the homogeneous equations $A(\partial, \tau)U = 0$ in $\Omega^+$ and $\Omega^-$ satisfying conditions* (2.207). *Then the following representation formula holds*

$$U(x) = W([U]_S)(x) - V([\mathcal{T}U]_S)(x), \quad x \in \Omega^+ \cup \Omega^-, \tag{3.65}$$

*where $[U]_S = \{U\}^+_S - \{U\}^-_S$ and $[\mathcal{T}U]_S = \{\mathcal{T}U\}^+_S - \{\mathcal{T}U\}^-_S$.*

*This formula can be extended to Lipschitz domains and to solution vector functions $U$ from the space $[W^1_{p,loc}(\Omega^-)]^6$ with $1 < p < \infty$ satisfying conditions* (2.207).

*Proof.* It immediately follows from Theorems 3.5 and 3.6. □

It is evident that representation formulas similar to (3.62), (3.64), and (3.65) hold also for domains $\Omega^{\pm}_{\Sigma}$ with interior cracks. For example, for a solution vector $U \in [W^1_p(\Omega^+_{\Sigma})]^6$ to the homogeneous equations $A(\partial, \tau)U = 0$ in $\Omega^+_{\Sigma}$ the following representation formula holds true

$$W_S(\{U\}^+_S)(x) - V_S(\{\mathcal{T}U\}^+_S)(x) + N_{\Omega^+_{\Sigma}}(A(\partial,\tau)U)(x) + W_{\Sigma}([U]_{\Sigma})(x) - V_{\Sigma}([\mathcal{T}U]_{\Sigma})(x) = \begin{cases} U(x) & \text{for } x \in \Omega^+_{\Sigma}, \\ 0 & \text{for } x \in \Omega^-. \end{cases} \tag{3.66}$$

Note that, if $U \in [W^1_p(\Omega^+_{\Sigma})]^6$ and $A(\partial, \tau)U \in [L_p(\Omega^+_{\Sigma})]^6$ or $U \in [W^1_{p,loc}(\Omega^-_{\Sigma})]^6$ and $A(\partial,\tau)U \in [L_{p,comp}(\Omega^-_{\Sigma})]^6$, then $U \in [W^2_p(\Omega)]^6$ for arbitrary $\overline{\Omega} \subset \Omega^{\pm}_{\Sigma}$ due to the interior regularity results, while by the conventional trace theorem for $\{U\}^{\pm}$ and the definition of the generalized traces of the stress vector $\{\mathcal{T}U\}^{\pm}$ we have

$$\{U\}^{\pm} \in \big[B^{1-\frac{1}{p}}_{p,p}(S)\big]^6, \;\; \{\mathcal{T}U\}^{\pm} \in \big[B^{-\frac{1}{p}}_{p,p}(S)\big]^6, \;\; [U]_{\Sigma} \in r_{\Sigma}\big[\widetilde{B}^{1-\frac{1}{p}}_{p,p}(\Sigma)\big]^6, \;\; [\mathcal{T}U]_{\Sigma} \in r_{\Sigma}\big[\widetilde{B}^{-\frac{1}{p}}_{p,p}(\Sigma)\big]^6. \tag{3.67}$$

Therefore, the surface integrals over the exterior boundary manifold $S$ or over the crack surface $\Sigma$, containing the traces of the generalized stress vector, should be understood in the sense of dualities which are well-defined between the corresponding pairs of the adjoint spaces $[B^{-\frac{1}{p}}_{p,p}(S)]^6$ and $[B^{\frac{1}{p}}_{p',p'}(S)]^6$, or $[\widetilde{B}^{-\frac{1}{p}}_{p,p}(\Sigma)]^6$ and $[B^{\frac{1}{p}}_{p',p'}(\Sigma)]^6$, respectively, with $1/p + 1/p' = 1$.

3.5. **Uniqueness results for exterior BVPs of statics.** Here we study the uniqueness of solutions to exterior BVPs of statics of thermo-electro-magneto-elasticity, which corresponds to the case $\tau = 0$. Throughout this subsection we assume that $S$, $\Sigma$, and $\partial\Sigma$ are Lipschitz if not otherwise stated. First we analyze the temperature field.

3.5.1. *Asymptotic behaviour of the temperature field at infinity.* As we have mentioned above, in Subsection 2.7.4, in the case of static problems the differential equation (see (2.35) and (2.45))

$$A_{66}(\partial_x, 0)\vartheta \equiv A^{(0)}_{66}(\partial_x)\vartheta \equiv \eta_{jl}\partial_j\partial_l\vartheta = \Phi_6 \tag{3.68}$$

and the corresponding boundary and crack type conditions for temperature field are separated. Here the right hand side function $\Phi_6$ has a compact support. Therefore, one can easily prove the corresponding uniqueness theorems for the homogenous BVPs for the temperature function $\vartheta \in W^1_{2,loc}(\Omega^-)$ or $\vartheta \in W^1_{2,loc}(\Omega^-_{\Sigma})$ satisfying the decay condition $\vartheta = o(1)$ at infinity. This decay condition automatically implies that

$$\partial^{\alpha}\vartheta(x) = \mathcal{O}(|x|^{-|\alpha|-1}) \;\text{ as }\; |x| \to \infty \tag{3.69}$$

for arbitrary multi-index $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ with $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3$.

For such solutions to the differential equation (3.68) we have the following integral representation formula (for the domain $\Omega^-_{\Sigma}$ say)

$$\begin{aligned} \vartheta(x) = &\int_S \Gamma^{(0)}_{66}(x-y)\{\partial_{n(y)}\vartheta(y)\}^- \, dS_y - \int_S \partial_{n(y)}\Gamma^{(0)}_{66}(x-y)\{\vartheta(y)\}^- \, dS_y \\ &- \int_{\Sigma} \Gamma^{(0)}_{66}(x-y)[\partial_{n(y)}\vartheta(y)]_{\Sigma} \, dS_y + \int_{\Sigma} \partial_{n(y)}\Gamma^{(0)}_{66}(x-y)[\vartheta(y)]_{\Sigma} \, dS_y \\ &+ \int_{\Omega^-_{\Sigma}} \Gamma^{(0)}_{66}(x-y)\Phi_6(y)dy, \quad x \in \Omega^-_{\Sigma}, \end{aligned} \tag{3.70}$$

where $\Gamma^{(0)}_{66}(x)$ is the fundamental solution of the operator $A_{66}(\partial,0)\equiv A^{(0)}_{66}(\partial)$ defined by (3.6), $\partial_{n(y)}=\mathcal{T}_{66}(\partial_y,n(y))=\eta_{kj}n_j(y)\partial_k$ denotes the co-normal derivative,

$$[\vartheta(y)]_\Sigma=\{\vartheta(y)\}^+-\{\vartheta(y)\}^-,\quad \big[\partial_{n(y)}\vartheta(y)\big]_\Sigma=\big\{\partial_{n(y)}\vartheta(y)\big\}^+-\big\{\partial_{n(y)}\vartheta(y)\big\}^- \ \text{ on } \ \Sigma.$$

If $\Omega^-$ does not contain an interior crack $\Sigma$, then in (3.70) we have not the surface integrals over $\Sigma$. Applying (3.70) we derive the following asymptotic relation

$$\vartheta(x)=\frac{\theta_0}{(Dx\cdot x)^{1/2}}+\mathcal{O}(|x|^{-2}) \ \text{ as } \ |x|\to\infty, \tag{3.71}$$

where $D=[d_{kj}]_{3\times3}$ is the same as in (3.6), $\theta_0$ is a real constant which is calculated explicitly

$$\theta_0=\lim_{|x|\to\infty}(Dx\cdot x)^{1/2}\vartheta(x)=-\frac{\alpha_0}{4\pi}\bigg[\int_S\{\partial_{n(y)}\vartheta(y)\}^-\,dS_y-\int_\Sigma[\partial_{n(y)}\vartheta(y)]_\Sigma\,dS_y+\int_{\Omega^-_\Sigma}\Phi(y)\,dy\bigg] \tag{3.72}$$

with $\alpha_0$ being defined in (3.6). Note that (3.71) can be differentiated any times with respect to $x_j$, $j=1,2,3$. In particular, in view of the symmetry property $d_{jl}=d_{lj}$, we have

$$\partial_j\vartheta(x)=-\frac{\theta_0d_{jl}x_l}{(Dx\cdot x)^{3/2}}+\mathcal{O}(|x|^{-3}) \ \text{ as } \ |x|\to\infty, \ \ j=1,2,3. \tag{3.73}$$

3.5.2. *General uniqueness results.* First, let us consider the exterior Dirichlet problem of statics of thermo-electro-magneto-elasticity:

$$A(\partial_x,0)U=\Phi \ \text{ in } \ \Omega^-, \tag{3.74}$$

$$\{U\}^-=g \ \text{ on } \ S=\partial\Omega^-, \tag{3.75}$$

where $U=(u,\varphi,\psi,\vartheta)^\top\in[W^1_{2,loc}(\Omega^-)]^6$ is a sought for vector and

$$\Phi=(\Phi_1,\dots,\Phi_6)^\top\in[L_{2,comp}(\Omega^-)]^6,\quad g=(g_1,\dots,g_6)^\top\in[H_2^{\frac12}(S)]^6.$$

Our goal is to establish asymptotic conditions at infinity which guarantee the uniqueness for the BVP (3.74), (3.75).

For the temperature function $\vartheta$ we have the separated exterior Dirichlet problem

$$A_{66}(\partial_x,0)\vartheta=\eta_{kj}\partial_k\partial_j\vartheta=\Phi_6 \ \text{ in } \ \Omega^-, \tag{3.76}$$

$$\{\vartheta\}^-=g_6 \ \text{ on } \ S=\partial\Omega^-. \tag{3.77}$$

We assume that

$$\vartheta(x)=\mathcal{O}(|x|^{-1}) \ \text{ as } \ |x|\to\infty. \tag{3.78}$$

Then the BVP (3.76)–(3.78) is uniquely solvable for arbitrary $\Phi_6$ and $g_6$, and there holds the asymptotic relation (3.71) with

$$\theta_0=\lim_{|x|\to\infty}(Dx\cdot x)^{1/2}\vartheta(x)=-\frac{\alpha_0}{4\pi}\bigg[\int_S\big\{\partial_{n(y)}\vartheta(y)\big\}^-\,dS_y+\int_{\Omega_0}\Phi(y)\,dy\bigg] \tag{3.79}$$

which follows from (3.6) and the representation formula

$$\begin{aligned}\vartheta(x)=&\int_S\Gamma^{(0)}_{66}(x-y)\{\partial_{n(y)}\vartheta(y)\}^-\,dS_y-\int_S\partial_{n(y)}\Gamma^{(0)}_{66}(x-y)\{\vartheta(y)\}^-\,dS_y\\&+\int_{\Omega_0}\Gamma^{(0)}_{66}(x-y)\Phi_6(y)\,dy,\ \ x\in\Omega^-,\end{aligned} \tag{3.80}$$

where $\Omega_0=\operatorname{supp}\Phi_6\subset\Omega^-$ is compact.

Since $\Phi_6$ has a compact support we see that outside of $\operatorname{supp}\Phi_6$ the temperature function $\vartheta$ is a real analytic function with respect to $x$ in $\Omega^-\setminus\overline{\Omega}_0$.

Thus, assuming that the temperature function is known we can substitute it in the first five equations in (3.74). Then from (3.74), (3.75) we obtain the following BVP for the unknown vector function $\widetilde U=(u,\psi,\varphi)^\top\in[W^1_{2,loc}(\Omega^-)]^5$

$$\widetilde A^{(0)}(\partial_x)\widetilde U=\widetilde\Psi+\widetilde\Phi \ \text{ in } \ \Omega^-, \tag{3.81}$$

$$\{\widetilde U\}^-=\widetilde g \ \text{ on } \ S=\partial\Omega^-, \tag{3.82}$$

where the differential operator $\widetilde{A}(\partial_x,0) = \widetilde{A}^{(0)}(\partial_x) = [\widetilde{A}^{(0)}_{kj}(\partial_x)]_{5\times 5}$ is defined by (2.380), $\widetilde{\Phi} = (\Phi_1,\dots,\Phi_5)^\top \in [L_{2,comp}(\Omega^-)]^5$, $\widetilde{g} = (g_1,\dots,g_5)^\top \in [H_2^{\frac{1}{2}}(S)]^5$, and

$$\widetilde{\Psi} = (\lambda_{1j}\partial_j\vartheta, \lambda_{2j}\partial_j\vartheta, \lambda_{3j}\partial_j\vartheta, p_j\partial_j\vartheta, m_j\partial_j\vartheta)^\top \in [L_2(\Omega^-)]^5. \tag{3.83}$$

Note that $\widetilde{\Psi}$ has not a compact support and due to formulas (3.73)

$$\widetilde{\Psi}(x) = \theta_0\widetilde{P}(x) + \widetilde{Q}(x), \tag{3.84}$$

where $\theta_0$ is defined in (3.79), $\widetilde{Q} \in [L_2(\Omega^-)]^5 \cap [C^\infty(\mathbb{R}^3 \setminus \operatorname{supp}\Phi_6)]^5$ and

$$\widetilde{Q}(x) = \mathcal{O}(|x|^{-3}) \;\text{ as }\; |x| \to \infty, \tag{3.85}$$

while $\widetilde{P}(x)$ is an odd, $C^\infty$-smooth homogeneous vector function of order $-2$,

$$\widetilde{P}(x) = -\frac{1}{(Dx,x)^{3/2}}\,(\lambda_{1j}d_{jl}x_l, \lambda_{2j}d_{jl}x_l, \lambda_{3j}d_{jl}x_l, p_jd_{jl}x_l, m_jd_{jl}x_l)^\top. \tag{3.86}$$

Therefore, it is easy to see that in a vicinity of infinity, more precisely, outside of $\operatorname{supp}\Phi$ the solution vector $\widetilde{U}$ of equation (3.81) is $C^\infty$-smooth but we can not assume that $\widetilde{U}$ decays at infinity, in general.

Now, we establish asymptotic properties of $\widetilde{U}(x)$ as $|x| \to \infty$. To this end, let us consider the equation

$$\widetilde{A}^{(0)}(\partial_x)\widetilde{U} = \theta_0\widetilde{P} \;\text{ in }\; \mathbb{R}^3 \setminus \{0\}, \tag{3.87}$$

where $\theta_0$ is given by (3.79). In view of (3.86) and in accordance with Lemma A.2 in Appendix A, equation (3.87) possesses a unique solution $\widetilde{W}^{(0)} \in [C^\infty(\mathbb{R}^3\setminus\{0\})]^5$ in the space of zero order homogeneous vector functions satisfying the condition

$$\int_{|x|=1} \widetilde{W}^{(0)}(x)\,dS = 0. \tag{3.88}$$

This solution reads as (cf. (A.17))

$$\widetilde{W}^{(0)}(x) = \theta_0\widetilde{U}^{(0)}(x) \;\text{ with }\; \widetilde{U}^{(0)}(x) := \mathcal{F}^{-1}_{\xi\to x}\big(\text{v.p.}[\widetilde{A}^{(0)}(-i\xi)]^{-1}\mathcal{F}\widetilde{P}(\xi)\big). \tag{3.89}$$

Equation (3.81) can be rewritten as

$$\widetilde{A}^{(0)}(\partial_x)\widetilde{U} = \theta_0\widetilde{P} + \widetilde{Q} + \widetilde{\Phi} \;\text{ in }\; \Omega^-, \tag{3.90}$$

and by Lemmas A.1–A.3 and Corollary A.5 (see Appendix A) we conclude that a solution of (3.90), which is bounded at infinity, has the form

$$\widetilde{U}(x) = C + \theta_0\widetilde{U}^{(0)}(x) + \widetilde{U}^*(x), \;\; x \in \Omega^-, \tag{3.91}$$

where $C = (C_1,\dots,C_5)^\top$ is an arbitrary constant vector, $\widetilde{U}^{(0)}$ is given by (3.89) and satisfies the condition (3.88), $\widetilde{U}^* \in [W^1_{2,loc}(\Omega^-)]^5 \cap [C^\infty(\mathbb{R}^3 \setminus \operatorname{supp}\Phi)]^5$ and

$$\partial^\alpha\widetilde{U}^*(x) = \mathcal{O}\big(|x|^{-1-|\alpha|}\ln|x|\big) \;\text{ as }\; |x| \to \infty \tag{3.92}$$

for arbitrary multi-index $\alpha = (\alpha_1,\alpha_2,\alpha_3)$.

Along with the boundedness at infinity, if we require that the mean value of a solution vector $\widetilde{U}$ over the sphere $\Sigma(O,R)$ tends to zero as $R \to \infty$, i.e.,

$$\lim_{R\to\infty}\frac{1}{4\pi R^2}\int_{\Sigma(O,R)} \widetilde{U}(x)\,d\Sigma(O,R) = 0, \tag{3.93}$$

then the constant summand $C$ in formula (3.91) vanishes and we arrive at the following assertion.

**Lemma 3.8.** *Let $S$ be Lipschitz and $\widetilde{U} \in [W^1_{2,loc}(\Omega^-)]^5$ be a solution of equation* (3.90)*, i.e., equation* (3.81)*, which is bounded at infinity and satisfies condition* (3.93)*. Then*

$$\widetilde{U}(x) = \theta_0\widetilde{U}^{(0)}(x) + \widetilde{U}^*(x), \;\; x \in \Omega^-, \tag{3.94}$$

*where $\widetilde{U}^{(0)}$ is given by* (3.89) *and $\widetilde{U}^*$ is as in* (3.91)*.*

Now, let us return to the exterior Dirichlet BVP (3.74), (3.75) and analyze the uniqueness question.

**Theorem 3.9.** *Let $S$ be Lipschitz. The exterior Dirichlet boundary value problem* (3.74), (3.75) *has at most one solution $U=(u,\varphi,\psi,\vartheta)^\top$ in the space $[W^1_{2,loc}(\Omega^-)]^6$, provided*

$$\vartheta(x)=\mathcal{O}(|x|^{-1}) \ \text{ as } \ |x|\to\infty, \tag{3.95}$$

*and $\widetilde{U}=(u,\varphi,\psi)^\top$ is bounded at infinity and satisfies the condition* (3.93).

*Proof.* Let $U^{(1)}=(u^{(1)},\varphi^{(1)},\psi^{(1)},\vartheta^{(1)})^\top$ and $U^{(2)}=(u^{(2)},\varphi^{(2)},\psi^{(2)},\vartheta^{(2)})^\top$ be two solutions of the problem under consideration with properties indicated in the theorem. Then the difference

$$V=(u',\varphi',\psi',\vartheta')^\top=U^{(1)}-U^{(2)}$$

solves the corresponding homogeneous problem.

Therefore, for the temperature function $\vartheta'$ we get the homogeneous Dirichlet problem of type (3.76), (3.77) (with $\Phi_6=0$, $g_6=0$) and since $\vartheta'$ satisfies the decay condition (3.95), it is identical zero in $\Omega^-$.

Consequently, the vector $\widetilde{V}=(u',\varphi',\psi')^\top$ is a solution of the homogeneous exterior Dirichlet problem

$$A^{(0)}(\partial)\widetilde{V}=0 \ \text{ in } \ \Omega^-, \tag{3.96}$$

$$\{\widetilde{V}\}^-=0 \ \text{ on } \ S=\partial\Omega^-. \tag{3.97}$$

Moreover, the vector $\widetilde{V}$ satisfies the condition (3.93) with $\widetilde{V}$ for $\widetilde{U}$ since both vector functions $\widetilde{U}^{(1)}=(u^{(1)},\varphi^{(1)},\psi^{(1)})^\top$ and $\widetilde{U}^{(2)}=(u^{(2)},\varphi^{(2)},\psi^{(2)})^\top$ satisfy the same condition.

In accordance with Lemma 3.8 then $\widetilde{V}$ is representable in the form (3.94),

$$\widetilde{V}(x)=\theta_0'\widetilde{U}^{(0)}(x)+\widetilde{V}^*(x), \ \ x\in\Omega^-,$$

where $\widetilde{U}^{(0)}$ is given by (3.89),

$$\partial^\alpha\widetilde{V}^*(x)=\mathcal{O}(|x|^{-1-|\alpha|}\ln|x|) \ \text{ as } \ |x|\to\infty$$

for arbitrary multi-index $\alpha$ and $\theta_0'=\lim\limits_{|x|\to\infty}(Dx\cdot x)^{1/2}\vartheta'(x)=0$ since $\vartheta'=0$ in $\Omega^-$ (cf. (3.79)). Therefore,

$$\partial^\alpha\widetilde{V}=\mathcal{O}(|x|^{-1-|\alpha|}\ln|x|) \ \text{ as } \ |x|\to\infty. \tag{3.98}$$

For vectors satisfying the decay conditions (3.98) we can easily derive the following Green's formula (cf. (2.382))

$$\int_{\Omega^-}\big[\widetilde{A}^{(0)}(\partial)\widetilde{V}\cdot\widetilde{V}+\widetilde{\mathcal{E}}(\widetilde{V},\widetilde{V})\big]\,dx=-\big\langle\{T\widetilde{V}\}^-,\{\widetilde{V}\}^-\big\rangle_{\partial\Omega^-}, \tag{3.99}$$

where $T(\partial,n)$ is given by (2.381) and

$$\widetilde{\mathcal{E}}(\widetilde{V},\widetilde{V})=c_{rjkl}\partial_l u_k'\partial_j u_r'+\varkappa_{jl}\partial_l\varphi'\partial_j\varphi'+a_{jl}(\partial_l\varphi'\partial_j\psi'+\partial_j\psi'\partial_l\varphi')+\mu_{jl}\partial_l\psi'\partial_j\psi'. \tag{3.100}$$

From (3.96), (3.97) and (3.99), (3.100) along with the inequalities (2.10) we get

$$\partial_l u_k'(x)+\partial_k u_l'(x)=0, \ \ \partial_k\varphi'(x)=0, \ \ \partial_k\psi'(x)=0, \ \ x\in\Omega^-, \ \ k,l=1,2,3,$$

implying $u'(x)=a\times x+b$, $\varphi'(x)=b_4$, $\psi'(x)=b_5$, $x\in\Omega^-$, where $a=(a_1,a_2,a_3)$ and $b=(b_1,b_2,b_3)$ are arbitrary constant vectors, and $b_4$ and $b_5$ are arbitrary constants. Now, in view of (3.98) we arrive at the equalities $u'(x)=0$, $\varphi'(x)=0$ and $\psi'(x)=0$ for $x\in\Omega^-$. Consequently, $U^{(1)}=U^{(2)}$ in $\Omega^-$. □

The proof of the following theorem is word for word.

**Theorem 3.10.** *Let $S$ be Lipschitz. The exterior Neumann and mixed boundary value problems of statics of thermo-electro-magneto-elasticity have at most one solution $U=(u,\varphi,\psi,\vartheta)^\top$ in the space $[W^1_{2,loc}(\Omega^-)]^6$, provided*

$$\vartheta(x)=\mathcal{O}(|x|^{-1}) \ \text{ as } \ |x|\to\infty, \tag{3.101}$$

*and $\widetilde{U}=(u,\varphi,\psi)^\top$ is bounded at infinity and satisfies the condition* (3.93).

Let us introduce the following class of vector functions.

**Definition 3.11.** We say that a vector function

$$U=(u,\varphi,\psi,\vartheta)^\top\equiv(U_1,\dots,U_6)^\top\in[W^1_{p,loc}(\Omega^-)]^6$$

belongs to the class $\mathbf{Z}(\Omega^-)$ if the components of $U$ satisfy the following asymptotic conditions:

$$\begin{gathered}\widetilde{U}(x):=(u(x),\varphi(x),\psi(x))^\top=\mathcal{O}(1)\ \text{ as }\ |x|\to\infty,\\ U_6(x)=\vartheta(x)=\mathcal{O}(|x|^{-1})\quad\text{as }\ |x|\to\infty,\\ \lim_{R\to\infty}\frac{1}{4\pi R^2}\int_{\Sigma_R}U_k(x)\,d\Sigma_R=0,\ \ k=1,2,\dots,5,\end{gathered}\tag{3.102}$$

where $\Sigma_R$ is the sphere centered at the origin and radius $R$.

*Remark* 3.12. Due to Definition 3.11 and Theorems 3.9 and 3.10, we see that for a Lipschitz domain $\Omega^-$ the homogeneous exterior Dirichlet, Neumann, and mixed BVPs of statics possess only the trivial solutions in the class $[W^1_{2,loc}(\Omega^-)]^6\cap\mathbf{Z}(\Omega^-)$.

## 4. Properties of Generalized Potentials

4.1. **Mapping properties.** Here we establish the mapping and regularity properties of the single and double layer potentials and the boundary pseudodifferential operators generated by them in the Hölder ($C^{k,\kappa}$), Sobolev–Slobodetskii ($W^s_p$), Bessel potential ($H^s_p$) and Besov ($B^s_{p,q}$) spaces. They can be established by standard methods (see, e.g., [57], [77], [100], [35], [30], [31], [91], [92], [75], [90], and [29]). We remark only that the layer potentials corresponding to the fundamental matrices with different values of the parameter $\tau$ ($\tau'$ and $\tau''$ say) have the same smoothness properties and possess the same jump relations, since the entries of the difference of the fundamental matrices $\Gamma(x,\tau')-\Gamma(x,\tau'')$ are bounded and their derivatives of order $m$ have a singularity of type $\mathcal{O}(|x|^{-m})$ in a neighbourhood of the origin. This implies that the boundary integral operators generated by the corresponding single layer potentials (respectively, by the double layer potentials) constructed with the help of the kernels $\Gamma(x,\tau')$ and $\Gamma(x,\tau'')$ differ by a compact perturbations. Therefore, using the technique and word for word arguments given in the references [57], [75], [30], [13], [12], and [29] we can prove the following theorems concerning the above introduced generalized potentials.

**Theorem 4.1.** *Let $S$, $m$, and $\kappa$ be as in* (3.58)*, $0<\kappa'<\kappa$, and let $k\le m-1$ be integer. Then the operators*

$$V:[C^{k,\kappa'}(S)]^6\longrightarrow[C^{k+1,\kappa'}(\overline{\Omega^\pm})]^6,\quad W:[C^{k,\kappa'}(S)]^6\longrightarrow[C^{k,\kappa'}(\overline{\Omega^\pm})]^6,\tag{4.1}$$

*are continuous.*

*For any $g\in[C^{0,\kappa'}(S)]^6$, $h\in[C^{1,\kappa'}(S)]^6$, and for any $x\in S$*

$$\{V(g)(x)\}^\pm=V(g)(x)=\mathcal{H}g(x),\tag{4.2}$$

$$\big\{\mathcal{T}(\partial_x,n(x),\tau)V(g)(x)\big\}^\pm=[\mp2^{-1}I_6+\mathcal{K}]g(x),\tag{4.3}$$

$$\{W(g)(x)\}^\pm=[\pm2^{-1}I_6+\mathcal{N}]g(x),\tag{4.4}$$

$$\big\{\mathcal{T}(\partial_x,n(x),\tau)W(h)(x)\big\}^+=\big\{\mathcal{T}(\partial_x,n(x),\tau)W(h)(x)\big\}^-=\mathcal{L}h(x),\ \ m\ge2,\tag{4.5}$$

*where*

$$\mathcal{H}g(x)\equiv\mathcal{H}_Sg(x):=\int_S\Gamma(x-y,\tau)g(y)\,dS_y,\ \ x\in S,\tag{4.6}$$

$$\mathcal{K}g(x)\equiv\mathcal{K}_Sg(x):=\int_S\big[\mathcal{T}(\partial_x,n(x),\tau)\Gamma(x-y,\tau)\big]\ g(y)\,dS_y,\ \ x\in S,\tag{4.7}$$

$$\mathcal{N}g(x)\equiv\mathcal{N}_Sg(x):=\int_S\big[\mathcal{P}(\partial_y,n(y),\overline{\tau})\Gamma^\top(x-y,\tau)\big]^\top g(y)\,dS_y,\ \ x\in S,\tag{4.8}$$

$$\mathcal{L}h(x)\equiv\mathcal{L}_Sh(x):=\lim_{\Omega^\pm\ni z\to x}\mathcal{T}(\partial_z,n(x),\tau)\int_S\big[\mathcal{P}(\partial_y,n(y),\overline{\tau})\Gamma^\top(z-y,\tau)\big]^\top h(y)\,dS_y,\ \ x\in S.\tag{4.9}$$

*Proof.* The proof of the relations (4.2)–(4.4) can be performed by standard arguments (see, e.g., [57, Ch. 5]). We demonstrate here only a simplified proof of relation (4.5), known as *Liapunov–Tauber* *type theorem.* Let $h \in [C^{1,\kappa'}(S)]^6$, $S \in C^{2,\kappa}$, and consider the double layer potential $U := W(h) \in [C^{1,\kappa'}(\overline{\Omega^\pm})]^6$. Then by Corollary 3.7 and the jump relations (4.4) we have

$$U(x) = W([U]_S)(x) - V([\mathcal{T}U]_S)(x), \ \ x \in \Omega^\pm,$$

i.e.,

$$W(h)(x) = W(h)(x) - V([\mathcal{T}W(h)]_S)(x), \ \ x \in \Omega^\pm,$$

since $[U]_S = \{W(h)\}^+ - \{W(h)\}^- = h$ on $S$ due to (4.4). Therefore $V([\mathcal{T}W(h)]_S) = 0$ in $\Omega^\pm$ and in view of (4.3) we conclude

$$\big\{\mathcal{T}V([\mathcal{T}W(h)]_S)\big\}^- - \big\{\mathcal{T}V([\mathcal{T}W(h)]_S)\big\}^+ = [\mathcal{T}W(h)]_S = \{\mathcal{T}W(h)\}^+ - \{\mathcal{T}W(h)\}^- = 0$$

on $S$, which completes the proof. □

Using the properties of the fundamental matrix $\Gamma(x-y,\tau)$ it can easily be shown that the operators $\mathcal{K}$ and $\mathcal{N}$ are singular integral operators, $\mathcal{H}$ is a smoothing (weakly singular) integral operator, while $\mathcal{L}$ is a singular integro-differential operator. For a $C^\infty$-smooth surface $S$ all these boundary operators can be treated as pseudodifferential operators on $S$ (cf., [1], [51], [31]). In contrast to the classical elasticity case, neither $\mathcal{H}$ and $\mathcal{L}$ are self-adjoint and nor $\mathcal{K}$ and $\mathcal{N}$ are mutually adjoint operators. For the adjoint operators $\mathcal{H}^*$, $\mathcal{K}^*$, and $\mathcal{N}^*$ we have

$$\mathcal{H}^* g(x) \equiv \mathcal{H}^*_\tau g(x) := \int_S \Gamma^*(x-y,\tau) g(y)\, dS_y, \ \ x \in S, \tag{4.10}$$

$$\mathcal{K}^* g(x) \equiv \mathcal{K}^*_\tau g(x) := \int_S \big[\mathcal{T}(\partial_y, n(y), \overline{\tau})[\Gamma^*(x-y,\tau)]^\top\big]^\top g(y)\, dS_y, \ \ x \in S, \tag{4.11}$$

$$\mathcal{N}^* g(x) \equiv \mathcal{N}^*_\tau g(x) := \int_S \mathcal{P}(\partial_x, n(x), \tau)\Gamma^*(x-y,\tau) g(y)\, dS_y, \ \ x \in S, \tag{4.12}$$

where $\Gamma^*(x-y,\tau) = [\overline{\Gamma(y-x,\tau)}]^\top$ is a fundamental matrix of the operator $A^*(\partial,\tau)$ (see Remark 3.4). These adjoint operators are defined by the duality relations for complex-valued function spaces (cf. (2.203))

$$\langle \mathcal{H}g, h\rangle_S = \langle g, \mathcal{H}^* h\rangle_S, \ \ \langle \mathcal{K}g, h\rangle_S = \langle g, \mathcal{K}^* h\rangle_S, \ \ \langle \mathcal{N}g, h\rangle_S = \langle g, \mathcal{N}^* h\rangle_S. \tag{4.13}$$

It is easy to see that the adjoint boundary operators are generated by the “adjoint” single layer and double layer potentials constructed with the help of the fundamental matrix $\Gamma^*(x-y,\tau)$. In particular, let

$$V^*_S(g)(x) = V^*(g)(x) := \int_S \Gamma^*(x-y,\tau) g(y)\, dS_y, \tag{4.14}$$

$$W^*_S(g)(x) = W^*(g)(x) = \int_S \big[\mathcal{T}(\partial_y, n(y), \overline{\tau})[\Gamma^*(x-y,\tau)]^\top\big]^\top g(y)\, dS_y. \tag{4.15}$$

Then for any solution $U^*$ of the equation $A^*(\partial_x,\tau)U^* = 0$ in $\Omega^+$ we have the representation formula

$$U^*(x) = W^*(\{U^*\}^+)(x) - V^*(\{\mathcal{P}U^*\}^+)(x), \ \ x \in \Omega^+, \tag{4.16}$$

which can be obtained by Green’s identity (2.200) (cf., Theorem 3.5). The right hand side expression in (4.16) vanishes for $x \in \Omega^-$. Clearly, the layer potential operators $V^*$ and $W^*$ have the same mapping properties as the operators $V$ and $W$, namely

$$V^* : [C^{k,\kappa'}(S)]^6 \longrightarrow [C^{k+1,\kappa'}(\overline{\Omega^\pm})]^6, \quad W^* : [C^{k,\kappa'}(S)]^6 \longrightarrow [C^{k,\kappa'}(\overline{\Omega^\pm})]^6, \tag{4.17}$$

where $S$, $m$, $\kappa$, $\kappa'$, and $k \le m-1$ are as in Theorem 4.1. Moreover, for $g \in [C^{0,\kappa'}(S)]^6$ and $h \in [C^{1,\kappa'}(S)]^6$ the following jump relations hold on $S$

$$\{V^*(g)(x)\}^\pm = V^*(g)(x) = \mathcal{H}^* g(x), \ \ x \in S, \tag{4.18}$$

$$\big\{\mathcal{P}(\partial_x, n(x), \tau)V^*(g)(x)\big\}^\pm = [\mp 2^{-1} I_6 + \mathcal{N}^*] g(x), \ \ x \in S, \tag{4.19}$$

$$\{W^*(g)(x)\}^\pm = [\pm 2^{-1} I_6 + \mathcal{K}^*] g(x), \ \ x \in S, \tag{4.20}$$

$$\big\{\mathcal{P}(\partial_x, n(x), \tau)W^*(h)(x)\big\}^+ = \big\{\mathcal{P}(\partial_x, n(x), \tau)W^*(h)(x)\big\}^- = \mathcal{L}^* h(x), \ m \ge 2, \ x \in S. \tag{4.21}$$

**Theorem 4.2.** *Let $S$ be a Lipschitz surface. The operators $V$, $W$, $V^*$, and $W^*$ can be extended to the continuous mappings*

$$V, V^* : [H_2^{-\frac{1}{2}}(S)]^6 \longrightarrow [H_2^1(\Omega^+)]^6 \quad \big[\, [H_2^{-\frac{1}{2}}(S)]^6 \longrightarrow [H_{2,loc}^1(\Omega^-)]^6 \,\big],$$
$$W, W^* : [H_2^{\frac{1}{2}}(S)]^6 \longrightarrow [H_2^1(\Omega^+)]^6 \quad \big[\, [H_2^{\frac{1}{2}}(S)]^6 \longrightarrow [H_{2,loc}^1(\Omega^-)]^6 \,\big].$$

*The jump relations* (4.2)–(4.5) *and* (4.18)–(4.21) *on $S$ remain valid for the extended operators in the corresponding function spaces.*

*Proof.* It is word for word of the proofs of the similar theorems in [25] and [75]. □

**Theorem 4.3.** *Let $S$, $m$, $\kappa$, $\kappa'$ and $k$ be as in Theorem* 4.1. *Then the operators*

$$\mathcal{H}, \mathcal{H}^* : [C^{k,\kappa'}(S)]^6 \longrightarrow [C^{k+1,\kappa'}(S)]^6, \ \ m \geq 1, \tag{4.22}$$
$$: [H_2^{-\frac{1}{2}}(S)]^6 \longrightarrow [H_2^{\frac{1}{2}}(S)]^6, \ \ m \geq 1, \tag{4.23}$$
$$\mathcal{K}, \mathcal{N}^* : [C^{k,\kappa'}(S)]^6 \longrightarrow [C^{k,\kappa'}(S)]^6, \ \ m \geq 1, \tag{4.24}$$
$$: [H_2^{-\frac{1}{2}}(S)]^6 \longrightarrow [H_2^{-\frac{1}{2}}(S)]^6, \ \ m \geq 1, \tag{4.25}$$
$$\mathcal{N}, \mathcal{K}^* : [C^{k,\kappa'}(S)]^6 \longrightarrow [C^{k,\kappa'}(S)]^6, \ \ m \geq 1, \tag{4.26}$$
$$: [H_2^{\frac{1}{2}}(S)]^6 \longrightarrow [H_2^{\frac{1}{2}}(S)]^6, \ \ m \geq 1, \tag{4.27}$$
$$\mathcal{L}, \mathcal{L}^* : [C^{k,\kappa'}(S)]^6 \longrightarrow [C^{k-1,\kappa'}(S)]^6, \ \ m \geq 2, \ \ k \geq 1, \tag{4.28}$$
$$: [H_2^{\frac{1}{2}}(S)]^6 \longrightarrow [H_2^{-\frac{1}{2}}(S)]^6 \ \ m \geq 2, \tag{4.29}$$

*are continuous. The operators* (4.23), (4.25), (4.27), *and* (4.29) *are bounded if $S$ is a Lipschitz surface and the following equalities hold true in appropriate function spaces:*

$$\mathcal{N}\mathcal{H} = \mathcal{H}\mathcal{K}, \quad \mathcal{L}\mathcal{N} = \mathcal{K}\mathcal{L}, \quad \mathcal{H}\mathcal{L} = -4^{-1}I_6 + \mathcal{N}^2, \quad \mathcal{L}\mathcal{H} = -4^{-1}I_6 + \mathcal{K}^2. \tag{4.30}$$

*Proof.* It is word for word of the proofs of the similar theorems in [57], [31], [25] and [75]. □

The next assertion is a consequence of the general theory of elliptic pseudodifferential operators on smooth manifolds without boundary (see, e.g., [1], [35], [100], [31], [29], [12], and the references therein).

**Theorem 4.4.** *Let $V$, $W$, $\mathcal{H}$, $\mathcal{K}$, $\mathcal{N}$, $\mathcal{L}$, $V^*$, $W^*$, $\mathcal{H}^*$, $\mathcal{K}^*$, $\mathcal{N}^*$, and $\mathcal{L}^*$ be as in Theorems* 4.1, 4.2 *and* 4.3 *and let $s \in \mathbb{R}$, $1 < p < \infty$, $1 \leq q \leq \infty$, $S \in C^\infty$. The layer potential operators* (4.1), (4.17) *and the boundary integral* (*pseudodifferential*) *operators* (4.22)–(4.29) *can be extended to the following continuous operators*

$$V, V^* : [B_{p,p}^s(S)]^6 \longrightarrow [H_p^{s+1+\frac{1}{p}}(\Omega^+)]^6 \quad \big[\, [B_{p,p}^s(S)]^6 \longrightarrow [H_{p,loc}^{s+1+\frac{1}{p}}(\Omega^-)]^6 \cap \mathbf{Z}_\tau(\Omega^-) \,\big],$$
$$: [B_{p,q}^s(S)]^6 \longrightarrow [B_{p,q}^{s+1+\frac{1}{p}}(\Omega^+)]^6 \quad \big[\, [B_{p,q}^s(S)]^6 \longrightarrow [B_{p,q,loc}^{s+1+\frac{1}{p}}(\Omega^-)]^6 \cap \mathbf{Z}_\tau(\Omega^-) \,\big],$$
$$W, W^* : [B_{p,p}^s(S)]^6 \longrightarrow [H_p^{s+\frac{1}{p}}(\Omega^+)]^6 \quad \big[\, [B_{p,p}^s(S)]^6 \longrightarrow [H_{p,loc}^{s+\frac{1}{p}}(\Omega^-)]^6 \cap \mathbf{Z}_\tau(\Omega^-) \,\big],$$
$$: [B_{p,q}^s(S)]^6 \longrightarrow [B_{p,q}^{s+\frac{1}{p}}(\Omega^+)]^6 \quad \big[\, [B_{p,q}^s(S)]^6 \longrightarrow [B_{p,q,loc}^{s+\frac{1}{p}}(\Omega^-)]^6 \cap \mathbf{Z}_\tau(\Omega^-) \,\big],$$
$$\mathcal{H}, \mathcal{H}^* : [H_p^s(S)]^6 \longrightarrow [H_p^{s+1}(S)]^6 \quad \big[\, [B_{p,q}^s(S)]^6 \longrightarrow [B_{p,q}^{s+1}(S)]^6 \,\big],$$
$$\mathcal{K}, \mathcal{N}^* : [H_p^s(S)]^6 \longrightarrow [H_p^s(S)]^6 \quad \big[\, [B_{p,q}^s(S)]^6 \longrightarrow [B_{p,q}^s(S)]^6 \,\big],$$
$$\mathcal{N}, \mathcal{K}^* : [H_p^s(S)]^6 \longrightarrow [H_p^s(S)]^6 \quad \big[\, [B_{p,q}^s(S)]^6 \to [B_{p,q}^s(S)]^6 \,\big],$$
$$\mathcal{L}, \mathcal{L}^* : [H_p^{s+1}(S)]^6 \longrightarrow [H_p^s(S)]^6 \quad \big[\, [B_{p,q}^{s+1}(S)]^6 \to [B_{p,q}^s(S)]^6 \,\big].$$

*The jump relations* (4.2)–(4.5) *and* (4.18)–(4.21) *remain valid in appropriate function spaces for arbitrary $g, h \in [B_{p,q}^s(S)]^6$ with $s \in \mathbb{R}$ if the limiting values* (*traces*) *on $S$ are understood in the sense described in* [100].

*In particular,*

(i) *if* $g \in [B_{p,q}^{-\frac{1}{p}}(S)]^6$, *then relations* (4.2) *and* (4.18) *remain valid in the sense of the space* $[B_{p,q}^{1-\frac{1}{p}}(S)]^6$, *while the relations* (4.3) *and* (4.19) *remain valid in the sense of the space* $[B_{p,q}^{-\frac{1}{p}}(S)]^6$;

(ii) *if* $g, h \in [B_{p,q}^{1-\frac{1}{p}}(S)]^6$, *then relations* (4.4) *and* (4.20) *remain valid in the sense of the space* $[B_{p,q}^{1-\frac{1}{p}}(S)]^6$, *while the relations* (4.5) *and* (4.21) *remain valid in the sense of the space* $[B_{p,q}^{-\frac{1}{p}}(S)]^6$.

*Proof.* It is word for word of the proofs of the similar theorems in [31], [12] and [29]. □

*Remark* 4.5. Let $\Phi \in [L_{p,comp}(\mathbb{R}^3)]^6$ with $p > 1$. Then the Newtonian volume potential $N_{\mathbb{R}^3}(\Phi)$ defined by (3.61) possesses the following properties (cf. [77, Ch. 11])

$$N_{\mathbb{R}^3}(\Phi) \in [W_{p,loc}^2(\mathbb{R}^3)]^6, \quad A(\partial, \tau)N_{\mathbb{R}^3}(\Phi) = \Phi \ \text{ almost everywhere in } \mathbb{R}^3.$$

Further, due to the properties of the fundamental matrices $\Gamma(x-y,\tau)$ and $\Gamma^*(x-y,\tau)$ at infinity (see (3.44), (3.45) and Remark 3.4) for $\tau = \sigma + i\omega$ with $\sigma > 0$ it follows that the single and double layer potentials, and the Newtonian volume potentials with compactly supported densities, associated with the differential operators $A(\partial_x, \tau)$ and $A^*(\partial_x, \tau)$, possess the asymptotic decay properties (2.207), i.e., they belong to the class $\mathbf{Z}_\tau(\Omega^-)$.

Moreover, for regular densities the volume potential operator $N_{\Omega^+}$ possesses the following properties (cf. [45], [78], [57], [3]): If $S = \partial\Omega^+ \in C^{2,\alpha}$, then the following operators are continuous,

$$N_{\Omega^+} : [L_\infty(\Omega^+)]^6 \longrightarrow [C^{1,\gamma}(\mathbb{R}^3)]^6 \ \text{ for all } \ 0 < \gamma < 1, \tag{4.31}$$

$$N_{\Omega^+} : [C^{0,\beta}(\overline{\Omega^+})]^6 \longrightarrow [C^{2,\beta}(\overline{\Omega^+})]^6, \ \ 0 < \beta < 1, \tag{4.32}$$

$$N_{\Omega^+} : [C^{1,\beta}(\overline{\Omega^+})]^6 \longrightarrow [C^{3,\beta}(\overline{\Omega^+})]^6, \ \ 0 < \beta < \alpha \leqslant 1. \tag{4.33}$$

4.2. **Coercivity and strong ellipticity properties of the operator $\mathcal{H}$.** Here we assume that either $\tau = \sigma + i\omega$ with $\sigma > 0$ and $\omega \in \mathbb{R}$ or $\tau = 0$, and establish that the boundary integral operator $\mathcal{H}$, defined by (4.6), satisfies Gårding type inequality. By $\mathcal{H}^{(0)}$, $\mathcal{K}^{(0)}$, $\mathcal{N}^{(0)}$ and $\mathcal{L}^{(0)}$ we denote the boundary operators generated by the single and double layer potentials constructed with the help of the fundamental matrix $\Gamma^{(0)}(\,\cdot\,)$ associated with the operator $A^{(0)}(\partial_x)$ (see (2.46) and Subsection 3.1). Note that $\Gamma^{(0)}(\,\cdot\,)$ is the principal singular part of the fundamental matrix $\Gamma(\,\cdot\,,\tau)$ (see Subsection 3.1). So, we have

$$\mathcal{H}^{(0)}h = \{V^{(0)}(h)\}^+ = \{V^{(0)}(h)\}^- \ \text{ on } \ S, \tag{4.34}$$

$$\big[\mp 2^{-1}I_6 + \mathcal{K}^{(0)}\big]g = \big\{\mathcal{T}^{(0)}(\partial_x, n(x))V^{(0)}(g)\big\}^\pm \ \text{ on } \ S, \tag{4.35}$$

$$\big[\pm 2^{-1}I_6 + \mathcal{N}^{(0)}\big]h = \{W^{(0)}(h)\}^\pm \ \text{ on } \ S, \tag{4.36}$$

$$\mathcal{L}^{(0)}g = \big\{\mathcal{T}^{(0)}(\partial_x, n)W^{(0)}(g)\big\}^+ = \big\{\mathcal{T}^{(0)}(\partial_x, n)W^{(0)}(g)\big\}^- \ \text{ on } \ S, \tag{4.37}$$

where

$$V^{(0)}(h)(x) = \int_S \Gamma^{(0)}(x-y)h(y)\,dS_y, \tag{4.38}$$

$$W^{(0)}(g)(x) = \int_S [\mathcal{P}^{(0)}(\partial_y, n(y))[\Gamma^{(0)}(x-y)]^\top]^\top g(y)\,dS_y. \tag{4.39}$$

Here the boundary differential operators $\mathcal{T}^{(0)}(\partial_x, n)$ and $\mathcal{P}^{(0)}(\partial_y, n(y))$ are defined by (2.59) and (2.60), respectively.

Evidently, Theorems 4.1–4.4 hold true for the layer potentials $V^{(0)}$ and $W^{(0)}$ and for the boundary operators generated by them.

For a Lipschitz surface $S$ the operators

$$\mathcal{H} - \mathcal{H}^{(0)} : [H_2^{-\frac{1}{2}}(S)]^6 \longrightarrow [H_2^{\frac{1}{2}}(S)]^6, \tag{4.40}$$

$$\mathcal{K} - \mathcal{K}^{(0)} : [H_2^{-\frac{1}{2}}(S)]^6 \longrightarrow [H_2^{-\frac{1}{2}}(S)]^6, \tag{4.41}$$

$$\mathcal{N} - \mathcal{N}^{(0)} : [H_2^{\frac{1}{2}}(S)]^6 \longrightarrow [H_2^{\frac{1}{2}}(S)]^6, \tag{4.42}$$

$$\mathcal{L} - \mathcal{L}^{(0)} : [H_2^{\frac{1}{2}}(S)]^6 \longrightarrow [H_2^{-\frac{1}{2}}(S)]^6, \tag{4.43}$$

are compact for arbitrary $\tau$ due to Lemma 3.3. Moreover, if $S$, $m$, $\kappa$, $\kappa'$ and $k$ are as in Theorem 4.1, then the operators

$$\mathcal{H} - \mathcal{H}^{(0)} : [C^{k,\kappa'}(S)]^6 \longrightarrow [C^{k+1,\kappa'}(S)]^6, \ \ m \geq 1, \tag{4.44}$$

$$\mathcal{K} - \mathcal{K}^{(0)} : [C^{k,\kappa'}(S)]^6 \longrightarrow [C^{k,\kappa'}(S)]^6, \ \ m \geq 1, \tag{4.45}$$

$$\mathcal{N} - \mathcal{N}^{(0)} : [C^{k,\kappa'}(S)]^6 \longrightarrow [C^{k,\kappa'}(S)]^6, \ \ m \geq 1, \tag{4.46}$$

$$\mathcal{L} - \mathcal{L}^{(0)} : [C^{k,\kappa'}(S)]^6 \longrightarrow [C^{k-1,\kappa'}(S)]^6, \ \ m \geq 2, \ \ k \geq 1, \tag{4.47}$$

are compact for arbitrary $\tau$ due to Lemma 3.3.

**Theorem 4.6.** *Let $\partial\Omega^+ = S$ be a Lipschitz surface. Then there is a positive constant $c$ such that for all $h \in [H_2^{-\frac{1}{2}}(S)]^6$ there holds the inequality*

$$\mathrm{Re}\langle -\mathcal{H}^{(0)}h, h\rangle_S \geq c\|h\|^2_{[H_2^{-\frac{1}{2}}(S)]^6}, \tag{4.48}$$

*where $\langle \cdot\,, \cdot\rangle_S$ denotes the duality between the spaces $[H_2^{\frac{1}{2}}(S)]^6$ and $[H_2^{-\frac{1}{2}}(S)]^6$.*

*Proof.* Note that, the single layer potential $V^{(0)}(h)$ with $h \in [H_2^{-\frac{1}{2}}(S)]^6$ belongs to the space $[W^1_{2,loc}(\mathbb{R}^3)]^6$, solves the homogeneous equation $A^{(0)}(\partial_x)V^{(0)}(h) = 0$ in $\Omega^\pm$ and possesses the following asymptotic property at infinity: $\partial^\alpha V^{(0)}(h)(x) = \mathcal{O}(|x|^{-1-|\alpha|})$ as $|x| \to \infty$ for arbitrary multi-index $\alpha = (\alpha_1, \alpha_2, \alpha_3)$. Therefore in view of formulas (2.378) and (2.382) we easily derive the following Green's identities

$$\int_{\Omega^+} \mathcal{E}^{(0)}(U,\overline{U})\,dx = \big\langle \{U\}^+, \{\mathcal{T}^{(0)}U\}^+\big\rangle_{\partial\Omega^+}, \quad \int_{\Omega^-} \mathcal{E}^{(0)}(U,\overline{U})\,dx = -\big\langle \{U\}^-, \{\mathcal{T}^{(0)}U\}^-\big\rangle_{\partial\Omega^-}, \tag{4.49}$$

with $U = (u, \varphi, \psi, \vartheta)^\top = V^{(0)}(h)$ and

$$\begin{aligned}\mathcal{E}^{(0)}(U,\overline{U}) = c_{rjkl}\partial_l u_k \overline{\partial_j u_r} + e_{lrj}(\partial_l\varphi\overline{\partial_j u_r} - \partial_j u_r\overline{\partial_l\varphi}) + q_{lrj}(\partial_l\psi\overline{\partial_j u_r} - \partial_j u_r\overline{\partial_l\psi}) \\ + \varkappa_{jl}\partial_l\varphi\overline{\partial_j\varphi} + a_{jl}(\partial_l\varphi\overline{\partial_j\psi} + \partial_j\psi\overline{\partial_l\varphi}) + \mu_{jl}\partial_l\psi\overline{\partial_j\psi} + \eta_{jl}\partial_l\vartheta\overline{\partial_j\vartheta}.\end{aligned} \tag{4.50}$$

Applying the properties of the single layer potential treated in Theorem 4.1, from (4.49) we get

$$\int_{\Omega^+\cup\Omega^-} \mathcal{E}^{(0)}(U,\overline{U})\,dx = \langle -\mathcal{H}^{(0)}h, h\rangle_S. \tag{4.51}$$

With the help of inequalities (2.10) and (2.15) we derive from (4.51)

$$\mathrm{Re}\langle -\mathcal{H}^{(0)}h, h\rangle_S = \mathrm{Re}\int_{\Omega^+\cup\Omega^-} \mathcal{E}^{(0)}(U,\overline{U})\,dx \geq c_1\int_{\Omega^+\cup\Omega^-} \big\{\varepsilon_{kj}\varepsilon_{kj} + |\nabla\varphi|^2 + |\nabla\psi|^2 + |\nabla\vartheta|^2\big\}\,dx, \tag{4.52}$$

where $\varepsilon_{kj} = 2^{-1}(\partial_k u_j + \partial_j u_k)$ and $c_1$ is a positive constant independent of $h$. Now, using the Korn's inequality for $\mathbb{R}^3$ (see [56]) we have

$$\mathrm{Re}\langle -\mathcal{H}^{(0)}h, h\rangle_S \geq c_2\bigg\{\sum_{k,j=1}^{3}\|\partial_j u_k\|^2_{L_2(\mathbb{R}^3)} + \|\nabla\varphi\|^2_{L_2(\mathbb{R}^3)} + \|\nabla\psi\|^2_{L_2(\mathbb{R}^3)} + \|\nabla\vartheta\|^2_{L_2(\mathbb{R}^3)}\bigg\}. \tag{4.53}$$

Due to the properties of the single layer potential $V^{(0)}(h)$ it follows that

$$V^{(0)}(h) \in BL(\mathbb{R}^3) := \Big\{U \in [W^1_{2,loc}(\mathbb{R}^3)]^6 : \ (1+|x|^2)^{-1/2}U_k \in L_2(\mathbb{R}^3), \ \nabla U_k \in [L_2(\mathbb{R}^3)]^3\Big\}, \tag{4.54}$$

where $BL(\mathbb{R}^3)$ denotes the Beppo-Levy type space (for details see [27, Ch. XI]). It is well known that the norm in this space defined by

$$\|U\|^2_{BL(\mathbb{R}^3)} := \big\|(1+|x|^2)^{-1/2}U\big\|^2_{[L_2(\mathbb{R}^3)]^6} + \sum_{k=1}^{6}\sum_{j=1}^{3}\|\partial_j U_k\|^2_{L_2(\mathbb{R}^3)} \tag{4.55}$$

is equivalent to the seminorm

$$\|U\|^2_{*BL(\mathbb{R}^3)} := \sum_{k=1}^{6}\sum_{j=1}^{3} \|\partial_j U_k\|^2_{L_2(\mathbb{R}^3)}. \tag{4.56}$$

Therefore, from (4.53) it follows that

$$\operatorname{Re}\langle -\mathcal{H}^{(0)}h, h\rangle_S \geq c_3 \|V^{(0)}(h)\|^2_{BL(\mathbb{R}^3)}. \tag{4.57}$$

v Since $A^{(0)}(\partial)V^{(0)}(h) = 0$ in $\Omega^+ \cup \Omega^-$ and $V^{(0)}(h) \in [W^1_{2,loc}(\mathbb{R}^3)]^6$, the generalized boundary functionals $[\mathcal{T}^{(0)}V^{(0)}(h)]^{\pm} \in [H_2^{-\frac{1}{2}}(S)]^6$ are defined correctly and the norms $\|[\mathcal{T}^{(0)}V^{(0)}(h)]^{\pm}\|_{[H_2^{-\frac{1}{2}}(S)]^6}$ can be controlled by the norm $\|V^{(0)}(h)\|_{BL(\mathbb{R}^3)}$ (see (2.5)). Consequently, there is a positive constant $c_5$ such that

$$\big\|\big\{\mathcal{T}^{(0)}V^{(0)}(h)\big\}^{\pm}\big\|_{[H_2^{-\frac{1}{2}}(S)]^6} \leq c_5 \|V^{(0)}(h)\|_{BL(\mathbb{R}^3)}. \tag{4.58}$$

Whence the inequality

$$\|h\|_{[H_2^{-\frac{1}{2}}(S)]^6} = \big\|\big\{\mathcal{T}^{(0)}V^{(0)}(h)\big\}^{-} - \big\{\mathcal{T}^{(0)}V^{(0)}(h)\big\}^{+}\big\|_{[H_2^{-\frac{1}{2}}(S)]^6} \leq c_6 \|V^{(0)}(h)\|_{BL(\mathbb{R}^3)} \tag{4.59}$$

follows immediately which along with (4.57) completes the proof. □

**Corollary 4.7.** *Let $\partial\Omega^+ = S$ be a Lipschitz surface. Then the operator*

$$\mathcal{H}^{(0)} : [H_2^{-\frac{1}{2}}(S)]^6 \longrightarrow [H_2^{\frac{1}{2}}(S)]^6$$

*is invertible.*

*Proof.* It follows from Theorem 4.6 and the Lax-Milgram theorem. □

**Corollary 4.8.** *Let $\partial\Omega^+ = S$ be a Lipschitz surface and either $\tau = \sigma + i\omega$ with $\sigma > 0$ and $\omega \in \mathbb{R}$ or $\tau = 0$. Then there is a positive constant $c_1$ such that for all $h \in [H_2^{-\frac{1}{2}}(S)]^6$ there holds the inequality*

$$\operatorname{Re}\big\langle(-\mathcal{H} + \mathcal{C})h, h\big\rangle_S \geq c_1 \|h\|^2_{[H_2^{-\frac{1}{2}}(S)]^6}, \tag{4.60}$$

*where $\mathcal{C} : [H_2^{-\frac{1}{2}}(S)]^6 \to [H_2^{\frac{1}{2}}(S)]^6$ is a compact operator. The operator*

$$\mathcal{H} : [H_2^{-\frac{1}{2}}(S)]^6 \longrightarrow [H_2^{\frac{1}{2}}(S)]^6 \tag{4.61}$$

*is invertible.*

*Proof.* The first part of the corollary follows from Theorem 4.6 and from the fact that the operator (4.40) is compact. In turn, (4.60) implies that operator (4.61) is Fredholm with zero index. On the other hand, from the uniqueness Theorem 2.25 for the Dirichlet BVP, we conclude that the null space of the operator (4.61) is trivial and, consequently, (4.61) is invertible. □

**Corollary 4.9.** *Let $\partial\Omega^{\pm} = S$ be a Lipschitz surface and either $\tau = \sigma + i\omega$ with $\sigma > 0$ and $\omega \in \mathbb{R}$ or $\tau = 0$. Further, let either $U \in [H^1_2(\Omega^+)]^6$ or $U \in [H^1_{2,loc}(\Omega^-)]^6$ be a solution to the homogeneous equation $A(\partial_x, \tau)U = 0$ in $\Omega^{\pm}$ possessing the property $\mathbf{Z}_\tau(\Omega^-)$ or $\mathbf{Z}(\Omega^-)$, respectively, in the case of exterior domain $\Omega^-$. Then $U$ is uniquely representable in the form*

$$U(x) = V\big(\mathcal{H}^{-1}\{U\}^{\pm}\big)(x), \ \ x \in \Omega^{\pm}, \tag{4.62}$$

*where $\{U\}^{\pm}$ are the interior and exterior traces of $U$ on $S$ from $\Omega^{\pm}$, respectively, and $\mathcal{H}^{-1}$ is the operator inverse to* (4.61)*.*

*Proof.* It follows from Corollary 4.8 and the uniqueness Theorems 2.25 and 2.26. □

*Remark* 4.10. If $S$ is a sufficiently smooth surface ($C^\infty$ regular surface say), then for arbitrary $\tau \in \mathbb{C}$, the operator $\mathcal{H}$ is a pseudodifferential operator of order $-1$ with the principal homogeneous symbol matrix $\mathfrak{S}(\mathcal{H}; x, \xi_1, \xi_2) = [\mathfrak{S}_{kj}(\mathcal{H}; x, \xi_1, \xi_2)]_{6\times 6}$ given by the following relation (see Subsection 3.1 and Appendix C)

$$\mathfrak{S}(\mathcal{H}; x, \xi_1, \xi_2) = \mathfrak{S}\big(\mathcal{H}^{(0)}; x, \xi_1, \xi_2\big) = H(x, \xi_1, \xi_2) = [H_{kj}(x, \xi_1, \xi_2)]_{6\times 6}$$

$$:= \begin{bmatrix} [H_{kj}(x,\xi_1,\xi_2)]_{5\times 5} & [0]_{5\times 1} \\ [0]_{1\times 5} & H_{66}(x,\xi_1,\xi_2) \end{bmatrix}_{6\times 6} =$$

$$= -\frac{1}{2\pi}\int_{-\infty}^{\infty} [A^{(0)}(B_n(x)\xi)]^{-1}\, d\xi_3 = -\frac{1}{2\pi}\int_{\ell^{\pm}} [A^{(0)}(B_n(x)\xi)]^{-1}\, d\xi_3, \tag{4.63}$$

$$B_n(x) = \begin{bmatrix} l_1(x) & m_1(x) & n_1(x) \\ l_2(x) & m_2(x) & n_2(x) \\ l_3(x) & m_3(x) & n_3(x) \end{bmatrix}, \quad x\in\partial\Omega,\ \ \xi=(\xi_1,\xi_2,\xi_3),\ \ \xi'=(\xi_1,\xi_2)\in\mathbb{R}^2\setminus\{0\}, \tag{4.64}$$

where $B_n(x)$ is an orthogonal matrix with $\det B_n(x) = 1$, $n(x) = (n_1(x), n_2(x), n_3(x))$ is the outward unit normal vectors to $S$, while $l(x) = (l_1(x), l_2(x), l_3(x))$ and $m(x) = (m_1(x), m_2(x), m_3(x))$ are orthogonal unit vectors in the tangential plane associated with some local chart at the point $x \in S$; here $\ell^+$ (respectively, $\ell^-$) is a closed contour in the upper (respectively, lower) complex half-plane $\operatorname{Re}\xi_3 > 0$ (respectively, $\operatorname{Re}\xi_3 < 0$), orientated counterclockwise (respectively, clockwise) and enclosing all the roots with positive (respectively, negative) imaginary parts of the equation $\det A^{(0)}(B_n\xi) = 0$ with respect to $\xi_3$; $\xi_1$ and $\xi_2$ are to be considered as real parameters.

From the representation (4.63) it follows that the entries of the principal homogeneous symbol matrix $\mathfrak{S}(\mathcal{H}; x, \xi_1, \xi_2)$ are odd, real valued and homogeneous of order $-1$ functions in $\xi' = (\xi_1, \xi_2) \in \mathbb{R}^2 \setminus \{0\}$,

$$\begin{gathered} \operatorname{Im}\mathfrak{S}_{kj}(\mathcal{H}; x,\xi_1,\xi_2) = 0, \quad \mathfrak{S}_{kj}(\mathcal{H}; x,-\xi_1,-\xi_2) = \mathfrak{S}_{kj}(\mathcal{H}; x,\xi_1,\xi_2), \\ \mathfrak{S}_{kj}(\mathcal{H}; x,t\xi_1,t\xi_2) = t^{-1}\mathfrak{S}_{kj}(\mathcal{H}; x,\xi_1,\xi_2) \ \text{ for all }\ t>0,\ \ k,j=1,\dots,6. \end{gathered} \tag{4.65}$$

In accordance with (3.5) we have

$$\mathfrak{S}_{6j}(\mathcal{H}; x,\xi_1,\xi_2) = \mathfrak{S}_{j6}(\mathcal{H}; x,\xi_1,\xi_2) = 0, \ \ j=1,\dots,5, \tag{4.66}$$

Moreover, with the help of the relations (2.48) and (4.63) one can show that the principal homogeneous symbol matrix $-\mathfrak{S}(\mathcal{H}; x, \xi_1, \xi_2)$ is strongly elliptic, i.e., there is a positive constant $C$ such that for all $\xi' = (\xi_1, \xi_2) \in \mathbb{R}^2 \setminus \{0\}$ and for all $\zeta \in \mathbb{C}^6$

$$\operatorname{Re}\big[-\mathfrak{S}(\mathcal{H}; x,\xi_1,\xi_2)\zeta\cdot\zeta\big] \geq C|\xi|^{-1}|\zeta|^2. \tag{4.67}$$

In addition, from the structure of the matrix $A^{(0)}(B_n(x)\xi)$ defined in (2.47) it follows that

$$\big\{[A^{(0)}(B_n(x)\xi)]^{-1}\big\}_{66} > 0 \ \text{ for }\ \xi\neq 0,$$

which in view of (4.63) implies that

$$-\mathfrak{S}_{66}(\mathcal{H}; x,\xi_1,\xi_2) = -\mathfrak{S}_{66}(\mathcal{H}; x,-\xi_1,-\xi_2) > 0 \ \text{ for }\ \xi\neq 0. \tag{4.68}$$

4.3. **Steklov–Poincaré type operators.** Now, we introduce the so called Steklov–Poincaré type operators $\mathcal{A}^{\pm}$ which map Dirichlet data to the corresponding Neumann data,

$$\mathcal{A}^+\{U\}^+ = \{\mathcal{T}U\}^+ \ \text{ and }\ \mathcal{A}^-\{U\}^- = \{\mathcal{T}U\}^- \ \text{ on }\ S, \tag{4.69}$$

where $U$ is a solution of the homogeneous equation $A(\partial,\tau)U = 0$ in $\Omega^{\pm}$ from the space $[W^1_p(\Omega^+)]^6$ or $[W^1_{p,loc}(\Omega^-)]^6 \cap \mathbf{Z}_\tau(\Omega^-)$, respectively.

From (4.62) and (4.3) it is evident that

$$\mathcal{A}^{\pm} := (\mp 2^{-1} I_6 + \mathcal{K})\mathcal{H}^{-1}. \tag{4.70}$$

**Lemma 4.11.** *Let $\partial\Omega^+ = S$ be a Lipschitz surface and $\tau = \sigma + i\omega$ with $\sigma > 0$ and $\omega \in \mathbb{R}$. Then there is a positive constant $C_1$ such that for all $h \in [H_2^{\frac12}(S)]^6$ there holds the inequality*

$$\operatorname{Re}\big\langle(\pm\mathcal{A}^{\pm} + \mathcal{C}_0)h, h\big\rangle_S \geq C_1\|h\|^2_{[H_2^{\frac12}(S)]^6}, \tag{4.71}$$

*where $\mathcal{C}_0 : [H_2^{\frac12}(S)]^6 \to [H_2^{-\frac12}(S)]^6$ is a compact operator. The operator*

$$\mathcal{A}^- : [H_2^{\frac12}(S)]^6 \longrightarrow [H_2^{-\frac12}(S)]^6, \tag{4.72}$$

*is invertible, while*

$$\mathcal{A}^+ : [H_2^{\frac12}(S)]^6 \longrightarrow [H_2^{-\frac12}(S)]^6 \tag{4.73}$$

*is a Fredholm operator of index zero and with the null space of dimension two.*

*Proof.* Mapping properties (4.72) and (4.73) follow from Theorem 4.3 and Corollary 4.8. With the help of Green's identities for the vector function $U = V(\mathcal{H}^{-1}h)$ with $h \in [H_2^{\frac{1}{2}}(S)]^6$ we get (see (2.198), (2.201), (2.211))

$$\langle \mathcal{A}^+ h, h\rangle_{\partial\Omega^+} = \int_{\Omega^+} \mathcal{E}_\tau(U,\overline{U})\,dx, \quad -\langle \mathcal{A}^- h, h\rangle_{\partial\Omega^-} = \int_{\Omega^-} \mathcal{E}_\tau(U,\overline{U})\,dx. \tag{4.74}$$

Applying the same reasoning as in the proofs of Theorem 4.6 and Corollary 4.8 we arrive at the inequalities (4.71) which in turn imply that the operators (4.72) and (4.73) are Fredholm and have zero index.

The null space of the operator (4.72) is trivial. Indeed, the homogeneous equation $\mathcal{A}^- h = 0$ corresponds to the exterior homogeneous Neumann type problem for the vector function $U = V(\mathcal{H}^{-1}h)$. Therefore by Theorem 2.26 we get $U = V(\mathcal{H}^{-1}h) = 0$ in $\Omega^-$. Hence $h = 0$ follows. Thus the operator (4.72) is invertible.

Now, we show that the operator (4.73) has two-dimensional null space. Set

$$\Psi = b_1\Psi^{(1)} + b_2\Psi^{(2)}, \tag{4.75}$$

where $b_1$ and $b_2$ are arbitrary constants and

$$\Psi^{(1)} = (0,0,0,1,0,0)^\top, \quad \Psi^{(2)} = (0,0,0,0,1,0)^\top. \tag{4.76}$$

Consider the vector $U^{(\mathcal{N})} := V(\mathcal{H}^{-1}\Psi)$ in $\Omega^+$. Since $[U^{(\mathcal{N})}]^+ = [V(\mathcal{H}^{-1}\Psi)]^+ = \Psi$ on $\partial\Omega^+$ and the interior Dirichlet problem possesses a unique solution we conclude that

$$U^{(\mathcal{N})} = V\big(\mathcal{H}^{-1}\Psi\big) = (0,0,0,b_1,b_2,0)^\top \ \text{ in } \ \Omega^+.$$

Therefore, $[\mathcal{T}(\partial_x,n)U^{(\mathcal{N})}]^+ \equiv \mathcal{A}^+\Psi = 0$ on $S$; hence it follows that $\dim\ker\mathcal{A}^+ \geq 2$, since $\Psi^{(1)}$ and $\Psi^{(2)}$ are linearly independent.

On the other hand, if $\mathcal{A}^+\psi = 0$ on $S$, then $[\mathcal{T}(\partial_x,n)V(\mathcal{H}^{-1}\psi)]^+ = 0$ on $S$, and by Theorem 2.25 we have $V(\mathcal{H}^{-1}\psi) = (0,0,0,b_1',b_2',0)^\top$ in $\Omega^+$, where $b_1'$ and $b_2'$ are arbitrary constants. Consequently,

$$\big[V(\mathcal{H}^{-1}\psi)\big]^+ = \psi = (0,0,0,b_1',b_2',0)^\top = b_1'\Psi^{(1)} + b_2'\Psi^{(2)} \ \text{ on } \ \partial\Omega^+ \tag{4.77}$$

and $\dim\ker\mathcal{A}^+ \leq 2$. Therefore $\dim\ker\mathcal{A}^+ = 2$. Moreover, from (4.77) it follows that the null space $\ker\mathcal{A}^+$ is the linear span of the vectors (4.76). □

*Remark* 4.12. If $S$ is a sufficiently smooth surface ($C^\infty$ regular surface say), then for arbitrary $\tau \in \mathbb{C}$, the operators $\pm\mathcal{A}^\pm$ are strongly elliptic pseudodifferential operators of order 1, i.e., there is a positive constant $C$ such that for all $\xi' = (\xi_1,\xi_2) \in \mathbb{R}^2 \setminus \{0\}$ and $x \in S$

$$\operatorname{Re}\big\{\mathfrak{S}(\pm\mathcal{A}^\pm;x,\xi_1,\xi_2)\zeta\cdot\zeta\big\} \geq C|\xi|\,|\zeta|^2, \tag{4.78}$$

which follow from equalities (4.74). Here $\mathfrak{S}\big(\pm\mathcal{A}^\pm;x,\xi_1,\xi_2)$ stand for the principal homogeneous symbol matrices of the operators $\pm\mathcal{A}^\pm$.

With the help of the strong elipticity property of the principal homogeneous symbol matrices $\mathfrak{S}\big(\pm\mathcal{H};x,\xi_1,\xi_2)$ and $\mathfrak{S}(\pm\mathcal{A}^\pm;x,\xi_1,\xi_2)$, by Corollary 4.8 and Lemma 4.11, and applying the general theory of pseudodifferential operators on manifolds without boundary (cf. [1, Ch. 3]), we infer that the operators

$$\mathcal{H} : [B^s_{p,q}(S)]^6 \longrightarrow [B^{s+1}_{p,q}(S)]^6, \ \ s \in \mathbb{R}, \ \ p > 1, \ \ q \geq 1, \tag{4.79}$$

$$\mathcal{A}^- : [B^{s+1}_{p,q}(S)]^6 \longrightarrow [B^{s}_{p,q}(S)]^6, \ \ s \in \mathbb{R}, \ \ p > 1, \ \ q \geq 1, \tag{4.80}$$

are invertible, while the operator

$$\mathcal{A}^+ : [B^{s+1}_{p,q}(S)]^6 \longrightarrow [B^{s}_{p,q}(S)]^6, \ \ s \in \mathbb{R}, \ \ p > 1, \ \ q \geq 1, \tag{4.81}$$

is Fredholm of zero index and the corresponding two-dimensional null space $\ker\mathcal{A}^+$ is a linear span of the vectors (4.76) as it is shown in the proof of Lemma 4.11.

*Remark* 4.13. The relations (4.78), (4.70), and (4.67) imply that the principal homogeneous symbol matrices of the operators $\pm 2^{-1} I_6 + \mathcal{K}$ are non-degenerated,

$$\det \mathfrak{S}(\pm 2^{-1} I_6 + \mathcal{K}; x, \xi_1, \xi_2) \neq 0, \quad x \in S, \quad \xi' = (\xi_1, \xi_2) \in \mathbb{R}^2 \setminus \{0\}. \tag{4.82}$$

By (4.82) and (4.67), in view of the second equality in (4.30), then we find that the principal homogeneous symbol matrix of the operator $\mathcal{L}$ is also non-degenerated,

$$\det \mathfrak{S}(\mathcal{L}; x, \xi_1, \xi_2) \neq 0, \quad x \in S, \quad \xi' = (\xi_1, \xi_2) \in \mathbb{R}^2 \setminus \{0\}. \tag{4.83}$$

Finally, (4.83), (4.67), and the first equality in (4.30) yield that the principal homogeneous symbol matrices of the operators $\pm 2^{-1} I_6 + \mathcal{N}$ are non-degenerated,

$$\det \mathfrak{S}(\pm 2^{-1} I_6 + \mathcal{N}; x, \xi_1, \xi_2) \neq 0, \quad x \in S, \quad \xi' = (\xi_1, \xi_2) \in \mathbb{R}^2 \setminus \{0\}. \tag{4.84}$$

In Subsection 5.5 we will show that the symbol matrix $\mathfrak{S}(\mathcal{L}; x, \xi_1, \xi_2)$ is strongly elliptic.

## 5. Investigation of Basic BVPs of Pseudo-Oscillations

Throughout this section we assume that $\operatorname{Re} \tau = \sigma > 0$ and investigate the Dirichlet $(\mathrm{D})^{\pm}_{\tau}$, Neumann $(\mathrm{N})^{\pm}_{\tau}$, and mixed $(\mathrm{M})^{+}_{\tau}$ boundary value problems for the pseudo-oscillation equation (2.155). Note that, with the help of the Newtonian volume potential $N_{\Omega^{\pm}}(\Phi)$ (see (3.61) and Remark 4.5 we can reduce the nonhomogeneous equation (2.155) to the homogeneous one. Therefore without loss of generality in what follows we consider the homogeneous differential equation (2.155) with $\Phi = 0$.

### 5.1. **The interior Dirichlet BVP: a regular case.** We assume that

$$S = \partial\Omega^{\pm} \in C^{m,\kappa} \ \text{ with integer } \ m \geq 2 \ \text{ and } \ 0 < \kappa \leq 1, \tag{5.1}$$

$$f \in [C^{k,\kappa'}(S)]^6, \quad 0 < \kappa' < \kappa, \quad 1 \leq k \leq m-1, \tag{5.2}$$

and look for a solution to the interior Dirichlet problem $(\mathrm{D})^{+}_{\tau}$ (see Subsection 2.3),

$$A(\partial, \tau) U = 0 \ \text{ in } \ \Omega^{+}, \tag{5.3}$$

$$\{U\}^{+} = f \ \text{ on } \ S, \tag{5.4}$$

in the form of double layer potential

$$U(x) = W(h)(x), \quad x \in \Omega^{+}, \tag{5.5}$$

where $h \in [C^{k,\kappa'}(S)]^6$ is an unknown density vector. By Theorem 4.1 and in view of the boundary condition (5.4) we get the following integral equation for the density vector function $h$

$$(2^{-1} I_6 + \mathcal{N}) h = f \ \text{ on } \ S, \tag{5.6}$$

where the singular integral operator $\mathcal{N}$ is defined by (4.8). Our goal is to prove that this integral equation is unconditionally solvable for an arbitrary right hand side vector function. To this end, we prove the following assertion.

**Theorem 5.1.** *Let conditions* (5.1) *and* (5.2) *be fulfilled. Then the singular integral operator*

$$2^{-1} I_6 + \mathcal{N} : [C^{k,\kappa'}(S)]^6 \longrightarrow [C^{k,\kappa'}(S)]^6, \quad 0 < \kappa' < \kappa, \tag{5.7}$$

*is invertible.*

*Proof.* The mapping property (5.7) follows from Theorem 4.3. With the help of the strong ellipticity property of the differential operator $A(\partial, \tau)$, by standard arguments (see, e.g., [91], [88], [53]) we can show that $2^{-1} I_6 + \mathcal{N}$ is a singular integral operator with elliptic principal homogeneous symbol matrix $\mathfrak{S}(2^{-1} I_6 + \mathcal{N}; x, \xi_1, \xi_2)$, i.e., $\det \mathfrak{S}(2^{-1} I_6 + \mathcal{N}; x, \xi_1, \xi_2) \neq 0$ for all $x \in S$ and $\xi' = (\xi_1, \xi_2) \in \mathbb{R}^2 \setminus \{0\}$ (see Remark 4.13). Therefore, (5.6) is normally solvable ([77], [57]).

Next, we show that the index of the operator

$$2^{-1} I_6 + \mathcal{N} : [L_2(S)]^6 \longrightarrow [L_2(S)]^6 \tag{5.8}$$

equals to zero. First we establish that the operator (5.8) is injective. Let $h_0 \in [L_2(S)]^6$ be a solution of the homogeneous equation $[2^{-1} I_6 + \mathcal{N}] h_0 = 0$ on $S$. Then by the embedding theorems (see, e.g., [57, Ch. IV]) we conclude that $h_0 \in [C^{k,\kappa'}(S)]^6$ and, consequently, the double layer potential $U_0(x) := W(h_0)(x)$ is a regular vector function of the class $[C^1(\overline{\Omega^{\pm}})]^6 \cap [C^2(\Omega^{\pm})]^6$ due to Theorem

4.1 and solves the homogeneous interior Dirichlet problem. By the uniqueness Theorem 2.25 then $U_0$ vanishes in $\Omega^+$ and in accordance with Theorem 4.1 we have

$$\big[\mathcal{T}(\partial_x, n(x), \tau)W(h_0)(x)\big]^- = \big[\mathcal{T}(\partial_x, n(x), \tau)W(h_0)(x)\big]^+ = 0, \ \ x \in S.$$

Thus $U_0$ solves the exterior homogeneous Neumann type problem in the domain $\Omega^-$ and possesses the decay property (2.207) at infinity. Therefore, by the uniqueness Theorem 2.26, it follows that $U_0$ vanishes in the exterior domain $\Omega^-$ as well. But then in view of the jump relation (4.4) we finally conclude $\{W(h_0)(x)\}^+ - \{W(h_0)(x)\}^- = h_0(x) = 0$ on $S$, whence the injectivity of the operator (5.7) follows.

Further, we show that the null space of the adjoint operator

$$2^{-1}I_6 + \mathcal{N}^* : [L_2(S)]^6 \longrightarrow [L_2(S)]^6 \tag{5.9}$$

is trivial as well. Indeed, let $h_0^* \in [L_2(S)]^6$ be a solution of the equation $[2^{-1}I_6 + \mathcal{N}^*]h_0^* = 0$ on $S$. Then, using again the embedding theorems we conclude that $h_0^* \in [C^{k,\kappa'}(S)]^6$. Evidently, the adjoint single layer potential (see (4.14)) $U_0^*(x) := V^*(h_0^*)(x)$ is a regular vector function of the class $[C^1(\overline{\Omega^\pm})]^6 \cap [C^2(\Omega^\pm)]^6$ due to the mapping property (4.17) and satisfies the decay conditions (2.207) at infinity (see Remark 4.5). Moreover, $U_0^*$ solves the exterior homogeneous Neumann type BVP for the adjoint differential equation (see Subsection 2.8 and formulas (2.49), (2.58), and (4.19))

$$\begin{gathered} A^*(\partial, \tau)U_0^* = 0 \ \text{ in } \ \Omega^-, \\ \big\{\mathcal{P}(\partial, n, \tau)U_0^*\big\}^- = [2^{-1}I_6 + \mathcal{N}^*]h_0^* = 0 \ \text{ on } \ S. \end{gathered} \tag{5.10}$$

Due to the uniqueness Theorem 2.30 we then have $U_0^*(x) = V^*(h_0^*)(x) = 0$, $x \in \Omega^-$. Now, by (4.18) we see that $U_0^* = V^*(h_0^*)$ is a solution to the interior homogeneous Dirichlet type auxiliary BVP for the adjoint differential equation,

$$\begin{gathered} A^*(\partial, \tau)U_0^* = 0 \ \text{ in } \ \Omega^+, \\ \{U_0^*\}^+ = 0 \ \text{ on } \ S. \end{gathered} \tag{5.11}$$

Again by Theorem 2.30 we find $U_0^*(x) = V^*(h_0^*) = 0$ in $\Omega^+$ and, consequently, in view of the jump relations (4.19), we finally get $h_0^* = 0$ on $S$. Thus the null spaces of the operators (5.8) and (5.9) are trivial and the index of the operator (5.8) equals to zero. Therefore, the operator (5.8) is invertible, which implies that the operator (5.7) is invertible as well. □

From the invertibility of the operator (5.7) the following existence result follows immediately.

**Theorem 5.2.** *Let $S$, $m$, $\kappa$, $\kappa'$ and $k$ be as in Theorem* 5.1. *Then the Dirichlet interior problem* (5.3), (5.4) *with $f \in [C^{k,\kappa'}(S)]^6$ is uniquely solvable in the class $[C^{k,\kappa'}(\overline{\Omega^+})]^6 \cap [C^\infty(\Omega^+)]^6$ and the solution is representable in the form of double layer potential* (5.5), *where the density vector $h$ is defined by the singular integral equation* (5.6).

5.2. **The exterior Dirichlet BVPs: a regular case.** We again assume that the conditions (5.1) and (5.2) are fulfilled and consider the exterior Dirichlet BVP:

$$A(\partial, \tau)U(x) = 0 \ \text{ in } \ \Omega^-, \tag{5.12}$$

$$\{U(x)\}^- = f(x) \ \text{ on } \ S = \partial\Omega^-. \tag{5.13}$$

We require that $U \in [C^{1,\kappa'}(\overline{\Omega^-})]^6$ and possesses the asymptotic properties (2.207).

We look for a solution to the exterior Dirichlet problem in the form of linear combination of the single and double layer potentials

$$U(x) = W(h)(x) + \alpha V(h)(x), \ \ x \in \Omega^-, \tag{5.14}$$

where $h \in [C^{k,\kappa'}(S)]^6$ is an unknown density vector and $\alpha > 0$ is a constant. In accordance with Remark 4.5 the asymptotic relations (2.207) are automatically satisfied, as well as equation (5.12). By Theorem 4.1 and in view of the boundary condition (5.13) we get the following integral equation for the density vector function $h$

$$(-2^{-1}I_6 + \mathcal{N} + \alpha\mathcal{H})h = f \ \text{ on } \ S, \tag{5.15}$$

where the singular integral operator $\mathcal{N}$ and weakly singular integral operator $\mathcal{H}$ are defined by (4.8) and (4.6), respectively.

**Theorem 5.3.** *Let conditions* (5.1) *and* (5.2) *be fulfilled. Then the singular integral operator*

$$-2^{-1}I_6 + \mathcal{N} + \alpha\mathcal{H} : [C^{k,\kappa'}(S)]^6 \longrightarrow [C^{k,\kappa'}(S)]^6, \ \ 0 < \kappa' < \kappa, \tag{5.16}$$

*is invertible.*

*Proof.* With the help of the strong ellipticity property of the differential operator $A(\partial,\tau)$, as in the previous case, by standard arguments (see, e.g., [91], [88], [53]) we can show that $-2^{-1}I_6+\mathcal{N}+\alpha\mathcal{H}$ is a singular integral operator with elliptic principal homogeneous symbol matrix $\mathfrak{S}\big(-2^{-1}I_6+\mathcal{N};x,\xi_1,\xi_2\big)$, i.e., $\det\mathfrak{S}(-2^{-1}I_6+\mathcal{N};x,\xi_1,\xi_2)\neq 0$ for all $x\in S$ and $\xi'=(\xi_1,\xi_2)\in\mathbb{R}^2\setminus\{0\}$ (see Remark 4.13). Note that, the summand $\alpha\mathcal{H}$ in (5.16) is a compact perturbation of the operator $-2^{-1}I_6+\mathcal{N}$.

Now, we show that the index of the operator

$$-2^{-1}I_6 + \mathcal{N} + \alpha\mathcal{H} : [L_2(S)]^6 \longrightarrow [L_2(S)]^6 \tag{5.17}$$

equals to zero. To this end, let us consider the homogeneous equation on $S$

$$(-2^{-1}I_6 + \mathcal{N} + \alpha\mathcal{H})h = 0. \tag{5.18}$$

By the embedding theorems we conclude that if $h_0\in[L_2(S)]^6$ solves equation (5.18), then $h_0\in[C^{k,\kappa'}(S)]^6$ and, consequently, the vector $U_0=W(h_0)+\alpha V(h_0)\in[C^1(\overline{\Omega^\pm})]^6\cap[C^2(\Omega^\pm)]^6$ is a regular solution of the homogenous equation $A(\partial,\tau)U_0=0$ in $\Omega^\pm$ satisfying the decay conditions (2.207) at infinity. In view of (5.18) we see that $U_0$ solves the exterior Dirichlet BVP and by the uniqueness Theorem 2.26 we have $U_0(x)=0$, $x\in\Omega^-$. Due to the jump relations for the layer potentials (see Theorem 4.1), we then have $\{U_0\}^+=h_0$ and $\{\mathcal{T}(\partial,n,\tau)U_0\}^+=-\alpha h_0$ on $S$, i.e.,

$$\big\{\mathcal{T}(\partial,n,\tau)U_0\big\}^+ + \alpha\{U_0\}^+ = 0 \ \text{ on } \ S. \tag{5.19}$$

With the help of Green's formula (2.198) we get

$$\int_{\Omega^+}\mathcal{E}_\tau(U_0,\overline{U'})\,dx + \alpha\int_S\{U_0\}^+\cdot\{U'\}^+\,dS = 0 \tag{5.20}$$

for arbitrary $U'\in[W_2^1(\Omega^+)]^6$. By the word for word arguments applied in the proof of Theorem 2.25 we conclude that $U_0(x)=0$, $x\in\Omega^+$. Therefore $h_0=0$ on $S$ and the null space of the operator (5.17) is trivial.

Quite similarly we can show that the null space of the adjoint operator

$$-2^{-1}I_6 + \mathcal{N}^* + \alpha\mathcal{H}^* : [L_2(S)]^6 \longrightarrow [L_2(S)]^6 \tag{5.21}$$

with $\mathcal{N}^*$ and $\mathcal{H}^*$ being defined in (4.12) and (4.10), is trivial. Indeed, if $h_0^*\in[L_2(S)]^6$ solves the homogeneous equation

$$(-2^{-1}I_6 + \mathcal{N}^* + \alpha\mathcal{H}^*)h_0^* = 0, \tag{5.22}$$

then $h_0^*\in[C^{k,\kappa'}(S)]^6$ by the embedding theorems and, consequently, the vector $U_0^*=V^*(h_0^*)\in[C^1(\overline{\Omega^\pm})]^6\cap[C^2(\Omega^\pm)]^6$ is a regular solution of the homogenous equation $A^*(\partial,\tau)U_0^*=0$ in $\Omega^\pm$ satisfying the decay conditions (2.207) at infinity. In view of (5.22) we find that $U_0^*$ satisfies the Robin type boundary condition on $S$ (see (4.18), (4.19))

$$\big\{\mathcal{P}(\partial,n,\tau)U_0^*\big\}^+ + \alpha\{U_0^*\}^+ = (-2^{-1}I_6+\mathcal{N}^*+\alpha\mathcal{H}^*)h_0^* = 0 \ \text{ on } \ S. \tag{5.23}$$

By Green's formula (2.199) we have

$$\int_{\Omega^+}\mathcal{E}_\tau(U,\overline{U_0^*})\,dx + \alpha\int_S\{U\}^+\cdot\{U_0^*\}^+\,dS = 0 \tag{5.24}$$

for arbitrary $U\in[W_2^1(\Omega^+)]^6$. By the same arguments as in the proof of the uniqueness Theorem 2.25 we derive that $U_0^*(x)=V^*(h_0^*)(x)=0$, $x\in\Omega^+$. Since the single layer potential is continuous across the surface $S$ (see (4.18)), we see that $U_0^*$ solves the homogeneous exterior Dirichlet type BVP for the operator $A^*(\partial,\tau)$ and satisfies the decay conditions (2.207) at infinity (see Remark (4.5)). Hence, with the help of Green's formula (2.209), it follows that $U_0^*(x)=V^*(h_0^*)(x)=0$, $x\in\Omega^-$. Due to the jump relations for the single layer potential (see (4.19)), we then have $h_0^*=0$ on $S$, i.e., the null space of the adjoint operator (5.21) is trivial.

Thus, the operator (5.17) is injective and has the zero index. Consequently, it is invertible. Then it follows that the operator (5.16) is invertible as well. □

This theorem leads to the following existence result for the exterior Dirichlet problem.

**Theorem 5.4.** *Let conditions* (5.1) *and* (5.2) *be fulfilled. Then the Dirichlet exterior problem* (5.12), (5.13) *with* $f\in[C^{k,\kappa'}(S)]^6$ *is uniquely solvable in the class of regular vector functions* $[C^{k,\kappa'}(\overline{\Omega^-})]^6$ *satisfying the asymptotic decay conditions* (2.207) *and the solution is representable in the form* (5.14), *where the density vector* $h$ *is defined by the uniquely solvable singular integral equation* (5.15).

5.3. **Single layer approach for the interior and exterior Dirichlet BVPs: a regular case.** Applying the results of Subsection 4.2, we can investigate the interior and exterior Dirichlet boundary value problems by means of the single layer potential and derive the corresponding existence results.

Let us look for solutions to the Dirichlet BVPs (5.3), (5.4) and (5.12), (5.13) with $f\in[C^{k,\kappa'}(S)]^6$ in the form of single layer potential

$$U(x)=V(h)(x),\ \ x\in\Omega^{\pm}, \tag{5.25}$$

where $h$ is a solution to the following equation

$$\mathcal{H}h(x)=f(x),\ \ x\in\partial\Omega^{\pm}. \tag{5.26}$$

Due to the results presented in Remark 4.10 and under the conditions (5.1) and (5.2) we see that the operator

$$\mathcal{H}:[C^{k,\kappa'}(S)]^6\longrightarrow[C^{k+1,\kappa'}(S)]^6,\ \ 1\leq k\leq m-1,\ \ m\geq 2, \tag{5.27}$$

is a strongly elliptic pseudodifferential operator of order $-1$ with index zero. Since the null space of the operator (5.27) is trivial we conclude that it is invertible and

$$\mathcal{H}^{-1}:[C^{k+1,\kappa'}(S)]^6\longrightarrow[C^{k,\kappa'}(S)]^6. \tag{5.28}$$

This leads to the following existence results and representation formulas of solutions.

**Theorem 5.5.** *Let conditions* (5.1) *and* (5.2) *be fulfilled. Then the Dirichlet interior and exterior boundary value problems are uniquely solvable in the space of regular vector functions* $[C^{k,\kappa'}(\overline{\Omega^{\pm}})]^6$ *and the solutions are representable in the form* (5.25), *where the density vector* $h$ *is defined by the uniquely solvable pseudodifferential equation* (5.26).

In the regular case under consideration, we have the following counterpart of Corollary 4.9.

**Corollary 5.6.** *Let condition* (5.1) *be fulfilled and* $U\in[C^{1,\kappa'}(\overline{\Omega^{\pm}})]^6$ *be a solution to the homogeneous equation* $A(\partial,\tau)U=0$ *in* $\Omega^{\pm}$, *satisfying the decay conditions* (2.207) *in the case of exterior domain* $\Omega^-$. *Then* $U$ *is uniquely representable in the form*

$$U(x)=V(\mathcal{H}^{-1}[U]^{\pm})(x),\ \ x\in\Omega^{\pm}, \tag{5.29}$$

*where* $[U]^{\pm}$ *are the interior and exterior limiting values (traces) of* $U$ *on* $S$ *from* $\Omega^{\pm}$, *respectively.*

5.4. **The interior and exterior Neumann BVPs: a regular case.** Here we assume that

$$\partial\Omega^{\pm}=S\in C^{m,\kappa},\ \ m\geq 2,\ \ 0<\kappa\leq 1, \tag{5.30}$$

$$F\in[C^{k,\kappa'}(S)]^6,\ \ 0\leq k\leq m-1,\ \ 0<\kappa'<\kappa, \tag{5.31}$$

and look for a solution of the interior Neumann BVP (see (2.155), (2.161)),

$$A(\partial,\tau)U(x)=0\ \text{ in }\ \Omega^{+}, \tag{5.32}$$

$$\big\{\mathcal{T}(\partial,n,\tau)U(x)\big\}^{+}=F(x)\ \text{ on }\ S=\partial\Omega^{+}, \tag{5.33}$$

in the form of single layer potential

$$U(x)=V(h)(x),\ \ x\in\Omega^{+}, \tag{5.34}$$

where $h\in[C^{k,\kappa'}(S)]^6$ is an unknown density vector function. By Theorem 4.1 and in view of the boundary condition (2.161) we get the following integral equation for the density vector $h$

$$(-2^{-1}I_6+\mathcal{K})h=F\ \text{ on }\ S. \tag{5.35}$$

**Theorem 5.7.** *Let* $S$ *and* $F=(F_1,\dots,F_6)^\top$ *satisfy the conditions* (5.30) *and* (5.31).

(i) *The operator*

$$-2^{-1}I_6+\mathcal{K}:[L_2(S)]^6\longrightarrow[L_2(S)]^6 \tag{5.36}$$

*is a singular integral operator of normal type with zero index and has a two-dimensional null space* $\Lambda(S):=\ker(-2^{-1}I_6+\mathcal{K})\subset[C^{m-1,\kappa'}(S)]^6$, *which represents a linear span of the vector functions*

$$h^{(1)},h^{(2)}\in\Lambda(S), \tag{5.37}$$

*such that*

$$V(h^{(1)})=\Psi^{(1)}:=(0,0,0,1,0,0)^\top\ \text{ and }\ V(h^{2)})=\Psi^{(2)}:=(0,0,0,0,1,0)^\top\ \text{ in }\ \Omega^+. \tag{5.38}$$

(ii) *The null space* $\Lambda^*(S)$ *of the operator adjoint to* (5.36),

$$-2^{-1}I_6+\mathcal{K}^*:[L_2(S)]^6\longrightarrow[L_2(S)]^6 \tag{5.39}$$

*is a linear span of the vectors* $\Psi^{(1)}=(0,0,0,1,0,0)^\top$ *and* $\Psi^{(2)}=(0,0,0,0,1,0)^\top$.

(iii) *Equation* (5.35) *is solvable if and only if*

$$\int_S F_4(x)\,dS=\int_S F_5(x)\,dS=0. \tag{5.40}$$

(iv) *If the conditions* (5.40) *are satisfied, then solutions to equation* (5.35) *belong to the space* $[C^{k,\kappa'}(S)]^6$ *and are defined modulo a linear combination of the vector functions* $h^{(1)}$ *and* $h^{(2)}$.

(v) *If the conditions* (5.40) *are satisfied, then the interior Neumann BVP is solvable in the space* $C^{k+1,\kappa'}(\overline{\Omega^+})$ *and the solutions are representable in the form of the single layer potential* (5.34), *where the density vector function* $h$ *is defined by the singular integral equation* (5.35). *A solution to the interior Neumann BVP is defined in* $\Omega^+$ *modulo a linear combination of the constant vector functions* $\Psi^{(1)}$ *and* $\Psi^{(2)}$ *given by* (5.38).

*Proof.* The mapping property (5.36) follows from Theorem 4.3. With the help of the strong ellipticity property of the differential operator $A(\partial,\tau)$, by standard arguments one can show that $-2^{-1}I_6+\mathcal{K}$ is a singular integral operator with elliptic principal homogeneous symbol matrix $\mathfrak{S}(-2^{-1}I_6+\mathcal{K};x,\xi_1,\xi_2)$, i.e., $\det\mathfrak{S}(-2^{-1}I_6+\mathcal{K};x,\xi_1,\xi_2)\neq0$ for all $x\in S$ and $\xi'=(\xi_1,\xi_2)\in\mathbb{R}^2\setminus\{0\}$ (for details see Remark 4.13). Therefore, (5.35) is normally solvable ([77], [57]).

Further, we prove that the index of the operator (5.36) equals to zero. To this end, let us consider the operator

$$-2^{-1}I_6+\mathcal{K}+\alpha\mathcal{H}:[L_2(S)]^6\longrightarrow[L_2(S)]^6 \tag{5.41}$$

with $\alpha>0$. Evidently, the operator (5.41) is a compact perturbation of the operator (5.36) due to Theorem 4.4 since $\mathcal{H}:[L_2(S)]^6\to[H_2^1(S)]^6$ and $[H_2^1(S)]^6$ is compactly embedded in $[L_2(S)]^6$. One can easily show that the homogeneous equation

$$(-2^{-1}I_6+\mathcal{K}+\alpha\mathcal{H})h_0=0\ \text{ on }\ S \tag{5.42}$$

has only the trivial solution in $[L_2(S)]^6$. Indeed, by the embedding theorem we have $h_0\in[C^{1,\kappa'}(S)]^6$ and the vector $U_0=V(h_0)\in[C^{2,\kappa'}(\overline{\Omega^\pm})]^6$ is a regular solution of the homogeneous equation $A(\partial,\tau)U_0=0$ in $\Omega^\pm$ and satisfies the following Robin type condition

$$\{\mathcal{T}U_0\}^++\alpha\{U_0\}^+=0\ \text{ on }\ S.$$

Therefore, by Green's formula (2.198) we derive $U_0(x)=V(h_0)(x)=0$ in $\Omega^+$, and, consequently, $h_0=0$, since $U_0=V(h_0)$ possesses the decay conditions (2.207). Thus $\ker(-2^{-1}I_6+\mathcal{K}+\alpha\mathcal{H})=\{0\}$.

Now, let us consider the adjoint homogeneous equation (see (4.10), (4.11))

$$(-2^{-1}I_6+\mathcal{K}^*+\alpha\mathcal{H}^*)h_0^*=0\ \text{ on }\ S. \tag{5.43}$$

Again by the embedding theorem we have that $h_0^*\in[C^{1,\kappa'}(S)]^6$ and the vector

$$U_0^*=W^*(h_0^*)+\alpha V^*(h_0)\in[C^{1,\kappa'}(\overline{\Omega^+})]^6 \tag{5.44}$$

is a regular solution to the homogeneous equation $A^*(\partial,\tau)U_0^*=0$ in $\Omega^\pm$, satisfying the decay conditions of type (2.207) at infinity and the homogeneous Dirichlet condition $\{U_0^*\}^-=0$ on $S$. Therefore $U_0^*=0$ in $\Omega^-$ by the uniqueness Theorem 2.30. In view of (5.44) and the jump relations for the layer potentials involved in (5.44), then it follows that $\{\mathcal{P}(\partial,n,\tau)U_0^*\}^+ + \alpha\{U_0^*\}^+ = 0$ on $S$ since $\{\mathcal{P}(\partial,n,\tau)U_0^*\}^+ - \{\mathcal{P}(\partial,n,\tau)U_0^*\}^- = -\alpha h_0^*$ and $\{U_0^*\}^+ - \{U_0^*\}^- = h_0^*$. As in the proof of Theorem 5.3 with the help of formula (5.24) we derive that $U_0^* = 0$ in $\Omega^+$ which implies $h_0^* = 0$ on $S$ and, consequently, $\ker(-2^{-1}I_6 + \mathcal{K}^* + \alpha\mathcal{H}^*) = \{0\}$. Thus the index of the operator (5.41) is zero. The same conclusion holds true for the operator (5.36) due to the above mentioned compactness property of the operator $\mathcal{H}$. Note that the operator (5.41) is invertible.

Now, we study the null spaces of the operator (5.36) and its adjoint one

$$-2^{-1}I_6 + \mathcal{K}^* : [L_2(S)]^6 \longrightarrow [L_2(S)]^6. \tag{5.45}$$

Evidently, $\dim\ker(-2^{-1}I_6 + \mathcal{K}) = \dim\ker(-2^{-1}I_6 + \mathcal{K}^*)$.

From the integral representation formula (4.16) it follows that for the vector

$$\Psi = (0,0,0,b_1',b_2',0)^\top = b_1'\Psi^{(1)} + b_2'\Psi^{(2)}, \tag{5.46}$$

where $b_1'$ and $b_2'$ are arbitrary constants and vector functions $\Psi^{(1)}$ and $\Psi^{(2)}$ are given by (5.38), the following formula

$$\Psi = W^*(\Psi) \ \text{ in } \ \Omega^+ \tag{5.47}$$

holds, since $A^*(\partial,\tau)\Psi = 0$ in $\mathbb{R}^3$ and $\mathcal{P}(\partial,n,\tau)\Psi = 0$ for arbitrary $n$ and $x \in \mathbb{R}^3$. From (5.47) we get

$$(-2^{-1}I_6 + \mathcal{K}^*)\Psi = 0 \ \text{ on } \ S. \tag{5.48}$$

Hence $\Psi \in \ker(-2^{-1}I_6 + \mathcal{K}^*)$ which shows that $\dim\ker(-2^{-1}I_6 + \mathcal{K}^*) \geq 2$. On the other hand, it is clear that if $\Phi \in \ker(-2^{-1}I_6 + \mathcal{K}) \equiv \Lambda(S)$, then $(-2^{-1}I_6 + \mathcal{K})\Phi = 0$ on $S$ which is equivalent to the relation $\{\mathcal{T}(\partial,n,\tau)V(\Phi)\}^+ = 0$ on $S$. Therefore $V(\Phi) = (0,0,0,b_1,b_2,0)^\top$ in $\Omega^+$ with arbitrary constants $b_1$ and $b_2$ due to Theorem 2.25, i.e., $V(\Phi) = b_1\Psi^{(1)} + b_2\Psi^{(2)}$, where $\Psi^{(1)}$ and $\Psi^{(2)}$ are given by (5.38). Since the operator (5.27) is invertible it follows that there are vector functions $h^{(1)} \in \Lambda(S)$ and $h^{(2)} \in \Lambda(S)$ such that $\mathcal{H}h^{(j)} = \Psi^{(j)}$ on $S$, $h^{(j)} \in [C^{m-1,\kappa'}(S)]^6$, $j = 1,2$, which in view of the uniqueness theorem for the interior Dirichlet problem lead to the equalities $V(h^{(j)}) = \Psi^{(j)}$ in $\Omega^+$ $j = 1,2$. In turn these formulas imply that

$$V(\Phi) = b_1V(h^{(1)}) + b_2V(h^{(2)}) \ \text{ in } \ \Omega^+, \quad \Phi = b_1\mathcal{H}^{-1}\Psi^{(1)} + b_2\mathcal{H}^{-1}\Psi^{(2)} \ \text{ on } \ S. \tag{5.49}$$

Therefore, $\dim\ker(-2^{-1}I_6 + \mathcal{K}) \leq 2$ since the vector functions $h^{(1)} := \mathcal{H}^{-1}\Psi^{(1)}$ and $h^{(2)} := \mathcal{H}^{-1}\Psi^{(2)}$ are linearly independent. Consequently, we finally get

$$\dim\ker(-2^{-1}I_6 + \mathcal{K}) = \dim\ker(-2^{-1}I_6 + \mathcal{K}^*) = 2,$$

and the vector functions $h^{(1)}$ and $h^{(2)}$ represent the basis of the null space $\Lambda(S)$, while the null space $\ker(-2^{-1}I_6 + \mathcal{K}^*)$ represents a linear span of the vector functions $\Psi^{(1)}$ and $\Psi^{(2)}$. From the above arguments the items (i) and (ii) of the theorem follow.

It is evident that the necessary and sufficient conditions for the integral equation (5.35) to be solvable reads then as (5.40) which proves the item (iii).

The item (iv) follows then from the embedding theorems (see, e.g., [57], Ch. IV), while the item (v) is a direct consequence of items (i)–(iv). □

In the class of vector functions satisfying the asymptotic conditions (2.207) the exterior Neumann BVP

$$A(\partial,\tau)U(x) = 0 \ \text{ in } \ \Omega^-, \tag{5.50}$$

$$\big\{\mathcal{T}(\partial,n,\tau)U(x)\big\}^- = F(x) \ \text{ on } \ S = \partial\Omega^-, \tag{5.51}$$

with $S$ and $F$ as in (5.30) and (5.31), can be studied quite similarly. Indeed, if we look for a solution again in the form of single layer potential

$$U(x) = V(h)(x), \ \ x \in \Omega^-, \tag{5.52}$$

we arrive at the following singular integral equation for the sought for density vector function $h$

$$(2^{-1}I_6 + \mathcal{K})h = F \ \text{ on } \ S. \tag{5.53}$$

The following assertion holds.

**Theorem 5.8.** *Let $S$ and $F = (F_1,\dots,F_6)^\top$ satisfy the conditions* (5.30) *and* (5.31).

(i) *The operators*

$$2^{-1}I_6+\mathcal{K}:[L_2(S)]^6\longrightarrow[L_2(S)]^6, \tag{5.54}$$

$$:[C^{k,\kappa'}(S)]^6\longrightarrow[C^{k,\kappa'}(S)]^6,\ \ 0\le k\le m-1, \tag{5.55}$$

*are singular integral operators of normal type with zero index and have the trivial null spaces.*

(ii) *Operators* (5.54) *and* (5.55) *are invertible, and equation* (5.53) *is uniquely solvable in the space* $[C^{k,\kappa'}(S)]^6$.

(iii) *The exterior Neumann BVP is uniquely solvable and the solution is representable in the form of single layer potential* (5.52)*, where the density vector function $h$ is defined by the singular integral equation* (5.53)*.*

*Proof.* Again, with the help of the strong ellipticity property of the differential operator $A(\partial,\tau)$, by standard arguments one can show that $2^{-1}I_6+\mathcal{K}$ is a singular integral operator of normal type ([77], [57]) with elliptic principal homogeneous symbol matrix $\mathfrak{S}(2^{-1}I_6+\mathcal{K};x,\xi_1,\xi_2)$, i.e., $\det\mathfrak{S}(2^{-1}I_6+\mathcal{K};x,\xi_1,\xi_2)\neq 0$ for all $x\in S$ and $\xi'=(\xi_1,\xi_2)\in\mathbb{R}^2\setminus\{0\}$ (see Remark 4.13).

Now, we show that the operator (5.54) and its adjoint one have trivial null spaces. Let $h_0\in[L_2(S)]^6$ be a solution to the homogeneous equation $[2^{-1}I_6+\mathcal{K}]h_0=0$ on $S$. Then by embedding theorems we conclude that $h_0\in[C^{m-1,\kappa'}(S)]^6$ and, consequently, the single layer potential $U_0(x)=V(h_0)(x)$ is a regular vector function of the class $[C^{m,\kappa'}(\overline{\Omega^-})]^6$ which solves the homogeneous exterior Neumann BVP and belongs to the class $\mathbf{Z}_\tau(\Omega)$. Therefore, $U_0=V(h_0)=0$ in $\Omega^-$ by Theorem 2.26. Due to continuity of the single layer potential we see that the vector function $U_0=V(h_0)$ solves then the homogeneous interior Dirichlet BVP in $\Omega^+$ and by Theorem 2.25 it vanishes identically in $\Omega^+$. In view of the jump formulas (4.3) we arrive at the equation $[\mathcal{T}(\partial,n,\tau)V(h_0)]^- - [\mathcal{T}(\partial,n,\tau)V(h_0)]^+ = h_0=0$ on $S$ implying that $\ker[2^{-1}I_6+\mathcal{K}]$ is trivial.

Now, let $h_0^*\in[L_2(S)]^6$ be a solution to the adjoint homogeneous equation $(2^{-1}I_6+\mathcal{K}^*)h_0^*=0$ on $S$. Then by embedding theorems we conclude that $h_0^*\in[C^{m-1,\kappa'}(S)]^6$ and, consequently, the double layer potential $U_0^*(x)=W^*(h_0^*)(x)$ is a regular vector function of the class $[C^{m-1,\kappa'}(\overline{\Omega^+})]^6$ which solves the homogeneous interior Dirichlet problem (2.400), (2.401) for the adjoint operator $A^*(\partial,\tau)$. Therefore, $U_0^*=W^*(h_0^*)=0$ in $\Omega^+$ by Theorem 2.30. Since $[\mathcal{P}(\partial,n,\tau)W^*(h_0^*)]^-=[\mathcal{P}(\partial,n,\tau)W^*(h_0^*)]^+=0$ on $S$ in view of (4.21), by the same Theorem 2.30 we get $U_0^*=W^*(h_0^*)=0$ in $\Omega^-$. Hence, in accordance with the jump relations (4.20) we finally derive $[W^*(h_0^*)]^+-[W^*(h_0^*)]^-=h_0$ on $S$, implying that $\ker[2^{-1}I_6+\mathcal{K}^*]$ is trivial.

From these results the items (i), (ii), and (iii) follow immediately. □

5.5. **Double layer approach for the interior and exterior Neumann BVPs: a regular case.** Let conditions (5.30) and (5.31) be satisfied with $1\le k\le m-1$ and look for a solution of the interior and exterior Neumann BVPs (5.32), (5.33) and (5.50), (5.51) in the form of double layer potential

$$U(x)=W(h)(x),\ \ x\in\Omega^\pm, \tag{5.56}$$

where $h\in[C^{k,\kappa'}(S)]^6$ is an unknown density vector function. By Theorem 4.1 (see (4.5) and 4.9) and in view of the boundary conditions (5.33) and (5.51) we get the following integral equation for the density vector $h$

$$\mathcal{L}h=F\ \text{ on }\ S. \tag{5.57}$$

The mapping properties of the operator $\mathcal{L}$ is described in Theorems 4.3 and 4.4. Due to the equalities

$$\mathcal{H}\mathcal{L}=-4^{-1}I_6+\mathcal{N}^2,\ \mathcal{L}\mathcal{H}=-4^{-1}I_6+\mathcal{K}^2,\ \mathcal{L}^*\mathcal{H}^*=-4^{-1}I_6+[\mathcal{N}^*]^2,\ \mathcal{H}^*\mathcal{L}^*=-4^{-1}I_6+[\mathcal{K}^*]^2, \tag{5.58}$$

with the help of Corollary 4.8 and Theorems 5.1 and 5.8, we see that

$$\ker\mathcal{L}=\ker(-2^{-1}I_6+\mathcal{N}),\quad \ker\mathcal{L}^*=\ker(-2^{-1}I_6+\mathcal{K}^*). \tag{5.59}$$

Now, we show that the null spaces of the operators $\mathcal{L}$ and $\mathcal{L}^*$ are the same and coincide with the linear span of the vectors $\Psi^{(1)}=(0,0,0,1,0,0)^\top$ and $\Psi^{(2)}=(0,0,0,0,1,0)^\top$ (see (5.38)).

From the integral representation formula (3.62) and Theorem 4.1 it follows that $\Psi^{(1)}$ and $\Psi^{(2)}$ are linearly independent solutions of the homogeneous equation $[-2^{-1}I_6+\mathcal{N}]h=0$ on $S$, since for the vector

$$\Psi=(0,0,0,b_1,b_2,0)^\top=b_1\Psi^{(1)}+b_2\Psi^{(2)}, \tag{5.60}$$

where $b_1$ and $b_2$ are arbitrary constants, we have $A(\partial,\tau)\Psi=0$ in $\mathbb{R}^3$ and $\mathcal{T}(\partial,n,\tau)\Psi=0$ for arbitrary $n$ and $x\in\mathbb{R}^3$. Consequently, in view of (3.62), the following formula

$$\Psi=W(\Psi) \ \text{ in } \ \Omega^+ \tag{5.61}$$

holds which implies $[-2^{-1}I_6+\mathcal{N}]\Psi=0$ on $S$. Hence $\Psi\in\ker(-2^{-1}I_6+\mathcal{N})$ which shows that $\dim\ker(-2^{-1}I_6+\mathcal{N})\geq 2$. On the other hand, it is clear that if $\Phi\in\ker(-2^{-1}I_6+\mathcal{N})$, then $\Phi\in[C^{m-1,\kappa'}(S)]^6$ and $(-2^{-1}I_6+\mathcal{N})\Phi=0$ on $S$ which is equivalent to the relation $\{W(\Phi)\}^-=0$ on $S$. Therefore $W(\Phi)=0$ in $\Omega^-$ due to Theorem 2.26 and $[\mathcal{T}(\partial,n)W(\Phi)]^+=[\mathcal{T}(\partial,n)W(\Phi)]^-=0$ by Theorem 4.1. In accordance with Theorem 2.25 the double layer potential $W(\Phi)$, as a solution to the interior homogeneous Neumann BVP in $\Omega^+$, belongs to the linear span of the vectors $\Psi^{(1)}$ and $\Psi^{(2)}$, i.e., $W(\Phi)=c_1\Psi^{(1)}+c_2\Psi^{(2)}$ in $\Omega^+$ with some constants $c_1$ and $c_2$. By the jump relations we derive $\Phi=[W(\Phi)]^+-[W(\Phi)]^-=[W(\Phi)]^+=c_1\Psi^{(1)}+c_2\Psi^{(2)}$ on $S$. Thus $\ker\mathcal{L}$ represents the linear span of the vectors $\Psi^{(1)}$ and $\Psi^{(2)}$.

By Theorem 5.7 the same holds for the null space of the operator $[-2^{-1}I_6+\mathcal{K}^*]$. Therefore

$$\ker\mathcal{L}=\ker(-2^{-1}I_6+\mathcal{N})=\ker\mathcal{L}^*=\ker(-2^{-1}I_6+\mathcal{K}^*)=\big\{c_1\Psi^{(1)}+c_2\Psi^{(2)},\ c_1,c_2\in\mathbb{C}\big\} \tag{5.62}$$

with $\Psi^{(1)}$ and $\Psi^{(2)}$ defined by (5.38). Consequently, the index of the operator $\mathcal{L}$ equals to zero.

Due to invertibility of the operator (5.27) from (5.58) we have the representation

$$\mathcal{L}=\mathcal{H}^{-1}(-4^{-1}I_6+\mathcal{N}^2). \tag{5.63}$$

Taking into account that the principal homogeneous symbol matrices of the pseudodifferential operators $\pm2^{-1}I_6+\mathcal{N}$, $\pm2^{-1}I_6+\mathcal{K}$ and $\mathcal{H}$ are elliptic (see Remark 4.13), we infer that $\mathcal{L}$ is an elliptic pseudodifferential operator of order $+1$ with the principal homogeneous symbol matrix

$$\mathfrak{S}(\mathcal{L};x,\xi_1,\xi_2)=\big[\mathfrak{S}(\mathcal{H};x,\xi_1,\xi_2)\big]^{-1}\mathfrak{S}(-4^{-1}I_6+\mathcal{N}^2;x,\xi_1,\xi_2) \tag{5.64}$$

for all $x\in S$ and $\xi'=(\xi_1,\xi_2)\in\mathbb{R}^2\setminus\{0\}$.

Note that, the entries of the matrix $\mathfrak{S}\big(\mathcal{H};x,\xi_1,\xi_2)$ are even and homogeneous functions of order $-1$ (see (4.65)), while the entries of the matrix $\mathfrak{S}(\mathcal{N};x,\xi_1,\xi_2)$ are odd functions of zero order since they represent the Fourier transforms of odd singular kernel functions. Therefore, from (5.64) we conclude that $\det\mathfrak{S}(\mathcal{L};x,\xi_1,\xi_2)\neq0$ for all $x\in S$ and $\xi'=(\xi_1,\xi_2)\in\mathbb{R}^2\setminus\{0\}$, and $\mathfrak{S}(\mathcal{L};x,\xi_1,\xi_2)$ is even and homogeneous of order $+1$ matrix function in $\xi'=(\xi_1,\xi_2)$ (see Remark 4.13 and Appendix C).

Further, let us show that the symbol matrix $\mathfrak{S}(\mathcal{L};x,\xi_1,\xi_2)=\mathfrak{S}(\mathcal{L}^{(0)};x,\xi_1,\xi_2)$ is strongly elliptic. To this end, we recall that the operator $\mathcal{L}^{(0)}:[H_2^{\frac12}(S)]^6\to[H_2^{-\frac12}(S)]^6$ is introduced in Subsection 4.2 and that the operator $\mathcal{L}-\mathcal{L}^{(0)}:[H_2^{\frac12}(S)]^6\to[H_2^{-\frac12}(S)]^6$ is compact.

Formulas (4.49) with $U=U^{(0)}=(u^{(0)},\varphi^{(0)},\psi^{(0)},\vartheta^{(0)})^\top:=W^{(0)}(g)$ imply (see Subsection 4.2)

$$\langle\mathcal{L}^{(0)}g,g\rangle_S=\int_{\Omega^+\cup\Omega^-}\mathcal{E}(U^{(0)},\overline{U^{(0)}})\,dx \tag{5.65}$$

for arbitrary $g\in[C^{1,\alpha}(S)]^6$. Note that $W^{(0)}(g)\in[H_2^1(\Omega^\pm)]^6$, but $W^{(0)}(g)\notin[H_2^1(\mathbb{R}^3)]^6$ if $g\neq0$. With the help of (4.50) and (5.65), using the Korn inequalities for the domains $\Omega^\pm$ (see [56]), the trace theorem, the boundedness properties and jump relations for the double layer potential $U^{(0)}=W^{(0)}(g)$ (see Theorems 4.1 and 4.2), we derive the following Gårding type inequality

$$\begin{aligned}\operatorname{Re}\langle\mathcal{L}^{(0)}g,g\rangle_S&\geq\int_{\Omega^+\cup\Omega^-}\big\{\varepsilon_{kj}^{(0)}\overline{\varepsilon_{kj}^{(0)}}+|\nabla\varphi^{(0)}|^2+|\nabla\psi^{(0)}|^2+|\nabla\vartheta^{(0)}|^2\big\}\,dx\\&\geq C_1\Big(\|U^{(0)}\|^2_{[H_2^1(\Omega^+)]^6}+\|U^{(0)}\|^2_{[H_2^1(\Omega^-)]^6}\Big)-C_2\|U^{(0)}\|^2_{[L_2(\Omega^+)]^6}\\&\geq C_3\Big(\big\|\{U^{(0)}\}^+\big\|^2_{[H_2^{1/2}(S)]^6}+\big\|\{U^{(0)}\}^-\big\|^2_{[H_2^{1/2}(S)]^6}\Big)-C_4\|U^{(0)}\|^2_{[L_2(\Omega^+)]^6}\\&\geq C_3\big\|\{U^{(0)}\}^+-\{U^{(0)}\}^-\big\|^2_{[H_2^{1/2}(S)]^6}-C_4\|U^{(0)}\|^2_{[L_2(\Omega^+)]^6}\\&\geq C_5\|g\|^2_{[H_2^{1/2}(S)]^6}-C_6\|g\|^2_{[H_2^{-1/2}(S)]^6},\end{aligned} \tag{5.66}$$

where $\varepsilon_{kj}^{(0)}=\frac12\,(\partial_k u_j^{(0)}+\partial_j u_k^{(0)})$ and $C_j$, $j=1,\dots,6$, are positive constants.

Next, we consider unbounded half-spaces $\mathbb{R}^3_+(n) := \{x_1n_1 + x_2n_2 + x_3n_3 < 0\}$ and $\mathbb{R}^3_-(n) := \{x_1n_1+x_2n_2+x_3n_3 > 0\}$ instead of $\Omega^+$ and $\Omega^-$, respectively, and assume that $n$ is the unit "outward" normal vector to the hyperplane $S_n := \{x_1n_1 + x_2n_2 + x_3n_3 = 0\}$ with respect to $\mathbb{R}^3_+(n)$. Evidently, $n$ is a constant vector. Further, let us note that the double layer potential $U^{(0)} = W^{(0)}(g)$ with the integration surface $S_n$ and the density $g$ being an arbitrary rapidly decreasing vector function of the Schwartz space, decays at infinity as $\mathcal{O}(|x|^{-2})$. Moreover, $\partial^\alpha W^{(0)}(g)(x) = \mathcal{O}(|x|^{-2-|\alpha|})$ as $|x| \to \infty$ for arbitrary multi-index $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ due to the homogeneity property of the fundamental matrix $\Gamma^{(0)}(x)$ given by (3.5) and since $g(\widetilde{x})$ decays at infinity faster than any negative power of $|\widetilde{x}|$, $\widetilde{x} \in S_n$. Therefore, $\partial^\alpha W^{(0)}(g) \in [L_2(\mathbb{R}^3_\pm(n))]^6$ for $|\alpha| \geq 0$ and with the help of the Korn inequalities for unbounded domains $\mathbb{R}^3_\pm(n)$ (see [56]), one can show a counterpart of formula (5.65) which yields the following relation (cf. (5.66))

$$\begin{aligned}
\mathrm{Re}\langle \mathcal{L}^{(0)}g, g\rangle_{S_n} &= \mathrm{Re}\int_{S_n} \mathcal{L}^{(0)}g\cdot g\, dS_n = \mathrm{Re}\int_{\mathbb{R}^3_+(n)\cup\mathbb{R}^3_-(n)} \geq \mathcal{E}(U^{(0)}, \overline{U^{(0)}})\, dx \\
&\geq \int_{\mathbb{R}^3_+(n)\cup\mathbb{R}^3_-(n)} \Big\{\varepsilon^{(0)}_{kj}\overline{\varepsilon^{(0)}_{kj}} + |\nabla\varphi^{(0)}|^2 + |\nabla\psi^{(0)}|^2 + |\nabla\vartheta^{(0)}|^2\Big\}\, dx \\
\geq C_1^*\Big(\|U^{(0)}\|^2_{[H^1_2(\mathbb{R}^3_+(n))]^6} &+ \|U^{(0)}\|^2_{[H^1_2(\mathbb{R}^3_-(n))]^6}\Big) \geq C_2^*\Big(\big\|\{U^{(0)}\}^+\big\|^2_{[H^{1/2}_2(S_n)]^6} + \big\|\{U^{(0)}\}^-\big\|^2_{[H^{1/2}_2(S_n)]^6}\Big) \\
&\geq C_2^*\big\|\{U^{(0)}\}^+ - \{U^{(0)}\}^-\big\|^2_{[H^{1/2}_2(S_n)]^6} \geq C_3^*\|g\|^2_{[H^{1/2}_2(S_n)]^6},
\end{aligned}$$

where $C_j^*$, $j = 1, 2, 3$, are some positive constants. Now, let us take into account that $\mathcal{L}^{(0)}$ is a convolution operator and perform an orthogonal transform $x = B(n)x'$ of the half-spaces $\mathbb{R}^3_\pm(n)$ onto the usual standard half-spaces $\mathbb{R}^3_\pm := \{x' \in \mathbb{R}^3 : \pm x'_3 \geq 0\}$ having the boundary $S = \mathbb{R}^2 := \{x' \in \mathbb{R}^3 : x'_3 = 0\}$. Here $B(n)$ is an orthogonal matrix given by (4.64) where $n = (n_1, n_2, n_3)$, $l = (l_1, l_2, l_3)$ and $m = (m_1, m_2, m_3)$ are mutually orthogonal constant unit vectors. Applying the Parseval equality we then easily deduce that the corresponding homogeneous symbol matrix $\mathfrak{S}(\mathcal{L}^{(0)}; \xi_1, \xi_2)$ is strongly elliptic, i.e., there is a positive constant $c$ such that

$$\mathrm{Re}\left[\mathfrak{S}(\mathcal{L}^{(0)}; \xi_1, \xi_2)\zeta\cdot\zeta\right] = \mathrm{Re}\left[\mathfrak{S}(\mathcal{L}; \xi_1, \xi_2)\zeta\cdot\zeta\right] \geq c|\xi'||\zeta|^2 \tag{5.67}$$

for arbitrary normal vector $n$ and for all $\zeta \in \mathbb{C}^6$, $\xi' = (\xi_1, \xi_2) \in \mathbb{R}^2 \setminus \{0\}$, and $x \in S_n$. The constant $c$ depends only on the material parameters.

Thus we have proved the following

**Lemma 5.9.** *Let condition* (5.30) *be satisfied and $\tau = \sigma + i\omega$ with $\sigma > 0$ and $\omega \in \mathbb{R}$. Then there is a positive constant $C_1$ such that for all $g \in [H_2^{\frac{1}{2}}(S)]^6$ there holds the inequality*

$$\mathrm{Re}\langle(\mathcal{L} + \mathcal{C}_0)g, g\rangle_S \geq C_1\|g\|^2_{[H_2^{\frac{1}{2}}(S)]^6}, \tag{5.68}$$

*where $\mathcal{C}_0 : [H_2^{\frac{1}{2}}(S)]^6 \to [H_2^{-\frac{1}{2}}(S)]^6$ is a compact operator. The operator*

$$\mathcal{L} : [H_2^{\frac{1}{2}}(S)]^6 \longrightarrow [H_2^{-\frac{1}{2}}(S)]^6, \tag{5.69}$$

*is a strongly elliptic pseudodifferential operator of index zero and the corresponding two dimensional null space is defined by* (5.62)*.*

*Equation* (5.57) *with $F \in [C^{k,\kappa'}(S)]^6$ is solvable if and only if*

$$\int_S F_4(x)\, dS = \int_S F_5(x)\, dS = 0 \tag{5.70}$$

*and solution $h \in [C^{k,\kappa'}(S)]^6$ is defined modulo a vector summand given by* (5.60)*.*

From Lemma 5.9 one can derive the corresponding existence results and representability of solutions to the Neumann BVPs by double layer potentials.

**Theorem 5.10.** *Let $S$ and $F = (F_1, \ldots, F_6)^\top$ satisfy the conditions* (5.30) *and* (5.31)*.*

(i) *If conditions* (5.70) *hold, then the interior Neumann BVP is solvable in the space of vector functions* $[C^{k,\kappa'}(\overline{\Omega^+})]^6$ *and its solutions are representable in the form of double layer potential* (5.56)*, where the density vector function h is defined by the pseudodifferential equation* (5.57) *and* $h \in [C^{k,\kappa'}(S)]^6$ *is defined modulo a vector summand given by* (5.60)*. A solutions to the interior Neumann BVP for the domain* $\Omega^+$ *is defined modulo a linear combination of the constant vector functions* $\Psi^{(1)} = (0,0,0,1,0,0)^\top$ *and* $\Psi^{(2)} = (0,0,0,0,1,0)^\top$.

(ii) *If conditions* (5.70) *hold, then the exterior Neumann BVP is solvable in the space of functions* $[C^{k,\kappa'}(\overline{\Omega^-})]^6$ *satisfying the decay conditions* (2.207) *at infinity and its solution is representable in the form of double layer potential* (5.56)*, where the density vector function h is defined by the pseudodifferential equation* (5.57) *and* $h \in [C^{k,\kappa'}(S)]^6$ *is defined modulo a vector summand given by* (5.60)*. A solution to the exterior Neumann BVP for the domain* $\Omega^-$ *is uniquely defined since the double layer potentials* $W(\Psi^{(j)})$*,* $j = 1, 2$*, vanish identically in* $\Omega^-$.

*Remark* 5.11. Note that, if we seek a solution to the exterior Neumann BVP (5.50), (5.51) in the form of linear combination of the single and double layer potentials

$$U(x) = W(h)(x) + \alpha V(h)(x), \;\; x \in \Omega^-, \;\; \alpha = const > 0, \tag{5.71}$$

we arrive at the equation (see Theorem 4.1, (4.7), (4.9))

$$\mathcal{L}h + \alpha(2^{-1}I_6 + \mathcal{K})h = F \;\text{ on }\; S. \tag{5.72}$$

It can be shown that the operator

$$\mathcal{L} + \alpha(2^{-1}I_6 + \mathcal{K}) : [H_2^{\frac{1}{2}}(S)]^6 \longrightarrow [H_2^{-\frac{1}{2}}(S)]^6, \tag{5.73}$$

is invertible. Indeed, since the index of the operator (5.73) is zero by Lemma 5.9, it suffices to prove that the corresponding null space is trivial. Let $\mathcal{L}h_0 + \alpha(2^{-1}I_6 + \mathcal{K})h_0 = 0$ on $S$. Then $h_0 \in [C^{1,\kappa'}(S)]^6$ and the regular vector $U_0 = W(h_0) + \alpha V(h_0) \in [C^{1,\kappa'}(\overline{\Omega^-})]^6$ solves the homogeneous exterior Neumann BVP. In view of the uniqueness Theorem 2.26 then $U_0 = 0$ in $\Omega^-$. Due to the jump relations we then get that $U_0$ solves the homogeneous Robin type BVP in $\Omega^+$,

$$\{\mathcal{T}U_0\}^+ + \alpha\{U_0\}^+ = 0 \;\text{ on }\; S. \tag{5.74}$$

As we have shown in the proof of Theorem 5.3 this problem has only the trivial solution, i.e., $U_0 = 0$ in $\Omega^+$. Therefore $h_0 = \{U_0\}^+ - \{U_0\}^- = 0$ on $S$ and, consequently, the null space of the operator (5.73) is trivial.

Thus (5.73) is invertible and equation (5.72) is uniquely solvable. This proves that a unique solution to the exterior Neumann BVP can be represented in the form (5.71) with the density $h \in [C^{k,\kappa'}(S)]^6$ defined by the pseudodifferential equation (5.72).

5.6. **The interior and exterior Dirichlet and Neumann BVPs in Bessel potential and Besov spaces.** If not otherwise stated, throughout this subsection we assume that

$$S \in C^\infty, \;\; p > 1, \;\; q \geq 1, \;\; s \in \mathbb{R}. \tag{5.75}$$

Applying the general theory of pseudodifferential equations on manifolds without boundary (see, e.g., [35], [101], [44]), we can generalize the existence results obtained in the previous subsections to more wide classes of boundary data. In particular, using Theorem 4.4 and the fact that the null spaces of strongly elliptic pseudodifferential operators acting in Bessel potential $H_p^s(S)$ and Besov $B_{p,q}^s(S)$ spaces actually do not depend on the parameters $s$, $p$, and $q$, we arrive at appropriate existence results.

**Theorem 5.12.** *Let condition* (5.75) *be fulfilled and* $g \in [B_{p,p}^{1-\frac{1}{p}}(S)]^6$*. Then the pseudodifferential operator*

$$2^{-1}I_6 + \mathcal{N} : [B_{p,p}^{1-\frac{1}{p}}(S)]^6 \longrightarrow [B_{p,p}^{1-\frac{1}{p}}(S)]^6 \tag{5.76}$$

*is invertible and the interior Dirichlet BVP* (5.3), (5.4) *is uniquely solvable in the space* $[H_p^1(\Omega^+)]^6$ *and the solution is representable in the form of double layer potential* $U = W(h)$ *with the density vector function* $h \in [B_{p,p}^{1-\frac{1}{p}}(S)]^6$ *being a unique solution of the equation*

$$(2^{-1}I_6 + \mathcal{N})h = g \;\text{ on }\; S. \tag{5.77}$$

*Proof.* The invertibility of the operator (5.76) immediately follows from the strong ellipticity of the operator $2^{-1}I_6+\mathcal{N}$ and the invertibility of the operator (5.8). Therefore equation (5.77) is uniquely solvable for arbitrary $g\in[B_{p,p}^{1-\frac{1}{p}}(S)]^6$. Moreover, due to Theorem 4.4, it is easy to see that the double layer potential $U=W(h)$ with density vector $h\in[B_{p,p}^{1-\frac{1}{p}}(S)]^6$ being a solution to equation (5.77), belongs to the space $[H_p^1(\Omega^+)]^6$ and solves the interior Dirichlet BVP. It remains to show that the homogenous interior Dirichlet BVP possesses only the trivial solution in the space $[H_p^1(\Omega^+)]^6$. Let $U_0\in[H_p^1(\Omega^+)]^6$ be a solution to the homogenous interior Dirichlet BVP. Due to the general integral representation formula (3.62) we then get

$$U_0=-V(\{\mathcal{T}U_0\}^+)\ \text{ in }\ \Omega^+, \tag{5.78}$$

where $\{\mathcal{T}U_0\}^+\in\big[B_{p,p}^{-\frac{1}{p}}(S)\big]^6$. Therefore, in view of the homogeneous Dirichlet condition on $S$, we have

$$\{U_0\}^+=-\mathcal{H}\{\mathcal{T}U_0\}^+=0\ \text{ on }\ S. \tag{5.79}$$

But the operator

$$\mathcal{H}:[B_{p,p}^{-\frac{1}{p}}(S)]^6\longrightarrow[B_{p,p}^{1-\frac{1}{p}}(S)]^6 \tag{5.80}$$

is invertible for arbitrary $p>1$, since for a particular value of the parameter $p=2$ it is invertible (see Corollary 4.8). Therefore (5.79) and (5.78) yield $U_0=0$ in $\Omega^+$, which completes the proof. □

The following assertions can be proved quite similarly.

**Theorem 5.13.** *Let condition* (5.75) *be fulfilled and* $g\in[B_{p,p}^{1-\frac{1}{p}}(S)]^6$. *Then the Dirichlet exterior problem* (5.12), (5.13), (2.207), *is uniquely solvable in the space* $[H_{p,loc}^1(\Omega^-)]^6$ *and the solution is representable in the form* $U=W(h)+\alpha V(h)$, *where the density vector function* $h\in\big[B_{p,p}^{1-\frac{1}{p}}(S)\big]^6$ *is defined by the uniquely solvable pseudodifferential equation*

$$(-2^{-1}I_6+\mathcal{N}+\alpha\mathcal{H})h=g\ \text{ on }\ S. \tag{5.81}$$

**Theorem 5.14.**

(i) *Let a vector function* $U\in[H_p^1(\Omega^+)]^6$ *solve the homogeneous differential equation* $A(\partial,\tau)U=0$ *in* $\Omega^+$. *Then it is uniquely representable in the form*

$$U(x)=V\big(\mathcal{H}^{-1}\{U\}^+\big)(x),\ \ x\in\Omega^+, \tag{5.82}$$

*where* $\mathcal{H}^{-1}$ *stands for the operator inverse to* (5.80), *while* $\{U\}^+$ *is the trace of* $U$ *on* $S$ *from* $\Omega^+$ *and belongs the space* $[B_{p,p}^{1-\frac{1}{p}}(S)]^6$.

(ii) *Let a vector function* $U\in[H_{p,loc}^1(\Omega^-)]^6$, *satisfy the decay conditions* (2.207), *and solve the homogeneous differential equation* $A(\partial,\tau)U=0$ *in* $\Omega^-$. *Then it is uniquely representable in the form*

$$U(x)=V\big(\mathcal{H}^{-1}\{U\}^-\big)(x),\ \ x\in\Omega^-, \tag{5.83}$$

*where* $\mathcal{H}^{-1}$ *again stands for the operator inverse to* (5.80), *while* $\{U\}^-$ *is the traces of* $U$ *on* $S$ *from* $\Omega^-$ *and belongs the space* $[B_{p,p}^{1-\frac{1}{p}}(S)]^6$.

Analogous propositions hold true for the interior and exterior Neumann BVPs. The following counterparts of Theorems 5.7 and 5.8 hold.

**Theorem 5.15.** *Let* (5.75) *be fulfilled and* $F=(F_1,\dots,F_6)^\top\in[B_{p,p}^{-\frac{1}{p}}(S)]^6$.

(i) *The operator*

$$-2^{-1}I_6+\mathcal{K}:[B_{p,p}^{-\frac{1}{p}}(S)]^6\longrightarrow[B_{p,p}^{-\frac{1}{p}}(S)]^6 \tag{5.84}$$

*is an elliptic pseudodifferential operator with zero index and has a two-dimensional null space* $\Lambda(S):=\ker(-2^{-1}I_6+\mathcal{K})\subset[C^\infty(S)]^6$, *which represents a linear spanof the vector functions*

$$h^{(1)},h^{(2)}\in\Lambda(S), \tag{5.85}$$

*such that*

$$V(h^{(1)}) = \Psi^{(1)} := (0,0,0,1,0,0)^\top \ \text{ and } \ V(h^{2)}) = \Psi^{(2)} := (0,0,0,0,1,0)^\top \ \text{ in } \ \Omega^+. \tag{5.86}$$

(ii) *The null space of the operator adjoint to* (5.84),

$$-2^{-1}I_6 + \mathcal{K}^* : [B_{p',p'}^{1-\frac{1}{p'}}(S)]^6 \longrightarrow [B_{p',p'}^{1-\frac{1}{p'}}(S)]^6, \ \ \frac{1}{p} + \frac{1}{p'} = 1, \tag{5.87}$$

*is a linear span of the vectors* $(0,0,0,1,0,0)^\top$ *and* $(0,0,0,0,1,0)^\top$.

(iii) *The equation*

$$(-2^{-1}I_6 + \mathcal{K})h = F \ \text{ on } \ S, \tag{5.88}$$

*is solvable if and only if*

$$\int_S F_4(x)\,dS = \int_S F_5(x)\,dS = 0. \tag{5.89}$$

(iv) *If the conditions* (5.89) *hold, then solutions to equation* (5.88) *are defined modulo a linear combination of the vector functions* $h^{(1)}$ *and* $h^{(2)}$.

(v) *If the conditions* (5.89) *hold, then the interior Neumann type boundary value problem* (5.32), (5.33) *is solvable in the space* $[H_p^1(\Omega^+)]^6$ *and solutions are representable in the form of single layer potential* $U = V(h)$, *where the density vector function* $h \in [B_{p,p}^{-\frac{1}{p}}(S)]^6$ *is defined by equation* (5.88). *A solutions to the interior Neumann BVP in* $\Omega^+$ *is defined modulo a linear combination of the constant vector functions* $\Psi^{(1)}$ *and* $\Psi^{(2)}$ *given by* (5.86).

**Theorem 5.16.** *Let* (5.75) *be fulfilled and* $F = (F_1, \dots, F_6)^\top \in [B_{p,p}^{-\frac{1}{p}}(S)]^6$.

(i) *The operator*

$$2^{-1}I_6 + \mathcal{K} : [B_{p,p}^{-\frac{1}{p}}(S)]^6 \longrightarrow [B_{p,p}^{-\frac{1}{p}}(S)]^6 \tag{5.90}$$

*is an invertible elliptic pseudodifferential operator.*

(ii) *The exterior Neumann type boundary value problem* (5.50), (5.51) *is uniquely solvable in the space of vector functions* $[H_p^1(\Omega^-)]^6$ *satisfying the decay conditions* (2.207) *and the solution is representable in the form of single layer potential* $U = V(h)$, *where the density vector function* $h \in [B_{p,p}^{-\frac{1}{p}}(S)]^6$ *is defined by the uniquely solvable pseudodifferential equation* $(2^{-1}I_6 + \mathcal{K})h = F$ *on* $S$.

*Remark* 5.17. From the general theory of pseudodifferential equations on $C^\infty$-smooth manifolds without boundary it follows that

(i) the elliptic pseudodifferential operators

$$\mathcal{H} : [B_{p,q}^s(S)]^6 \longrightarrow [B_{p,q}^{s+1}(S)]^6, \tag{5.91}$$

$$2^{-1}I_6 + \mathcal{N} : [B_{p,q}^{s+1}(S)]^6 \longrightarrow [B_{p,q}^{s+1}(S)]^6, \tag{5.92}$$

$$2^{-1}I_6 + \mathcal{K} : [B_{p,q}^s(S)]^6 \longrightarrow [B_{p,q}^s(S)]^6, \tag{5.93}$$

are invertible for arbitrary $s \in \mathbb{R}$, $p > 1$, $q \geq 1$, since (5.91), (5.92), and (5.93) are invertible for the following particular values of the parameters $s = -\frac{1}{2}$ and $p = q = 2$ due to Corollary 4.9, Theorem 5.12 and Theorem 5.16, respectively;

(ii) the elliptic pseudodifferential operators

$$-2^{-1}I_6 + \mathcal{N} : [B_{p,q}^s(S)]^6 \longrightarrow [B_{p,q}^s(S)]^6, \quad -2^{-1}I_6 + \mathcal{K} : [B_{p,q}^s(S)]^6 \longrightarrow [B_{p,q}^s(S)]^6,$$

have zero index for arbitrary $s \in \mathbb{R}$, $p > 1$, $q \geq 1$, and their two-dimensional null spaces do not depend on $s$, $p$, $q$.

*Remark* 5.18. From the results obtained in Subsections 4.2 and 4.3 it follows that Theorems 5.13–5.16 with $p = 2$ hold true for Lipschitz domains (cf. [75, Ch. 7]).

5.7. **Basic mixed type BVPs.** Having in hand the results obtained in the previous subsections, we can investigate the mixed type boundary value problems formulated in Subsection 2.3 (see (2.166)–(2.173)). In general, solutions to the mixed type BVPs are not $C^\alpha$-Hölder continuous with $\alpha > \frac{1}{2}$ at the exceptional curves $\ell_m$ where different boundary conditions collide. Therefore we are not allowed to look for solutions in the space of regular vector functions even for $C^\infty$ smooth boundary surfaces and $C^\infty$ smooth boundary data.

Here we study in detail the *basic interior mixed type boundary value problems* associated with a simple dissection of the boundary $\partial\Omega^+$ into the Dirichlet and Neumann parts, $\partial\Omega^+ = \overline{S_D} \cup \overline{S_N}$. The exterior problems can be treated quite similarly.

In particular, in this subsection, we will analyze the following mixed type BVP: Find a solution vector $U = (u, \varphi, \psi, \vartheta)^\top \in [W^1_p(\Omega^+)]^6$ to the homogeneous system of pseudo-oscillation equation

$$A(\partial, \tau)U = 0 \ \text{ in } \ \Omega^+, \tag{5.94}$$

which satisfies the mixed Dirichlet–Neumann type boundary conditions

$$\{U\}^+ = f^{(D)} \ \text{ on } \ S_D, \tag{5.95}$$

$$\big\{\mathcal{T}(\partial, n)U\big\}^+ = F^{(N)} \ \text{ on } \ S_N. \tag{5.96}$$

Here

$$f^{(D)} \in [B^{1-\frac{1}{p}}_{p,p}(S_D)]^6, \quad F^{(N)} \in [B^{-\frac{1}{p}}_{p,p}(S_N)]^6. \tag{5.97}$$

For simplicity, throughout this subsection we assume that $S$ and $\partial S_D = \partial S_N = \ell_m$ are $C^\infty$-smooth. Denote by $f^{(e)}$ a fixed extension of the vector function $f^{(D)}$ from $S_D$ onto the whole of $S$ preserving the functional space,

$$f^{(e)} \in [B^{1-\frac{1}{p}}_{p,p}(S)]^6, \ \ r_{_{S_D}} f^{(e)} = f^{(D)} \ \text{ on } \ S_D. \tag{5.98}$$

Recall that $r_{_\mathcal{M}}$ denotes the restriction operator onto $\mathcal{M}$.

*Remark* 5.19. If $f^{(D)} = 0$ on $S_D$, we always choose in the role of a fixed extension preserving the space the zero function $f^{(e)} = 0$ on $S$.

Clearly, an arbitrary extension $f$ of $f^{(D)}$ onto the whole of $S$, which preserves the functional space, can be then represented as

$$f = f^{(e)} + \widetilde{f} \ \text{ with } \ \widetilde{f} \in [\widetilde{B}^{1-\frac{1}{p}}_{p,p}(S_N)]^6. \tag{5.99}$$

In accordance with Theorem 5.14, we can seek a solution in the form

$$U = V\big(\mathcal{H}^{-1}(f^{(e)} + \widetilde{f})\big), \tag{5.100}$$

where $\widetilde{f} \in [\widetilde{B}^{1-\frac{1}{p}}_{p,p}(S_N)]^6$ is an unknown vector function and $\mathcal{H}^{-1}$ is a strongly elliptic pseudodifferential operator inverse to the operator (5.80) (see Remark 5.17),

$$\mathcal{H}^{-1} : [B^{1-\frac{1}{p}}_{p,p}(S)]^6 \longrightarrow [B^{-\frac{1}{p}}_{p,p}(S)]^6. \tag{5.101}$$

In view of (5.98) and (5.99), it is easy to check that the Dirichlet condition (5.95) on $S_D$ is satisfied automatically. It remains only to satisfy the Neumann condition (5.96) on $S_N$, which leads to the pseudodifferential equation

$$(-2^{-1}I_6 + \mathcal{K})\mathcal{H}^{-1}(f^{(e)} + \widetilde{f}) = F^{(N)} \tag{5.102}$$

on the open subsurface $S_N$ for the unknown vector function $\widetilde{f}$.

We recall that

$$\mathcal{A}^+ = (-2^{-1}I_6 + \mathcal{K})\mathcal{H}^{-1} \tag{5.103}$$

is the Steklov–Poincaré operator introduced and studied in Subsection 4.3 for $p = 2$. In view of Remark 4.12 it is clear that

$$\mathcal{A}^+ : [B^{1-\frac{1}{p}}_{p,p}(S)]^6 \longrightarrow [B^{-\frac{1}{p}}_{p,p}(S)]^6 \tag{5.104}$$

is a strongly elliptic pseudodifferential operator of order 1 with index equal to zero.

Denote

$$F^{(0)} := F^{(N)} - r_{_{S_N}} \mathcal{A}^+ f^{(e)} \in [B^{-\frac{1}{p}}_{p,p}(S_N)]^6 \tag{5.105}$$

and rewrite equation (5.102) as

$$r_{_{S_N}}\mathcal{A}^+\widetilde{f}=F^{(0)} \ \text{ on } \ S_N, \tag{5.106}$$

which is a pseudodifferential equation on the submanifold $S_N$ with boundary $\partial S_N$. We would like to investigate the solvability of equation (5.106). To this end, we proceed as follows.

Denote by $\mathfrak{S}(\mathcal{A}^+;x,\xi_1,\xi_2)$ the principal homogeneous symbol matrix of the operator $\mathcal{A}^+$ in some local coordinate system at the point $x\in\overline{S_N}$ and let $\xi'=(\xi_1,\xi_2)\in\mathbb{R}^2\setminus\{0\})$. In view of Remark 4.12 the symbol matrix $\mathfrak{S}(\mathcal{A}^+;x,\xi_1,\xi_2)$ is strongly elliptic. Let $\lambda_1(x),\dots,\lambda_6(x)$ be the eigenvalues of the matrix

$$M_{\mathcal{A}^+}(x):=\big[\mathfrak{S}(\mathcal{A}^+;x,0,+1)\big]^{-1}\mathfrak{S}(\mathcal{A}^+;x,0,-1), \ \ x\in\partial S_N. \tag{5.107}$$

Introduce the notation

$$\delta_j(x)=\operatorname{Re}\big[(2\pi i)^{-1}\ln\lambda_j(x)\big], \ \ j=1,\dots,6, \tag{5.108}$$

$$a_1=\inf_{x\in\ell_m,1\le j\le 6}\delta_j(x), \quad a_2=\sup_{x\in\ell_m,1\le j\le 6}\delta_j(x); \tag{5.109}$$

here $\ln\zeta$ denotes the branch of the logarithm analytic in the complex plane cut along $(-\infty,0]$. Note that the numbers $\delta_j(x)$ do not depend on the choice of the local coordinate system (see Appendix B). Due to the strong ellipticity of the operator $\mathcal{A}^+$ we have the strict inequalities $-\frac12<\delta_j(x)<\frac12$ for $x\in\overline{S_N}$, $j=1,\dots,6$. Therefore

$$-\frac{1}{2}<a_1\le a_2<\frac{1}{2}\,. \tag{5.110}$$

Moreover, from the structure of the principal homogeneous symbol matrix of the operator $\mathcal{A}^+$,

$$\mathfrak{S}(\mathcal{A}^+;x,\xi_1,\xi_2)=\mathfrak{S}(-2^{-1}I_6+\mathcal{K};x,\xi_1,\xi_2)\mathfrak{S}(\mathcal{H}^{-1};x,\xi_1,\xi_2),$$

it follows that (see Appendix C and Remarks C.5)

$$\begin{aligned}\big[\mathfrak{S}(\mathcal{A}^+;x,\xi_1,\xi_2)\big]_{j6}&=\big[\mathfrak{S}(\mathcal{A}^+;x,\xi_1,\xi_2)\big]_{6j}=0, \ \ j=1,\dots,5,\\ \big[\mathfrak{S}(\mathcal{A}^+;x,\xi_1,\xi_2)\big]_{66}&=-2^{-1}\big[\mathfrak{S}(\mathcal{H}^{-1};x,\xi_1,\xi_2)\big]_{66}\\ &=-2^{-1}\big[\mathfrak{S}(\mathcal{H}^{-1};x,-\xi_1,-\xi_2)]_{66}=\big[\mathfrak{S}(\mathcal{A}^+;x,-\xi_1,-\xi_2)\big]_{66}>0,\end{aligned}$$

in accordance with equalities (4.68) and $[\mathfrak{S}(-2^{-1}I_6+\mathcal{K};x,\xi_1,\xi_2)]_{66}=-\frac12$. Therefore, we find that $[M_{\mathcal{A}^+}(x)]_{66}=1$ and due to Remark 4.12 and Lemma C.4, it follows that one of the eigenvalues of the matrix $M_{\mathcal{A}^+}(x)$ defined in (5.107) equals to 1, say $\lambda_6=1$, implying $\delta_6=0$.

Consequently, we have the following estimates

$$-\frac{1}{2}<a_1\leqslant 0\leqslant a_2<\frac{1}{2}\,. \tag{5.111}$$

**Lemma 5.20.** *The operators*

$$r_{_{S_N}}\mathcal{A}^+:[\widetilde{H}^s_p(S_N)]^6\longrightarrow[H^{s-1}_p(S_N)]^6, \tag{5.112}$$

$$r_{_{S_N}}\mathcal{A}^+:[\widetilde{B}^s_{p,q}(S_N)]^6\longrightarrow[B^{s-1}_{p,q}(S_N)]^6, \ \ q\ge 1 \tag{5.113}$$

*are invertible if*

$$\frac{1}{p}-\frac{1}{2}+a_2<s<\frac{1}{p}+\frac{1}{2}+a_1, \tag{5.114}$$

*where $a_1$ and $a_2$ are given by* (5.109).

*Proof.* The mapping properties (5.112) and (5.113) follow from Theorem 4.4 and Remark 5.17. To prove the invertibility of the operators (5.112) and (5.113), we first consider the particular values of the parameters $s=1/2$ and $p=q=2$, which fall into the region defined by the inequalities (5.114), and show that the null space of the operator

$$r_{_{S_N}}\mathcal{A}^+:[\widetilde{H}^{\frac12}_2(S_N)]^6\longrightarrow[H^{-\frac12}_2(S_N)]^6 \tag{5.115}$$

is trivial, i.e., the equation

$$r_{_{S_N}}\mathcal{A}^+\widetilde{f}=0 \ \text{ on } \ S_N \tag{5.116}$$

admits only the trivial solution in the space $[\widetilde{H}^{\frac12}_2(S_N)]^6$. Recall that $\widetilde{H}^s_2(S_N)=\widetilde{B}^s_{2,2}(S_N)$ and $H^s_2(S_N)=B^s_{2,2}(S_N)$ for $s\in\mathbb{R}$.

Let $\widetilde{f} \in [\widetilde{H}^{\frac{1}{2}}(S_N)]^6$ be a solution of the homogeneous equation (5.116). It is evident that the vector $U = V(\mathcal{H}^{-1}\widetilde{f})$ belongs to the space $[H_2^1(\Omega^+)]^6 = [W_2^1(\Omega^+)]^6$ and solves the homogeneous mixed BVP (5.94)–(5.96) with $f^{(D)} = 0$ and $F^{(N)} = 0$. Therefore, $U(x) = V(\mathcal{H}^{-1}\widetilde{f})(x) = 0$ for $x \in \Omega^+$, due to Theorem 2.25 and, consequently, $\{U\}^+ = \widetilde{f} = 0$ on $S$.

Since the principal homogeneous symbol matrix of the operator $\mathcal{A}^+$ is strongly elliptic, by Theorem B.1 (see Appendix B) we conclude that for all values of the parameters satisfying the inequalities (5.114), the operators (5.112) and (5.113) are Fredholm with zero index and with trivial null spaces. Thus they are invertible. □

With the help of this lemma we can prove the following main existence result.

**Theorem 5.21.** *Let the conditions* (5.97) *be fulfilled,* $a_1$ *and* $a_2$ *be defined by* (5.109)*, and*

$$\frac{4}{3-2a_2} < p < \frac{4}{1-2a_1}\,. \tag{5.117}$$

*Then the mixed boundary value problem* (5.94)–(5.96) *has a unique solution* $U \in [W_p^1(\Omega^+)]^6$ *which is representable in the form of single layer potential* (5.100),

$$U = V\big(\mathcal{H}^{-1}(f^{(e)} + \widetilde{f})\big), \tag{5.118}$$

*where* $f^{(e)} \in [B_{p,p}^{1-1/p}(S)]^6$ *is a fixed extension of the vector function* $f^{(D)} \in [B_{p,p}^{1-1/p}(S_D)]^g$ *from* $S_D$ *onto* $S$ *preserving the functional space and* $\widetilde{f} \in [\widetilde{B}_{p,p}^{1-1/p}(S_N)]^6$ *is defined by the uniquely solvable pseudodifferential equation*

$$r_{_{S_N}}\mathcal{A}^+\widetilde{f} = F^{(0)} \ \ \text{on} \ \ S_N \tag{5.119}$$

*with*

$$F^{(0)} := F^{(N)} - r_{_{S_N}}\mathcal{A}^+ f^{(e)} \in [B_{p,p}^{-1/p}(S_N)]^6.$$

*Proof.* First we note that in accordance with Lemma 5.20, equation (5.119) is uniquely solvable for $s = 1 - \frac{1}{p}$ where $p$ satisfies the inequality (5.117), since the inequalities (5.114) are fulfilled. This implies that the mixed boundary value problem (5.94)–(5.96) is solvable in the space $[W_p^1(\Omega^+)]^6$ with $p$ as in (5.117).

Next, we show the uniqueness of solution in the space $[W_p^1(\Omega^+)]^6$ for arbitrary $p$ satisfying (5.117). Note that $p = 2$ belongs to the interval defined by the inequality (5.117) and for $p = 2$ the uniqueness has been proved in Theorem 2.25. Now, let $U \in [W_p^1(\Omega^+)]^6$ be some solution of the homogeneous mixed boundary value problem. Evidently, then

$$\{U\}^+ \in [\widetilde{B}_{p,p}^{1-1/p}(S_N)]^6. \tag{5.120}$$

By Theorem 5.14, we have the representation

$$U(x) = V(\mathcal{H}^{-1}\{U\}^+)(x), \ \ x \in \Omega^+.$$

Since $U$ satisfies the homogeneous Neumann condition (5.96) on $S_N$, we arrive at the equation

$$r_{_{S_N}}\mathcal{A}^+\{U\}^+ = 0 \ \ \text{on} \ \ S_N,$$

whence $\{U\}^+ = 0$ on $S$ follows due to the inclusion (5.120), Lemma 5.20, and inequality (5.117) implying the conditions (5.114). Therefore, $U = 0$ in $\Omega^+$. □

Further, we prove almost the best regularity results for solutions to the mixed type boundary value problems.

**Theorem 5.22.** *Let inclusions* (5.97) *hold and let*

$$\frac{4}{3-2a_2} < p < \frac{4}{1-2a_1}, \ \ 1 < r < \infty, \ \ 1 \le q \le \infty, \ \ \frac{1}{r} - \frac{1}{2} + a_2 < s < \frac{1}{r} + \frac{1}{2} + a_1, \tag{5.121}$$

*with* $a_1$ *and* $a_2$ *defined by* (5.109).

*Further, let* $U \in [W_p^1(\Omega^+)]^6$ *be a unique solution to the mixed boundary value problem* (5.94)–(5.96)*. Then the following hold:*

(i) *if*

$$f^{(D)} \in [B^{s}_{r,r}(S_D)]^6, \quad F^{(D)} \in [B^{s-1}_{r,r}(S_N)]^6,$$

*then* $U \in [H_r^{s+\frac{1}{r}}(\Omega^+)]^6$*;*

(ii) *if*

$$f^{(D)} \in [B^{s}_{r,q}(S_D)]^6, \quad F^{(D)} \in [B^{s-1}_{r,q}(S_N)]^6,$$

*then*

$$U \in [B^{s+\frac{1}{r}}_{r,q}(\Omega^+)]^6; \tag{5.122}$$

(iii) *if* $\alpha > 0$ *is not integer and*

$$f^{(D)} \in [C^{\alpha}(\overline{S_D})]^6, \quad F^{(D)} \in [B^{\alpha-1}_{\infty,\infty}(S_N)]^6, \tag{5.123}$$

*then*

$$U \in \bigcap_{\alpha'<\kappa_m} [C^{\alpha'}(\overline{\Omega^+})]^6,$$

*where* $0 < \kappa_m = \min\{\alpha,\ a_1 + \frac{1}{2}\} \leqslant \frac{1}{2}$.

*Proof.* The proofs of items (i) and (ii) follow easily from Lemma 5.20, and Theorems 5.21, and B.1.

To prove (iii) we use the following embedding (see, e.g., [108], [109])

$$C^{\alpha}(\mathcal{M}) = B^{\alpha}_{\infty,\infty}(\mathcal{M}) \subset B^{\alpha-\varepsilon}_{\infty,1}(\mathcal{M}) \subset B^{\alpha-\varepsilon}_{\infty,q}(\mathcal{M}) \subset B^{\alpha-\varepsilon}_{r,q}(\mathcal{M}) \subset C^{\alpha-\varepsilon-k/r}(\mathcal{M}), \tag{5.124}$$

where $\varepsilon$ is an arbitrary small positive number, $\mathcal{M} \subset \mathbb{R}^3$ is a compact $k$-dimensional $(k = 2,3)$ smooth manifold with smooth boundary, $1 \le q \le \infty$, $1 < r < \infty$, $\alpha - \varepsilon - \frac{k}{r} > 0$, and $\alpha$ and $\alpha - \varepsilon - \frac{k}{r}$ are not integers.

From (5.123) and the embedding (5.124) the condition (5.122) follows with any $s \le \alpha - \varepsilon$.

Bearing in mind (5.121) and taking $r$ sufficiently large and $\varepsilon$ sufficiently small, we can put

$$s = \alpha - \varepsilon \ \text{ if } \ \frac{1}{r} - \frac{1}{2} + a_2 < \alpha - \varepsilon < \frac{1}{r} + \frac{1}{2} + a_1, \tag{5.125}$$

and

$$s \in \Big(\frac{1}{r} - \frac{1}{2} + a_2, \frac{1}{r} + \frac{1}{2} + a_1\Big) \ \text{ if } \ \frac{1}{r} + \frac{1}{2} + a_1 < \alpha - \varepsilon. \tag{5.126}$$

By (5.122) for the solution vector we have $U \in [B^{s+\frac{1}{r}}_{r,q}(\Omega^+)]^6$ with

$$s + \frac{1}{r} = \alpha - \varepsilon + \frac{1}{r}$$

if (5.125) holds, and with

$$s + \frac{1}{r} \in \Big(\frac{2}{r} - \frac{1}{2} + a_2, \frac{2}{r} + \frac{1}{2} + a_1\Big)$$

if (5.126) holds. In the last case we can take

$$s + \frac{1}{r} = \frac{2}{r} + \frac{1}{2} + a_1 - \varepsilon.$$

Therefore, we have either

$$U \in [B^{\alpha-\varepsilon+\frac{1}{r}}_{r,q}(\Omega^+)]^6 \quad \text{or} \quad U \in [B^{\frac{1}{2}+\frac{2}{r}+a_1-\varepsilon}_{r,q}(\Omega^+)]^5,$$

in accordance with the inequalities (5.125) and (5.126). The last embedding in (5.124) (with $k = 3$) yields then that either

$$U \in [C^{\alpha-\varepsilon-\frac{2}{r}}(\overline{\Omega^+})]^6 \quad \text{or} \quad U \in [C^{\frac{1}{2}-\varepsilon+a_1-\frac{1}{r}}(\overline{\Omega^+})]^6.$$

These relations lead to the inclusions

$$U \in [C^{\kappa_m-\varepsilon-\frac{2}{r}}(\overline{\Omega^+})]^6, \tag{5.127}$$

where $\kappa_m = \min\{\alpha, a_1 + \frac{1}{2}\}$ and $0 < \kappa_m \leqslant \frac{1}{2}$ due to the inequalities (5.111). Since $r$ is sufficiently large and $\varepsilon$ is sufficiently small, the inclusions (5.127) accomplish the proof. □

*Remark* 5.23. Using exactly the same arguments, it can be shown that the similar uniqueness, existence and regularity results hold also true for the exterior mixed boundary value problem $(\mathrm{M})_\tau^-$. We note only that the solution $U = (u,\varphi,\psi,\vartheta)^\top \in [W_p^1(\Omega^-)]^6$, satisfying the decay condition (2.207), is representable again in the form of the single layer potential (5.100), where $f^{(e)} \in [B_{p,p}^{1-\frac{1}{p}}(S)]^6$ is the same as above, and $\widetilde{f} \in [\widetilde{B}_{p,p}^{1-\frac{1}{p}}(S_N)]^6$ is the unique solution of the pseudodifferential equation

$$r_{S_N}\mathcal{A}^-\widetilde{g} = \widetilde{F}^{(0)} \ \text{ on } \ S_N, \tag{5.128}$$

where

$$\mathcal{A}^- := [2^{-1}I_6 + \mathcal{K}]\mathcal{H}^{-1}, \quad \widetilde{F}^{(0)} := F^{(N)} - r_{S_N}\mathcal{A}^- f^{(e)}.$$

The operator $r_{S_N}\mathcal{A}^-$ has the same Fredholm properties as $r_{S_N}\mathcal{A}^+$, in particular, Lemma 5.20 holds with $r_{S_N}\mathcal{A}^-$ for $r_{S_N}\mathcal{A}^+$.

*Remark* 5.24. Lemma 5.20 with $p=q=2$ and $s=\frac{1}{2}$ and Theorems 5.21 with $p=2$ remain valid for Lipschitz domains due to Lemma 4.11 and the uniqueness Theorem 2.25.

*Remark* 5.25. The asymptotic expansions derived in [16] of solutions imply that for sufficiently smooth boundary data (e.g., $C^\infty$-smooth data say) the solution vector $U$ to the mixed boundary value problem $(\mathrm{M})_\tau^+$ belongs to the class of semi-regular functions described in Definition 2.2:

$$U \in [\mathbf{C}(\widetilde{\Omega}_\ell;\alpha)]^6,$$

where $\alpha = \frac{1}{2} - a_1 + \varepsilon$ with $a_1$ defined in (5.109) and $\varepsilon$ being an arbitrarily small positive number. Due to the relations (5.110), it is evident that $\frac{1}{2} < \alpha < 1$ if $0 < \varepsilon < \frac{1}{4} + \frac{1}{2}a_1$.

Moreover, the dominant terms of the asymptotic expansion of the solution vector $U$ near the curve $\ell_m$ can be represented as the product of $C^\infty$-smooth vector-functions and factors of the form $[\ln\varrho(x)]^{m_j-1}[\varrho(x)]^{\kappa_j+i\nu_j}$, where $\varrho(x)$ is the distance from a reference point $x$ to the curve $\ell_m$ (for details see [16, Subsection 6.2]). Therefore, near the curve $\ell_m$ the dominant singular terms of the corresponding generalized stress vector $\mathcal{T}U$ is represented as the product of $C^\infty$-smooth vector-functions and the singular factors $[\ln\varrho(x)]^{m_j-1}[\varrho(x)]^{-1+\kappa_j+i\nu_j}$. The numbers $\nu_j$ are different from zero, in general, and describe the oscillating character of the stress singularities.

The exponents $\kappa_j + i\nu_j$ are related to the corresponding eigenvalues $\lambda_j$ of the matrix (5.107) by the equalities

$$\kappa_j = \frac{1}{2} + \frac{\arg\lambda_j}{2\pi}, \quad \nu_j = -\frac{\ln|\lambda_j|}{2\pi}, \ \ j=1,2,\dots,6. \tag{5.129}$$

In the above expressions the parameter $m_j$ denotes the multiplicity of the eigenvalue $\lambda_j$.

It is evident that at the curve $\ell_m$ the components of the generalized stress vector $\mathcal{T}U$ behave like $\mathcal{O}([\ln\varrho(x)]^{m_0-1}[\varrho(x)]^{-\frac{1}{2}+a_1})$, where $m_0$ denotes the maximal multiplicity of the eigenvalues $\lambda_j$, $j=,\dots,6$. This is a global singularity effect for the first order derivatives of the vectors $U$. In contrast to the classical pure elasticity case (where $a_1 = 0$), here $a_1$ depends on the material parameters and is different from zero, in general. Since $a_1 \leqslant 0$, we see that the stress singularity exponents may become less than $-\frac{1}{2}$, in general.

## 6. Investigation of Model Crack Type Problems of Pseudo-Oscillations

In this section, first we investigate in detail model crack type problems. To describe principal qualitative aspects of the crack problems, for simplicity, first we assume that an elastic solid occupies a whole space $\mathbb{R}^3$ and contains an interior crack which coincides with a two-dimensional, two-sided smooth manifold $\Sigma$ with the crack edge $\ell_c := \partial\Sigma$. Denote $\mathbb{R}^3_\Sigma := \mathbb{R}^3 \setminus \overline{\Sigma}$. As in Subsection 2.3, the crack surface $\Sigma$ is considered as a submanifold of a closed surface $S_0$ surrounding a bounded domain $\overline{\Omega}_0$. We choose the direction of the unit normal vector on the fictional surface $S_0$ such that it is outward with respect to the domain $\Omega_0$. This agreement defines uniquely the direction of the normal vector on the crack surface $\Sigma$.

For the domain $\mathbb{R}^3_\Sigma$ we treat two model problems when on the crack surface $\Sigma$ there are prescribed either the Neumann type crack boundary conditions $(\text{CR-N})_\tau$ (see (2.176), (2.177)) or mixed

Neumann-Transmission crack type conditions $(\text{CR-NT})_\tau$ (see (2.178)–(2.184)). We prove unique solvability of the problems $(\text{CR-N})_\tau$ and $(\text{CR-NT})_\tau$ by the potential method and analyze regularity properties of solutions.

Afterwards, for a bounded domain $\Omega^+_\Sigma$ containing an interior crack $\Sigma$, we investigate in detail the crack type BVPs $(\text{D-CR-N})^+_\tau$ and $(\text{M-CR-N})^+_\tau$ with the Dirichlet and basic mixed boundary conditions on the exterior boundary $S$ of $\Omega^+_\Sigma$ associated with the dissection $S=\overline{S_D}\cup\overline{S_N}$. The BVPs $(\text{D-CR-N})^-_\tau$, $(\text{M-CR-N})^-_\tau$, $(\text{D-CR-NT})^\pm_\tau$, $(\text{M-CR-NT})^\pm_\tau$, $(\text{N-CR-N})^\pm_\tau$, and $(\text{N-CR-NT})^\pm_\tau$, can be studied by the same approach.

For simplicity, throughout this section we assume that $\Sigma$, $\ell_c=\partial\Sigma$, $S=\partial\Omega^\pm$, and $\ell_m=\partial S_D=\partial S_N$ are $C^\infty$-smooth if not otherwise stated.

6.1. **Crack type problem $(\text{CR-N})_\tau$.** We have to find a solution vector $U=(u,\varphi,\psi,\vartheta)^\top\in[W^1_{p,loc}(\mathbb{R}^3_\Sigma)]^6$ to the equation

$$A(\partial,\tau)U=0 \ \text{ in } \ \mathbb{R}^3_\Sigma, \tag{6.1}$$

possessing the decay properties (2.207) and satisfying the crack type boundary conditions on the faces of surface $\Sigma$ (cf. (2.176), (2.177)):

$$\big\{\mathcal{T}(\partial,n,\tau)U\big\}^+=F^{(+)} \ \text{ on } \ \Sigma, \tag{6.2}$$

$$\big\{\mathcal{T}(\partial,n,\tau)U\big\}^-=F^{(-)} \ \text{ on } \ \Sigma, \tag{6.3}$$

where $F^{(\pm)}=(F^{(\pm)}_1,\dots,F^{(\pm)}_6)^\top\in[B^{-\frac1p}_{p,p}(\Sigma)]^6$ are given vector functions on $\Sigma$ satisfying the following compatibility condition

$$F^{(+)}-F^{(-)}\in[\widetilde{B}^{-\frac1p}_{p,p}(\Sigma)]^6. \tag{6.4}$$

The imbedding (6.4) means that the extension of the vector function $F^{(+)}-F^{(-)}$ form $\Sigma$ onto the whole of $S_0$ by zero preserves the functional space $[\widetilde{B}^{-\frac1p}_{p,p}(\Sigma)]^6$. It is easy to see that (6.4) is a necessary condition for the problem $(\text{CR-N})_\tau$ to be solvable in the space $[W^1_{p,loc}(\mathbb{R}^3_\Sigma)]^6$.

Recall that these boundary conditions correspond to the case when the crack gap is *thermally insulated and electrically impermeable* (see Subsection 2.2).

Let us rewrite the boundary conditions (6.2), (6.3) in the following equivalent form

$$\big\{[\mathcal{T}(\partial,n,\tau)U]\big\}^+ + \big\{[\mathcal{T}(\partial,n,\tau)U]\big\}^- = F^{(+)}+F^{(-)} \ \text{ on } \ \Sigma, \tag{6.5}$$

$$\big\{[\mathcal{T}(\partial,n,\tau)U]\big\}^+ - \big\{[\mathcal{T}(\partial,n,\tau)U]\big\}^- = F^{(+)}-F^{(-)} \ \text{ on } \ \Sigma. \tag{6.6}$$

Due to Corollary 3.7 we look for a solution to the crack type BVP $(\text{CR-N})_\tau$ in the form

$$U=W(g)-V(h) \ \text{ in } \ \mathbb{R}^3_\Sigma, \tag{6.7}$$

where $W(g)=W_\Sigma(g)$ and $V(h)=V_\Sigma(h)$ are double and single layer potentials defined by (3.60) and (3.59), respectively, with $\Sigma$ for $S$,

$$g=[U]_\Sigma=\{U\}^+-\{U\}^-\in[\widetilde{B}^{1-\frac1p}_{p,p}(\Sigma)]^6 \tag{6.8}$$

is an unknown vector function on $\Sigma$, while

$$h=[\mathcal{T}U]_\Sigma=\{\mathcal{T}U\}^+-\{\mathcal{T}U\}^-=F^{(+)}-F^{(-)}\in[\widetilde{B}^{-\frac1p}_{p,p}(\Sigma)]^6 \tag{6.9}$$

is a given vector function on $\Sigma$.

In view of Theorem 4.4 and jump relations (4.2)–(4.5), it is evident that the vector function (6.7) with $g$ and $h$ as in (6.8) and (6.9) belongs to the space $[W^1_{p,loc}(\mathbb{R}^3_\Sigma)]^6$, satisfies the decay condition (2.207) and the boundary condition (6.6) on $\Sigma$. The remaining boundary condition (6.5) leads then to the pseudodifferential equation for the unknown vector function $g$

$$r_{\Sigma}\mathcal{L}g=F \ \text{ on } \ \Sigma, \tag{6.10}$$

where $\mathcal{L}$ is a pseudodifferential operator defined by (4.9) with $\Sigma$ for $S$ and

$$F=\frac12\,[F^{(+)}+F^{(-)}]+r_{\Sigma}\mathcal{K}h\in[B^{-\frac1p}_{p,p}(\Sigma)]^6 \tag{6.11}$$

with $h$ given by (6.9) and where the operator $\mathcal{K}$ is defined by (4.7). In what follows we show that the pseudodifferential equation (6.10) is uniquely solvable for arbitrary right hand side. To this end, we first formulate the following propositions.

**Lemma 6.1.** *Let the crack manifold $\Sigma$ and the crack edge $\partial\Sigma$ be Lipschitz. Then the crack problems* $(\text{CR-N})_\tau$ *and* $(\text{CR-NT})_\tau$ *possess at most one solution in the space* $[W^1_{2,loc}(\mathbb{R}^3_\Sigma)]^6$ *satisfying the decay properties* (2.207).

*Proof.* It is word for word of the proof of the uniqueness Theorem 2.26. □

**Lemma 6.2.** *Let $s \in \mathbb{R}$, $p > 1$, and $q \geq 1$. The operators*

$$r_{\Sigma}\mathcal{L} : [\widetilde{H}^s_p(\Sigma)]^6 \longrightarrow [H^{s-1}_p(\Sigma)]^6, \tag{6.12}$$

$$r_{\Sigma}\mathcal{L} : [\widetilde{B}^s_{p,q}(\Sigma)]^6 \longrightarrow [B^{s-1}_{p,q}(\Sigma)]^6, \tag{6.13}$$

*are invertible if*

$$\frac{1}{p} - \frac{1}{2} < s < \frac{1}{p} + \frac{1}{2}\,. \tag{6.14}$$

*Proof.* In Subsection 5.5, we have shown that the principal homogeneous symbol matrix $\mathfrak{S}(\mathcal{L}; x, \xi')$ of the operator $\mathcal{L}$ is even and homogeneous of order $+1$ in $\xi' = (\xi_1, \xi_2)$. Moreover, in the same subsection we have established that the symbol matrix $\mathfrak{S}(\mathcal{L}; x, \xi_1, \xi_2)$ is strongly elliptic. So we can apply the theory of strongly elliptic pseudodifferential equations on manifolds with boundary (see Appendix B, Theorem B.1).

Note that, since the principal homogeneous symbol matrix $\mathfrak{S}(\mathcal{L}; x, \xi_1, \xi_2)$ is even in $\xi' = (\xi_1, \xi_2)$, we have

$$M_{\mathcal{L}}(x) := [\mathfrak{S}(\mathcal{L}; x, 0, +1)]^{-1}[\mathfrak{S}(\mathcal{L}; x, 0, -1)] = I_6, \quad x \in \overline{\Sigma}. \tag{6.15}$$

Therefore all the eigenvalues $\lambda_1(x), \ldots, \lambda_6(x)$ of the matrix $M_{\mathcal{L}}(x)$ equal to 1 and

$$\delta_j(x) = \operatorname{Re}\big[(2\pi i)^{-1} \ln \lambda_j(x)\big] = 0, \quad j = 1, \ldots, 6, \tag{6.16}$$

where $\ln\zeta$ denotes the branch of the logarithm analytic in the complex plane cut along $(-\infty, 0]$ (see Appendix B). Therefore, by Theorem B.1 the operators (6.12) and (6.13) are Fredholm with zero index if the conditions (6.14) hold. Moreover, in view of Theorem B.1, it remains to show that for some particular values of the parameters $s$, $p$, and $q$, satisfying the inequalities (6.14), they are invertible. Let us take $s = 1/2$, $p = q = 2$ and recall that $\widetilde{H}^s_2(\Sigma) = \widetilde{B}^s_{2,2}(\Sigma)$ and $H^s_2(\Sigma) = B^s_{2,2}(\Sigma)$ for $s \in \mathbb{R}$. Thus the operators (6.12) and (6.13) coincide for the chosen particular values of the parameters and actually we have to prove that the null space of the operator $r_{\Sigma}\mathcal{L} : [\widetilde{H}^{\frac{1}{2}}_2(\Sigma)]^6 \to [H^{-\frac{1}{2}}_2(\Sigma)]^6$ is trivial. Indeed, let $g_0 \in [\widetilde{H}^{\frac{1}{2}}_2(\Sigma)]^6$ be a solution of the homogeneous equation $r_{\Sigma}\mathcal{L}g_0 = 0$ on $\Sigma$ and construct the vector function $U_0 = W(g_0)$. By Theorem 4.2 we see that $U_0 = W(g_0) \in [W^1_{2,loc}(\mathbb{R}^3_\Sigma)]^6$ and satisfies the decay conditions (2.207). Moreover, it is also easy to see that $U_0$ satisfies the homogeneous crack conditions (6.5), (6.6) due to Theorem 4.2. Therefore $U_0 = 0$ in $\mathbb{R}^3_\Sigma$ by Lemma 6.1. Consequently, $\{U_0\}^+ - \{U_0\}^- = g_0 = 0$ on $\Sigma$ which implies that the null space of the operator $r_{\Sigma}\mathcal{L} : [\widetilde{H}^{\frac{1}{2}}_2(\Sigma)]^6 \to [H^{-\frac{1}{2}}_2(\Sigma)]^6$ is trivial. Therefore the null spaces of the operators (6.12) and (6.13) are trivial as well due to Theorem B.1 which completes the proof. □

Now, we can prove the following existence result.

**Theorem 6.3.** *Let $F^{(\pm)} = (F^{(\pm)}_1, \ldots, F^{(\pm)}_6)^\top \in [B^{-\frac{1}{p}}_{p,p}(\Sigma)]^6$, the compatibility conditions* (6.4) *be fulfilled and*

$$\frac{4}{3} < p < 4. \tag{6.17}$$

*Then the crack type BVP* $(\text{CR-N})_\tau$ *has a unique solution $U \in [W^1_{p,loc}(\mathbb{R}^3_\Sigma)]^6 \cap \mathbf{Z}_\tau(\mathbb{R}^3_\Sigma)$ which is representable in the form*

$$U = W(g) - V(F^{(+)} - F^{(-)}) \ \ in \ \ \mathbb{R}^3_\Sigma, \tag{6.18}$$

*where $g \in [\widetilde{B}^{1-\frac{1}{p}}_{p,p}(\Sigma)]^6$ is defined by the uniquely solvable pseudodifferential equation* (6.10), (6.11).

*Proof.* First we note that in accordance with Lemma 6.2, the pseudodifferential equation (6.10), (6.11) is uniquely solvable for $s = 1 - \frac{1}{p}$ with $p$ from the interval (6.17), since the inequalities (6.14) are fulfilled. This implies that the crack type BVP $(\text{CR-N})_\tau$ is solvable in the space $[W^1_{p,loc}(\mathbb{R}^3_\Sigma)]^6 \cap \mathbf{Z}_\tau(\mathbb{R}^3_\Sigma)$ with $p$ satisfying inequalities (6.17).

Next, we show the uniqueness of solution in the space $[W^1_{p,loc}(\mathbb{R}^3_\Sigma)]^6 \cap \mathbf{Z}_\tau(\mathbb{R}^3_\Sigma)$ with $p$ satisfying (6.17). Note that, $p = 2$ belongs to the interval (6.17) and for $p = 2$ the uniqueness holds due to Lemma 6.1. Now, let $U \in [W^1_{p,loc}(\mathbb{R}^3_\Sigma)]^6 \cap \mathbf{Z}_\tau(\mathbb{R}^3_\Sigma)$ be some solution of the homogeneous crack type BVP $(\text{CR-N})_\tau$. Then

$$\begin{aligned} &\{U\}^\pm \in [B^{1-1/p}_{p,p}(\Sigma)]^6, \quad \{\mathcal{T}U\}^\pm \in [B^{-1/p}_{p,p}(\Sigma)]^6, \\ &\{U\}^+ - \{U\}^- \in [\widetilde{B}^{1-1/p}_{p,p}(\Sigma)]^6, \quad \{\mathcal{T}U\}^+ - \{\mathcal{T}U\}^- \in [\widetilde{B}^{-1/p}_{p,p}(\Sigma)]^6, \end{aligned} \tag{6.19}$$

since actually $U \in [W^1_{p,loc}(\mathbb{R}^3_\Sigma)]^6 \cap [C^\infty(\mathbb{R}^3_\Sigma)]^6$ due to the interior regularity results.

In accordance with Corollary 3.7 for the solution vector $U$ of the homogeneous crack type BVP $(\text{CR-N})_\tau$ we have then the representation

$$U = W_\Sigma(g) \equiv W(g) \ \text{ in } \ \mathbb{R}^3_\Sigma, \tag{6.20}$$

with $g = [U]_\Sigma = \{U\}^+ - \{U\}^- \in [\widetilde{B}^{1-\frac{1}{p}}_{p,p}(\Sigma)]^6$. Since $U$ satisfies the homogeneous crack type conditions on $\Sigma$, we arrive at the equation $r_{\Sigma}\{\mathcal{T}U\}^\pm = r_{\Sigma}\mathcal{L}g = 0$ on $\Sigma$, whence $g = 0$ on $\Sigma$ follows due to Lemma 6.2 in view of the inequality (6.17). Therefore, $U = 0$ in $\mathbb{R}^3_\Sigma$. □

As in the case of mixed type BVP $(\text{M})^+_\tau$ (see Theorem 5.22), we can prove almost the best regularity results for solutions to the crack type BVP $(\text{CR-N})_\tau$.

**Theorem 6.4.** *Let the inclusions* $F^{(\pm)} = (F^{(\pm)}_1, \dots, F^{(\pm)}_6)^\top \in [B^{-\frac{1}{p}}_{p,p}(\Sigma)]^6$ *and the compatibility conditions* (6.4) *hold and let*

$$\frac{4}{3} < p < 4, \ \ 1 < r < \infty, \ \ 1 \le q \le \infty, \ \ \frac{1}{r} - \frac{1}{2} < s < \frac{1}{r} + \frac{1}{2}\,. \tag{6.21}$$

*Further, let* $U \in [W^1_{p,loc}(\mathbb{R}^3_\Sigma)]^6 \cap \mathbf{Z}_\tau(\mathbb{R}^3_\Sigma)$ *be a unique solution to the crack type BVP* $(\text{CR-N})_\tau$. *Then the following hold:*

(i) *if*

$$F^{(\pm)} \in [B^{s-1}_{r,r}(\Sigma)]^6, \ \ F^{(+)} - F^{(-)} \in [\widetilde{B}^{s-1}_{r,r}(\Sigma)]^6,$$

*then* $U \in [H^{s+\frac{1}{r}}_{r,loc}(\mathbb{R}^3_\Sigma)]^6$;

(ii) *if*

$$F^{(\pm)} \in [B^{s-1}_{r,q}(\Sigma)]^6, \ \ F^{(+)} - F^{(-)} \in [\widetilde{B}^{s-1}_{r,q}(\Sigma)]^6,$$

*then*

$$U \in [B^{s+\frac{1}{r}}_{r,q,loc}(\mathbb{R}^3_\Sigma)]^6; \tag{6.22}$$

(iii) *if* $\alpha > 0$ *is not integer and*

$$F^{(\pm)} \in [B^{\alpha-1}_{\infty,\infty}(\Sigma)]^6, \ \ F^{(+)} - F^{(-)} \in [\widetilde{B}^{\alpha-1}_{\infty,\infty}(\Sigma)]^6, \tag{6.23}$$

*then*

$$U \in \bigcap_{\alpha' < \kappa_c} [C^{\alpha'}(\overline{\Omega})]^6,$$

*with* $\kappa_c = \min\{\alpha, \frac{1}{2}\} > 0$; *here* $\Omega$ *is either* $\Omega_0$ *or* $\mathbb{R}^3 \setminus \overline{\Omega_0}$, *where* $\Omega_0$ *is a domain with* $C^\infty$ *regular boundary* $S_0 = \partial\Omega_0$ *which contains the crack surface* $\Sigma$ *as a proper part.*

*Proof.* It is word for word of the proof of Theorem 5.22 with $a_1 = a_2 = 0$. □

*Remark* 6.5. If we compare the regularity results exposed in Theorems 5.22 and 6.4 for solutions of mixed $(\text{M})^\pm_\tau$ and crack type $(\text{CR-N})_\tau$ BVPs near the exceptional curves, i.e., near the curve $\ell_m$ where the Dirichlet and Neumann conditions collide and near the crack edge $\ell_c$, we see that the Hölder smoothness exponent for solution vectors at the curve $\ell_c$ is greater than the Hölder smoothness exponent at the curve $\ell_m$, in general. In particular, if boundary data are sufficiently smooth, $\alpha > 1/2$

say, due to Theorem 5.22(iii) solutions to mixed BVPs belong then to the class $\bigcap\limits_{\alpha'<\kappa_m} C^{\alpha'}$ at the curve $\ell_m$ where the positive upper bound $\kappa_m = a_1 + \frac{1}{2}$ depends on the material parameters essentially and may take an arbitrary value from the interval $(0, \frac{1}{2}]$, since in general $a_1$ may take an arbitrary value from the interval $(-\frac{1}{2}, 0]$ depending on the material parameters.

In the case of crack type BVPs with $\alpha > 1/2$, due to Theorem 6.4(iii) solutions belong to the class $\bigcap\limits_{\alpha'<1/2} C^{\alpha'}$ at the crack edge $\ell_c$ and as we see the upper bound $\kappa_c = 1/2$ does not depend on the material parameters. Thus $\kappa_m \le \kappa_c$, which proves that, in general, solutions to the crack type BVPs possess higher regularity near the crack edge $\ell_c$ than solutions to the mixed type BVPs at the exceptional collision curve $\ell_m$ (cf. [15], [10], [11]).

*Remark* 6.6. The asymptotic expansions of solutions at the crack edge derived in [16] imply that for sufficiently smooth boundary data (e.g., $C^\infty$-smooth data say) the solution vector $U$ to the crack problem (CR-N)$_\tau$ belongs to the class of semi-regular functions described in Definition 2.2,

$$U \in [\mathbf{C}(\mathbb{R}^3_\Sigma; \alpha)]^6 \ \text{ with } \ \alpha = \frac{1}{2}\,.$$

Moreover, the dominant terms of the asymptotic expansion of the solution vector $U$ near the curve $\ell_c$ has the form $\mathcal{O}(\varrho^{\frac{1}{2}}(x))$, where $\varrho(x)$ is the distance from a reference point $x$ to the curve $\ell_c$, while the dominant singular terms of the corresponding generalized stress vector $\mathcal{T}U$ are estimated by the expressions of type $\mathcal{O}(\varrho^{-\frac{1}{2}}(x))$ (for details see [16, Subsection 6.1]).

6.2. **Crack type problem (CR-NT)$_\tau$.** In this case we have to find a solution vector $U = (u, \varphi, \psi, \vartheta)^\top \in [W^1_{p,loc}(\mathbb{R}^3_\Sigma)]^6$ to the equation $A(\partial, \tau)U = 0$ in $\mathbb{R}^3_\Sigma$ possessing the decay properties (2.207) and satisfying the boundary conditions (2.178)–(2.184) on the crack faces. Recall that these boundary conditions correspond to the case when the crack gap is *thermally and electrically conductive* (see Subsection 2.2).

As in the previous case, we first reformulate the crack conditions (2.178)–(2.184) on $\Sigma$ equivalently,

$$\big\{[\mathcal{T}(\partial,n)U]_k\big\}^+ - \big\{[\mathcal{T}(\partial,n)U]_k\big\}^- = F^{**}_k := F^{(+)}_k - F^{(-)}_k, \ \ k = 1,2,3, \tag{6.24}$$

$$\big\{[\mathcal{T}(\partial,n)U]_j\big\}^+ - \big\{[\mathcal{T}(\partial,n)U]_j\big\}^- = F^{**}_j, \ \ j = 4,5,6, \tag{6.25}$$

$$\{U_j\}^+ - \{U_j\}^- = f^{**}_j, \ \ j = 4,5,6, \tag{6.26}$$

$$\big\{[\mathcal{T}(\partial,n)U]_k\big\}^+ + \big\{[\mathcal{T}(\partial,n)U]_k\big\}^- = F^{(+)}_k + F^{(-)}_k, \ \ k = 1,2,3, \tag{6.27}$$

where

$$\begin{aligned} F^{(\pm)}_k \in B^{-\frac{1}{p}}_{p,p}(\Sigma), \quad F^{**}_k := F^{(+)}_k - F^{(-)}_k \in \widetilde{B}^{-\frac{1}{p}}_{p,p}(\Sigma), \\ f^{**}_j \in \widetilde{B}^{1-\frac{1}{p}}_{p,p}(\Sigma), \quad F^{**}_j \in \widetilde{B}^{-\frac{1}{p}}_{p,p}(\Sigma), \end{aligned} \qquad k = 1,2,3, \ \ j = 4,5,6. \tag{6.28}$$

Again, due to Corollary 3.7 we look for a solution to the crack type BVP (CR-NT)$_\tau$ in the form

$$U = W(g) - V(h) \ \text{ in } \ \mathbb{R}^3_\Sigma, \tag{6.29}$$

where $W(g) = W_\Sigma(g)$ and $V(h) = V_\Sigma(h)$ are double and single layer potentials defined by (3.60) and (3.59), respectively, with $\Sigma$ for $S$, and where $g$ and $h$ are related to the solution vector $U$ by the relations:

$$g = (g_1, g_2, \dots, g_6)^\top = \{U\}^+ - \{U\}^- \in [\widetilde{B}^{1-\frac{1}{p}}_{p,p}(\Sigma)]^6, \tag{6.30}$$

$$h = (h_1, h_2, \dots, h_6)^\top = \{\mathcal{T}U\}^+ - \{\mathcal{T}U\}^- \in [\widetilde{B}^{-\frac{1}{p}}_{p,p}(\Sigma)]^6. \tag{6.31}$$

It is easy to see that the vector function $h$ is defined explicitly from the boundary conditions (6.24), (6.25),

$$h_j = F^{**}_j \in \widetilde{B}^{-\frac{1}{p}}_{p,p}(\Sigma), \ \ j = 1,\dots,6. \tag{6.32}$$

In view of (6.26) and (6.30) the components $g_4$, $g_5$, and $g_5$ of the vector function $g$ are also explicitly defined

$$g_j = f^{**}_j \in \widetilde{B}^{1-\frac{1}{p}}_{p,p}(\Sigma), \ \ j = 4,5,6. \tag{6.33}$$

So as we see, if (6.32) and (6.33) hold, then the vector function $U$ defined by (6.29) automatically satisfies all conditions of the crack type BVP (CR-NT)$_\tau$ except the three boundary conditions in

(6.27). Keeping in mind that in the representation formula (6.29) the only unknowns remain the functions $g_1$, $g_2$, and $g_3$, the first three components of the vector $g$, and taking into account the boundary conditions (6.27), we arrive at the following pseudodifferential equations

$$r_{\Sigma}\sum_{j=1}^{3}\mathcal{L}_{kj}g_j = F_k \ \text{ on } \ \Sigma, \ \ k=1,2,3, \tag{6.34}$$

where $\mathcal{L}=[\mathcal{L}_{kj}]_{6\times 6}$ is a pseudodifferential operator defined by (4.9) with $\Sigma$ for $S$ and

$$F_k = \frac{1}{2}\,[F_k^{(+)}+F_k^{(-)}] - r_{\Sigma}\sum_{j=4}^{6}\mathcal{L}_{kj}f_j^{**} + r_{\Sigma}\sum_{j=1}^{6}\mathcal{K}_{kj}F_j^{**} \in B_{p,p}^{-\frac{1}{p}}(\Sigma), \ \ k=1,2,3, \tag{6.35}$$

where $f_j^{**}$, $j=1,2,3$, and $F_j^{**}$, $j=1,2,\dots,6$, are given functions from the spaces shown in (6.28) and the pseudodifferential operator $\mathcal{K}=[\mathcal{K}_{kj}]_{6\times 6}$ is defined by (4.7) with $\Sigma$ for $S$.

Let us introduce the matrix pseudodifferential operator

$$\mathfrak{L} := [\mathcal{L}_{kj}]_{3\times 3}, \ \ 1\le k,j\le 3, \tag{6.36}$$

which coincides with the first basic $3\times 3$ block of the matrix pseudodifferential operator $\mathcal{L}=[\mathcal{L}_{kj}]_{6\times 6}$ defined by (4.9).

Further, we rewrite the system of equation (6.34) in matrix form

$$r_{\Sigma}\mathfrak{L}g^{(3)} = F \ \text{ on } \ \Sigma, \tag{6.37}$$

where

$$g^{(3)} = (g_1,g_2,g_3)^\top \in [\widetilde{B}_{p,p}^{1-\frac{1}{p}}(\Sigma)]^3 \tag{6.38}$$

is the unknown vector function and

$$F = (F_1,F_2,F_3)^\top \in [B_{p,p}^{-\frac{1}{p}}(\Sigma)]^3 \tag{6.39}$$

is a known right hand side defined in (6.35).

From the properties of the operator $\mathcal{L}$, described in Lemma 5.9, it follows immediately that the pseudodifferential operator $\mathfrak{L}$ is strongly elliptic as well and the principal homogeneous symbol matrix $\mathfrak{S}(\mathfrak{L};x,\xi') = [\mathfrak{S}_{kj}(\mathfrak{L};x,\xi')]_{3\times 3}$ of the operator $\mathfrak{L}$ is even and homogeneous of order $+1$ in $\xi'=(\xi_1,\xi_2)$. Therefore, we can apply Theorem B.1 and prove the following counterpart of Lemma 6.2.

**Lemma 6.7.** *Let $s\in\mathbb{R}$, $p>1$, and $q\ge 1$. The operators*

$$r_{\Sigma}\mathfrak{L} : [\widetilde{H}_p^s(\Sigma)]^3 \longrightarrow H_p^{s-1}(\Sigma)]^3, \tag{6.40}$$

$$r_{\Sigma}\mathfrak{L} : [\widetilde{B}_{p,q}^s(\Sigma)]^3 \longrightarrow [B_{p,q}^{s-1}(\Sigma)]^3, \tag{6.41}$$

*are invertible if*

$$\frac{1}{p}-\frac{1}{2} < s < \frac{1}{p}+\frac{1}{2}\,. \tag{6.42}$$

*Proof.* By the same arguments as in the proof of Lemma 6.2 we easily derive that the operators (6.40) and (6.41) are Fredholm with zero index if the inequalities (6.42) hold. Therefore we need only to prove that the operator (6.40) has the trivial null space if $s=1/2$, $p=q=2$. Let $g_0 = (g_{01},g_{02},g_{03})^\top \in [\widetilde{H}_2^{\frac{1}{2}}(\Sigma)]^3$ be a solution of the homogeneous equation $r_{\Sigma}\mathfrak{L}g_0=0$ on $\Sigma$. We set $f_0 := (g_0,0,0,0)^\top = (g_{01},g_{02},g_{03},0,0,0)^\top$ and construct the vector function $U_0 = W(f_0)\equiv W_{\Sigma}(f_0)$. By Theorem 4.2 we see that $U_0 = W(f_0)\in[W^1_{2,loc}(\mathbb{R}^3_{\Sigma})]^6$ and $U_0$ satisfies the decay conditions (2.207). Moreover, it is also easy to see that $U_0$ satisfies the homogeneous crack conditions (6.24)–(6.27) due to Theorem 4.2 and the homogeneous equation for $g_0$ on $\Sigma$. By the uniqueness Lemma 6.1 we conclude $U_0=0$ in $\mathbb{R}^3_{\Sigma}$. Consequently, $\{U_0\}^+ - \{U_0\}^- = f_0 = 0$ on $\Sigma$ implying that the null space of the operator $r_{\Sigma}\mathfrak{L} : [\widetilde{H}_2^{\frac{1}{2}}(\Sigma)]^3 \to [H_2^{-\frac{1}{2}}(\Sigma)]^3$ is trivial. Therefore under the condition (6.42), the null spaces of the operators (6.40) and (6.41) are trivial as well due to Theorem B.1 which completes the proof. □

Lemma 6.7 immediately leads to the following existence and regularity results which can be proved by means of exactly the same arguments as Theorems 6.3 and 6.4.

**Theorem 6.8.** *Let conditions* (6.28) *be fulfilled and* $\frac{4}{3} < p < 4$. *Then the crack type BVP* $(\text{CR-NT})_\tau$ *has a unique solution* $U \in [W^1_{p,loc}(\mathbb{R}^3_\Sigma)]^6 \cap \mathbf{Z}_\tau(\mathbb{R}^3_\Sigma)$ *which is representable in the form*

$$U = W(g^*) + W(f^{**}) - V(F^{**}) \ \ in \ \ \mathbb{R}^3_\Sigma, \tag{6.43}$$

*where* $F^{**} := (F_1^{**}, \dots, F_6^{**})^\top$ *and* $f^{**} := (0,0,0,f_4^{**},f_5^{**},f_6^{**})^\top$ *are given boundary data defined in* (6.24)–(6.28), *while the unknown vector function* $g^* = (g^{(3)},0,0,0)^\top$ *with* $g^{(3)} = (g_1,g_2,g_3)^\top \in [\widetilde{B}^{1-\frac{1}{p}}_{p,p}(\Sigma)]^3$ *is defined by the uniquely solvable pseudodifferential equation* (6.37), *i.e., the equations* (6.34), (6.35).

**Theorem 6.9.** *Let conditions* (6.28) *be fulfilled and let*

$$\frac{4}{3} < p < 4, \ \ 1 < r < \infty, \ \ 1 \le q \le \infty, \ \ \frac{1}{r} - \frac{1}{2} < s < \frac{1}{r} + \frac{1}{2}\,, \tag{6.44}$$

*Further, let* $U \in [W^1_{p,loc}(\mathbb{R}^3_\Sigma)]^6 \cap \mathbf{Z}_\tau(\mathbb{R}^3_\Sigma)$ *be a unique solution to the crack type BVP* $(\text{CR-NT})_\tau$. *Then the following hold:*

(i) *if*

$$\begin{gathered} F_k^{(\pm)} \in B^{s-1}_{r,r}(\Sigma), \ \ F_k^{**} := F_k^{(+)} - F_k^{(-)} \in \widetilde{B}^{s-1}_{r,r}(\Sigma), \\ f_j^{**} \in \widetilde{B}^{s}_{r,r}(\Sigma), \ \ F_j^{**} \in \widetilde{B}^{s-1}_{r,r}(\Sigma), \ \ k=1,2,3, \ \ j=4,5,6, \end{gathered} \tag{6.45}$$

*then* $U \in [H^{s+\frac{1}{r}}_{r,loc}(\mathbb{R}^3_\Sigma)]^6$;

(ii) *if*

$$\begin{gathered} F_k^{(\pm)} \in B^{s-1}_{r,q}(\Sigma), \ \ F_k^{**} := F_k^{(+)} - F_k^{(-)} \in \widetilde{B}^{s-1}_{r,q}(\Sigma), \\ f_j^{**} \in \widetilde{B}^{s}_{r,q}(\Sigma), \ \ F_j^{**} \in \widetilde{B}^{s-1}_{r,q}(\Sigma), \ \ k=1,2,3, \ \ j=4,5,6, \end{gathered} \tag{6.46}$$

*then*

$$U \in [B^{s+\frac{1}{r}}_{r,q,loc}(\mathbb{R}^3_\Sigma)]^6; \tag{6.47}$$

(iii) *if* $\alpha > 0$ *is not integer and*

$$\begin{gathered} F_k^{(\pm)} \in B^{\alpha-1}_{\infty,\infty}(\Sigma), \ \ F_k^{**} := F_k^{(+)} - F_k^{(-)} \in \widetilde{B}^{\alpha-1}_{\infty,\infty}(\Sigma), \ \ F_j^{**} \in \widetilde{B}^{\alpha-1}_{\infty,\infty}(\Sigma), \\ f_j^{**} \in C^\alpha(\Sigma), \ \ r_{\partial\Sigma} f_j^{**} = 0, \ \ k=1,2,3, \ \ j=4,5,6, \end{gathered} \tag{6.48}$$

*then*

$$U \in \bigcap_{\alpha' < \kappa_c} [C^{\alpha'}(\overline{\Omega})]^6,$$

*with* $\kappa_c = \min\{\alpha, \frac{1}{2}\} > 0$; *here* $\Omega$ *is either* $\Omega_0$ *or* $\mathbb{R}^3 \setminus \overline{\Omega_0}$, *where* $\Omega_0$ *is a domain with* $C^\infty$ *regular boundary* $S_0 = \partial\Omega_0$ *which contains the crack surface* $\Sigma$ *as a proper part.*

*Remark* 6.10. If we compare the regularity results exposed in Theorems 5.22 and 6.9 for solutions of mixed $(\text{M})^\pm_\tau$ and crack type $(\text{CR-NT})_\tau$ BVPs near the exceptional curves, i.e., near the curve $\ell_m$ where the Dirichlet and Neumann conditions collide and near the crack edge $\ell_c$, we see that the Hölder smoothness exponent for solution vectors at the curve $\ell_c$ is greater than the Hölder smoothness exponent at the curve $\ell_m$, since $\kappa_m \le \kappa_c$, in general (cf. Remark 6.5).

*Remark* 6.11. By the arguments applied in the proof of Lemma 5.9, we can show that inequality (5.68) holds true for an arbitrary Lipschitz surface $S$ (cf. [75, Ch. 7]), which implies that if the crack surface $\Sigma$ and the crack edge $\partial\Sigma$ are Lipschitz, then the operator

$$\mathcal{L} : [\widetilde{H}^{\frac{1}{2}}_2(\Sigma)]^6 \longrightarrow [H^{-\frac{1}{2}}_2(\Sigma)]^6 \tag{6.49}$$

is Fredholm operator with zero index ([75, Ch. 2]). Then by the uniqueness Lemma 6.1 we can prove that the null space of the operator (6.49) is trivial and, consequently, it is invertible. Therefore it is easy to show that Lemmas 6.2 and 6.7 with $p = q = 2$ and $s = \frac{1}{2}$, and Theorems 6.3 and 6.8 with $p = 2$ remain valid if the crack surface $\Sigma$ and the crack edge $\partial\Sigma$ are Lipschitz.

*Remark* 6.12. Applying again the asymptotic expansions of solutions at the crack edge derived in [16] imply that for sufficiently smooth boundary data (e.g., $C^\infty$-smooth data say) the solution vector $U$ to the crack problem $(\text{CR-NT})_\tau$ belongs to the class of semi-regular functions described in Definition 2.5,

$$U \in [\mathbf{C}(\widetilde{\mathbb{R}}^3_\Sigma;\alpha)]^6 \ \text{ with } \ \alpha = \frac{1}{2}\,.$$

Moreover, the dominant terms of the asymptotic expansion of the solution vector $U$ near the curve $\ell_c$ has the form $\mathcal{O}(\varrho^{\frac{1}{2}}(x))$, where $\varrho(x)$ is the distance from a reference point $x$ to the curve $\ell_c$, while the dominant singular terms of the corresponding generalized stress vector $\mathcal{T}U$ are estimated by the expressions of type $\mathcal{O}(\varrho^{-\frac{1}{2}}(x))$ (for details see [16, Subsection 6.1]).

6.3. **Interior crack type problem $(\text{D-CR-N})^+_\tau$.** Let an elastic solid occupy a bounded domain $\Omega^+$ with boundary $S = \partial\Omega^+$ and contain an interior crack $\Sigma \subset \Omega^+$, $S \cap \overline{\Sigma} = \varnothing$. As above, here we assume that the crack surface $\Sigma$ is a submanifold of a closed surface $S_0$ surrounding a bounded domain $\overline{\Omega}_0 \subset \Omega^+$. Moreover, we again assume that $S$, $\Sigma$, and $\ell_c = \partial\Sigma$ are $C^\infty$ regular if not otherwise stated.

The problem we would like to study in this subsection can be reformulated as follows (see Subsection 2.3): Find a solution $U = (u,\varphi,\psi,\vartheta)^\top \in [W^1_p(\Omega^+_\Sigma)]^6$ to the equation

$$A(\partial,\tau)U = 0 \ \text{ in } \ \Omega^+_\Sigma := \Omega^+ \setminus \overline{\Sigma}, \tag{6.50}$$

satisfying the Dirichlet boundary condition on the exterior surface $S = \partial\Omega^+$

$$\{U\}^+ = f \ \text{ on } \ S, \tag{6.51}$$

and $(\text{CR-N})_\tau$ type conditions on the crack faces (see (6.2), (6.3) and (6.5), (6.6))

$$\big\{[\mathcal{T}(\partial,n)U]\big\}^+ + \big\{[\mathcal{T}(\partial,n)U]\big\}^- = F^{(+)} + F^{(-)} \ \text{ on } \ \Sigma, \tag{6.52}$$

$$\big\{[\mathcal{T}(\partial,n)U]\big\}^+ - \big\{[\mathcal{T}(\partial,n)U]\big\}^- = F^{(+)} - F^{(-)} \ \text{ on } \ \Sigma, \tag{6.53}$$

where $f \in [B^{1-\frac{1}{p}}_{p,p}(S)]^6$ and $F^{(\pm)} = (F^{(\pm)}_1,\dots,F^{(\pm)}_6)^\top \in [B^{-\frac{1}{p}}_{p,p}(\Sigma)]^6$ are given vector functions on $S$ and $\Sigma$, respectively. We assume the following compatibility condition

$$F := F^{(+)} - F^{(-)} \in [\widetilde{B}^{-\frac{1}{p}}_{p,p}(\Sigma)]^6. \tag{6.54}$$

We look for a solution vector in the form

$$U = V_S(\mathcal{H}^{-1}_S g) + W_\Sigma(h) - V_\Sigma(F) \ \text{ in } \ \Omega^+_\Sigma, \tag{6.55}$$

where $V_S$, $V_\Sigma$, and $W_\Sigma$ are single and double layer potentials defined by (3.59) and (3.60), $\mathcal{H}_S$ is a pseudodifferential operator defined by (4.6) and $\mathcal{H}^{-1}_S$ is the inverse to the operator (5.80), $g = (g_1,\dots,g_6)^\top \in [B^{1-\frac{1}{p}}_{p,p}(S)]^6$ and $h = (h_1,\dots,h_6)^\top \in [\widetilde{B}^{1-\frac{1}{p}}_{p,p}(\Sigma)]^6$ are unknown vector functions.

It is evident that the differential equation (6.50) and the crack condition (6.53) are satisfied automatically, while the conditions (6.51) and (6.52) lead to the system of pseudodifferential equations

$$g + r_{_S} W_\Sigma(h) = \Phi^{(1)} \ \text{ on } \ S, \tag{6.56}$$

$$r_{_\Sigma}\mathcal{T}(\partial,n)V_S(\mathcal{H}^{-1}_S g) + r_{_\Sigma}\mathcal{L}_\Sigma h = \Phi^{(2)} \ \text{ on } \ \Sigma, \tag{6.57}$$

where

$$\Phi^{(1)} := f + r_{_S} V_\Sigma(F) \in [B^{1-\frac{1}{p}}_{p,p}(S)]^6, \quad \Phi^{(2)} := \frac{1}{2}F + r_{_\Sigma}\mathcal{K}_\Sigma F \in [B^{-\frac{1}{p}}_{p,p}(\Sigma)]^6. \tag{6.58}$$

Here $\mathcal{K}_\Sigma$ and $\mathcal{L}_\Sigma$ are pseudodifferential operators defined by (4.7) and (4.9) with $\Sigma$ for $S$.

Denote the operator generated by the left hand side expressions in (6.56), (6.57) by $\mathcal{D}$ which acts on the pair of the sought for vectors $(g,h)^\top$,

$$\mathcal{D} := \begin{bmatrix} I_6 & r_{_S} W_\Sigma \\ r_{_\Sigma}\mathcal{T}(\partial,n)V_S(\mathcal{H}^{-1}_S) & r_{_\Sigma}\mathcal{L}_\Sigma \end{bmatrix}_{12\times 12}. \tag{6.59}$$

Evidently, the operators $r_{_S} W_\Sigma$ and $r_{_\Sigma}\mathcal{T}(\partial,n)V_S(\mathcal{H}^{-1}_S)$ are smoothing operators, since the manifolds $S$ and $\Sigma$ are disjoint.

Set

$$\Psi = (g,h)^\top, \quad \Phi = (\Phi^{(1)}, \Phi^{(2)})^\top,$$

and rewrite equations (6.56), (6.57) in matrix form

$$\mathcal{D}\Psi = \Phi. \tag{6.60}$$

Theorem 4.4 yields the following mapping properties

$$\mathcal{D} : \mathbf{X}^s_p \longrightarrow \mathbf{Y}^s_p, \tag{6.61}$$

$$\mathcal{D} : \mathbf{X}^s_{p,t} \longrightarrow \mathbf{Y}^s_{p,t}, \tag{6.62}$$

where $s \in \mathbb{R}$, $1 < p < \infty$, $1 \le t \le \infty$, and

$$\mathbf{X}^s_p := [H^s_p(S)]^6 \times [\widetilde{H}^s_p(\Sigma)]^6, \quad \mathbf{Y}^s_p := [H^s_p(S)]^6 \times [H^{s-1}_p(\Sigma)]^6,$$

$$\mathbf{X}^s_{p,t} := [B^s_{p,t}(S)]^6 \times [\widetilde{B}^s_{p,t}(\Sigma)]^6, \quad \mathbf{Y}^s_{p,t} := [B^s_{p,t}(S)]^6 \times [B^{s-1}_{p,t}(\Sigma)]^6.$$

Further, let us consider the operator

$$\widetilde{\mathcal{D}} := \begin{bmatrix} I_6 & 0 \\ 0 & r_\Sigma \mathcal{L}_\Sigma \end{bmatrix}_{12\times 12}. \tag{6.63}$$

It is evident that $\widetilde{\mathcal{D}}$ has the same mapping properties as the operator $\mathcal{D}$ and the operator $\mathcal{D} - \widetilde{\mathcal{D}}$, with the same domain and range spaces as in (6.61), (6.62), is a compact operator. Moreover, in view of Lemma 6.2 the operators

$$\widetilde{\mathcal{D}} : \mathbf{X}^s_p \longrightarrow \mathbf{Y}^s_p, \quad \widetilde{\mathcal{D}} : \mathbf{X}^s_{p,t} \longrightarrow \mathbf{Y}^s_{p,t}, \tag{6.64}$$

are invertible if

$$\frac{1}{p} - \frac{1}{2} < s < \frac{1}{p} + \frac{1}{2}\,. \tag{6.65}$$

Therefore the operators (6.61) and (6.62) are Fredholm with zero index for arbitrary $p$ and $s$ satisfying the inequalities (6.65) hold (see Theorem B.1).

**Lemma 6.13.** *The operators* (6.61) *and* (6.62) *are invertible if the inequalities* (6.65) *hold.*

*Proof.* Since the operators (6.61) and (6.62) are Fredholm with zero index for arbitrary $s$ and $p$ satisfying (6.65), in accordance with Theorem B.1 we need only to show that the corresponding null-spaces are trivial for some particular values of the parameters $s$ and $p$ satisfying the inequalities (6.65). Let us take $s = \frac{1}{2}$ and $p = 2$, and prove that the homogeneous system $\mathcal{D}\Psi = 0$, i.e., the equations (6.56), (6.57) with $\Phi^{(1)} = \Phi^{(2)} = 0$ have only the trivial solution in the space $\mathbf{X}^{\frac{1}{2}}_2 := [H^{\frac{1}{2}}_2(S)]^6 \times [\widetilde{H}^{\frac{1}{2}}_2(\Sigma)]^6$. Indeed, let $\Psi_0 = (g_0, h_0)^\top \in [H^{\frac{1}{2}}_2(S)]^6 \times [\widetilde{H}^{\frac{1}{2}}_2(\Sigma)]^6$ be a solution of the homogeneous system (6.56), (6.57) and construct the vector

$$U_0(x) = V_S(\mathcal{H}^{-1}_S g_0)(x) + W_\Sigma(h_0)(x), \ \ x \in \mathbb{R}^3 \setminus (S \cup \overline{\Sigma}).$$

Now, it is easy to show that the embedding $U_0 \in [W^1_2(\Omega^+_\Sigma)]^6$ holds and $U_0$ solves the homogeneous BVP (D-CR-N)$^+_\tau$. By Theorem 2.25 we conclude $U_0 = 0$ in $\Omega^+_\Sigma$. Hence $\{U_0\}^+_\Sigma - \{U_0\}^-_\Sigma = h_0 = 0$ on $\Sigma$ follows immediately. Therefore, we get $U_0 = V_S(\mathcal{H}^{-1}_S g_0) = 0$ in $\Omega^+_\Sigma$ which implies $g_0 = \{U_0\}^+_S = 0$ on $S$. Thus $\Psi_0 = (g_0, h_0)^\top = 0$ and the operator $\mathcal{D}$ has a trivial null space in $\mathbf{X}^{\frac{1}{2}}_2$. Taking into account the relations $H^s_2(S) = B^s_{2,2}(S)$ and $\widetilde{H}^s_2(\Sigma) = \widetilde{B}^s_{2,2}(\Sigma)$, by Theorem B.1 we now conclude that the null spaces of the operators (6.61) and (6.62) are trivial under the condition (6.65) and therefore they are invertible. □

These invertibility properties for the operator $\mathcal{D}$ lead to the following existence results for Problem (D-CR-N)$^+_\tau$.

**Theorem 6.14.** *Let* $4/3 < p < 4$ *and*

$$f \in [B^{1-\frac{1}{p}}_{p,p}(S)]^6, \ \ F^{(\pm)} \in [B^{-\frac{1}{p}}_{p,p}(\Sigma)]^6, \ \ F = F^{(+)} - F^{(-)} \in [\widetilde{B}^{-\frac{1}{p}}_{p,p}(\Sigma)]^6.$$

*Then the crack type BVP* (D-CR-N)$^+_\tau$ *possesses a unique solution* $U \in [W^1_p(\Omega^+_\Sigma)]^6$ *which can be represented by formula* (6.55)*, where the pair* $(g,h)^\top \in [B^{1-\frac{1}{p}}_{p,p}(S)]^6 \times [\widetilde{B}^{1-\frac{1}{p}}_{p,p}(\Sigma)]^6$ *is a unique solution of the system of boundary pseudodifferential equations* (6.56)*,* (6.57)*.*

*Proof.* Existence of solutions directly follows from Lemma 6.13 since the condition (6.65) is fulfilled for $s = 1 - \frac{1}{p}$ and $\frac{4}{3} < p < 4$. Uniqueness for $p = 2$ follows from Theorem 2.25. Let us now show uniqueness of solutions for arbitrary $p \in (\frac{4}{3}, 4)$.

Let $U \in [W^1_p(\Omega^+_\Sigma)]^6$ be a solution to the homogeneous BVP (D-CR-N)$^+_\tau$. Then by the general integral representation formula (3.66) we get

$$U(x) = -V_S(\{\mathcal{T}U\}^+_S)(x) + W_\Sigma([U]_\Sigma)(x), \;\; x \in \Omega^+_\Sigma, \tag{6.66}$$

due to the homogeneous Dirichlet condition on $S$ and the homogeneous crack type conditions on $\Sigma$. Recall that $[U]_\Sigma$ stands for the jump of a vector $U$ across the surface $\Sigma$. Note that

$$\{\mathcal{T}U\}^+_S \in [B^{-\frac{1}{p}}_{p,p}(S)]^6, \quad [U]_\Sigma \in [\widetilde{B}^{1-\frac{1}{p}}_{p,p}(\Sigma)]^6. \tag{6.67}$$

Since $U$ solves the homogeneous BVP (D-CR-N)$^+_\tau$ we arrive at the following pseudodifferential equations

$$-\mathcal{H}_S\{\mathcal{T}U\}^+_S + W_\Sigma([U]_\Sigma) = 0 \;\text{ on }\; S, \quad -\mathcal{T}V_S(\{\mathcal{T}U\}^+_S) + \mathcal{L}_\Sigma[U]_\Sigma = 0 \;\text{ on }\; \Sigma, \tag{6.68}$$

which can be rewritten as

$$\mathcal{D}\widetilde{\Psi} = 0, \tag{6.69}$$

where $\mathcal{D}$ is given by (6.59) and $\widetilde{\Psi} := (\widetilde{g}, \widetilde{h})^\top$ with

$$\widetilde{g} := -\mathcal{H}_S\{\mathcal{T}U\}^+_S \in [B^{1-\frac{1}{p}}_{p,p}(S)]^6, \quad \widetilde{h} = [U]_\Sigma \in [\widetilde{B}^{1-\frac{1}{p}}_{p,p}(\Sigma)]^6.$$

Clearly, $\widetilde{\Psi} \in \mathbf{X}^{1-\frac{1}{p}}_{p,p} := [B^{1-\frac{1}{p}}_{p,p}(S)]^6 \times [\widetilde{B}^{1-\frac{1}{p}}_{p,p}(\Sigma)]^6$. Now, by Lemma 6.13 we conclude that $\widetilde{\Psi} = 0$ since the conditions (6.65) are fulfilled for $s = 1 - \frac{1}{p}$ and $\frac{4}{3} < p < 4$. Consequently, $\{\mathcal{T}U\}^+_S = 0$ on $S$ in view of invertibility of the operator $\mathcal{H}_S$ (see Remark 5.17) and $[U]_\Sigma = 0$ on $\Sigma$. But then (6.66) yields $U = 0$ in $\Omega^+_\Sigma$ which completes the proof. □

*Remark* 6.15. By the same arguments as in Remark 6.13 we can verify that Lemma 6.13 and Theorem 6.14 with $p = t = 2$ and $s = \frac{1}{2}$ remain valid for Lipschitz domains if the surface $S$, crack surface $\Sigma$ and the crack edge $\ell_c = \partial\Sigma$ are Lipschitz.

*Remark* 6.16. Applying again the asymptotic expansions of solutions at the crack edge derived in [16] imply that for sufficiently smooth boundary data (e.g., $C^\infty$-smooth data say) the solution vector $U$ to the crack problem (D-CR-NT)$_\tau$ belongs to the class of semi-regular functions described in Definition 2.5, $U \in [\mathbf{C}(\widetilde{\Omega}^+_\Sigma; \alpha)]^6$ with $\alpha = 1/2$. Moreover, the dominant terms of the asymptotic expansion of the solution vector $U$ near the curve $\ell_c$ has the form $\mathcal{O}(\varrho^{\frac{1}{2}}(x))$, where $\varrho(x)$ is the distance from a reference point $x$ to the curve $\ell_c$, while the dominant singular terms of the corresponding generalized stress vector $\mathcal{T}U$ are estimated by the expressions of type $\mathcal{O}(\varrho^{-\frac{1}{2}}(x))$ (for details see [16, Subsection 6.1]).

6.4. **Interior crack type problem (M-CR-N)$^+_\tau$.** We reformulate the problem (M-CR-N)$^+_\tau$ as follows (see Subsection 2.3): Find a solution $U = (u, \varphi, \psi, \vartheta)^\top \in [W^1_p(\Omega^+_\Sigma)]^6$ to the equation

$$A(\partial, \tau)U = 0 \;\text{ in }\; \Omega^+_\Sigma := \Omega^+ \setminus \overline{\Sigma} \tag{6.70}$$

satisfying the mixed Dirichlet–Neumann type boundary conditions on the exterior surface $S = \partial\Omega^+ = \overline{S_D} \cap \overline{S_N}$

$$\{U\}^+ = f^{(D)} \;\text{ on }\; S_D, \tag{6.71}$$

$$\big\{\mathcal{T}(\partial, n)U\big\}^+ = F^{(N)} \;\text{ on }\; S_N, \tag{6.72}$$

and (CR-N)$_\tau$ type conditions on the crack faces

$$\big\{[\mathcal{T}(\partial, n)U]\big\}^+ + \big\{[\mathcal{T}(\partial, n)U]\big\}^- = F^{(+)} + F^{(-)} \;\text{ on }\; \Sigma, \tag{6.73}$$

$$\big\{[\mathcal{T}(\partial, n)U]\big\}^+ - \big\{[\mathcal{T}(\partial, n)U]\big\}^- = F^{(+)} - F^{(-)} \;\text{ on }\; \Sigma, \tag{6.74}$$

where $f^{(D)} \in [B^{1-\frac{1}{p}}_{p,p}(S_D)]^6$, $F^{(N)} \in [B^{-\frac{1}{p}}_{p,p}(S_N)]^6$, and $F^{(\pm)} = (F^{(\pm)}_1, \dots, F^{(\pm)}_6)^\top \in [B^{-\frac{1}{p}}_{p,p}(\Sigma)]^6$ are given vector functions on $S$ and $\Sigma$, respectively. We assume the following compatibility condition

$$F := F^{(+)} - F^{(-)} \in [\widetilde{B}^{-\frac{1}{p}}_{p,p}(\Sigma)]^6. \tag{6.75}$$

As in Subsection 5.7, we denote by $f^{(e)}$ a fixed extension of the vector function $f^{(D)}$ from $S_D$ onto the whole of $S$ preserving the functional space,

$$f^{(e)} \in [B_{p,p}^{1-\frac{1}{p}}(S)]^6, \;\; r_{S_D} f^{(e)} = f^{(D)} \;\text{ on }\; S_D. \tag{6.76}$$

Evidently, an arbitrary extension $f$ of $f^{(D)}$ onto the whole of $S$, which preserves the functional space, can be then represented as

$$f = f^{(e)} + g \;\text{ with }\; g \in [\widetilde{B}_{p,p}^{1-\frac{1}{p}}(S_N)]^6. \tag{6.77}$$

In accordance with Theorem 5.14, we can seek a solution to the BVP (M-CR-N)$_\tau^+$ in the form

$$U = V_S\big(\mathcal{H}_S^{-1}\big[f^{(e)} + g - W_\Sigma(h) + V_\Sigma(F)\big]\big) + W_\Sigma(h) - V_\Sigma(F) \;\text{ in }\; \Omega_\Sigma^+, \tag{6.78}$$

where $F$ and $f^{(e)}$ are the above introduced given vector functions with properties (6.75) and (6.76); $V_S$, $V_\Sigma$, and $W_\Sigma$ are single and double layer potentials defined by (3.59) and (3.60), $\mathcal{H}_S$ is a pseudodifferential operator defined by (4.6) and $\mathcal{H}_S^{-1}$ is the inverse to the operator (5.80); $g = (g_1, \dots, g_6)^\top \in [\widetilde{B}_{p,p}^{1-\frac{1}{p}}(S_N)]^6$ and $h = (h_1, \dots, h_6)^\top \in [\widetilde{B}_{p,p}^{1-\frac{1}{p}}(\Sigma)]^6$ are unknown vector functions.

It is evident that the differential equation (6.70), the Dirichlet condition (6.71) and the crack condition (6.74) are satisfied automatically, while the Neumann condition (6.72) and the condition (6.73) on the crack surface lead to the system of pseudodifferential equations

$$\big[-2^{-1}I_6 + \mathcal{K}_S\big]\mathcal{H}_S^{-1}\big[f^{(e)} + g - W_\Sigma(h) + V_\Sigma(F)\big] + \mathcal{T}W_\Sigma(h) - \mathcal{T}V_\Sigma(F) = F^{(N)} \;\text{ on }\; S_N,$$

$$2\mathcal{T}(\partial,n)V_S\big(\mathcal{H}_S^{-1}\big[f^{(e)} + g - W_\Sigma(h) + V_\Sigma(F)\big]\big) + 2\mathcal{L}_\Sigma h - 2\mathcal{K}_\Sigma F = F^+ + F^- \;\text{ on }\; \Sigma,$$

which can be rewritten as

$$r_{S_N}\mathcal{A}_S^+ g - r_{S_N}\mathcal{A}_S^+ W_\Sigma(h) + r_{S_N}\mathcal{T}W_\Sigma(h) = Q^{(1)} \;\text{ on }\; S_N, \tag{6.79}$$

$$r_\Sigma\mathcal{T}(\partial,n)V_S(\mathcal{H}_S^{-1}g) + r_\Sigma\mathcal{L}_\Sigma h - r_\Sigma\mathcal{T}V_S(\mathcal{H}_S^{-1}W_\Sigma(h)) = Q^{(2)} \;\text{ on }\; \Sigma, \tag{6.80}$$

where $\mathcal{A}_S^+ := [-2^{-1}I_6 + \mathcal{K}_S]\mathcal{H}_S^{-1}$ is the Steklov–Poincaré operator (see Subsection 4.3) and

$$Q^{(1)} := F^{(N)} - r_{S_N}\mathcal{A}_S^+[f^{(e)} + V_\Sigma(F)] + r_{S_N}\mathcal{T}V_\Sigma(F) \in [B_{p,p}^{-\frac{1}{p}}(S_N)]^6, \tag{6.81}$$

$$Q^{(2)} := \frac{1}{2}\,[F^+ + F^-] - r_\Sigma\mathcal{T}(\partial,n)V_S\big(\mathcal{H}_S^{-1}[f^{(e)} + V_\Sigma(F)]\big) + r_\Sigma\mathcal{K}_\Sigma F \in [B_{p,p}^{-\frac{1}{p}}(\Sigma)]^6. \tag{6.82}$$

Here $\mathcal{K}_\Sigma$ and $\mathcal{L}_\Sigma$ are pseudodifferential operators defined by (4.7) and (4.9) with $\Sigma$ for $S$.

Denote by $\mathcal{M}$ the pseudodifferential matrix operator generated by the left hand side expressions in (6.79), (6.80),

$$\mathcal{M} := \begin{bmatrix} r_{S_N}\mathcal{A}_S^+ & -r_{S_N}\mathcal{A}_S^+ W_\Sigma + r_{S_N}\mathcal{T}W_\Sigma \\ r_\Sigma\mathcal{T}(\partial,n)V_S(\mathcal{H}_S^{-1}) & r_\Sigma\mathcal{L}_\Sigma - r_\Sigma\mathcal{T}V_S(\mathcal{H}_S^{-1}W_\Sigma) \end{bmatrix}_{12\times 12}. \tag{6.83}$$

Evidently, the operators

$$r_{S_N}\mathcal{A}_S^+ W_\Sigma, \quad r_{S_N}\mathcal{A}_S^+ V_\Sigma, \quad r_{S_N}\mathcal{T}W_\Sigma, \quad r_{S_N}\mathcal{T}V_\Sigma,$$
$$r_\Sigma\mathcal{T}(\partial,n)V_S(\mathcal{H}_S^{-1}), \quad r_\Sigma\mathcal{T}V_S(\mathcal{H}_S^{-1}W_\Sigma), \quad r_\Sigma\mathcal{T}V_S(\mathcal{H}_S^{-1}V_\Sigma),$$

are smoothing operators, since the manifolds $S$ and $\Sigma$ are disjoint.

Set

$$\Psi = (g,h)^\top, \quad Q = (Q^{(1)}, Q^{(2)})^\top,$$

and rewrite equations (6.79), (6.80) in matrix form

$$\mathcal{M}\Psi = Q. \tag{6.84}$$

For $C^\infty$ regular $S$ and $\Sigma$, Theorem 4.4 yields the following mapping properties

$$\mathcal{M} : \mathbb{X}_p^s \longrightarrow \mathbb{Y}_p^{s-1}, \tag{6.85}$$

$$\mathcal{M} : \mathbb{X}_{p,t}^s \longrightarrow \mathbb{Y}_{p,t}^{s-1}, \tag{6.86}$$

where $s \in \mathbb{R}$, $1 < p < \infty$, $1 \le t \le \infty$, and

$$\mathbb{X}_p^s := [\widetilde{H}_p^s(S_N)]^6 \times [\widetilde{H}_p^s(\Sigma)]^6, \quad \mathbb{Y}_p^s := [H_p^s(S_N)]^6 \times [H_p^s(\Sigma)]^6,$$
$$\mathbb{X}_{p,t}^s := [\widetilde{B}_{p,t}^s(S_N)]^6 \times [\widetilde{B}_{p,t}^s(\Sigma)]^6, \quad \mathbb{Y}_{p,t}^s := [B_{p,t}^s(S_N)]^6 \times [B_{p,t}^s(\Sigma)]^6.$$

Further, let us consider the operator

$$\widetilde{\mathcal{M}} := \begin{bmatrix} r_{S_N}\mathcal{A}_S^+ & 0 \\ 0 & r_{\Sigma}\mathcal{L}_{\Sigma} \end{bmatrix}_{12\times 12}. \tag{6.87}$$

It is evident that $\widetilde{\mathcal{M}}$ has the same mapping properties as $\mathcal{M}$ and the operator $\mathcal{M}-\widetilde{\mathcal{M}}$ with the same domain and range spaces as in (6.85) and (6.86) is a compact operator. Moreover, in view of Lemmas 5.20 and 6.2 the operators

$$\widetilde{\mathcal{M}} : \mathbb{X}_p^s \longrightarrow \mathbb{Y}_p^{s-1}, \quad \widetilde{\mathcal{M}} : \mathbb{X}_{p,t}^s \longrightarrow \mathbb{Y}_{p,t}^{s-1}, \tag{6.88}$$

are invertible if the following inequalities

$$\frac{1}{p}-\frac{1}{2} < s < \frac{1}{p}+\frac{1}{2}, \quad \frac{1}{p}-\frac{1}{2}+a_2 < s < \frac{1}{p}+\frac{1}{2}+a_1, \tag{6.89}$$

are satisfied, where $a_1$ and $a_2$ are defined by relations (5.109).

Therefore the operators (6.85) and (6.86) are Fredholm with zero index if the inequalities (6.89) hold.

**Lemma 6.17.** *The operators* (6.85) *and* (6.86) *are invertible if the inequalities* (6.89) *hold.*

*Proof.* As in the case of Lemma 6.13, we need only to show that the corresponding null-spaces are trivial for some particular values of the parameters $s$ and $p$ satisfying the inequalities (6.89). We again take $s=\frac{1}{2}$ and $p=2$, and prove that the homogeneous system $\mathcal{M}\Psi=0$, i.e., the equations (6.79), (6.80) with $Q^{(1)}=Q^{(2)}=0$ have only the trivial solution. Indeed, let $\Psi_0=(g_0,h_0)^\top \in [\widetilde{H}_2^{\frac{1}{2}}(S_N)]^6 \times [\widetilde{H}_2^{\frac{1}{2}}(\Sigma)]^6$ be a solution of the homogeneous system (6.79), (6.80) and construct the vector

$$U_0(x) = V_S\big(\mathcal{H}_S^{-1}[g_0 - W_\Sigma(h_0)]\big)(x) + W_\Sigma(h_0)(x), \ \ x\in\mathbb{R}^3\setminus(S\cup\overline{\Sigma}).$$

It is easy to see that $U_0\in[W_2^1(\Omega_\Sigma^+)]^6$ and $U_0$ solves the homogeneous BVP (M-CR-N)$_\tau^+$. By Theorem 2.25 we then have $U_0=0$ in $\Omega_\Sigma^+$. Hence $\{U_0\}_\Sigma^+ - \{U_0\}_\Sigma^- = h_0 = 0$ on $\Sigma$ follows immediately. Therefore we get $U_0=V_S(\mathcal{H}_S^{-1}g_0)=0$ in $\Omega_\Sigma^+$ which implies $g_0=\{U_0\}_S^+=0$ on $S$. Thus $\Psi_0=(g_0,h_0)^\top=0$. This implies that the null spaces of the operators

$$\mathcal{M} : \mathbb{X}_2^{\frac{1}{2}} \longrightarrow \mathbb{Y}_2^{-\frac{1}{2}}, \quad \mathcal{M} : \mathbb{X}_{2,2}^{\frac{1}{2}} \longrightarrow \mathbb{Y}_{2,2}^{-\frac{1}{2}},$$

are trivial since $\mathbb{X}_2^{\frac{1}{2}}=\mathbb{X}_{2,2}^{\frac{1}{2}}$ and $\mathbb{Y}_2^{-\frac{1}{2}}=\mathbb{Y}_{2,2}^{-\frac{1}{2}}$ due to the relations $\widetilde{H}_2^s(S_N)=\widetilde{B}_{2,2}^s(S_N)$ and $\widetilde{H}_2^s(\Sigma)=\widetilde{B}_{2,2}^s(\Sigma)$ for $s\in\mathbb{R}$. Actually these operators are the same. Therefore, by Theorem B.1 the null spaces of the operators (6.85) and (6.86) are also trivial if the conditions (6.89) hold. This completes the proof of the lemma. □

From Lemma 6.17 the following existence result follows directly.

**Theorem 6.18.** *Let $a_1$ and $a_2$ be defined by relations* (5.109) *and*

$$\frac{4}{3} < p < 4, \quad \frac{4}{3-2a_2} < p < \frac{4}{1-2a_1}, \tag{6.90}$$

$$f^{(D)}\in[B_{p,p}^{1-\frac{1}{p}}(S_D)]^6, \ F^{(N)}\in\big[B_{p,p}^{-\frac{1}{p}}(S_N)\big]^6, \ F^{(\pm)}\in[B_{p,p}^{-\frac{1}{p}}(\Sigma)]^6, \ F=F^{(+)}-F^{(-)}\in[\widetilde{B}_{p,p}^{-\frac{1}{p}}(\Sigma)]^6.$$

*Then the crack type BVP* (M-CR-N)$_\tau^+$ (6.70)–(6.74) *possesses a unique solution $U\in[W_p^1(\Omega_\Sigma^+)]^6$ which can be represented by formula* (6.78), *where the pair $(g,h)^\top\in[\widetilde{B}_{p,p}^{1-\frac{1}{p}}(S_N)]^6\times[\widetilde{B}_{p,p}^{1-\frac{1}{p}}(\Sigma)]^6$ is a unique solution of the system of boundary pseudodifferential equations* (6.79), (6.80).

*Proof.* Existence of a solution directly follows from Lemma 6.17 since the conditions (6.89) are fulfilled for $s=1-1/p$ where $p$ is restricted by the inequalities (6.90). Uniqueness for $p=2$ follows from Theorem 2.25. Note that for $p=2$ both relations in (6.90) hold, since $-\frac{1}{2}<a_1\le a_2<\frac{1}{2}$.

Now, let $p$ satisfy the inequalities (6.90) and let $U_0\in[W_p^1(\Omega_\Sigma^+)]^6$ be a solution to the homogeneous BVP (M-CR-N)$^+$. We have to show that $U_0$ vanishes identically in $\Omega_\Sigma^+$. We proceed as follows.

Keeping in mind the homogeneous crack type conditions (6.73), (6.74) on $\Sigma$, from the general integral representation formula (3.66) we get

$$U_0(x) = W_S(\{U_0\}_S^+)(x) - V_S(\{\mathcal{T}U_0\}_S^+)(x) + W_\Sigma([U_0]_\Sigma)(x), \quad x \in \Omega_\Sigma^+. \tag{6.91}$$

Recall that here $[U_0]_\Sigma$ stands for the jump of the vector $U_0$ across the crack surface $\Sigma$ and

$$h_0 := [U_0]_\Sigma \in [\widetilde{B}_{p,p}^{1-\frac{1}{p}}(\Sigma)]^6. \tag{6.92}$$

Note also that due to the homogeneous mixed boundary conditions (6.71), (6.72) we have

$$g_0 := \{U_0\}_S^+ \in [\widetilde{B}_{p,p}^{1-\frac{1}{p}}(S_N)]^6, \quad G_0 := \{\mathcal{T}U_0\}_S^+ \in [\widetilde{B}_{p,p}^{-\frac{1}{p}}(S_D)]^6. \tag{6.93}$$

With the help of Theorem 5.14 and the relation $\mathcal{N}_S\mathcal{H}_S = \mathcal{H}_S\mathcal{K}_S$ (see Theorem 4.3) we conclude that the double layer potential $W_S(\{U_0\}_S^+)$ can be represented in the form of a single layer potential uniquely

$$W_S(\{U_0\}_S^+) = W_S(g_0) = V_S(\mathcal{A}_S^- g_0) \ \text{ in } \ \Omega_\Sigma^+, \tag{6.94}$$

where $\mathcal{A}_S^- := [2^{-1}I_6 + \mathcal{K}_S]\mathcal{H}_S^{-1}$ is the Steklov–Poincaré operator (see Subsection 4.3). Indeed, one can easily check that the layer potentials $W_S(g_0)$ and $V_S(\mathcal{A}_S^- g_0)$ have the same Dirichlet data on the boundary $S$,

$$\begin{aligned}\{W_S(g_0)\}^+ = \Big[\frac{1}{2}I_6 + \mathcal{N}_S\Big]g_0 = \Big[\frac{1}{2}I_6 + \mathcal{N}_S\Big]\mathcal{H}_S\mathcal{H}_S^{-1}g_0 \\ = \mathcal{H}_S\Big[\frac{1}{2}I_6 + \mathcal{K}_S\Big]\mathcal{H}_S^{-1}g_0 = \mathcal{H}_S\mathcal{A}_S^- g_0 = \{V_S(\mathcal{A}_S^- g_0)\}^+.\end{aligned}$$

Therefore $W_S(g_0) = V_S(\mathcal{A}_S^- g_0)$ in $\Omega^+$ by the uniqueness Theorem 2.25. Consequently, from (6.91) it follows that $U_0$ is representable in the form

$$U_0 = V_S(\chi) + W_\Sigma([U_0]_\Sigma) \ \text{ in } \ \Omega_\Sigma^+ \ \text{ with } \ \chi := \mathcal{A}_S^-\{U_0\}_S^+ - \{\mathcal{T}U_0\}_S^+. \tag{6.95}$$

In turn, (6.95) yields $\{U_0\}_S^+ = \mathcal{H}_S\chi + W_\Sigma([U_0]_\Sigma)$ on $S$. Whence $\chi = \mathcal{H}_S^{-1}[\{U_0\}_S^+ - W_\Sigma([U_0]_\Sigma)]$ on $S$, and finally, in view of (6.95), we arrive at the representation (cf. (6.78))

$$U_0 = V_S\big(\mathcal{H}_S^{-1}\big[\{U_0\}_S^+ - W_\Sigma([U_0]_\Sigma)\big]\big) + W_\Sigma([U_0]_\Sigma) = V_S\big(\mathcal{H}_S^{-1}\big[g_0 - W_\Sigma(h_0)\big]\big) + W_\Sigma(h_0) \ \text{ in } \ \Omega_\Sigma^+, \tag{6.96}$$

where $g_0$ and $h_0$ are given by relations (6.93) and (6.92).

Now recall that by assumption $U_0$ solves the homogeneous BVP $(\text{M-CR-N})_\tau^+$. As we see from the representation (6.96), the vector $U_0$ satisfies the homogeneous boundary conditions (6.71) and (6.74) with $f^{(D)} = 0$ and $F = F^{(+)} - F^{(-)} = 0$, while the homogeneous conditions (6.72) and (6.73) with $F^{(N)} = 0$ and $F^{(+)} + F^{(-)} = 0$ give the following relations (cf. (6.79), (6.80)):

$$r_{S_N}\mathcal{A}_S^+ g_0 - r_{S_N}\mathcal{A}_S^+ W_\Sigma(h_0) + r_{S_N}\mathcal{T}W_\Sigma(h_0) = 0 \ \text{ on } \ S_N, \tag{6.97}$$

$$r_\Sigma\mathcal{T}(\partial,n)V_S(\mathcal{H}_S^{-1}g_0) + r_\Sigma\mathcal{L}_\Sigma h_0 - r_\Sigma\mathcal{T}V_S(\mathcal{H}_S^{-1}W_\Sigma(h_0)) = 0 \ \text{ on } \ \Sigma, \tag{6.98}$$

where $\mathcal{A}_S^+ = [-2^{-1}I_6 + \mathcal{K}_S]\mathcal{H}_S^{-1}$ is the Steklov–Poincaré operator (see Subsection 4.3).

It is easy to see that this system is equivalent to the homogeneous equation

$$\mathcal{M}\widetilde{\Psi}_0 = 0, \tag{6.99}$$

where $\mathcal{M}$ is given by (6.83) and $\widetilde{\Psi}_0 := (g_0, h_0)^\top \in \mathbb{X}_{p,p}^{1-\frac{1}{p}} := [\widetilde{B}_{p,p}^{1-\frac{1}{p}}(S_N)]^6 \times [\widetilde{B}_{p,p}^{1-\frac{1}{p}}(\Sigma)]^6$. By Lemma 6.17 we then conclude that $\widetilde{\Psi}_0 = 0$ since the conditions (6.89) are fulfilled if $p$ satisfies inequalities (6.90) and $s = 1 - \frac{1}{p}$. Consequently, $g_0 := \{U_0\}_S^+ = 0$ and $h_0 := [U_0]_\Sigma = 0$. But then (6.96) yields $U_0 = 0$ in $\Omega_\Sigma^+$ which completes the proof. □

*Remark* 6.19. By the same arguments as in Remark 6.13 we can verify that Lemma 6.17 and Theorem 6.18 with $p = t = 2$ and $s = \frac{1}{2}$ remain valid for Lipschitz domains if the surface $S$, crack surface $\Sigma$ and the crack edge $\partial\Sigma$ are Lipschitz.

*Remark* 6.20. Note that, smoothness properties of solutions to the BVPs $(\text{D-CR-N})_\tau^+$ and $(\text{M-CR-N})_\tau^+$ near the crack edge $\ell_c$ and near the collision curve $\ell_m$ are described by Theorems 5.22 and 6.4.

*Remark* 6.21. In view of Remarks 5.25 and 6.16 we conclude that for sufficiently smooth boundary data (e.g., $C^\infty$-smooth data say) the solution vector $U$ to the boundary value problem $(\text{M-CR-N})^+_\tau$ belongs to the class of semi-regular functions near the collision curve $\ell_m$ with the exponent $\alpha = \frac{1}{2} - a_1 + \varepsilon$, while near the crack edge $\ell_C$ it is semi-regular with the exponent $\alpha = \frac{1}{2}$. Evidently, the generalized stress vector $\mathcal{T}U$ possesses different singularities at the collision curve $\ell_m$ and the crack edge $\ell_C$ described in Remarks 5.25 and 6.16.

## 7. Basic BVPs of Statics

We demonstrate our approach for the interior and exterior Neumann-type boundary-value problems of statics (cf. [84], [82], [83]). The Dirichlet and mixed type BVPs of statics can be studied quite analogously and in some sense their analysis is even simpler since they are unconditionally solvable due to the corresponding uniqueness results.

7.1. **Formulation of problems.** The basic differential equations of statics of the theory of thermo-electro-magneto-elasticity read as follows (cf. (2.35) and (2.45)):

$$\begin{aligned} c_{rjkl}\partial_j\partial_l u_k(x) + e_{lrj}\partial_j\partial_l\varphi(x) + q_{lrj}\partial_j\partial_l\psi(x) - \lambda_{rj}\partial_j\vartheta(x) &= -\varrho F_r(x), \quad r=1,2,3,\\ -e_{jkl}\partial_j\partial_l u_k(x) + \varkappa_{jl}\partial_j\partial_l\varphi(x) + a_{jl}\partial_j\partial_l\psi(x) - p_j\partial_j\vartheta(x) &= -\varrho_e(x),\\ -q_{jkl}\partial_j\partial_l u_k(x) + a_{jl}\partial_j\partial_l\varphi(x) + \mu_{jl}\partial_j\partial_l\psi(x) - m_j\partial_j\vartheta(x) &= -\varrho_c(x),\\ \eta_{jl}\partial_j\partial_l\vartheta(x) &= -\varrho T_0^{-1}Q(x). \end{aligned} \tag{7.1}$$

In matrix form these equations can be written as $A(\partial)U(x) = \Phi(x)$, where

$$\begin{aligned} U &= (u_1,u_2,u_3,u_4,u_5,u_6)^\top := (u,\varphi,\psi,\vartheta)^\top,\\ \Phi &= (\Phi_1,\dots,\Phi_6)^\top := (-\varrho F_1, -\varrho F_2, -\varrho F_3, -\varrho_e, -\varrho_c, -\varrho T_0^{-1}Q)^\top \end{aligned}$$

and $A(\partial)$ is the matrix differential operator generated by equations (7.1) (see (2.35))

$$\begin{aligned} A(\partial) = A(\partial,0) &= [A_{pq}(\partial)]_{6\times 6}\\ &:= \begin{bmatrix} [c_{rjkl}\partial_j\partial_l]_{3\times3} & [e_{lrj}\partial_j\partial_l]_{3\times1} & [q_{lrj}\partial_j\partial_l]_{3\times1} & [-\lambda_{rj}\partial_j]_{3\times1}\\ [-e_{jkl}\partial_j\partial_l]_{1\times3} & \varkappa_{jl}\partial_j\partial_l & a_{jl}\partial_j\partial_l & -p_j\partial_j\\ [-q_{jkl}\partial_j\partial_l]_{1\times3} & a_{jl}\partial_j\partial_l & \mu_{jl}\partial_j\partial_l & -m_j\partial_j\\ [0]_{1\times3} & 0 & 0 & \eta_{jl}\partial_j\partial_l \end{bmatrix}_{6\times6}. \end{aligned} \tag{7.2}$$

**Neumann problems $(\mathbf{N})^\pm$:** *Find a regular solution vector* $U = (u,\varphi,\psi,\vartheta)^\top \in [C^1(\overline{\Omega^\pm})]^6 \cap [C^2(\Omega^\pm)]^6$ *to the system of equations*

$$A(\partial)U = \Phi \ \text{ in } \ \Omega^\pm,$$

*satisfying the Neumann-type boundary condition*

$$\{\mathcal{T}U\}^\pm = f \ \text{ on } \ S,$$

*where* $A(\partial)$ *is a nonselfadjoint strongly elliptic matrix partial differential operator defined in* (7.2), *while* $\mathcal{T}(\partial,n) := \mathcal{T}(\partial,n,0)$ *is the matrix boundary stress operator of statics defined in* (2.374).

In what follows we always assume that in the case of exterior boundary-value problems of statics for the domain $\Omega^-$ a solution vector $U$ possesses $\mathbf{Z}(\Omega^-)$ property introduced in Subsection 3.5.2 (see Definition 3.11).

7.2. **Potentials of statics and their properties.** From the results obtained in Subsection 3.3 it follows that the fundamental matrix $\Gamma(x) := \Gamma(x,0) = [\Gamma_{kj}(x)]_{6\times6}$ which solves the equation $A(\partial)\Gamma(x) = I_6\delta(x)$, where $\delta(\,\cdot\,)$ is the Dirac delta distribution and $I_6$ stands for the unit $6\times6$ matrix, can be represented in the form (see (3.51))

$$\Gamma(x) = \mathcal{F}^{-1}_{\xi\to x}[A^{-1}(-i\xi)], \tag{7.3}$$

where $\mathcal{F}^{-1}$ is the generalized inverse Fourier transform and $A^{-1}(-i\xi)$ is the matrix inverse to $A(-i\xi)$. Moreover, as we have shown in Subsection 3.3, the entries of the fundamental matrix $\Gamma(x)$ are homogeneous functions in $x$ and at the origin and at infinity the following asymptotic relations hold (see (3.57))

$$\Gamma(x) = \begin{bmatrix} [\mathcal{O}(|x|^{-1})]_{5\times5} & [\mathcal{O}(1)]_{5\times1} \\ [0]_{1\times5} & \mathcal{O}(|x|^{-1}) \end{bmatrix}_{6\times6}. \tag{7.4}$$

From the relations (7.4) and (3.56) it easily follows that the columns of the matrix $\Gamma(x)$ possess the property $\mathbf{Z}(\mathbb{R}^3 \setminus \{0\})$.

With the help of this fundamental matrix we construct the generalized single and double layer potentials, and the Newton-type volume potentials of statics (the potentials of statics are equipped with the subscript "zero" showing that they correspond to the above introduced pseudo-oscillation potentials with $\tau = 0$)

$$V_0(h)(x) = V_{S,0}(h)(x) = \int_S \Gamma(x-y)h(y)\, dS_y, \quad x \in \mathbb{R}^3 \setminus S, \tag{7.5}$$

$$W_0(h)(x) = W_{S,0}(h)(x) = \int_S \big[\mathcal{P}(\partial_y, n(y))\Gamma^\top(x-y)\big]^\top h(y)\, dS_y, \quad x \in \mathbb{R}^3 \setminus S, \tag{7.6}$$

$$N_{\Omega^\pm,0}(g)(x) = \int_{\Omega^\pm} \Gamma(x-y)g(y)\, dy, \quad x \in \mathbb{R}^3, \tag{7.7}$$

where $h = (h_1, \dots, h_6)^\top$ and $g = (g_1, \dots, g_6)^\top$ are density vector functions defined, respectively, on $S$ and in $\Omega^\pm$; the so called *generalized stress operator* $\mathcal{P}(\partial, n)$, associated with the adjoint differential operator of statics $A^*(\partial) = A^\top(-\partial)$, reads as (cf. (2.58))

$$\mathcal{P}(\partial, n) = \mathcal{P}(\partial, n, 0) = \big[\mathcal{P}_{pq}(\partial, n)\big]_{6\times6}$$
$$= \begin{bmatrix} [c_{rjkl}n_j\partial_l]_{3\times3} & [-e_{lrj}n_j\partial_l]_{3\times1} & [-q_{lrj}n_j\partial_l]_{3\times1} & [0]_{3\times1} \\ [e_{jkl}n_j\partial_l]_{1\times3} & \varkappa_{jl}n_j\partial_l & a_{jl}n_j\partial_l & 0 \\ [q_{jkl}n_j\partial_l]_{1\times3} & a_{jl}n_j\partial_l & \mu_{jl}n_j\partial_l & 0 \\ [0]_{1\times3} & 0 & 0 & \eta_{jl}n_j\partial_l \end{bmatrix}. \tag{7.8}$$

The following properties of layer potentials of statics immediately follow from their definition and the results presented in Sections 3 and 4.

**Theorem 7.1.** *The generalized single and double layer potentials of statics solve the homogeneous differential equation $A(\partial)U = 0$ in $\mathbb{R}^3 \setminus S$ and possess the property $\mathbf{Z}(\Omega^-)$.*

With the help of Green's formulas, one can derive general integral representation formulas of solutions to the homogeneous equation $A(\partial)U = 0$ in $\Omega^+$. In particular, the following counterpart of Theorem 3.5 holds.

**Theorem 7.2.** *Let $S = \partial\Omega^+ \in C^{1,\kappa}$ with $0 < \kappa \le 1$ and $U$ be a regular solution to the homogeneous equation $A(\partial)U = 0$ in $\Omega^+$ of the class $[C^1(\overline{\Omega^+})]^6 \cap [C^2(\Omega^+)]^6$. Then there holds the integral representation formula*

$$W_0(\{U\}^+)(x) - V_0(\{\mathcal{T}U\}^+)(x) = \begin{cases} U(x) & \text{for } x \in \Omega^+, \\ 0 & \text{for } x \in \Omega^-. \end{cases} \tag{7.9}$$

Now, we show that for the exterior unbounded domain the similar general integral representation formula holds also true for solutions to the homogeneous static equation satisfying the $\mathbf{Z}(\Omega^-)$ condition. As we will see the proof is not as trivial as the proof of Theorem 3.6.

**Theorem 7.3.** *Let $S = \partial\Omega^-$ be $C^{1,\kappa}$-smooth with $0 < \kappa \le 1$ and let $U$ be a regular solution to the homogeneous equation $A(\partial)U = 0$ in $\Omega^-$ of the class $[C^1(\overline{\Omega^-})]^6 \cap [C^2(\Omega^-)]^6$ having the property $\mathbf{Z}(\Omega^-)$. Then there holds the integral representation formula*

$$-W_0(\{U\}^-)(x) + V_0(\{\mathcal{T}U\}^-)(x) = \begin{cases} 0 & \text{for } x \in \Omega^+, \\ U(x) & \text{for } x \in \Omega^-. \end{cases} \tag{7.10}$$

*Proof.* Let $\Omega_R^- := B(0,R)\setminus\overline{\Omega^+}$ and $\Sigma_R = \partial B(0,R)$, where $B(0,R)$ is a ball centered at the origin and radius $R$. We assume that the origin belongs to the domain $\Omega^+$ and $R$ is sufficiently large, such that $\overline{\Omega^+}\subset B(0,R)$. Then in view of (7.9) we have

$$U(x) = -W_{S,0}(\{U\}_S^-)(x) + V_{S,0}\big(\{\mathcal{T}U\}_S^-\big)(x) + \Phi_R(x),\ \ x\in\Omega_R^-, \tag{7.11}$$

$$0 = -W_{S,0}(\{U\}_S^-)(x) + V_{S,0}\big(\{\mathcal{T}U\}_S^-\big)(x) + \Phi_R(x),\ \ x\in\Omega^+, \tag{7.12}$$

where the directions of the normal vectors are outward with respect to $S$ and $\Sigma_R$ and

$$\Phi_R(x) := W_{\Sigma_R,0}(U)(x) - V_{\Sigma_R,0}(\mathcal{T}U)(x). \tag{7.13}$$

Here $V_{\mathcal{M},0}$ and $W_{\mathcal{M},0}$ denote the single and double layer potential operators of statics (7.5) and (7.6) with integration surface $\mathcal{M}$. Evidently,

$$A(\partial)\Phi_R(x) = 0,\ \ |x|<R. \tag{7.14}$$

In turn, from (7.11) and (7.12) we get

$$\begin{aligned}
&\Phi_R(x) = U(x) + W_{S,0}(\{U\}_S^-)(x) - V_{S,0}(\{\mathcal{T}U\}_S^-)(x),\ \ x\in\Omega_R^-,\\
&\Phi_R(x) = W_{S,0}(\{U\}_S^-)(x) - V_{S,0}(\{\mathcal{T}U\}_S^-)(x),\ \ x\in\Omega^+,
\end{aligned} \tag{7.15}$$

whence the equality $\Phi_{R_1}(x) = \Phi_{R_2}(x)$ follows for $|x|<R_1<R_2$. We assume that $R_1$ and $R_2$ are sufficiently large numbers. Therefore, for an arbitrary fixed point $x\in\mathbb{R}^3$ the following limit exists

$$\Phi(x) := \lim_{R\to\infty}\Phi_R(x) = \begin{cases} U(x) + W_{S,0}(\{U\}_S^-)(x) - V_{S,0}(\{\mathcal{T}U\}_S^-)(x), & x\in\Omega^-,\\ W_{S,0}(\{U\}_S^-)(x) - V_{S,0}(\{\mathcal{T}U\}_S^-)(x), & x\in\Omega^+, \end{cases} \tag{7.16}$$

and $A(\partial)\Phi(x) = 0$ for all $x\in\Omega^+\cup\Omega^-$. On the other hand, for arbitrary fixed point $x\in\mathbb{R}^3$ and a number $R_1$, such that $|x|<R_1$ and $\overline{\Omega^+}\subset B(0,R_1)$, from (7.15) we have

$$\Phi(x) = \lim_{R\to\infty}\Phi_R(x) = \Phi_{R_1}(x).$$

Now, from (7.13), (7.14) we deduce

$$A(\partial)\Phi(x) = 0\ \ \forall\, x\in\mathbb{R}^3. \tag{7.17}$$

Since $U_{S,0}, W_{S,0}, V_{S,0}\in\mathbf{Z}(\Omega^-)$ we conclude from (7.16) that $\Phi(x)\in\mathbf{Z}(\mathbb{R}^3)$. In particular, we have

$$\lim_{R\to\infty}\frac{1}{4\pi R^2}\int_{\Sigma_R}\Phi(x)\,d\Sigma_R = 0. \tag{7.18}$$

Our goal is to show that $\Phi(x) = 0\ \ \forall\, x\in\mathbb{R}^3$.
Applying the generalized Fourier transform to equation (7.61) we get $A(-i\xi)\widehat{\Phi}(\xi) = 0$, $\ \xi\in\mathbb{R}^3$, where $\widehat{\Phi}(\xi)$ is the Fourier transform of $\Phi(x)$. Taking into account that $\det A(-i\xi)\neq 0$ for all $\xi\in\mathbb{R}^3\setminus\{0\}$, we conclude that the support of the generalized functional $\widehat{\Phi}(\xi)$ is the origin and, consequently,

$$\widehat{\Phi}(\xi) = \sum_{|\alpha|\le M} c_\alpha\delta^{(\alpha)}(\xi),$$

where $\alpha$ is a multi-index, $c_\alpha$ are constant vectors and $M$ is some nonnegative integer. Then it follows that $\Phi(x)$ is a polynomial in $x$ (see, e.g., [35, Ch. 1, Example 2.2]),

$$\Phi(x) = \sum_{|\alpha|\le M} c_\alpha x^\alpha,$$

and due to the inclusion $\Phi\in\mathbf{Z}(\Omega^-)$, the vector $\Phi(x)$ is bounded at infinity, i.e., $\Phi(x) = const$ in $\mathbb{R}^3$. Therefore (7.18) implies that $\Phi(x)$ vanishes identically in $\mathbb{R}^3$. This proves that formula (7.10) holds. □

By standard limiting procedure, formulas (7.9) and (7.10) can be extended to Lipschitz domains and to solution vectors from the spaces $[W_p^1(\Omega^+)]^6$ and $[W_{p,loc}^1(\Omega^-)]^6\cap\mathbf{Z}(\Omega^-)$ with $1<p<\infty$ (cf., [51], [75], [91]).

The qualitative and mapping properties of the layer potentials are described by the following theorems (see Subsection 4.1).

**Theorem 7.4.** *Let $S = \partial\Omega^\pm \in C^{m,\kappa}$ with integers $m \geq 1$ and $k \leq m-1$, and $0 < \kappa' < \kappa \leq 1$. Then the operators*

$$V_0 : [C^{k,\kappa'}(S)]^6 \longrightarrow [C^{k+1,\kappa'}(\overline{\Omega^\pm})]^6, \quad W_0 : [C^{k,\kappa'}(S)]^6 \longrightarrow [C^{k,\kappa'}(\overline{\Omega^\pm})]^6 \tag{7.19}$$

*are continuous and $V_0, W_0 \in \mathbf{Z}(\Omega^-)$.*

*For any $g \in [C^{0,\kappa'}(S)]^6$, $h \in [C^{1,\kappa'}(S)]^6$, and any $x \in S$ we have the following jump relations:*

$$\{V_0(g)(x)\}^\pm = V_0(g)(x) = \mathcal{H}_0 g(x), \tag{7.20}$$

$$\big\{\mathcal{T}(\partial_x, n(x))V_0(g)(x)\big\}^\pm = \big[\mp 2^{-1} I_6 + \mathcal{K}_0\big] g(x), \tag{7.21}$$

$$\{W_0(g)(x)\}^\pm = \big[\pm 2^{-1} I_6 + \mathcal{N}_0\big] g(x), \tag{7.22}$$

$$\big\{\mathcal{T}(\partial_x, n(x))W_0(h)(x)\big\}^+ = \{\mathcal{T}(\partial_x, n(x))W_0(h)(x)\}^- = \mathcal{L}_0 h(x), \ \ m \geq 2, \tag{7.23}$$

*where $\mathcal{H}_0$ is a weakly singular integral operator, $\mathcal{K}_0$ and $\mathcal{N}_0$ are singular integral operators, and $\mathcal{L}_0$ is a singular integro-differential operator,*

$$\begin{aligned}
\mathcal{H}_0 g(x) &:= \int_S \Gamma(x-y) g(y)\, dS_y, \\
\mathcal{K}_0 g(x) &:= \int_S \mathcal{T}(\partial_x, n(x))\Gamma(x-y)\ g(y)\, dS_y, \\
\mathcal{N}_0 g(x) &:= \int_S \big[\mathcal{P}(\partial_y, n(y))\Gamma^\top(x-y)\big]^\top g(y)\, dS_y, \\
\mathcal{L}_0 h(x) &:= \lim_{\Omega^\pm \ni z \to x \in S} \mathcal{T}(\partial_z, n(x)) \int_S \big[\mathcal{P}(\partial_y, n(y))\Gamma^\top(z-y)\big]^\top h(y)\, dS_y.
\end{aligned} \tag{7.24}$$

**Theorem 7.5.** *Let $S$ be a Lipschitz surface. The operators $V_0$ and $W_0$ can be extended to the continuous mappings*

$$\begin{aligned}
V_0 &: [H_2^{-\frac{1}{2}}(S)]^6 \longrightarrow [H_2^1(\Omega^+)]^6, \\
V_0 &: [H_2^{-\frac{1}{2}}(S)]^6 \longrightarrow [H_{2,loc}^1(\Omega^-)]^6 \cap \mathbf{Z}(\Omega^-), \\
W_0 &: [H_2^{\frac{1}{2}}(S)]^6 \longrightarrow [H_2^1(\Omega^+)]^6, \\
W_0 &: [H_2^{\frac{1}{2}}(S)]^6 \longrightarrow [H_{2,loc}^1(\Omega^-)]^6 \cap \mathbf{Z}(\Omega^-).
\end{aligned}$$

*The jump relations* (7.20)–(7.23) *on $S$ remain valid for the extended operators in the corresponding function spaces.*

**Theorem 7.6.** *Let $S$, $m$, $\kappa$, $\kappa'$ and $k$ be as in Theorem 7.4. Then the operators*

$$\mathcal{H}_0 : [C^{k,\kappa'}(S)]^6 \longrightarrow [C^{k+1,\kappa'}(S)]^6, \ \ m \geq 1, \tag{7.25}$$

$$: [H_2^{-\frac{1}{2}}(S)]^6 \longrightarrow [H_2^{\frac{1}{2}}(S)]^6, \ \ m \geq 1, \tag{7.26}$$

$$\mathcal{K}_0 : [C^{k,\kappa'}(S)]^6 \longrightarrow [C^{k,\kappa'}(S)]^6, \ \ m \geq 1, \tag{7.27}$$

$$: [H_2^{-\frac{1}{2}}(S)]^6 \longrightarrow [H_2^{-\frac{1}{2}}(S)]^6, \ \ m \geq 1, \tag{7.28}$$

$$\mathcal{N}_0 : [C^{k,\kappa'}(S)]^6 \longrightarrow [C^{k,\kappa'}(S)]^6, \ \ m \geq 1, \tag{7.29}$$

$$: [H_2^{\frac{1}{2}}(S)]^6 \longrightarrow [H_2^{\frac{1}{2}}(S)]^6, \ \ m \geq 1, \tag{7.30}$$

$$\mathcal{L}_0 : [C^{k,\kappa'}(S)]^6 \longrightarrow [C^{k-1,\kappa'}(S)]^6, \ \ m \geq 2, \ \ k \geq 1, \tag{7.31}$$

$$: [H_2^{\frac{1}{2}}(S)]^6 \longrightarrow [H_2^{-\frac{1}{2}}(S)]^6, \ \ m \geq 2, \tag{7.32}$$

*are continuous. The operators* (7.26), (7.28), (7.30), *and* (7.32) *are bounded if $S$ is a Lipschitz surface.*

Proofs of the above formulated theorems are word for word proofs of the similar theorems in [15], [25], [30], [31], [29], [53], [54], [57], [75], [88], [92]).

From Corollary 4.9 and the uniqueness Theorem 2.28 for the Dirichlet problem of static we can deduce the following assertion.

**Theorem 7.7.** *Let $S$, $m \geq 1$, $\kappa$, $\kappa'$ and $k$ be as in Theorem 7.4. Then the operators*

$$\mathcal{H}_0 : [C^{k,\kappa'}(S)]^6 \longrightarrow [C^{k+1,\kappa'}(S)]^6, \tag{7.33}$$

$$: [H_2^{-\frac{1}{2}}(S)]^6 \longrightarrow [H_2^{\frac{1}{2}}(S)]^6 \tag{7.34}$$

*are invertible.*

The next assertion is a consequence of the general theory of elliptic pseudodifferential operators on smooth manifolds without boundary (see, e.g., [1], [13], [29], [51], [100], and the references therein).

**Theorem 7.8.** *Let $V_0$, $W_0$, $\mathcal{H}_0$, $\mathcal{K}_0$, $\mathcal{N}_0$, and $\mathcal{L}_0$ be as in Theorem* 7.4 *and let $s \in \mathbb{R}$, $1 < p < \infty$, $1 \leq q \leq \infty$, $S \in C^\infty$. The layer potential operators* (7.19) *and the boundary integral* (*pseudodifferential*) *operators* (7.25), (7.27), (7.29), *and* (7.31) *can be extended to the following continuous operators*

$$V_0 : [B^s_{p,p}(S)]^6 \longrightarrow [H_p^{s+1+\frac{1}{p}}(\Omega^+)]^6, \qquad W_0 : [B^s_{p,p}(S)]^6 \longrightarrow [H_p^{s+\frac{1}{p}}(\Omega^+)]^6,$$
$$V_0 : [B^s_{p,p}(S)]^6 \longrightarrow [H_{p,loc}^{s+1+\frac{1}{p}}(\Omega^-)]^6 \cap \mathbf{Z}(\Omega^-), \quad W_0 : [B^s_{p,p}(S)]^6 \longrightarrow [H_{p,loc}^{s+\frac{1}{p}}(\Omega^-)]^6 \cap \mathbf{Z}(\Omega^-),$$
$$\mathcal{H}_0 : [H^s_p(S)]^6 \longrightarrow [H_p^{s+1}(S)]^6, \qquad \mathcal{K}_0 : [H^s_p(S)]^6 \longrightarrow [H^s_p(S)]^6,$$
$$\mathcal{N}_0 : [H^s_p(S)]^6 \longrightarrow [H^s_p(S)]^6, \qquad \mathcal{L}_0 : [H^{s+1}_p(S)]^6 \longrightarrow [H^s_p(S)]^6.$$

*The jump relations* (7.20)–(7.22) *remain valid in appropriate function spaces for arbitrary $g \in [B^s_{p,q}(S)]^6$ with $s \in \mathbb{R}$ if the limiting values* (*traces*) *on $S$ are understood in the sense described in* [100].

*In particular,*

(i) *if $g \in [B_{p,q}^{-\frac{1}{p}}(S)]^6$, then relations* (7.20) *remains valid in the sense of the function space $[B_{p,q}^{1-\frac{1}{p}}(S)]^6$, while the relations* (7.21) *remains valid in the sense of the space $[B_{p,q}^{-\frac{1}{p}}(S)]^6$;*

(ii) *if $g, h \in [B_{p,q}^{1-\frac{1}{p}}(S)]^6$, then relations* (7.22) *remains valid in the sense of the space $[B_{p,q}^{1-\frac{1}{p}}(S)]^6$, while the relations* (7.23) *remains valid in the sense of the space $[B_{p,q}^{-\frac{1}{p}}(S)]^6$.*

*Remark* 7.9. Let either $\Phi \in [L_p(\Omega^+)]^6$ or $\Phi \in [L_{p,comp}(\Omega^-)]^6$, $p > 1$. Then the Newtonian volume potentials $N_{\Omega^\pm,0}(\Phi)$ defined in (7.7) possess the following properties (see, e.g., [77]):

$$N_{\Omega^+,0}(\Phi) \in [W_p^2(\Omega^+)]^6, \;\; N_{\Omega^-,0}(\Phi) \in [W^2_{p,loc}(\Omega^-)]^6 \cap \mathbf{Z}(\Omega^-), \tag{7.35}$$

$$A(\partial) N_{\Omega^\pm,0}(\Phi) = \Phi \;\text{ almost everywhere in }\; \Omega^\pm. \tag{7.36}$$

If $\Phi \in [C^{0,\kappa}(\Omega^\pm)]^6$ with $\kappa > 0$, then $N_{\Omega^\pm,0}(\Phi) \in [C^{1,\kappa}(\overline{\Omega^\pm})]^6 \cap [C^{2,\kappa}(\Omega^\pm)]^6$ and equation (7.36) holds for all $x \in \Omega^\pm$.

Moreover, for regular densities the volume potential operator $N_{\Omega^+}$ possesses the following properties (cf. [45], [78], [57], [3]): If $S = \partial\Omega^+ \in C^{2,\alpha}$, then the following operators are continuous,

$$N_{\Omega^+,0} : [L_\infty(\Omega^+)]^6 \longrightarrow [C^{1,\gamma}(\mathbb{R}^3)]^6 \;\text{ for all }\; 0 < \gamma < 1,$$
$$N_{\Omega^+,0} : [C^{0,\beta}(\overline{\Omega^+})]^6 \longrightarrow [C^{2,\beta}(\overline{\Omega^+})]^6, \;\; 0 < \beta < 1,$$
$$N_{\Omega^+,0} : [C^{1,\beta}(\overline{\Omega^+})]^6 \longrightarrow [C^{3,\beta}(\overline{\Omega^+})]^6, \;\; 0 < \beta < \alpha \leqslant 1.$$

7.3. **Investigation of the exterior Neumann BVP.** We start with the regular setting of the exterior Neumann-type BVP for the domain $\Omega^-$:

$$A(\partial)U(x) = 0, \;\; x \in \Omega^-, \tag{7.37}$$

$$\big\{\mathcal{T}(\partial, n)U(x)\big\}^- = F(x), \;\; x \in S, \tag{7.38}$$

$$U \in [C^{1,\kappa'}(\overline{\Omega^-})]^6 \cap [C^2(\Omega^-)]^6 \cap \mathbf{Z}(\Omega^-). \tag{7.39}$$

In accordance with Remark 7.9, without loss of generality, we assume that the right hand side function in the differential equation (7.37) vanishes, since a particular solution to a nonhomogeneous differential equation $A(\partial)U(x) = \Phi$ in $\Omega^-$ with a compactly supported right hand side $\Phi \in [C^{0,\kappa}(\Omega^-)]^6$ always can be written explicitly in the form of the Newtonian potential $N_{\Omega^-,0}(\Phi)$.

Moreover, we assume that $S \in C^{1,\kappa}$ and $F \in C^{0,\kappa'}(S)$ with $0 < \kappa' < \kappa \leq 1$.

As we have shown in Subsection 3.5.2 the homogeneous version of the exterior Neumann-type problem possesses at most one solution.

To prove the existence result, we look for a solution of the problem (7.37), (7.38) as the single layer potential

$$U(x) \equiv V_0(h)(x) = \int_S \Gamma(x-y)h(y)\,dS_y, \tag{7.40}$$

where $h = (h_1, \dots, h_6)^\top \in [C^{0,\kappa'}(S)]^6$ is the unknown density. By Theorem 7.4 and in view of the boundary condition (7.38), we get the following integral equation for the density vector

$$[2^{-1}I_6 + \mathcal{K}_0]h = F \ \text{ on } \ S, \tag{7.41}$$

where $\mathcal{K}_0$ is a singular integral operator defined by (7.24). Note that, the operators

$$2^{-1}I_6 + \mathcal{K}_0 : [C^{0,\kappa'}(S)]^6 \longrightarrow [C^{0,\kappa'}(S)]^6, \tag{7.42}$$

$$: [L_2(S)]^6 \longrightarrow [L_2(S)]^6, \tag{7.43}$$

are compact perturbations of their counterpart operators associated with the pseudo-oscillation equations which are studied in Section 5. Therefore $2^{-1}I_6 + \mathcal{K}_0$ is a singular integral operator of normal type (i.e., its principal homogeneous symbol matrix is non-degenerate) and its index equals to zero.

Let us show that the operators (7.42) and (7.43) have trivial null spaces. To this end, it suffices to prove that the corresponding homogeneous integral equation

$$[2^{-1}I_6 + \mathcal{K}_0]h = 0 \ \text{ on } \ S \tag{7.44}$$

has only the trivial solution in the appropriate space. Let $h^{(0)} \in [L_2(S)]^6$ be a solution to equation (7.44). By the embedding theorems (see, e.g., [57, Ch. 4]), we actually have that $h^{(0)} \in [C^{0,\kappa'}(S)]^6$. Now, we construct the single layer potential $U_0(x) = V_0(h^{(0)})(x)$. Evidently, $U_0 \in [C^{1,\kappa'}(\overline{\Omega^\pm})]^6 \cap [C^2(\Omega^\pm)]^6 \cap \mathbf{Z}(\Omega^-)$ and the equation $A(\partial)U_0 = 0$ in $\Omega^\pm$ is automatically satisfied due to Theorems 7.1 and 7.4. Since $h^{(0)}$ solves equation (7.44), we have

$$\{\mathcal{T}(\partial,n)U_0\}^- = [2^{-1}I_6 + \mathcal{K}_0]h^{(0)} = 0 \ \text{ on } \ S.$$

Therefore $U_0$ is a solution to the homogeneous exterior Neumann problem satisfying the property $\mathbf{Z}(\Omega^-)$. Consequently, $U_0 = 0$ in $\Omega^-$ by the uniqueness Theorem 3.10. Applying the continuity property of the single layer potential we find $0 = \{U_0\}^- = \{U_0\}^+$ on $S$, yielding that the vector $U_0 = V_0(h^{(0)})$ represents a solution to the homogeneous interior Dirichlet problem of statics. Now, by the uniqueness Theorem 2.28 we deduce that $U_0 = 0$ in $\Omega^+$. Thus $U_0 = 0$ in $\Omega^\pm$ and by virtue of the jump formula we get

$$\big\{\mathcal{T}(\partial,n)U_0\big\}^+ - \big\{\mathcal{T}(\partial,n)U_0\big\}^- = -h^{(0)} = 0 \ \text{ on } \ S.$$

Therefore the null space of the operator $2^{-1}I_6 + \mathcal{K}_0$ is trivial and the operators (7.42) and (7.43) are invertible. As a ready consequence, we finally conclude that the non-homogeneous integral equation (7.41) is uniquely solvable in $[C^{0,\kappa'}(S)]^6$ for arbitrary right hand side vector $F \in [C^{0,\kappa'}(S)]^6$, which implies the following existence result.

**Theorem 7.10.** *Let $m \geq 0$ be a nonnegative integer and $0 < \kappa' < \kappa \leq 1$. Further, let $S \in C^{m+1,\kappa}$ and $F \in [C^{m,\kappa'}(S)]^6$. Then the exterior Neumann-type BVP* (7.37)–(7.39) *is uniquely solvable in the space of regular vector functions, $[C^{m+1,\kappa'}(\overline{\Omega^-})]^6 \cap [C^2(\Omega^-)]^6 \cap \mathbf{Z}(\Omega^-)$, and the solution is representable by the single layer potential $U(x) = V_0(h)(x)$ with the density $h = (h_1, \dots, h_6)^\top \in [C^{m,\kappa'}(S)]^6$ being a unique solution of the integral equation* (7.41).

Applying the same arguments as in Section 5, we can prove the existence theorem for the exterior Neumann problem of statics in the Sobolev spaces (cf. Theorem 5.16).

**Theorem 7.11.** *Let $S \in C^\infty$, $p > 1$, $s \geqslant 0$, and $F = (F_1, \dots, F_6)^\top \in [B_{p,p}^{s-\frac{1}{p}}(S)]^6$. Then*

(i) *the elliptic singular integral operator*

$$2^{-1}I_6 + \mathcal{K}_0 : \big[B_{p,p}^{s-\frac{1}{p}}(S)\big]^6 \longrightarrow \big[B_{p,p}^{s-\frac{1}{p}}(S)\big]^6 \tag{7.45}$$

*is invertible;*

(ii) *the exterior Neumann type boundary value problem is uniquely solvable in the space of vector functions* $[B_{p,p}^{s+1}(\Omega^-)]^6 \cap \mathbf{Z}(\Omega^-) \equiv [W_p^{s+1}(\Omega^-)]^6 \cap \mathbf{Z}(\Omega^-)$ *and the solution is representable in the form of single layer potential* $U = V(h)$, *where the density vector function* $h \in [B_{p,p}^{s-\frac{1}{p}}(S)]^6$ *is defined by the uniquely solvable pseudodifferential equation* $(2^{-1}I_6 + \mathcal{K}_0)h = F$ *on* $S$.

*Remark* 7.12. As in Section 5 (see Remark 5.18) we can obtain existence results for static problems in the case of non-smooth domains. In particular, if $S$ is Lipschitz surface and $F \in [H^{-1/2}(S)]^6$, then

(i) the integral equation (7.41) is uniquely solvable in the space $[H^{-1/2}(S)]^6$;

(ii) the exterior Neumann-type BVP is uniquely solvable in the space $[H^1_{2,loc}(\Omega^-)]^6 \cap \mathbf{Z}(\Omega^-)$ and the solution is representable by the single layer potential (7.40), where the density vector $h \in [H^{-1/2}(S)]^6$ solves the integral equation (7.41).

7.4. **Investigation of the interior Neumann BVP.** Before we go over to the interior Neumann problem we prove some preliminary assertions needed in our analysis.

7.4.1. *Some auxiliary results.* Let us consider the adjoint operator $A^*(\partial)$ to the operator $A(\partial)$,

$$A^*(\partial) := \begin{bmatrix} [c_{kjrl}\partial_j\partial_l]_{3\times 3} & [-e_{jkl}\partial_j\partial_l]_{3\times 1} & [-q_{jkl}\partial_j\partial_l]_{3\times 1} & [0]_{3\times 1} \\ [e_{lrj}\partial_j\partial_l]_{1\times 3} & \varkappa_{jl}\partial_j\partial_l & a_{jl}\partial_j\partial_l & 0 \\ [q_{lrj}\partial_j\partial_l]_{1\times 3} & a_{jl}\partial_j\partial_l & \mu_{jl}\partial_j\partial_l & 0 \\ [\lambda_{rj}\partial_j]_{1\times 3} & p_j\partial_j & m_j\partial_j & \eta_{jl}\partial_j\partial_l \end{bmatrix}_{6\times 6}. \tag{7.46}$$

The corresponding matrix of fundamental solutions $\Gamma^*(x-y) = [\Gamma(y-x)]^\top$ has the following property at infinity

$$\Gamma^*(x-y) = \Gamma^\top(y-x) := \begin{bmatrix} [\mathcal{O}(|x|^{-1})]_{5\times 5} & [0]_{5\times 1} \\ [\mathcal{O}(1)]_{1\times 5} & \mathcal{O}(|x|^{-1}) \end{bmatrix}_{6\times 6} \tag{7.47}$$

as $|x| \to \infty$. With the help of the fundamental matrix $\Gamma^*(x-y)$ we construct the single and double layer potentials, and the Newtonian volume potentials

$$V_0^*(h^*)(x) = V_{S,0}^*(h^*)(x) = \int_S \Gamma^*(x-y)h^*(y)\,dS_y, \quad x \in \mathbb{R}^3 \setminus S, \tag{7.48}$$

$$W_0^*(h^*)(x) = W_{S,0}^*(h^*)(x) = \int_S \big[\mathcal{T}(\partial_y, n(y))[\Gamma^*(x-y)]^\top\big]^\top h^*(y)\,dS_y, \quad x \in \mathbb{R}^3 \setminus S, \tag{7.49}$$

$$N_{\Omega^\pm,0}^*(g^*)(x) = \int_{\Omega^\pm} \Gamma^*(x-y)g^*(y)dy, \quad x \in \mathbb{R}^3,$$

where $\mathcal{T}(\partial_y, n(y))$ is defined by (2.374), the density vector $h^* = (h_1^*, \dots, h_6^*)^\top$ is defined on $S$, while $g^* = (g_1^*, \dots, g_6^*)^\top$ is defined in $\Omega^\pm$. We assume that in the case of the domain $\Omega^-$ the vector $g^*$ has a compact support.

It can be shown that the layer potentials $V_0^*$ and $W_0^*$ possess exactly the same mapping properties and jump relations as the potentials $V_0$ and $W_0$ (see Theorems 7.4–7.8). In particular,

$$\{V_0^*(h^*)\}^+ = \{V_0^*(h^*)\}^- = \mathcal{H}_0^* h^*,$$

$$\{W_0^*(h^*)\}^\pm = \pm 2^{-1}h^* + \mathcal{K}_0^* h^*, \tag{7.50}$$

$$\big\{\mathcal{P}V_0^*(h^*)\big\}^\pm = \mp 2^{-1}h^* + \mathcal{N}_0^* h^*, \tag{7.51}$$

where $\mathcal{H}_0^*$ is a weakly singular integral operator, while $\mathcal{K}_0^*$ and $\mathcal{N}_0^*$ are singular integral operators,

$$\begin{aligned} \mathcal{H}_0^* h^*(x) &:= \int_S \Gamma^*(x-y)h^*(y)\,dS_y, \\ \mathcal{K}_0^* h^*(x) &:= \int_S \big[\mathcal{T}(\partial_y, n(y))[\Gamma^*(x-y)]^\top\big]^\top h^*(y)\,dS_y, \\ \mathcal{N}_0^* h^*(x) &:= \int_S \big[\mathcal{P}(\partial_x, n(x))\Gamma^*(x-y)\big]h^*(y)\,dS_y. \end{aligned} \tag{7.52}$$

Now, we introduce a special class of vector functions which is a counterpart of the class $\mathbf{Z}(\Omega^-)$.

**Definition 7.13.** We say that a vector function $U^* = (u^*, \varphi^*, \psi^*, \vartheta^*)^\top \in [W^1_{p,loc}(\Omega^-)]^6$ has the property $\mathbf{Z}^*(\Omega^-)$ in the domain $\Omega^-$, if the following conditions are satisfied

$$\widetilde{U}^*(x) = \big(u^*(x), \varphi^*(x), \psi^*(x)\big)^\top = \mathcal{O}(|x|^{-1}) \text{ as } |x| \to \infty, \quad \vartheta^*(x) = \mathcal{O}(1) \text{ as } |x| \to \infty,$$

$$\lim_{R\to\infty} \frac{1}{4\pi R^2} \int_{\Sigma_R} \vartheta^*(x)\, d\Sigma_R = 0,$$

where $\Sigma_R$ is a sphere centered at the origin and radius $R$.

The following assertion easily follows from the relations (7.47) and (3.56).

**Theorem 7.14.** *The generalized single and double layer potentials, defined by* (7.48) *and* (7.49)*, solve the homogeneous differential equation* $A^*(\partial)U^* = 0$ *in* $\mathbb{R}^3 \setminus S$ *and possess the property* $\mathbf{Z}^*(\Omega^-)$.

For an arbitrary regular solution to the equation $A^*(\partial)U^*(x) = 0$ in $\Omega^+$ one can derive the following integral representation formula with the help of Green's formula (2.200) with $\tau = 0$

$$W_0^*(\{U^*\}^+)(x) - V_0^*(\{\mathcal{P}U^*\}^+)(x) = \begin{cases} U^*(x) & \text{for } x \in \Omega^+, \\ 0 & \text{for } x \in \Omega^-. \end{cases} \tag{7.53}$$

Similar integral representation formula holds also for an arbitrary regular solution to the equation $A^*(\partial)U^*(x) = 0$ in $\Omega^-$ possessing the property $\mathbf{Z}^*(\Omega^-)$:

$$-W_0^*(\{U^*\}_S^-)(x) + V_0^*(\{\mathcal{P}U^*\}_S^-)(x) = \begin{cases} U^*(x), & x \in \Omega^-, \\ 0, & x \in \Omega^+. \end{cases} \tag{7.54}$$

The proof of (7.54) is quite similar to the proof of Theorem 7.3. Indeed, introduce again the notation $\Omega_R^- := B(0,R) \setminus \overline{\Omega^+}$, where $B(0,R)$ is a ball centered at the origin and radius $R$. Then in view of (7.53) we have

$$U^*(x) = -W_{S,0}^*(\{U^*\}_S^-)(x) + V_{S,0}^*(\{\mathcal{P}U^*\}_S^-)(x) + \Phi_R^*(x), \ \ x \in \Omega_R^-, \tag{7.55}$$

$$0 = -W_{S,0}^*(\{U^*\}_S^-)(x) + V_{S,0}^*(\{\mathcal{P}U^*\}_S^-)(x) + \Phi_R^*(x), \ \ x \in \Omega^+, \tag{7.56}$$

where

$$\Phi_R^*(x) := W_{\Sigma_R,0}^*(U^*)(x) - V_{\Sigma_R,0}^*(\mathcal{P}U^*)(x). \tag{7.57}$$

Here $V_{\mathcal{M},0}^*$ and $W_{\mathcal{M},0}^*$ denote the single and double layer potential operators (7.48) and (7.49) with integration surface $\mathcal{M}$. Evidently,

$$A^*(\partial)\Phi_R^*(x) = 0, \ \ |x| < R. \tag{7.58}$$

In turn, from (7.55) and (7.56) we get

$$\begin{aligned} &\Phi_R^*(x) = U^*(x) + W_{S,0}^*(\{U^*\}_S^-)(x) - V_{S,0}^*(\{\mathcal{P}U^*\}_S^-)(x), \ \ x \in \Omega_R^-, \\ &\Phi_R^*(x) = W_{S,0}^*(\{U^*\}_S^-)(x) - V_{S,0}^*(\{\mathcal{P}U^*\}_S^-)(x), \ \ x \in \Omega^+, \end{aligned} \tag{7.59}$$

whence the equality $\Phi_{R_1}^*(x) = \Phi_{R_2}^*(x)$ follows for $|x| < R_1 < R_2$. We assume that $R_1$ and $R_2$ are sufficiently large numbers. Therefore, for an arbitrary fixed point $x \in \mathbb{R}^3$ the following limit exists

$$\Phi^*(x) := \lim_{R\to\infty} \Phi_R^*(x) = \begin{cases} U^*(x) + W_{S,0}^*(\{U^*\}_S^-)(x) - V_{S,0}^*(\{\mathcal{P}U^*\}_S^-)(x), & x \in \Omega^-, \\ W_{S,0}^*(\{U^*\}_S^-)(x) - V_{S,0}^*(\{\mathcal{P}U^*\}_S^-)(x), & x \in \Omega^+, \end{cases} \tag{7.60}$$

and $A^*(\partial)\Phi^*(x) = 0$ for all $x \in \Omega^+ \cup \Omega^-$. On the other hand, for arbitrary fixed point $x \in \mathbb{R}^3$ and a number $R_1$, such that $|x| < R_1$ and $\overline{\Omega^+} \subset B(0,R_1)$, from (7.59) we have

$$\Phi^*(x) = \lim_{R\to\infty} \Phi_R^*(x) = \Phi_{R_1}^*(x).$$

Now, from (7.57), (7.58), we deduce

$$A^*(\partial)\Phi^*(x) = 0 \ \ \forall\, x \in \mathbb{R}^3. \tag{7.61}$$

Since $U^*, W_{S,0}^*, V_{S,0}^* \in \mathbf{Z}^*(\Omega^-)$ we conclude from (7.60) that $\Phi^*(x) \in \mathbf{Z}^*(\mathbb{R}^3)$. In particular, we have

$$\lim_{R\to\infty} \frac{1}{4\pi R^2} \int_{\Sigma_R} \Phi^*(x)\, d\Sigma_R = 0. \tag{7.62}$$

Our goal is to show that $\Phi^*(x)=0\ \ \forall\, x\in\mathbb{R}^3$. Applying the generalized Fourier transform to equation (7.61) we get $A^*(-i\xi)\widehat{\Phi}^*(\xi)=0,\ \ \xi\in\mathbb{R}^3$, where $\widehat{\Phi}^*(\xi)$ is the Fourier transform of $\Phi^*(x)$. Taking into account that $\det A^*(-i\xi)\neq 0$ for all $\xi\in\mathbb{R}^3\setminus\{0\}$, we conclude that the support of the generalized functional $\widehat{\Phi}^*(\xi)$ is the origin and, consequently,

$$\widehat{\Phi}^*(\xi)=\sum_{|\alpha|\le M} c_\alpha\delta^{(\alpha)}(\xi),$$

where $\alpha$ is a multi-index, $c_\alpha$ are constant vectors and $M$ is some nonnegative integer. Then it follows that $\Phi^*(x)$ is polynomial in $x$,

$$\Phi^*(x)=\sum_{|\alpha|\le M} c_\alpha x^\alpha,$$

and due to the inclusion $\Phi^*\in\mathbf{Z}^*(\Omega^-)$, $\Phi^*(x)$ is bounded at infinity, i.e., $\Phi^*(x)=const$ in $\mathbb{R}^3$. Therefore (7.62) implies that $\Phi^*(x)$ vanishes identically in $\mathbb{R}^3$. This proves that formula (7.54) holds.

**Theorem 7.15.** *Let $S\in C^{2,\kappa}$ and $h\in[C^{1,\kappa'}(S)]^6$ with $0<\kappa'<\kappa\le 1$. Then for the double layer potential $W_0^*$ defined by* (7.49) *there holds the following formula* (*generalized Lyapunov–Tauber relation*)

$$\big\{\mathcal{P}W_0^*(h)\big\}^+=\big\{\mathcal{P}W_0^*(h)\big\}^- \ \ \text{on}\ \ S, \tag{7.63}$$

*where the operator $\mathcal{P}$ is given by* (7.8).

*For a Lipschitz $S$ and $h\in[H_2^{\frac12}(S)]^6$, the relation* (7.63) *also holds true in the space $[H_2^{-\frac12}(S)]^6$.*

*Proof.* Since $h\in\big[C^{1,\kappa'}(S)\big]^6$, evidently $U^*:=W_0^*(h)\in[C^{1,\kappa'}(\overline{\Omega^\pm})]^6\cap\mathbf{Z}^*(\Omega^-)$. It is clear that, the vector $U^*$ is a solution of the homogeneous equation $A^*(\partial)U^*(x)=0$ in $\Omega^+\cup\Omega^-$, where the operator $A^*(\partial)$ is defined by (7.46). With the help of (7.53) and (7.54), for the vector function $U^*$ we derive the following representation formula

$$U^*(x)=W_0^*([U^*]_S)(x)-V_0^*([\mathcal{P}U^*]_S)(x),\ \ x\in\Omega^+\cup\Omega^-, \tag{7.64}$$

where

$$[U^*]_S\equiv\{U^*\}^+-\{U^*\}^- \ \text{ and }\ [\mathcal{P}U^*]_S\equiv\{\mathcal{P}U^*\}^+-\{\mathcal{P}U^*\}^- \ \text{ on }\ S.$$

In view of the equality $U^*=W_0^*(h)$, from (7.64) we get

$$W_0^*(h)(x)=W_0^*([W_0^*(h)]_S)(x)-V_0^*([\mathcal{P}W_0^*(h)]_S)(x),\ \ x\in\Omega^+\cup\Omega^-.$$

Using the jump relation (7.50), we find $[U^*]_S=[W_0^*(h)]_S=\{W_0^*(h)\}^+-\{W_0^*(h)\}^-=h$. Therefore,

$$W_0^*(h)(x)=W_0^*(h)(x)-V_0^*([\mathcal{P}W_0^*(h)]_S)(x),\ \ x\in\Omega^+\cup\Omega^-,$$

i.e., $V_0^*(\Phi^*)(x)=0$ in $\Omega^+\cup\Omega^-$, where $\Phi^*:=[\mathcal{P}W_0^*(h)]_S$. With the help of the jump relation (7.51) finally we arrive at the equation

$$0=\{\mathcal{P}V_0^*(\Phi^*)\}^- -\{\mathcal{P}V_0^*(\Phi^*)\}^+=\Phi^*=[\mathcal{P}W_0^*(h)]_S=\{\mathcal{P}W_0^*(h)\}^+-\{\mathcal{P}W_0^*(h)\}^-$$

on $S$, which completes the proof for the regular case.

The second part of the theorem can be proved by standard limiting procedure. □

Let us consider the interior and exterior homogeneous Dirichlet BVPs for the adjoint differential operator $A^*(\partial)$

$$A^*(\partial)U^*=0\ \text{ in }\ \Omega^\pm, \tag{7.65}$$

$$\{U^*\}^\pm=0\ \text{ on }\ S. \tag{7.66}$$

In the case of the interior problem, we assume that either $U^*$ is a regular vector of the class $[C^{1,\kappa'}(\overline{\Omega^+})]^6$ or $U^*\in[W_2^1(\Omega^+)]^6$, while in the case of the exterior problem, we assume that either $U^*\in[C^{1,\kappa'}(\overline{\Omega^-})]^6\cap\mathbf{Z}^*(\Omega^-)$ or $U^*\in[W_{2,loc}^1(\Omega^-)]^6\cap\mathbf{Z}^*(\Omega^-)$.

**Theorem 7.16.** *The interior and exterior homogeneous Dirichlet type BVPs* (7.65), (7.66) *have only the trivial solution in the appropriate spaces.*

*Proof.* First, we treat the exterior Dirichlet problem. In view of the structure of the operator $A^*(\partial)$, it is easy to see that we can consider separately the BVP for the vector function $\widetilde{U}^* = (u^*, \varphi^*, \psi^*)^\top$, constructed by the first five components of the solution vector $U^*$,

$$\widetilde{A}^*(\partial)\widetilde{U}^*(x) = 0, \quad x \in \Omega^-, \tag{7.67}$$

$$\{\widetilde{U}^*(x)\}^- = 0, \quad x \in S, \tag{7.68}$$

where $\widetilde{A}^*(\partial)$ is the $5\times 5$ matrix differential operator, obtained from $A^*(\partial)$ by deleting the sixth column and the sixth row,

$$\widetilde{A}^*(\partial) := \begin{bmatrix} [c_{kjrl}\partial_j\partial_l]_{3\times 3} & [-e_{jkl}\partial_j\partial_l]_{3\times 1} & [-q_{jkl}\partial_j\partial_l]_{3\times 1} \\ [e_{lrj}\partial_j\partial_l]_{1\times 3} & \varkappa_{jl}\partial_j\partial_l & a_{jl}\partial_j\partial_l \\ [q_{lrj}\partial_j\partial_l]_{1\times 3} & a_{jl}\partial_j\partial_l & \mu_{jl}\partial_j\partial_l \end{bmatrix}_{5\times 5}. \tag{7.69}$$

With the help of Green's identity in $\Omega_R^- = B(0,R) \setminus \overline{\Omega^+}$, we have

$$\int_{\Omega_R^-} \big[\widetilde{U}^* \cdot \widetilde{A}^*(\partial)\widetilde{U}^* + \widetilde{\mathcal{E}}(\widetilde{U}^*, \widetilde{U}^*)\big]\, dx = -\int_S \{\widetilde{U}^*\}^- \cdot \{\widetilde{P}(\partial,n)\widetilde{U}^*\}^-\, dS + \int_{\Sigma_R} \widetilde{U}^* \cdot \widetilde{P}(\partial,n)\widetilde{U}^*\, d\Sigma_R, \tag{7.70}$$

where

$$\widetilde{\mathcal{P}}(\partial,n) := \begin{bmatrix} [c_{rjkl}n_j\partial_l]_{3\times 3} & [-e_{lrj}n_j\partial_l]_{3\times 1} & [-q_{lrj}n_j\partial_l]_{3\times 1} \\ [e_{jkl}n_j\partial_l]_{1\times 3} & \varkappa_{jl}n_j\partial_l & a_{jl}n_j\partial_l \\ [q_{jkl}n_j\partial_l]_{1\times 3} & a_{jl}n_j\partial_l & \mu_{jl}n_j\partial_l \end{bmatrix}_{5\times 5}, \tag{7.71}$$

and

$$\widetilde{\mathcal{E}}(\widetilde{U}^*, \widetilde{U}^*) = c_{rjkl}\partial_l u_k^* \partial_j u_r^* + \varkappa_{jl}\partial_l\varphi^*\partial_j\varphi^* + a_{jl}(\partial_l\varphi^*\partial_j\psi^* + \partial_j\psi^*\partial_l\varphi^*) + \mu_{jl}\partial_l\psi^*\partial_j\psi^*. \tag{7.72}$$

Due to the fact that $U^*$ possesses the property $\mathbf{Z}^*(\Omega^-)$, it follows that $\widetilde{U}^* = \mathcal{O}(|x|^{-1})$ and $\partial_j\widetilde{U}^* = \mathcal{O}(|x|^{-2})$ as $|x| \to \infty$, $j = 1,2,3$. Therefore,

$$\bigg|\int_{\Sigma_R} \widetilde{U}^* \cdot \widetilde{P}(\partial,n)\widetilde{U}^*\, d\Sigma_R\bigg| \le \int_{\Sigma_R} \frac{C}{R^3}\, d\Sigma_R = \frac{C}{R^3} 4\pi R^2 = \frac{4\pi C}{R} \to 0 \ \text{ as } \ R \to \infty. \tag{7.73}$$

Taking into account that $\widetilde{\mathcal{E}}(\widetilde{U}^*, \widetilde{U}^*) \ge 0$, applying the relations (7.67), (7.68), and (7.73), from (7.70) we conclude that $\widetilde{\mathcal{E}}(\widetilde{U}^*, \widetilde{U}^*) = 0$ in $\Omega^-$. Hence in view of (2.10) and (2.15) it follows that (see the proof of Theorem 2.28) $\widetilde{U}^* = (a \times x + b, b_4, b_5)$, where $a$ and $b$ are arbitrary constant vectors, and $b_4$ and $b_5$ are arbitrary scalar constants. Recall that the symbol $\times$ denotes the cross product operation. Due to the boundary condition (7.68) we get then $a = b = 0$ and $b_4 = b_5 = 0$, from which we derive that $\widetilde{U}^* = 0$. Since $\widetilde{U}^*$ vanishes in $\Omega^-$, from (7.65), (7.66) we arrive at the following boundary-value problem for $\vartheta^*$,

$$\eta_{kj}\partial_k\partial_j\vartheta^* = 0 \ \text{ in } \ \Omega^-, \quad \{\vartheta^*\}^- = 0 \ \text{ on } \ S. \tag{7.74}$$

From boundedness of $\vartheta^*$ at infinity and from (7.74) one can derive that $\vartheta^*(x) = C + \mathcal{O}(|x|^{-1})$, where $C$ is an arbitrary constant (see Lemma A.1 in Appendix A). In view of $U^* \in \mathbf{Z}^*(\Omega^-)$ we have $C = 0$ and $\vartheta^*(x) = \mathcal{O}(|x|^{-1})$, $\partial_j\vartheta^*(x) = \mathcal{O}(|x|^{-2})$, $j = 1,2,3$. Therefore, we can apply Green's formula

$$\int_{\Omega_R^-} \big[\vartheta^*\eta_{kj}\partial_k\partial_j\vartheta^* + \eta_{kj}\partial_k\vartheta^*\partial_j\vartheta^*\big]\, dx = -\int_S \{\vartheta^*\}^- \big\{\eta_{kj}n_k\partial_j\vartheta^*\big\}^-\, dS + \int_{\Sigma_R} \vartheta^*\eta_{kj}n_k\partial_j\vartheta^*\, d\Sigma_R.$$

Passing to the limit as $R \to \infty$, we get

$$\int_{\Omega^-} \eta_{kj}\partial_k\vartheta^*\partial_j\vartheta^*\, dx = 0.$$

Using the fact that the matrix $[\eta_{kj}]_{3\times 3}$ is positive definite, we conclude that $\vartheta^* = C_1 = const$ and since $\vartheta^*(x) = \mathcal{O}(|x|^{-1})$ as $|x| \to \infty$, finally we get $\vartheta^* = 0$ in $\Omega^-$. Thus $U^* = 0$ in $\Omega^-$ which completes the proof for the exterior problem. The interior problem can be treated quite similarly. □

7.4.2. *Existence results for the interior Neumann BVP.* First, let us treat the uniqueness question. To this end, we consider the homogeneous interior Neumann-type BVP

$$A(\partial)U(x) = 0, \ \ x \in \Omega^+, \tag{7.75}$$

$$\big\{\mathcal{T}(\partial, n)U(x)\big\}^+ = 0, \ \ x \in S = \partial\Omega^+. \tag{7.76}$$

As it is shown in Subsection 2.7, a general solution to the problem (7.75), (7.76) can be represented in the form (see Theorem 2.29)

$$U = \sum_{k=1}^{9} C_k U^{(k)} \ \text{ in } \ \Omega^+, \tag{7.77}$$

where $C_k$ are arbitrary scalar constants and $\{U^{(k)}\}_{k=1}^{9}$ is the basis in the space of solution vectors of the homogeneous problem (7.75), (7.76). They read as

$$U^{(k)} = \big(\widetilde{V}^{(k)}, 0\big)^\top, \ \ k = 1, \dots, 8, \ \ U^{(9)} = \big(\widetilde{V}^{(9)}, 1\big)^\top, \tag{7.78}$$

where $U^{(k)} = (u^{(k)}, \varphi^{(k)}, \psi^{(k)}, \vartheta^{(k)})^\top$, $\widetilde{V}^{(k)} = (u^{(k)}, \varphi^{(k)}, \psi^{(k)})^\top$,

$$\widetilde{V}^{(1)} = (0, -x_3, x_2, 0, 0)^\top, \ \ \widetilde{V}^{(2)} = (x_3, 0, -x_1, 0, 0)^\top, \ \ \widetilde{V}^{(3)} = (-x_2, x_1, 0, 0, 0)^\top, \ \ \widetilde{V}^{(4)} = (1, 0, 0, 0, 0)^\top,$$

$$\widetilde{V}^{(5)} = (0, 1, 0, 0, 0)^\top, \ \ \widetilde{V}^{(6)} = (0, 0, 1, 0, 0)^\top, \ \ \widetilde{V}^{(7)} = (0, 0, 0, 1, 0)^\top, \ \ \widetilde{V}^{(8)} = (0, 0, 0, 0, 1)^\top,$$

and $\widetilde{V}^{(9)}$ is defined as

$$\widetilde{V}^{(9)} = (u^{(9)}, \varphi^{(9)}, \psi^{(9)})^\top, \ \ u_k^{(9)} = b_{kq} x_q, \ \ k = 1, 2, 3, \quad \varphi^{(9)} = c_q x_q, \ \ \psi^{(9)} = d_q x_q,$$

with the twelve coefficients $b_{kq} = b_{qk}$, $c_q$ and $d_q$, $k, q = 1, 2, 3$, defined by the uniquely solvable linear algebraic system of equations (see Subsection 2.7.4)

$$\begin{aligned} &c_{rjkl} b_{kl} + e_{lrj} c_l + q_{lrj} d_l = \lambda_{rj}, \ \ r, j = 1, 2, 3, \\ &-e_{jkl} b_{kl} + \varkappa_{jl} c_l + a_{jl} d_l = p_j, \ \ j = 1, 2, 3, \\ &-q_{jkl} b_{kl} + a_{jl} c_l + \mu_{jl} d_l = m_j, \ \ j = 1, 2, 3. \end{aligned}$$

We have shown in the proof of Theorem 2.29 that the vector $U$ given by (7.77) can be rewritten as

$$U = (\widetilde{V}, 0)^\top + b_6 (\widetilde{V}^{(9)}, 1)^\top,$$

where $\widetilde{V} = (a \times x + b, b_4, b_5)^\top$, and $a = (a_1, a_2, a_3)^\top$ and $b = (b_1, b_2, b_3)^\top$ are arbitrary constant vectors, while $b_4$, $b_5$, $b_6$ are arbitrary scalar constants.

Now, let us consider the non-homogeneous interior Neumann-type BVP

$$A(\partial)U(x) = 0, \ \ x \in \Omega^+, \tag{7.79}$$

$$\big\{\mathcal{T}(\partial, n)U(x)\big\}^+ = F(x), \ \ x \in S, \tag{7.80}$$

where $U \in [C^{1,\kappa'}(\overline{\Omega^+})]^6 \cap [C^2(\Omega^+)]^6$ is a sought for vector and $F \in [C^{0,\kappa'}(S)]^6$ is a given vector. It is clear that if the problem (7.79), (7.80) is solvable, then a solution is defined within a summand vector of type (7.77).

We look for a solution to the problem (7.79), (7.80) in the form of the single layer potential,

$$U(x) = V_0(h)(x), \ \ x \in \Omega^+, \tag{7.81}$$

where $h = (h_1, \dots, h_6)^\top \in [C^{0,\kappa'}(S)]^6$ is an unknown density. From the boundary condition (7.80) and by virtue of the jump relation (7.21) (see Theorem 7.4) we get the following integral equation for the density vector $h$

$$(-2^{-1} I_6 + \mathcal{K}_0) h = F \ \text{ on } \ S, \tag{7.82}$$

where $\mathcal{K}_0$ is a singular integral operator defined in (7.24). Note that $-2^{-1} I_6 + \mathcal{K}_0$ is a singular integral operator of normal type with index zero due to Lemma 3.3, Remark 4.13, and Theorem 5.7.

Now, we investigate the null space $\operatorname{Ker}(-2^{-1} I_6 + \mathcal{K}_0)$. To this end, we consider the homogeneous equation

$$(-2^{-1} I_6 + \mathcal{K}_0) h = 0 \ \text{ on } \ S \tag{7.83}$$

and assume that a vector $h^{(0)}$ is a solution to (7.83), i.e., $h^{(0)} \in \operatorname{Ker}(-2^{-1} I_6 + \mathcal{K}_0)$. Since $h^{(0)} \in [C^{0,\kappa'}(S)]^6$, it is evident that the corresponding single layer potential $U_0(x) = V_0(h^{(0)})(x)$ belongs to the space of regular vector functions and solves the homogeneous equation $A(\partial)U_0(x) = 0$ in $\Omega^+$. Moreover, $\{\mathcal{T}(\partial,n)U_0(x)\}^+ = -2^{-1}h^{(0)} + \mathcal{K}_0h^{(0)} = 0$ on $S$ due to (7.83), i.e., $U_0(x)$ solves the homogeneous interior Neumann problem. Therefore, in accordance with the above results $U_0(x) = \sum\limits_{k=1}^{9} C_kU^{(k)}(x)$ in $\Omega^+$, where $C_k$, $k = 1,\dots,9$, are some constants, and the vectors $U^{(k)}(x)$ are defined by (7.78). Hence, we have $V_0(h^{(0)})(x) = \sum\limits_{k=1}^{9} C_kU^{(k)}(x)$, $x \in \Omega^+$, which implies

$$\{V_0(h^{(0)})(x)\}^+ \equiv \mathcal{H}_0(h^{(0)})(x) = \sum_{k=1}^{9} C_kU^{(k)}(x), \quad x \in S. \tag{7.84}$$

Keeping in mind that the operators

$$\mathcal{H}_0 : [H^{-\frac{1}{2}}(S)]^6 \longrightarrow [H^{\frac{1}{2}}(S)]^6, \quad \mathcal{H}_0 : [C^{0,\kappa'}(S)]^6 \longrightarrow [C^{1,\kappa'}(S)]^6$$

are invertible (see Subsection 4.2, Corollary 4.8), from (7.84) we obtain

$$h^{(0)} = \sum_{k=1}^{9} C_kh^{(k)}(x) \ \text{ with } \ h^{(k)}(x) := \mathcal{H}_0^{-1}(U^{(k)})(x), \quad x \in S, \quad k = 1,\dots,9. \tag{7.85}$$

Further, we show that the system of vectors $\{h^{(k)}\}_{k=1}^{9}$ is linearly independent. Let us assume the opposite. Then there exist constants $c_k$, $k = 1,\dots,9$, such that $\sum\limits_{k=1}^{9} |c_k| \neq 0$ and the following relation $\sum\limits_{k=1}^{9} c_kh^{(k)} = 0$ on $S$ holds, i.e., $\sum\limits_{k=1}^{9} c_k\mathcal{H}_0^{-1}(U^{(k)}) = 0$ on $S$. Hence we get $\mathcal{H}_0^{-1}\big(\sum\limits_{k=1}^{9} c_kU^{(k)}\big) = 0$ on $S$, and, consequently,

$$\sum_{k=1}^{9} c_kU^{(k)}(x) = 0, \quad x \in S. \tag{7.86}$$

Now, consider the vector $U^*(x) \equiv \sum\limits_{k=1}^{9} c_kU^{(k)}(x)$, $x \in \Omega^+$. Since the vectors $U^{(k)}$ are solutions of the homogeneous equation (7.79), in view of (7.86) we have

$$A(\partial)U^*(x) = 0, \quad x \in \Omega^+, \quad \{U^*(x)\}^+ = \Big\{\sum_{k=1}^{9} c_kU^{(k)}(x)\Big\}^+ = 0, \quad x \in S.$$

That is, $U^*$ is a solution of the homogeneous interior Dirichlet problem and in accordance with the uniqueness theorem for the interior Dirichlet BVP we conclude $U^*(x) = 0$ in $\Omega^+$, i.e.,

$$\sum_{k=1}^{9} c_kU^{(k)}(x) = 0, \quad x \in \Omega^+.$$

This contradicts to linear independence of the system $\{U^{(k)}\}_{k=1}^{9}$. Thus, the system of the vectors $\{h^{(k)}\}_{k=1}^{9}$ is linearly independent which implies that $\dim\operatorname{Ker}(-2^{-1}I_6 + \mathcal{K}_0) \geq 9$. Next, we show that $\dim\operatorname{Ker}(-2^{-1}I_6 + \mathcal{K}_0) \leq 9$. Let the equation $(-2^{-1}I_6 + \mathcal{K}_0)h = 0$ have a solution $h^{(10)}$ which is not representable in the form of a linear combination of the system $\{h^{(k)}\}_{k=1}^{9}$. Then the system $\{h^{(k)}\}_{k=1}^{10}$ is linearly independent. It is easy to show that the system of the corresponding single layer potentials $V^{(k)}(x) := V_0(h^{(k)})(x)$, $k = 1,\dots,10$, $x \in \Omega^+$, is linearly independent as well. Indeed, let us assume the opposite. Then there are constants $a_k$, such that

$$U(x) := \sum_{k=1}^{10} a_kV^{(k)}(x) = 0, \quad x \in \Omega^+, \tag{7.87}$$

with $\sum\limits_{k=1}^{10} |a_k| \neq 0$. From (7.87) we then derive that $\{U(x)\}^+ = 0$, $x \in S$. Therefore,

$$\{U\}^+ = \sum_{k=1}^{10} a_k\{V^{(k)}\}^+ = \sum_{k=1}^{10} a_k\mathcal{H}_0(h^{(k)}) = \mathcal{H}_0\Big(\sum_{k=1}^{10} a_kh^{(k)}\Big) = 0 \ \text{ on } \ S.$$

Whence, due to the invertibility of the operator $\mathcal{H}_0$, we get $\sum\limits_{k=1}^{10} a_k h^{(k)} = 0$ on $S$, which contradicts to the linear independence of the system $\{h^{(k)}\}_{k=1}^{10}$. Thus the system $\{V_0(h^{(k)})(x)\}_{k=1}^{10}$ is linearly independent.

On the other hand, we have

$$A(\partial)V^{(k)}(x) = 0, \quad x \in \Omega^+,$$

$$\big\{\mathcal{T}V^{(k)}\big\}^+ = (-2^{-1}I_6 + \mathcal{K}_0)h^{(k)} = 0, \quad x \in S,$$

since $h^{(k)}$, $k = 1, \dots, 10$, are solutions to the homogeneous equation (7.83). Therefore, the vectors $V^{(k)}$, $k = 1, \dots, 10$, are solutions to the homogeneous interior Neumann-type BVP and they can be expressed by linear combinations of the vectors $U^{(j)}$, $j = 1, \dots, 9$, defined in (7.78). Whence it follows that the system $\{V^{(k)}\}_{k=1}^{10}$ is linearly dependent and so is the system $\{h^{(k)}\}_{k=1}^{10}$ for an arbitrary solution $h^{(10)}$ of the equation (7.83). Consequently, $\dim \operatorname{Ker}(-2^{-1}I_6 + \mathcal{K}_0) \le 9$ implying that $\dim \operatorname{Ker}(-2^{-1}I_6 + \mathcal{K}_0) = 9$. We can consider the system $h^{(1)}, \dots, h^{(9)}$ defined in (7.85) as basis vectors of the null space of the operator $-2^{-1}I_6 + \mathcal{K}_0$. If $h_0$ is a particular solution to the nonhomogeneous integral equation (7.82), then a general solution of the same equation is represented as $h = h_0 + \sum\limits_{k=1}^{9} c_k h^{(k)}$, where $c_k$ are arbitrary constants.

For our further analysis we need also to study the homogeneous interior Neumann-type BVP for the adjoint operator $A^*(\partial)$,

$$A^*(\partial)U^* = 0 \ \text{ in } \ \Omega^+, \tag{7.88}$$

$$\{\mathcal{P}U^*\}^+ = 0 \ \text{ on } \ S = \partial\Omega^+; \tag{7.89}$$

here the adjoint operator $A^*(\partial)$ and the boundary operator $\mathcal{P}$ are defined by (7.46) and (7.8), respectively.

Note that, in the case of the problem (7.88), (7.89) we get also two separated problems:

(a) For the vector function $\widetilde{U}^* \equiv (u^*, \varphi^*, \psi^*)^\top$,

$$\widetilde{A}^*(\partial)\widetilde{U}^* = 0 \ \text{ in } \ \Omega^+, \tag{7.90}$$

$$\big\{\widetilde{\mathcal{P}}\widetilde{U}^*\big\}^+ = 0 \ \text{ on } \ S, \tag{7.91}$$

where $\widetilde{A}^*$ and $\widetilde{\mathcal{P}}$ are defined by (7.69) and (7.71), respectively, and

(b) For the function $U_6^* \equiv \vartheta^*$

$$\lambda_{rj}\partial_j u_r^* + p_j\partial_j\varphi^* + m_j\partial_j\psi^* + \eta_{jl}\partial_j\partial_l\vartheta^* = 0 \ \text{ in } \ \Omega^+, \tag{7.92}$$

$$\eta_{jl}n_j\partial_l\vartheta^* = 0 \ \text{ on } \ S. \tag{7.93}$$

For a regular solution vector $\widetilde{U}^*$ of the problem (7.90), (7.91) we can write the following Green's identity

$$\int_{\Omega^+} \big[\widetilde{U}^* \cdot \widetilde{A}^*(\partial)\widetilde{U}^* + \widetilde{\mathcal{E}}(\widetilde{U}^*, \widetilde{U}^*)\big]\,dx = \int_{\partial\Omega^+} \{\widetilde{U}^*\}^+ \cdot \big\{\widetilde{\mathcal{P}}(\partial, n)\widetilde{U}^*\big\}^+\,dS, \tag{7.94}$$

where $\widetilde{\mathcal{E}}$ is given by (7.72). If we take into account the conditions (7.90)–(7.91), from (7.94) we obtain

$$\int_{\Omega^+} \widetilde{\mathcal{E}}(\widetilde{U}^*, \widetilde{U}^*)\,dx = 0,$$

whence we get $\partial_j\varphi^* = 0$, $\partial_j\psi^* = 0$, $j = 1, 2, 3$, and $\partial_l u_k^* + \partial_j u_r^* = 0$ in $\Omega^+$. Therefore, $u^*(x) = a \times x + b$ is a rigid displacement vector, $\varphi^* = b_4$ and $\psi^* = b_5$ are arbitrary constants in $\Omega^+$. It is evident that

$$\lambda_{rj}\partial_j u_r^* = \frac{1}{2}\,\lambda_{rj}(\partial_j u_r^* + \partial_r u_j^*) = 0$$

and $p_j\partial_j\varphi^* = m_j\partial_j\psi^* = 0$. Then from (7.92), (7.93) we get the following BVP for the scalar function $\vartheta^*$,

$$\eta_{jl}\partial_j\partial_l\vartheta^* = 0 \ \text{ in } \ \Omega^+, \quad \eta_{jl}n_j\partial_l\vartheta^* = 0 \ \text{ on } \ S.$$

Using the following Green's identity

$$\int_{\Omega^+} \eta_{jl}\partial_j\partial_l\vartheta^*\vartheta^*\,dx = -\int_{\Omega^+} \eta_{jl}\partial_l\vartheta^*\partial_j\vartheta^*\,dx + \int_{\partial\Omega^+} \{\eta_{jl}n_j\partial_l\vartheta^*\}^+\{\partial_j\vartheta^*\}^+\,dS,$$

we find

$$\int_{\Omega^+} \eta_{jl}\partial_l\vartheta^*\partial_j\vartheta^*\,dx = 0,$$

and by the positive definiteness of the matrix $[\eta_{jl}]_{3\times 3}$ we get $\partial_j\vartheta^* = 0$, $j = 1,2,3$, in $\Omega^+$, i.e., $\vartheta^* = b_6 = const$ in $\Omega^+$. Consequently, a general solution $U^* = (u^*,\varphi^*,\psi^*,\vartheta^*)^\top$ of the adjoint homogeneous BVP (7.88), (7.89) can be represented as

$$U^*(x) = \sum_{k=1}^{9} C_k U^{*(k)}(x), \quad x \in \Omega^+,$$

where $C_k$ are arbitrary scalar constants and

$$\begin{gathered} U^{*(1)} = (0,-x_3,x_2,0,0,0)^\top, \quad U^{*(2)} = (x_3,0,-x_1,0,0,0)^\top, \quad U^{*(3)} = (-x_2,x_1,0,0,0,0)^\top, \\ U^{*(4)} = (1,0,0,0,0,0)^\top, \quad U^{*(5)} = (0,1,0,0,0,0)^\top, \quad U^{*(6)} = (0,0,1,0,0,0)^\top, \\ U^{*(7)} = (0,0,0,1,0,0)^\top, \quad U^{*(8)} = (0,0,0,0,1,0)^\top, \quad U^{*(9)} = (0,0,0,0,0,1)^\top. \end{gathered} \tag{7.95}$$

As we see, $U^{*(k)} = U^{(k)}$, $k = 1,\dots,8$, where $U^{(k)}$, $k = 1,\dots,8$, is given in (7.78). One can easily check that the system $\{U^{*(k)}\}_{k=1}^9$ is linearly independent. As a result we get the following assertion.

**Lemma 7.17.** *The space of solutions of the adjoint homogeneous BVP* (7.88), (7.89) *is nine dimensional and an arbitrary solution can be represented as a linear combination of the vectors* $\{U^{*(k)}\}_{k=1}^9$, *i.e., the system* $\{U^{*(k)}\}_{k=1}^9$ *is a basis in the space of solutions to the homogeneous BVP* (7.88), (7.89).

Now, we return to the nonhomogeneous equation (7.82). Note that $\mathcal{K}_0^*$ and $\mathcal{K}_0$ are mutually adjoint operators in the sense of the duality relation

$$(\mathcal{K}_0 h, h^*)_{L_2(S)} = (h, \mathcal{K}_0^* h^*)_{L_2(S)}, \quad \forall\, h, h^* \in [L_2(S)]^6.$$

Consider the corresponding homogeneous adjoint equation

$$(-2^{-1}I_6 + \mathcal{K}_0^*)h^* = 0 \ \text{ on } \ S,$$

In what follows we prove that $\dim\operatorname{Ker}(-\frac{1}{2}I_6 + \mathcal{K}_0^*) = 9$. Indeed, in accordance with Lemma 7.17 we have that $A^*(\partial)U^{*(k)} = 0$ in $\Omega^+$ and $\{\mathcal{P}U^{*(k)}\}^+ = 0$ on $S$. Therefore from (7.53) we have

$$U^{*(k)}(x) = W_0^*(\{U^{*(k)}\}^+)(x), \quad x \in \Omega^+. \tag{7.96}$$

Denote

$$h^{*(k)} := \{U^{*(k)}\}^+, \quad k = 1,\dots,9. \tag{7.97}$$

By the jump relations (7.50) we get from (7.96)

$$h^{*(k)} = 2^{-1}h^{*(k)} + \mathcal{K}_0^* h^{*(k)} \ \text{ on } \ S,$$

whence it follows that

$$(-2^{-1}I_6 + \mathcal{K}_0^*)h^{*(k)} = 0, \quad k = 1,\dots,9.$$

By Theorem 7.16 and the relations (7.96) and (7.97) we conclude that the system $\{h^{*(k)}\}_{k=1}^9$ is linearly independent, and therefore $\dim\operatorname{Ker}\big(-2^{-1}I_6 + \mathcal{K}_0^*\big) \geq 9$. Now, let $h^{*(0)} \in \operatorname{Ker}(-2^{-1}I_6 + \mathcal{K}_0^*)$, i.e., $(-2^{-1}I_6 + \mathcal{K}_0^*)h^{*(0)} = 0$. The corresponding double layer potential $U_0^*(x) := W_0^*(h^{*(0)})(x)$ is a solution to the homogeneous equation $A^*(\partial)U_0^* = 0$ in $\Omega^+$. Moreover, $\{W_0^*(h^{*(0)})\}^- = -2^{-1}h^{*(0)} + \mathcal{K}_0^* h^{*(0)} = 0$ on $S$. Consequently, $U_0^*$ is a solution of the homogeneous exterior Dirichlet BVP possessing the property $\mathbf{Z}^*(\Omega^-)$. With the help of the uniqueness Theorem 7.16 we conclude that $W_0^*(h^{*(0)}) = 0$ in $\Omega^-$. Further, $\{\mathcal{P}W_0^*(h^{*(0)})\}^+ = \{\mathcal{P}W_0^*(h^{*(0)})\}^- = 0$ due to Theorem 7.15, and for the vector function $U_0^*$ we arrive at the following BVP,

$$A^*(\partial)U_0^* = 0 \ \text{ in } \ \Omega^+, \quad \{\mathcal{P}U_0^*\}^+ = 0 \ \text{ on } \ S.$$

Using Lemma 7.17 we can write $U_0^*(x) = W_0^*(h^{*(0)})(x) = \sum_{k=1}^{9} c_k U^{*(k)}(x)$, $x \in \Omega^+$, where $c_k$ are some constants. The jump relation for the double layer potential then gives

$$\big\{W_0^*(h^{*(0)})(x)\big\}^+ - \big\{W_0^*(h^{*(0)})(x)\big\}^- = h^{*(0)}(x) = \sum_{k=1}^{9} c_k \big\{U^{*(k)}(x)\big\}^+ = \sum_{k=1}^{9} c_k h^{*(k)}(x), \quad x \in S,$$

which implies that the system $\{h^{*(k)}\}_{k=1}^{9}$ represents a basis of the null space $\operatorname{Ker}(-2^{-1}I_6 + \mathcal{K}^*)$. Thus $\dim \operatorname{Ker}(-2^{-1}I_6 + \mathcal{K}_0^*) = 9$.

Now, we can formulate the following basic existence theorem for the integral equation (7.82) and the interior Neumann-type BVP.

**Theorem 7.18.** *Let $m \geq 0$ be a nonnegative integer and $0 < \kappa' < \kappa \leq 1$. Further, let $S \in C^{m+1,\kappa}$ and $F \in [C^{m,\kappa'}(S)]^6$. The necessary and sufficient conditions for the integral equation* (7.82) *and the interior Neumann-type BVP* (7.79), (7.80) *to be solvable read as*

$$\int_S F(x) \cdot h^{*(k)}(x)\, dS = 0, \quad k = 1, \dots, 9, \tag{7.98}$$

*where the system $\{h^{*(k)}\}_{k=1}^{9}$ is defined explicitly by* (7.97) *and* (7.95).

*If these conditions are satisfied, then a solution vector to the interior Neumann-type BVP is representable by the single layer potential* (7.81), *where the density vector $h \in [C^{m,\kappa'}(S)]^6$ is defined by the integral equation* (7.82).

*A solution vector function $U \in [C^{m+1,\kappa'}(\overline{\Omega^+})]^6$ is defined modulo a linear combination of the vector functions $\{U^{(k)}\}_{k=1}^{9}$ given by* (7.78).

*Remark* 7.19. Similar to the exterior problem, if $S$ is a Lipschitz surface, $F \in [H^{-1/2}(S)]^6$, and the conditions (7.98) are fulfilled, then

(i) the integral equation (7.82) is solvable in the space $[H^{-1/2}(S)]^6$;
(ii) the interior Neumann-type BVP (7.79), (7.80) is solvable in the space $[H_2^1(\Omega^+)]^6$ and solutions are representable by the single layer potential (7.81), where the density vector $h \in \big[H^{-1/2}(S)\big]^6$ solves the integral equation (7.82);
(iii) a solution $U \in [H_2^1(\Omega^+)]^6$ to the interior Neumann-type BVP (7.79), (7.80) is defined modulo a linear combination of the vector functions $\{U^{(k)}\}_{k=1}^{9}$ given by (7.78).

## 8. Transmission Problems of Pseudo-Oscillations

Throughout this section, we assume that $\operatorname{Re}\tau = \sigma > 0$ and investigate the boundary-transmission problems for the pseudo-oscillation equations for piecewise homogeneous elastic composite bodies. Along with the classical regular transmission problems, we study the mixed transmission problems arising in the case when a composite body under consideration contains interface cracks and study the smoothness properties of weak solutions.

Note that depending on the physical properties of the interface layer, which is assumed to be very thin and is mathematically modeled as a two-dimensional interface surface between adjacent elastic regions of the composite solid, one can consider different boundary-transmission conditions for the thermo-mechanical and electro-magnetic fields adequately describing the physical process. For illustration, we treat the so called rigid transmission conditions which correspond to complete bonding of different adjacent parts of the composite body and the mixed transmission conditions for the case when a composite body contains an interfacial crack (for information concerning the formulation of the boundary and transmission conditions and their relationship to the above mentioned physical properties see the material presented in Subsection 2.2 before setting of crack dynamical problems).

Note that, the transmission problems of statics (i.e., when $\tau = 0$) can be studied by the same approach as the corresponding pseudodifferential problems.

### 8.1. Formulation of the Transmission problems of pseudo-oscillations and uniqueness theorems.

First, let us consider the so called *basic transmission problem* for the whole piecewise homogeneous space when $\mathbb{R}^3$ is divided into two regions, a bounded domain $\Omega^{(1)} := \Omega^+$ and its unbounded complement $\Omega^{(2)} := \Omega^- = \mathbb{R}^3 \setminus \overline{\Omega^+}$. For simplicity, we assume that the interface $S = \partial\Omega^+ = \partial\Omega^-$ is an infinitely smooth simply connected manifold if not otherwise stated. As above $n(x)$ denotes the unit normal vector at the point $x \in S$ directed outwards the domain $\Omega^+$. Further, we assume that the domains $\Omega^{(1)}$ and $\Omega^{(2)}$ are occupied by anisotropic homogeneous materials possessing different thermo-electro-magneto-elastic properties described in Section 2. This means that their material parameters are different constants, in general. The thermo-mechanical and electro-magnetic characteristics associated with the domain $\Omega^{(\beta)}$ for $\beta = 1, 2$, we equip with the superscript $(\beta)$ and employ the following notation

- for displacement vectors, strain and stress tensors:

$$u^{(\beta)} = (u_1^{(\beta)}, u_2^{(\beta)}, u_3^{(\beta)})^\top, \ \ \varepsilon_{kj}^{(\beta)}, \ \ \sigma_{kj}^{(\beta)};$$

- for electric and magnetic potentials, electric displacements and magnetic inductions:

$$\varphi^{(\beta)}, \ \ \psi^{(\beta)}, \ \ D^{(\beta)} = (D_1^{(\beta)}, D_2^{(\beta)}, D_3^{(\beta)})^\top, \ \ B^{(\beta)} = (B_1^{(\beta)}, B_2^{(\beta)}, B_3^{(\beta)})^\top;$$

- for temperature functions and heat fluxes:

$$\vartheta^{(\beta)}, \ \ q^{(\beta)} = (q_1^{(\beta)}, q_2^{(\beta)}, q_3^{(\beta)})^\top;$$

- for material constants:

$$c_{rjkl}^{(\beta)}, \ \ e_{rjk}^{(\beta)}, \ \ q_{rjk}^{(\beta)}, \ \ \lambda_{jk}^{(\beta)}, \ \ a_{jk}^{(\beta)}, \ \ \varkappa_{jk}^{(\beta)}, \ \ p_j^{(\beta)}, \ \ m_j^{(\beta)}, \ \ \nu_0^{(\beta)}, \ \ h_0^{(\beta)}, \ \ a_0^{(\beta)}, \ \ d_0^{(\beta)} \ \text{etc.}$$

For the region $\Omega^{(\beta)}$, we use the notation introduced in the previous sections for the basic field equations, differential and boundary operators, as well as the fundamental solutions, single layer, double layer and volume potentials and the corresponding boundary integral operators, but now equipped with the superscript $(\beta)$, e.g., $A^{(\beta)}(\partial, \tau)$, $\mathcal{T}^{(\beta)}(\partial, n, \tau)$, $\mathcal{P}^{(\beta)}(\partial, n, \tau)$, $V^{(\beta)}$, $W^{(\beta)}$, $N^{(\beta)}$, $\mathcal{H}^{(\beta)}$, $\mathcal{K}^{(\beta)}$, $\mathcal{N}^{(\beta)}$, $\mathcal{L}^{(\beta)}$ etc.

**The basic transmission problem of pseudo-oscillations** $(\mathbf{T})_\tau$ in the classical pointwise setting reads as follows:

Find regular solutions

$$U^{(\beta)} = (u^{(\beta)}, \varphi^{(\beta)}, \psi^{(\beta)}, \vartheta^{(\beta)})^\top \in [C^1(\overline{\Omega^{(\beta)}})]^6 \cap [C^2(\Omega^{(\beta)})]^6, \ \ \beta = 1, 2, \tag{8.1}$$

to the pseudo-oscillation equations of the GTEME theory

$$A^{(\beta)}(\partial_x, \tau) U^{(\beta)}(x) = \Phi^{(\beta)}(x), \ \ x \in \Omega^{(\beta)}, \ \ \beta = 1, 2, \tag{8.2}$$

satisfying the transmission conditions on the interface $S = \partial\Omega^{(1)} = \partial\Omega^{(2)}$:

$$\{U^{(1)}(x)\}^+ - \{U^{(2)}(x)\}^- = f(x), \ \ x \in S, \tag{8.3}$$

$$\big\{\mathcal{T}^{(1)}(\partial_x, n, \tau) U^{(1)}(x)\big\}^+ - \big\{\mathcal{T}^{(2)}(\partial_x, n, \tau) U^{(2)}(x)\big\}^- = F(x), \ \ x \in S, \tag{8.4}$$

where the differential operator $A^{(\beta)}(\partial_x, \tau)$ and the generalized stress operator $\mathcal{T}^{(\beta)}(\partial_x, n, \tau)$ are given by (2.45) and (2.57) with the material constants associated with the region $\Omega^{(\beta)}$, $\beta = 1, 2$, $\Phi^{(\beta)} = (\Phi_1^{(\beta)}, \dots, \Phi_6^{(\beta)})^\top$, $f = (f_1, \dots, f_6)^\top$, and $F = (F_1, \dots, F_6)^\top$ are given smooth vector functions from the appropriate spaces; in addition, we assume that the vector function $\Phi^{(2)}$ has a compact support and the sought for vector function $U^{(2)} = (u^{(2)}, \varphi^{(2)}, \psi^{(2)}, \vartheta^{(2)})^\top$ belongs to the class $\mathbf{Z}_\tau(\Omega^{(2)})$, i.e., at infinity (as $|x| \to \infty$) the following asymptotic relations hold:

$$\begin{aligned} &u_k^{(2)}(x) = \mathcal{O}(|x|^{-2}), \quad \partial_j u_k^{(2)}(x) = \mathcal{O}(|x|^{-2}),\\ &\varphi^{(2)}(x) = \mathcal{O}(|x|^{-1}), \quad \partial_j \varphi^{(2)}(x) = \mathcal{O}(|x|^{-2}),\\ &\psi^{(2)}(x) = \mathcal{O}(|x|^{-1}), \quad \partial_j \psi^{(2)}(x) = \mathcal{O}(|x|^{-2}),\\ &\vartheta^{(2)}(x) = \mathcal{O}(|x|^{-2}), \quad \partial_j \vartheta^{(2)}(x) = \mathcal{O}(|x|^{-2}), \ \ k, j = 1, 2, 3. \end{aligned} \tag{8.5}$$

In the case of **weak formulation of the basic transmission problem** $(\mathbf{T})_\tau$, we look for vector functions

$$U^{(1)} \in [W_p^1(\Omega^{(1)})]^6, \quad U^{(2)} \in [W_{p,loc}^1(\Omega^{(2)})]^6 \cap \mathbf{Z}_\tau(\Omega^{(2)}), \tag{8.6}$$

satisfying the differential equations (8.2) of pseudo-oscillations in the distributional sense, the Dirichlet type transmission condition (8.3) in the usual trace sense, and the Neumann type transmission condition (8.4) in the generalized functional sense defined by Green's formulas (2.202) and (2.211). Moreover, in the case of weak setting, we assume that

$$\Phi^{(1)} \in [L_p(\Omega^{(1)})]^6, \quad \Phi^{(2)} \in [L_{p,comp}(\Omega^{(2)})]^6, \quad f \in [B_{p,p}^{1-\frac{1}{p}}(S)]^6, \quad F \in [B_{p,p}^{-\frac{1}{p}}(S)]^6. \tag{8.7}$$

Now, let us consider the case when the composite body contains an interfacial crack, which mathematically means that the interface surface $S$ is divided into two proper, non-overlapping, open submanifolds, $S = \overline{S_T} \cup \overline{S_C}$, where $S_T$ is the transmission part of the interface and $S_C$ is the crack surface with infinitely smooth boundary curve $\ell = \partial S_T = \partial S_C$ which is also the interfacial crack edge. Let $\widetilde{\Omega}_\ell^{(\beta)} := \overline{\Omega^{(\beta)}} \setminus \ell$.

**Mixed type interfacial crack problems (MTC-D)$_\tau$, (MTC-N)$_\tau$, and (MTC-M)$_\tau$**, in the classical setting are formulated as follows: Find semi-regular solutions

$$U^{(\beta)} = (u^{(\beta)}, \varphi^{(\beta)}, \psi^{(\beta)}, \vartheta^{(\beta)})^\top \in [C^1(\widetilde{\Omega}_\ell^{(\beta)})]^6 \cap [C^2(\Omega^{(\beta)})]^6, \ \ \beta = 1,2, \tag{8.8}$$

to the pseudo-oscillation equations of the GTEME theory (8.2) satisfying the decay conditions at infinity (8.5), the rigid transmission conditions on $S_T$:

$$\{U^{(1)}(x)\}^+ - \{U^{(2)}(x)\}^- = f^{(1)}(x), \ \ x \in S_T, \tag{8.9}$$

$$\big\{\mathcal{T}^{(1)}(\partial_x, n, \tau)U^{(1)}(x)\big\}^+ - \big\{\mathcal{T}^{(2)}(\partial_x, n, \tau)U^{(2)}(x)\big\}^- = F^{(1)}(x), \ \ x \in S_T, \tag{8.10}$$

and one of the following pairs of boundary conditions on the crack surface $S_C$:

(a) screen type conditions (**Problem (MTC-D)$_\tau$**):

$$\{U^{(1)}(x)\}^+ = f^{(+)}(x), \ \ x \in S_C, \tag{8.11}$$

$$\{U^{(2)}(x)\}^- = f^{(-)}(x), \ \ x \in S_C, \tag{8.12}$$

or

(b) crack type conditions (**Problem (MTC-N)$_\tau$**):

$$\{\mathcal{T}^{(1)}(\partial_x, n, \tau)U^{(1)}(x)\}^+ = F^{(+)}(x), \ \ x \in S_C, \tag{8.13}$$

$$\big\{\mathcal{T}^{(2)}(\partial_x, n, \tau)U^{(2)}(x)\big\}^- = F^{(-)}(x), \ \ x \in S_C, \tag{8.14}$$

or

(c) mixed crack type conditions (**Problem (MTC-M)$_\tau$**):

$$\{[\mathcal{T}^{(1)}(\partial_x, n, \tau)U^{(1)}(x)]_r\}^+ = F_r^{(+)}(x), \ \ x \in S_C, \ \ r = 1,2,3, \tag{8.15}$$

$$\big\{[\mathcal{T}^{(2)}(\partial_x, n, \tau)U^{(2)}(x)]_r\big\}^- = F_r^{(-)}(x), \ \ x \in S_C, \ \ r = 1,2,3, \tag{8.16}$$

$$\{\varphi^{(1)}(x)\}^+ - \{\varphi^{(2)}(x)\}^- = f_4^*(x), \ \ x \in S_C, \tag{8.17}$$

$$\big\{[\mathcal{T}^{(1)}(\partial_x, n, \tau)U^{(1)}(x)]_4\big\}^+ - \big\{[\mathcal{T}^{(2)}(\partial_x, n, \tau)U^{(2)}(x)]_4\big\}^- = F_4^*(x), \ \ x \in S_C, \tag{8.18}$$

$$\{\psi^{(1)}(x)\}^+ - \{\psi^{(2)}(x)\}^- = f_5^*(x), \ \ x \in S_C, \tag{8.19}$$

$$\big\{[\mathcal{T}^{(1)}(\partial_x, n, \tau)U^{(1)}(x)]_5\big\}^+ - \big\{[\mathcal{T}^{(2)}(\partial_x, n, \tau)U^{(2)}(x)]_5\big\}^- = F_5^*(x), \ \ x \in S_C, \tag{8.20}$$

$$\{\vartheta^{(1)}(x)\}^+ - \{\vartheta^{(2)}(x)\}^- = f_6^*(x), \ \ x \in S_C, \tag{8.21}$$

$$\big\{[\mathcal{T}^{(1)}(\partial_x, n, \tau)U^{(1)}(x)]_6\big\}^+ - \big\{[\mathcal{T}^{(2)}(\partial_x, n, \tau)U^{(2)}(x)]_6\big\}^- = F_6^*(x), \ \ x \in S_C, \tag{8.22}$$

where

$$f^{(1)} = (f_1^{(1)}, \dots, f_6^{(1)})^\top, \ \ F^{(1)} = (F_1^{(1)}, \dots, F_6^{(1)})^\top, \ \ f^{(\pm)} = (f_1^{(\pm)}, \dots, f_6^{(\pm)})^\top,$$
$$F^{(\pm)} = (F_1^{(\pm)}, \dots, F_6^{(\pm)})^\top, \ \ f_j^*, \ \ F_j^*, \ \ j = 4,5,6,$$

are given smooth functions form appropriate spaces.

In the case of **weak formulation of the mixed type interfacial crack problems (MTC-D)$_\tau$, (MTC-N)$_\tau$, and (MTC-M)$_\tau$**, we look for vector functions $U^{(1)} \in [W_p^1(\Omega^{(1)})]^6$ and $U^{(2)} \in [W_{p,loc}^1(\Omega^{(2)})]^6 \cap \mathbf{Z}_\tau(\Omega^{(2)})$ satisfying the differential equations (8.2) of pseudo-oscillations in the distributional sense, the Dirichlet type conditions (8.9), (8.11), (8.12), (8.17), (8.19), and (8.21) in the usual trace sense, and the Neumann type transmission condition (8.10), (8.13), (8.14), (8.15), (8.16) (8.18), (8.20), and (8.22) in the generalized functional sense defined by Green's appropriate formulas. Moreover, in the case of weak setting, we assume that

$$\begin{gathered}\Phi^{(1)} \in [L_{p,comp}(\Omega^{(1)})]^6, \quad \Phi^{(2)} \in [L_p(\Omega^{(2)})]^6, \quad f_k^{(1)} \in B_{p,p}^{1-\frac{1}{p}}(S_T), \quad F_k^{(1)} \in B_{p,p}^{-\frac{1}{p}}(S_T), \\ f_k^{(\pm)} \in B_{p,p}^{1-\frac{1}{p}}(S_C), \quad F_k^{(\pm)} \in B_{p,p}^{-\frac{1}{p}}(S_C), \quad f_j^* \in B_{p,p}^{1-\frac{1}{p}}(S_C), \quad F_j^* \in B_{p,p}^{-\frac{1}{p}}(S_C), \\ k=1,\dots,6, \;\; j=4,5,6.\end{gathered} \tag{8.23}$$

Further, we prove the uniqueness theorem for weak solutions of the above formulated interfacial crack problems of pseudo-oscillations in the case of $p=2$.

**Theorem 8.1.** *Let the interface surface $S$ and its submanifolds $S_T$ and $S_C$ along with the curve $\ell = \partial S_T = \partial S_C$ be Lipschitz and $\tau = \sigma + i\omega$ with $\sigma > \sigma_0 \geqslant 0$ and $\omega \in \mathbb{R}$. The homogeneous basic interface problem (T)$_\tau$ and homogeneous interfacial crack problems* (MTC-D)$_\tau$, (MTC-N)$_\tau$, *and* (MTC-M)$_\tau$ *possess only the trivial weak solution in the spaces $[W_2^1(\Omega^{(1)})]^6$ and $[W_{2,loc}^1(\Omega^{(2)})]^6 \cap \mathbf{Z}_\tau(\Omega^{(2)})$, assuming that the time relaxation parameters $\nu_0^{(1)}$ and $\nu_0^{(2)}$ are the same,*

$$\nu_0^{(1)} = \nu_0^{(2)} =: \nu_0. \tag{8.24}$$

*Proof.* Let the vector functions

$$\begin{aligned} U^{(1)} &= (u^{(1)}, \varphi^{(1)}, \psi^{(1)}, \vartheta^{(1)})^\top \in [W_2^1(\Omega^{(1)})]^6, \\ U^{(2)} &= (u^{(2)}, \varphi^{(2)}, \psi^{(2)}, \vartheta^{(2)})^\top \in [W_{2,loc}^1(\Omega^{(2)})]^6 \cap \mathbf{Z}_\tau(\Omega^{(2)}), \end{aligned}$$

solve one of the homogeneous transmission problems listed in the theorem. For arbitrary $U' = (u', \varphi', \psi', \vartheta')^\top \in [W_2^1(\Omega^{(1)})]^6$ or $U' = (u', \varphi', \psi', \vartheta')^\top \in [W_{2,loc}^1(\Omega^{(2)})]^6$ satisfying the decay conditions (8.5), from Green's formulas (2.202) and (2.211) then we have

$$\int_{\Omega^{(1)}} \mathcal{E}_\tau^{(1)}(U^{(1)}, \overline{U'})\, dx = \big\langle \big\{\mathcal{T}^{(1)}(\partial, n, \tau) U^{(1)}\big\}^+, \{U'\}^+ \big\rangle_S, \tag{8.25}$$

$$\int_{\Omega^{(2)}} \mathcal{E}_\tau^{(2)}(U^{(2)}, \overline{U'})\, dx = -\big\langle \big\{\mathcal{T}^{(2)}(\partial, n, \tau) U^{(2)}\big\}^-, \{U'\}^- \big\rangle_S, \tag{8.26}$$

where $\mathcal{E}_\tau^{(\beta)}(\,\cdot\,,\,\cdot\,)$ is defined by the relation (2.201) with the material constants associated with the region $\Omega^{(\beta)}$, $\beta = 1,2$,

$$\begin{aligned} \mathcal{E}_\tau^{(\beta)}(U^{(\beta)}, \overline{U'}) &:= c_{rjkl}^{(\beta)} \partial_l u_k^{(\beta)} \overline{\partial_j u_r'} + \varrho^{(\beta)} \tau^2 u_r^{(\beta)} \overline{u_r'} + e_{lrj}^{(\beta)} \big(\partial_l \varphi^{(\beta)} \overline{\partial_j u_r'} - \partial_j u_r^{(\beta)} \overline{\partial_l \varphi'}\big) \\ &+ q_{lrj}^{(\beta)} \big(\partial_l \psi^{(\beta)} \overline{\partial_j u_r'} - \partial_j u_r^{(\beta)} \overline{\partial_l \psi'}\big) + \varkappa_{jl}^{(\beta)} \partial_l \varphi^{(\beta)} \overline{\partial_j \varphi'} + a_{jl}^{(\beta)} \big(\partial_l \varphi^{(\beta)} \overline{\partial_j \psi'} + \partial_j \psi^{(\beta)} \overline{\partial_l \varphi'}\big) \\ &+ \mu_{jl}^{(\beta)} \partial_l \psi^{(\beta)} \overline{\partial_j \psi'} + \lambda_{kj}^{(\beta)} \big[\tau \partial_j u_k^{(\beta)} \overline{\vartheta'} - (1+\nu_0^{(\beta)}\tau) \vartheta^{(\beta)} \overline{\partial_j u_k'}\big] - p_l^{(\beta)} \big[\tau \partial_l \varphi^{(\beta)} \overline{\vartheta'} + (1+\nu_0^{(\beta)}\tau) \vartheta^{(\beta)} \overline{\partial_l \varphi'}\big] \\ &- m_l^{(\beta)} \big[\tau \partial_l \psi^{(\beta)} \overline{\vartheta'} + (1+\nu_0^{(\beta)}\tau) \vartheta^{(\beta)} \overline{\partial_l \psi'}\big] + \eta_{jl}^{(\beta)} \partial_l \vartheta^{(\beta)} \overline{\partial_j \vartheta'} + \tau \big(h_0^{(\beta)} \tau + d_0^{(\beta)}\big) \vartheta^{(\beta)} \overline{\vartheta'}. \end{aligned} \tag{8.27}$$

Recall that the summation over the repeated Latin indices is meant from 1 to 3 if not otherwise stated. If in (8.25) and (8.26) we substitute successively the vectors

$$\begin{gathered} \big(u_1^{(2)}, u_2^{(2)}, u_3^{(2)}, 0, 0, 0\big)^\top, \;\; \big(0, 0, 0, \varphi^{(2)}, 0, 0\big)^\top, \;\; \big(0, 0, 0, 0, \psi^{(2)}, 0\big)^\top, \;\; \Big(0, 0, 0, 0, 0, \frac{1+\nu_0^{(2)}\tau}{\overline{\tau}} \vartheta^{(2)}\Big)^\top, \\ \big(u_1^{(1)}, u_2^{(1)}, u_3^{(1)}, 0, 0, 0\big)^\top, \;\; \big(0, 0, 0, \varphi^{(1)}, 0, \big)^\top, \;\; \big(0, 0, 0, 0, \psi^{(1)}, 0\big)^\top, \;\; \Big(0, 0, 0, 0, 0, \frac{1+\nu_0^{(1)}\tau}{\overline{\tau}} \vartheta^{(1)}\Big)^\top \end{gathered}$$

in the place of the vector $U'$, we get the following relations for the vectors $U^{(1)}$ and $U^{(2)}$:

$$\int_{\Omega^{(1)}}\Big[c^{(1)}_{rjkl}\partial_l u^{(1)}_k\overline{\partial_j u^{(1)}_r}+\varrho^{(1)}\tau^2 u^{(1)}_r\overline{u^{(1)}_r}+e^{(1)}_{lrj}\partial_l\varphi^{(1)}\overline{\partial_j u^{(1)}_r}+q^{(1)}_{lrj}\partial_l\psi^{(1)}\overline{\partial_j u^{(1)}_r}$$
$$-(1+\nu^{(1)}_0\tau)\lambda^{(1)}_{kj}\vartheta^{(1)}\overline{\partial_j u^{(1)}_k}\Big]\,dx=\Big\langle\big\{\big[\mathcal{T}^{(1)}(\partial,n,\tau)U^{(1)}\big]_r\big\}^+,\{u^{(1)}_r\}^+\Big\rangle_S, \tag{8.28}$$

$$\int_{\Omega^{(1)}}\Big[-e^{(1)}_{lrj}\partial_j u^{(1)}_r\overline{\partial_l\varphi^{(1)}}+\varkappa^{(1)}_{jl}\partial_l\varphi^{(1)}\overline{\partial_j\varphi^{(1)}}+a^{(1)}_{jl}\partial_j\psi^{(1)}\overline{\partial_l\varphi^{(1)}}$$
$$-(1+\nu^{(1)}_0\tau)p^{(1)}_l\vartheta^{(1)}\overline{\partial_l\varphi^{(1)}}\Big]\,dx=\Big\langle\big\{\big[\mathcal{T}^{(1)}(\partial,n,\tau)U^{(1)}\big]_4\big\}^+,\{\varphi^{(1)}\}^+\Big\rangle_S, \tag{8.29}$$

$$\int_{\Omega^{(1)}}\Big[-q^{(1)}_{lrj}\partial_j u^{(1)}_r\overline{\partial_l\psi^{(1)}}+a^{(1)}_{jl}\partial_l\varphi^{(1)}\overline{\partial_j\psi^{(1)}}+\mu^{(1)}_{jl}\partial_l\psi^{(1)}\overline{\partial_j\psi^{(1)}}$$
$$-(1+\nu^{(1)}_0\tau)m^{(1)}_l\vartheta^{(1)}\overline{\partial_l\psi^{(1)}}\Big]\,dx=\Big\langle\big\{\big[\mathcal{T}^{(1)}(\partial,n,\tau)U^{(1)}\big]_5\big\}^+,\{\psi^{(1)}\}^+\Big\rangle_S, \tag{8.30}$$

$$\int_{\Omega^{(1)}}\Big\{(1+\nu^{(1)}_0\overline{\tau})\Big[\lambda^{(1)}_{kj}\overline{\vartheta^{(1)}}\partial_j u^{(1)}_k-p^{(1)}_l\overline{\vartheta^{(1)}}\partial_l\varphi^{(1)}-m^{(1)}_l\overline{\vartheta^{(1)}}\partial_l\psi^{(1)}+(h^{(1)}_0\tau+d^{(1)}_0)|\vartheta^{(1)}|^2\Big]$$
$$+\frac{1+\nu^{(1)}_0\overline{\tau}}{\tau}\,\eta^{(1)}_{jl}\partial_l\vartheta^{(1)}\overline{\partial_j\vartheta^{(1)}}\Big\}\,dx=\frac{1+\nu^{(1)}_0\overline{\tau}}{\tau}\Big\langle\big\{\big[\mathcal{T}^{(1)}(\partial,n,\tau)U^{(1)}\big]_6\big\}^+,\{\vartheta^{(1)}\}^+\Big\rangle_S \tag{8.31}$$

and

$$\int_{\Omega^{(2)}}\Big[c^{(2)}_{rjkl}\partial_l u^{(2)}_k\overline{\partial_j u^{(2)}_r}+\varrho^{(2)}\tau^2 u^{(2)}_r\overline{u^{(2)}_r}+e^{(2)}_{lrj}\partial_l\varphi^{(2)}\overline{\partial_j u^{(2)}_r}+q^{(2)}_{lrj}\partial_l\psi^{(2)}\overline{\partial_j u^{(2)}_r}$$
$$-(1+\nu^{(2)}_0\tau)\lambda^{(2)}_{kj}\vartheta^{(2)}\overline{\partial_j u^{(2)}_k}\Big]\,dx=-\Big\langle\big\{\big[\mathcal{T}^{(2)}(\partial,n,\tau)U^{(2)}\big]_r\big\}^-,\{u^{(2)}_r\}^-\Big\rangle_S, \tag{8.32}$$

$$\int_{\Omega^{(2)}}\Big[-e^{(2)}_{lrj}\partial_j u^{(2)}_r\overline{\partial_l\varphi^{(2)}}+\varkappa^{(2)}_{jl}\partial_l\varphi^{(2)}\overline{\partial_j\varphi^{(2)}}+a^{(2)}_{jl}\partial_j\psi^{(2)}\overline{\partial_l\varphi^{(2)}}$$
$$-(1+\nu^{(2)}_0\tau)p^{(2)}_l\vartheta^{(2)}\overline{\partial_l\varphi^{(2)}}\Big]\,dx=-\Big\langle\big\{\big[\mathcal{T}^{(2)}(\partial,n,\tau)U^{(2)}\big]_4\big\}^-,\{\varphi^{(2)}\}^-\Big\rangle_S, \tag{8.33}$$

$$\int_{\Omega^{(2)}}\Big[-q^{(2)}_{lrj}\partial_j u^{(2)}_r\overline{\partial_l\psi^{(2)}}+a^{(2)}_{jl}\partial_l\varphi^{(2)}\overline{\partial_j\psi^{(2)}}+\mu^{(2)}_{jl}\partial_l\psi^{(2)}\overline{\partial_j\psi^{(2)}}$$
$$-(1+\nu^{(2)}_0\tau)m^{(2)}_l\vartheta^{(2)}\overline{\partial_l\psi^{(2)}}\Big]\,dx=-\Big\langle\big\{\big[\mathcal{T}^{(2)}(\partial,n,\tau)U^{(2)}\big]_5\big\}^-,\{\psi^{(2)}\}^-\Big\rangle_S, \tag{8.34}$$

$$\int_{\Omega^{(2)}}\Big\{(1+\nu^{(2)}_0\overline{\tau})\Big[\lambda^{(2)}_{kj}\overline{\vartheta^{(2)}}\partial_j u^{(2)}_k-p^{(2)}_l\overline{\vartheta^{(2)}}\partial_l\varphi^{(2)}-m^{(2)}_l\overline{\vartheta^{(2)}}\partial_l\psi^{(2)}+(h^{(2)}_0\tau+d^{(2)}_0)|\vartheta^{(2)}|^2\Big]$$
$$+\frac{1+\nu^{(2)}_0\overline{\tau}}{\tau}\,\eta^{(2)}_{jl}\partial_l\vartheta^{(2)}\overline{\partial_j\vartheta^{(2)}}\Big\}\,dx=-\frac{1+\nu^{(2)}_0\overline{\tau}}{\tau}\Big\langle\big\{\big[\mathcal{T}^{(2)}(\partial,n,\tau)U^{(2)}\big]_6\big\}^-,\{\vartheta^{(2)}\}^-\Big\rangle_S. \tag{8.35}$$

Note that, due to (8.24), the coefficients in the right hand side expressions in equalities (8.31) and (8.35) coincide.

Now, if we add to equation (8.28) (respectively, to (8.32)) the complex conjugate of equations (8.29)–(8.31) (respectively, to (8.33)–(8.35)), and take into account equality (8.24), the symmetry properties (2.9) of coefficients and the homogeneous boundary conditions of the interfacial crack problems $(\text{MTC-D})_\tau$, $(\text{MTC-N})_\tau$, and $(\text{MTC-M})_\tau$, we find

$$\sum_{\beta=1}^{2}\int_{\Omega^{(\beta)}}\Big\{c^{(\beta)}_{rjkl}\partial_l u^{(\beta)}_k\overline{\partial_j u^{(\beta)}_r}+\varrho^{(\beta)}\tau^2|u^{(\beta)}|^2+\varkappa^{(\beta)}_{jl}\partial_l\varphi^{(\beta)}\overline{\partial_j\varphi^{(\beta)}}+a^{(\beta)}_{jl}\big(\partial_l\psi^{(\beta)}\overline{\partial_j\varphi^{(\beta)}}+\partial_j\varphi^{(\beta)}\overline{\partial_l\psi^{(\beta)}}\big)$$
$$+\mu^{(\beta)}_{jl}\partial_l\psi^{(\beta)}\overline{\partial_j\psi^{(\beta)}}-2\,\mathrm{Re}\big[p^{(\beta)}_l(1+\nu_0\tau)\vartheta^{(\beta)}\overline{\partial_l\varphi^{(\beta)}}\big]-2\,\mathrm{Re}\big[m^{(\beta)}_l(1+\nu_0\tau)\vartheta^{(\beta)}\overline{\partial_l\psi^{(\beta)}}\big]$$
$$+(1+\nu_0\tau)(h^{(\beta)}_0\overline{\tau}+d^{(\beta)}_0)|\vartheta^{(\beta)}|^2+\frac{1+\nu_0\tau}{\overline{\tau}}\eta^{(\beta)}_{jl}\partial_l\vartheta^{(\beta)}\overline{\partial_j\vartheta^{(\beta)}}\Big\}\,dx=0. \tag{8.36}$$

Now, it is evident that this equality is a counterpart of the relation (2.361) and by the word-for-word arguments applied in the proof of Theorem 2.25 we deduce that

$$u^{(\beta)}=0,\ \ \varphi^{(\beta)}=b^{(\beta)}_1=const,\ \ \psi^{(\beta)}=b^{(\beta)}_2=const,\ \ \vartheta^{(\beta)}=0\ \text{ in }\ \Omega^{(\beta)},\ \ \beta=1,2, \tag{8.37}$$

for all $\tau=\sigma+i\omega$ with $\sigma>\sigma_0\geqslant 0$ and $\omega\in\mathbb{R}$. Since the vector $U^{(1)}$ satisfies the decay conditions (8.5) at infinity and either on the whole interface or on the submanifold $S_T$ the Dirichlet type homogeneous transmission conditions are satisfied, finally we conclude that $b_1^{(\beta)}=b_2^{(\beta)}=0$, $\beta=1,2$, which completes the proof. □

Throughout this and forthcoming sections when we treat transmission problems for composite bodies, we always assume that the condition (8.24) is satisfied.

8.2. **Existence and regularity results for transmission problems.** As in the previous sections without loss of generality here we consider the homogeneous differential equations of pseudo-oscillations (8.2) with $\Phi^{(\beta)}(x)=0$ in $\Omega^{(\beta)}$, $\beta=1,2$, and prove the existence theorems of solutions to the above formulated basic transmission and interfacial crack problems. As we will see the qualitative properties of solutions near the edges of the interior cracks and near the edges of the interfacial cracks, are essentially different. In particular, as we will see the Hölder continuity exponents of solutions to the interfacial cracks problems essentially depend on material parameters and for sufficiently smooth problem data they are not close to 0.5, as it was in the case of interior cracks (see Theorem 6.4 and Remark 6.5).

8.2.1. *Existence and regularity of solutions to the basic transmission problem.* We consider the weak formulation of the basic transmission problem for the homogeneous differential equations of pseudo-oscillations of the GTEME theory,

$$A^{(\beta)}(\partial_x,\tau)U^{(\beta)}=0 \ \text{ in } \ \Omega^{(\beta)}, \ \ \beta=1,2, \tag{8.38}$$

$$\{U^{(1)}\}^+-\{U^{(2)}\}^-=f \ \text{ on } \ S, \tag{8.39}$$

$$\big\{\mathcal{T}^{(1)}(\partial_x,n,\tau)U^{(1)}\big\}^+-\big\{\mathcal{T}^{(2)}(\partial_x,n,\tau)U^{(2)}\big\}^-=F \ \text{ on } \ S, \tag{8.40}$$

where $U^{(1)}\in[W_p^1(\Omega^{(1)})]^6$, $U^{(2)}\in[W_{p,loc}^1(\Omega^{(2)})]^6\cap\mathbf{Z}_\tau(\Omega^{(2)})$, and

$$f\in[B_{p,p}^{1-\frac1p}(S)]^6, \quad F\in[B_{p,p}^{-\frac1p}(S)]^6, \ \ p>1. \tag{8.41}$$

Let us look for solution vectors $U^{(2)}$ and $U^{(1)}$ of the problem (8.38)–(8.40) in the form of single layer potentials associated with the operator $A^{(\beta)}(\partial_x,\tau)$ and constructed by the corresponding fundamental matrix $\Gamma^{(\beta)}(x-y,\tau)$ (see Section 3)

$$U^{(1)}(x)=V^{(1)}g^{(1)}(x) \ \text{ in } \ \Omega^{(1)}, \quad U^{(2)}(x)=V^{(2)}g^{(2)}(x) \ \text{ in } \ \Omega^{(2)}, \tag{8.42}$$

where $g^{(\beta)}\in[B_{p,p}^{-1/p}(S)]^6$, $\beta=1,2$, are unknown density vectors. The transmission conditions (8.39) and (8.40) and properties of the single layer potentials lead then to the following system of the pseudodifferential equations for $g^{(1)}$ and $g^{(2)}$,

$$\mathcal{H}^{(1)}g^{(1)}-\mathcal{H}^{(2)}g^{(2)}=f \ \text{ on } \ S, \tag{8.43}$$

$$\Big(-\frac12\,I_6+\mathcal{K}^{(1)}\Big)g^{(1)}-\Big(\frac12\,I_6+\mathcal{K}^{(2)}\Big)g^{(2)}=F \ \text{ on } \ S, \tag{8.44}$$

where the integral operators $\mathcal{H}^{(\beta)}$ and $\mathcal{K}^{(\beta)}$ are generated by the single layer potential $V^{(\beta)}$ and are defined by (4.6) and (4.7), respectively. Note that, the operators

$$\mathcal{H}^{(\beta)}:[B_{p,p}^{-\frac1p}(S)]^6\longrightarrow[B_{p,p}^{1-\frac1p}(S)]^6, \ \ \beta=1,2, \tag{8.45}$$

are invertible for arbitrary $p>1$ (see Remarks 4.12 and 5.17). Therefore we can introduce new unknown vectors $\widetilde{g}^{(2)}$ and $\widetilde{g}^{(1)}$ by the relations

$$\widetilde{g}^{(1)}=\mathcal{H}^{(1)}g^{(1)}\in[B_{p,p}^{1-\frac1p}(S)]^6, \quad \widetilde{g}^{(2)}=\mathcal{H}^{(2)}g^{(2)}\in[B_{p,p}^{1-\frac1p}(S)]^6. \tag{8.46}$$

Then

$$g^{(1)}=[\mathcal{H}^{(1)}]^{-1}\widetilde{g}^{(1)}, \quad g^{(2)}=[\mathcal{H}^{(2)}]^{-1}\widetilde{g}^{(2)}, \tag{8.47}$$

where $[\mathcal{H}^{(\beta)}]^{-1}$ is the operator inverse to $\mathcal{H}^{(\beta)}$ in (8.45). It is evident that the system (8.43)–(8.44) is then equivalent via the relations (8.46) and (8.47) to the following simultaneous equations

$$\widetilde{g}^{(1)}-\widetilde{g}^{(2)}=f \ \text{ on } \ S, \tag{8.48}$$

$$\mathcal{A}^{(1)}\widetilde{g}^{(1)} - \mathcal{A}^{(2)}\widetilde{g}^{(2)} = F \ \text{ on } \ S, \tag{8.49}$$

where

$$\mathcal{A}^{(1)} := \Big(-\frac{1}{2}I_6 + \mathcal{K}^{(1)}\Big)[\mathcal{H}^{(1)}]^{-1}, \quad \mathcal{A}^{(2)} := \Big(\frac{1}{2}I_6 + \mathcal{K}^{(2)}\Big)[\mathcal{H}^{(2)}]^{-1} \tag{8.50}$$

are Steklov–Poincaré type operators associated with the domains $\Omega^{(1)}$ and $\Omega^{(2)}$, respectively (see Subsection 4.3). Due to Remark 4.12 the pseudodifferential operators

$$\mathcal{A}^{(1)} : [B^{1-\frac{1}{p}}_{p,p}(S)]^6 \longrightarrow [B^{-\frac{1}{p}}_{p,p}(S)]^6, \quad -\mathcal{A}^{(2)} : [B^{1-\frac{1}{p}}_{p,p}(S)]^6 \longrightarrow [B^{-\frac{1}{p}}_{p,p}(S)]^6, \tag{8.51}$$

are strongly elliptic Fredholm operators with index equal to zero, and moreover, $\mathcal{A}^{(2)}$ is invertible. By Lemma 4.11 the operator

$$\mathcal{A}^{(1)} - \mathcal{A}^{(2)} : [B^{1-\frac{1}{p}}_{p,p}(S)]^6 \longrightarrow [B^{-\frac{1}{p}}_{p,p}(S)]^6 \tag{8.52}$$

is strongly elliptic Fredholm operator with zero index as well. Let us show that (8.52) is invertible. To this end we prove that it has the trivial null space for $p = 2$ implying invertibility of (8.52) for $p = 2$, which in turn implies then invertibility for arbitrary $p > 1$ due to the general theory of pseudodifferential equations on manifolds without boundary.

Let $\widetilde{g} \in [B^{\frac{1}{2}}_{2,2}(S)]^6 = [H^{\frac{1}{2}}_2(S)]^6$ be a solution to the homogeneous equation

$$[\mathcal{A}^{(1)} - \mathcal{A}^{(2)}]\widetilde{g} = 0 \ \text{ on } \ S, \tag{8.53}$$

and construct the vectors

$$U^{(1)}(x) = V^{(1)}\big([\mathcal{H}^{(1)}]^{-1}\widetilde{g}\big)(x) \ \text{ in } \ \Omega^{(1)}, \quad U^{(2)}(x) = V^{(2)}\big([\mathcal{H}^{(2)}]^{-1}\widetilde{g}\big)(x) \ \text{ in } \ \Omega^{(2)}. \tag{8.54}$$

On account of (8.53) and since $[\mathcal{H}^{(\beta)}]^{-1}\widetilde{g} \in [H_2^{-\frac{1}{2}}(S)]^6$, applying the properties of the single layer potential operators, it is easy to verify that these vectors satisfy the conditions:

$$U^{(1)}(x) \in [W^1_2(\Omega^{(1)})]^6, \quad U^{(2)}(x) \in [W^1_{2,loc}(\Omega^{(2)})]^6 \cap \mathbf{Z}_\tau(\Omega^{(2)}), \tag{8.55}$$

$$\{U^{(1)}\}^+ - \{U^{(2)}\}^- = 0 \ \text{ on } \ S, \tag{8.56}$$

$$\big\{\mathcal{T}^{(1)}(\partial_x, n, \tau)U^{(1)}\big\}^+ - \big\{\mathcal{T}^{(2)}(\partial_x, n, \tau)U^{(2)}\big\}^- = 0 \ \text{ on } \ S. \tag{8.57}$$

Therefore, the vectors $U^{(1)}$ and $U^{(2)}$ solve the homogenous transmission problem (8.38)–(8.40) for $p = 2$ and by the uniqueness Theorem 8.1 they equal to zero in the corresponding regions, respectively. Consequently, $\{U^{(1)}\}^+ = \{U^{(2)}\}^- = \widetilde{g} = 0$, which completes the proof of the invertibility of the operator (8.52).

Let us denote by $\mathbb{A}$ and $\widetilde{\mathbb{A}}$ the operators generated by the left hand side expresions of the systems (8.43), (8.44) and (8.48), (8.49), respectively:

$$\mathbb{A} := \begin{bmatrix} \mathcal{H}^{(1)} & -\mathcal{H}^{(2)} \\ \frac{1}{2}\,I_6 + \mathcal{K}^{(1)} & -\frac{1}{2}\,I_6 - \mathcal{K}^{(2)} \end{bmatrix}_{12\times 12}, \quad \widetilde{\mathbb{A}} := \begin{bmatrix} I_6 & -I_6 \\ \mathcal{A}^{(1)} & -\mathcal{A}^{(2)} \end{bmatrix}_{12\times 12} \tag{8.58}$$

Applying the ellipticity of the symbol matrices $\mathfrak{S}(\mathcal{A}^{(1)} - \mathcal{A}^{(2)})$ and $\mathfrak{S}(\mathcal{H}^{(j)})$ for $j = 1, 2$, we can easily show that the symbol matrices $\mathfrak{S}(\mathbb{A})$ and $\mathfrak{S}(\widetilde{\mathbb{A}})$ are elliptic.

**Theorem 8.2.** *The elliptic pseudodifferential operators*

$$\mathbb{A} : [B^{-\frac{1}{p}}_{p,p}(S)]^6 \times [B^{-\frac{1}{p}}_{p,p}(S)]^6 \longrightarrow [B^{1-\frac{1}{p}}_{p,p}(S)]^6 \times [B^{-\frac{1}{p}}_{p,p}(S)]^6, \tag{8.59}$$

$$\widetilde{\mathbb{A}} : [B^{1-\frac{1}{p}}_{p,p}(S)]^6 \times [B^{1-\frac{1}{p}}_{p,p}(S)]^6 \longrightarrow [B^{-\frac{1}{p}}_{p,p}(S)]^6 \times [B^{-\frac{1}{p}}_{p,p}(S)]^6, \tag{8.60}$$

*are invertible and the systems* (8.43), (8.44) *and* (8.48), (8.49) *are uniquely solvable. The solution vectors* $g^{(\beta)} \in [B^{-\frac{1}{p}}_{p,p}(S)]^6$ *and* $\widetilde{g}^{(\beta)} \in [B^{1-\frac{1}{p}}_{p,p}(S)]^6$, $\beta = 1, 2$, *of the system* (8.43), (8.44) *are representable in the form*

$$\widetilde{g}^{(1)} = [\mathcal{A}^{(1)} - \mathcal{A}^{(2)}]^{-1}F - [\mathcal{A}^{(1)} - \mathcal{A}^{(2)}]^{-1}\mathcal{A}^{(2)}f, \tag{8.61}$$

$$\widetilde{g}^{(2)} = [\mathcal{A}^{(1)} - \mathcal{A}^{(2)}]^{-1}F - [\mathcal{A}^{(1)} - \mathcal{A}^{(2)}]^{-1}\mathcal{A}^{(1)}f. \tag{8.62}$$

*and*

$$g^{(1)} = [\mathcal{H}^{(1)}]^{-1}[\mathcal{A}^{(1)} - \mathcal{A}^{(2)}]^{-1}F - [\mathcal{H}^{(1)}]^{-1}[\mathcal{A}^{(1)} - \mathcal{A}^{(2)}]^{-1}\mathcal{A}^{(2)}f, \tag{8.63}$$

$$g^{(2)} = [\mathcal{H}^{(2)}]^{-1}[\mathcal{A}^{(1)} - \mathcal{A}^{(2)}]^{-1}F - [\mathcal{H}^{(2)}]^{-1}[\mathcal{A}^{(1)} - \mathcal{A}^{(2)}]^{-1}\mathcal{A}^{(1)}f. \tag{8.64}$$

*Proof.* It immediately follows from the invertibility of the operator (8.52) and the relations (8.47)–(8.49). □

**Theorem 8.3.** *If conditions* (8.41) *hold, then the basic transmission problem* $(\mathrm{T})_\tau$ (8.38)–(8.40) *is uniquely solvable in the space* $[W^1_p(\Omega^{(1)})]^6 \times ([W^1_{p,loc}(\Omega^{(2)})]^6 \cap \mathbf{Z}_\tau(\Omega^{(2)}))$ *for arbitrary* $p > 1$ *and the solution vectors are representable in the form*

$$U^{(1)} = V^{(1)}\Big([\mathcal{H}^{(1)}]^{-1}[\mathcal{A}^{(1)} - \mathcal{A}^{(2)}]^{-1}F - [\mathcal{H}^{(1)}]^{-1}[\mathcal{A}^{(1)} - \mathcal{A}^{(2)}]^{-1}\mathcal{A}^{(2)}f\Big) \ \ in \ \ \Omega^{(1)}, \tag{8.65}$$

$$U^{(2)} = V^{(2)}\Big([\mathcal{H}^{(2)}]^{-1}[\mathcal{A}^{(1)} - \mathcal{A}^{(2)}]^{-1}F - [\mathcal{H}^{(2)}]^{-1}[\mathcal{A}^{(1)} - \mathcal{A}^{(2)}]^{-1}\mathcal{A}^{(1)}f\Big) \ \ in \ \ \Omega^{(1)}. \tag{8.66}$$

*Proof.* Due to conditions (8.41) and the mapping properties of the single layer potential operators and the boundary integral operators involved in formulas (8.65) and (8.66) (see Section 4) it follows that $U^{(1)} \in [W^1_p(\Omega^{(1)})]^6$ and $U^{(2)} \in [W^1_{p,loc}(\Omega^{(2)})]^6 \cap \mathbf{Z}_\tau(\Omega^{(2)})$.

Evidently, the vectors (8.65) and (8.66) solve the homogeneous differential equations (8.38). Using relations (4.3), formulas (8.50), and the equality

$$\mathcal{A}^{(1)}[\mathcal{A}^{(2)} - \mathcal{A}^{(1)}]^{-1}\mathcal{A}^{(2)} = \mathcal{A}^{(2)}[\mathcal{A}^{(2)} - \mathcal{A}^{(1)}]^{-1}\mathcal{A}^{(1)}, \tag{8.67}$$

by direct calculation one can easily verify that the Dirichlet transmission conditions (8.39) and the Neumann transmission conditions (8.40) hold as well. Thus the vectors (8.65) and (8.66) satisfy all conditions of the basic transmission problem (8.38)–(8.40). It remains to show that for arbitrary $p > 1$ the problem possesses a unique solution, i.e., the homogeneous problem possesses only the trivial solution. We proceed as follows. Let $U^{(1)}_0 \in [W^1_p(\Omega^{(1)})]^6$ and $U^{(2)}_0 \in [W^1_{p,loc}(\Omega^{(2)})]^6 \cap \mathbf{Z}_\tau(\Omega^{(2)})$ be a solution to the homogeneous basic transmission problem (8.38)–(8.40) with $f = F = 0$. Denote

$$\begin{aligned} \{U^{(1)}_0\}^+ = \{U^{(2)}_0\}^- &=: g \in [B^{1-\frac{1}{p}}_{p,p}(S)]^6, \\ \big\{\mathcal{T}^{(1)}(\partial_x, n, \tau)U^{(1)}\big\}^+ = \big\{\mathcal{T}^{(2)}(\partial_x, n, \tau)U^{(2)}\big\}^- &=: h \in [B^{-\frac{1}{p}}_{p,p}(S)]^6. \end{aligned} \tag{8.68}$$

Due to the general integral representation formulas derived in Subsection 3.4, see Theorems 3.5–3.6, we have

$$U^{(1)}_0 = W^{(1)}(g) - V^{(1)}(h) \ \text{ in } \ \Omega^{(1)}, \quad U^{(2)}_0 = -W^{(2)}(g) + V^{(2)}(h) \ \text{ in } \ \Omega^{(2)}, \tag{8.69}$$

leading to the relations upon taking the traces from the corresponding regions,

$$\mathcal{H}^{(1)}h = \Big(-\frac{1}{2}\,I_6 + \mathcal{N}^{(1)}\Big)g, \quad \mathcal{H}^{(2)}h = \Big(\frac{1}{2}\,I_6 + \mathcal{N}^{(2)}\Big)g, \tag{8.70}$$

where $\mathcal{N}^{(\beta)}g$ is the direct value of the double layer potential $W^{(\beta)}(g)$ on the surface $S$ (see (4.4) and (4.8)). By invertibility of the operator (8.45), from (8.70) it follows that

$$h = [\mathcal{H}^{(1)}]^{-1}\Big(-\frac{1}{2}\,I_6 + \mathcal{N}^{(1)}\Big)g, \quad h = [\mathcal{H}^{(2)}]^{-1}\Big(\frac{1}{2}\,I_6 + \mathcal{N}^{(2)}\Big)g, \tag{8.71}$$

and, consequently,

$$\Big[[\mathcal{H}^{(1)}]^{-1}\Big(-\frac{1}{2}\,I_6 + \mathcal{N}^{(1)}\Big) - [\mathcal{H}^{(2)}]^{-1}\Big(\frac{1}{2}\,I_6 + \mathcal{N}^{(2)}\Big)\Big]g = 0. \tag{8.72}$$

The first relation in (4.30) implies

$$[\mathcal{H}^{(1)}]^{-1}\mathcal{N}^{(1)} = \mathcal{K}^{(1)}[\mathcal{H}^{(1)}]^{-1}, \quad [\mathcal{H}^{(2)}]^{-1}\mathcal{N}^{(2)} = \mathcal{K}^{(2)}[\mathcal{H}^{(2)}]^{-1},$$

and therefore (8.72) can be rewritten equivalently as

$$\Big[\Big(-\frac{1}{2}I_6 + \mathcal{K}^{(1)}\Big)[\mathcal{H}^{(1)}]^{-1} - \Big(\frac{1}{2}I_6 + \mathcal{K}^{(2)}\Big)[\mathcal{H}^{(2)}]^{-1}\Big]g = 0, \tag{8.73}$$

that is,

$$(\mathcal{A}^{(1)} - \mathcal{A}^{(2)})g = 0. \tag{8.74}$$

In view of the inclusion (8.68) and invertibility of the operator (8.52) for arbitrary $p > 1$, we deduce from(8.74) that $g = 0$ on $S$, implying by (8.71) $h = 0$ on $S$, which finally leads to the equalities $U_0^{(1)} = 0$ in $\Omega^{(1)}$ and $U_0^{(2)} = 0$ in $\Omega^{(2)}$ due to (8.69). This completes the proof. □

For the regular setting, due to the imbedding theorems and the invertibility of the corresponding integral operators in the Hölder continuous spaces, we have the following existence theorem.

**Theorem 8.4.** *Let $S \in C^{k+2,\kappa}$, $f \in [C^{k+1,\kappa'}(S)]^6$, and $F \in [C^{k,\kappa'}(S)]^6$ with a nonnegative integer $k$ and $0 < \kappa' < \kappa \leqslant 1$. Then the basic transmission problem $(T)_\tau$ possesses a unique regular solution*

$$U^{(\beta)} = (u^{(\beta)}, \varphi^{(\beta)}, \psi^{(\beta)}, \vartheta^{(\beta)})^\top \in [C^{k+1,\kappa'}(\overline{\Omega^{(\beta)}})]^6 \cap [C^\infty(\Omega^{(\beta)})]^6, \;\; \beta = 1, 2,$$

*representable in the form* (8.65), (8.66).

*Proof.* It follows from the invertibility of the strongly elliptic pseudodifferential operators in smooth spaces

$$\mathcal{H}^{(\beta)} : [C^{k,\kappa'}(S)]^6 \longrightarrow [C^{k+1,\kappa'}(S)]^6, \tag{8.75}$$

$$\mathcal{A}^{(1)} - \mathcal{A}^{(2)} : [C^{k+1,\kappa'}(S)]^6 \longrightarrow [C^{k,\kappa'}(S)]^6, \tag{8.76}$$

which follows from the results obtained in Section 5. □

8.2.2. *Existence and regularity of solutions to the interfacial crack problem* (MTC-D)$_\tau$. Let us assume that the conditions (cf. (8.23))

$$f^{(1)} \in [B_{p,p}^{1-\frac{1}{p}}(S_T)]^6, \quad F^{(1)} \in [B_{p,p}^{-\frac{1}{p}}(S_T)]^6, \quad f^{(\pm)} \in [B_{p,p}^{1-\frac{1}{p}}(S_C)]^6, \tag{8.77}$$

are satisfied and reformulate equivalently the screen type conditions (8.11) and (8.12) in the setting of the crack problem (MTC-D)$_\tau$ in the following form:

$$\{U^{(1)}\}^+ - \{U^{(2)}\}^- = f^{(1)} \;\text{ on }\; S_T, \tag{8.78}$$

$$\big\{\mathcal{T}^{(1)}(\partial_x, n, \tau)U^{(1)}\big\}^+ - \big\{\mathcal{T}^{(2)}(\partial_x, n, \tau)U^{(2)}\big\}^- = F^{(1)} \;\text{ on }\; S_T, \tag{8.79}$$

$$\{U^{(1)}\}^+ - \{U^{(2)})\}^- = f^{(+)} - f^{(-)} \;\text{ on }\; S_C, \tag{8.80}$$

$$\{U^{(1)}\}^+ + \{U^{(2)}\}^- = f^{(+)} + f^{(-)} \;\text{ on }\; S_C. \tag{8.81}$$

Define the vector function

$$\widetilde{f} := \begin{cases} f^{(1)} & \text{on } S_T, \\ f^{(+)} - f^{(-)} & \text{on } S_C. \end{cases} \tag{8.82}$$

We assume that the following natural compatibility condition is satisfied

$$\widetilde{f} \in [B_{p,p}^{1-\frac{1}{p}}(S)]^6. \tag{8.83}$$

We see that the weak formulation of the crack problem (MTC-D)$_\tau$ now reads as follows: Find vector functions $U^{(1)} \in [W_p^1(\Omega^{(1)})]^6$ and $U^{(2)} \in [W^1_{p,loc}(\Omega^{(2)})]^6 \cap \mathbf{Z}_\tau(\Omega^{(2)})$ satisfying the homogeneous differential equations (8.2) of pseudo-oscillations in the distributional sense with $\Phi^{(\beta)} = 0$, $\beta = 1, 2$, and

$$\{U^{(1)}\}^+ - \{U^{(2)}\}^- = \widetilde{f} \;\text{ on }\; S, \tag{8.84}$$

$$\big\{\mathcal{T}^{(1)}(\partial_x, n, \tau)U^{(1)}\big\}^+ - \big\{\mathcal{T}^{(2)}(\partial_x, n, \tau)U^{(2)}\big\}^- = F^{(1)} \;\text{ on }\; S_T, \tag{8.85}$$

$$\{U^{(1)}\}^+ + \{U^{(2)}\}^- = f^{(+)} + f^{(-)} \;\text{ on }\; S_C. \tag{8.86}$$

Let $\widetilde{F} \in \big[B_{p,p}^{-\frac{1}{p}}(S)\big]^6$ be a fixed extension of the vector function $F^{(1)}$ from $S_T$ onto the whole of $S$. Then an arbitrary extension preserving the space has the form $F = \widetilde{F} + g$, where $g \in [\widetilde{B}_{p,p}^{-\frac{1}{p}}(S_C)]^6$. Evidently, $r_{S_T} F = F^{(1)}$ on $S_T$.

Motivated by the results obtained in Subsection 8.2.1 (see Theorem 8.3), we look for a solution of the problem (MTC-D)$_\tau$ in the form

$$U^{(1)} = V^{(1)}\Big([\mathcal{H}^{(1)}]^{-1}[\mathcal{A}^{(1)} - \mathcal{A}^{(2)}]^{-1}(\widetilde{F} + g) - [\mathcal{H}^{(1)}]^{-1}[\mathcal{A}^{(1)} - \mathcal{A}^{(2)}]^{-1}\mathcal{A}^{(2)}\widetilde{f}\Big) \;\text{ in }\; \Omega^{(2)}, \tag{8.87}$$

$$U^{(2)} = V^{(2)}\Big([\mathcal{H}^{(2)}]^{-1}[\mathcal{A}^{(1)} - \mathcal{A}^{(2)}]^{-1}(\widetilde{F} + g) - [\mathcal{H}^{(2)}]^{-1}[\mathcal{A}^{(1)} - \mathcal{A}^{(2)}]^{-1}\mathcal{A}^{(1)}\widetilde{f}\Big) \ \text{ in } \ \Omega^{(1)}. \tag{8.88}$$

It can easily be verified that all the conditions of the problem $(\text{MTC-D})_\tau$ are satisfied automatically, except the condition (8.86), which leads to the following pseudodifferential equation on $S_C$ with respect to the unknown vector $g \in [\widetilde{B}_{p,p}^{-\frac{1}{p}}(S_C)]^6$,

$$2[\mathcal{A}^{(1)} - \mathcal{A}^{(2)}]^{-1}(\widetilde{F} + g) - [\mathcal{A}^{(1)} - \mathcal{A}^{(2)}]^{-1}[\mathcal{A}^{(1)} + \mathcal{A}^{(2)}]\widetilde{f} = f^{(+)} + f^{(-)} \ \text{ on } \ S_C,$$

that is,

$$r_{S_C}\mathcal{A}g = G^{(D)} \ \text{ on } \ S_C, \tag{8.89}$$

where

$$\mathcal{A} := [\mathcal{A}^{(1)} - \mathcal{A}^{(2)}]^{-1}, \tag{8.90}$$

$$G^{(D)} := \frac{1}{2}\,(f^{(+)} + f^{(-)}) + \frac{1}{2}\,r_{S_C}\mathcal{A}[\mathcal{A}^{(1)} + \mathcal{A}^{(2)}]\widetilde{f} - r_{S_C}\mathcal{A}\widetilde{F} \in [\widetilde{B}_{p,p}^{1-\frac{1}{p}}(S_C)]^6. \tag{8.91}$$

As we have shown in Subsection 8.2.1 the operator (8.52) is strongly elliptic invertible pseudodifferential operator implying that the principal homogeneous symbol matrix of the inverse operator, $\mathfrak{S}(\mathcal{A}; x, \xi_1, \xi_2)$, $x \in S$, $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$, is also strongly elliptic,

$$\operatorname{Re}\big[\mathfrak{S}(\mathcal{A}; x, \xi_1, \xi_2)\eta \cdot \eta\big] \geq c|\xi|^{-1}|\eta|^2 \ \text{ for all } \ x \in S, \ \ (\xi_1, \xi_2) \in \mathbb{R}^2 \setminus \{0\}, \ \ \eta \in \mathbb{C}^6, \tag{8.92}$$

and, consequently, the operators

$$r_{S_C}\mathcal{A} : [\widetilde{H}_p^{-\frac{1}{p}}(S_C)]^6 \longrightarrow [H_p^{1-\frac{1}{p}}(S_C)]^6, \quad r_{S_C}\mathcal{A} : [\widetilde{B}_{p,p}^{-\frac{1}{p}}(S_C)]^6 \longrightarrow [B_{p,p}^{1-\frac{1}{p}}(S_C)]^6,$$

are bounded. To analyse the invertibility of these operators, in accordance with Theorem B.1 in Appendix B, let us consider the matrix

$$\mathbf{M}_{\mathcal{A}}(x) := [\mathfrak{S}(\mathcal{A}; x, 0, +1)]^{-1}[\mathfrak{S}(\mathcal{A}; x, 0, -1)], \ \ x \in \ell = \partial S_C, \tag{8.93}$$

constructed by the principal homogeneous symbol matrix $\mathfrak{S}(\mathcal{A}; x, \xi_1, \xi_2)$ of the pseudodifferential operator $\mathcal{A}$. Let $\lambda_1(x), \dots, \lambda_6(x)$ be the eigenvalues of the matrix (8.93) and

$$\delta_j(x) = \operatorname{Re}\big[(2\pi i)^{-1}\ln\lambda_j(x)\big], \ \ j = 1, \dots, 6, \tag{8.94}$$

$$a_3 = \inf_{x\in\ell, 1\leq j\leq 6}\delta_j(x), \quad a_4 = \sup_{x\in\ell, 1\leq j\leq 6}\delta_j(x); \tag{8.95}$$

here $\ln\zeta$ denotes the branch of the logarithm analytic in the complex plane cut along $(-\infty, 0]$. Due to the strong ellipticity of the operator $\mathcal{A}$ we have the strict inequalities $-\frac{1}{2} < \delta_j(x) < \frac{1}{2}$ for $x \in \overline{S_C}$, $j = 1, \dots, 6$. Therefore

$$-\frac{1}{2} < a_3 \leq a_4 < \frac{1}{2}\,. \tag{8.96}$$

Moreover, by the same arguments as in Subsection 5.7, we can show that one of the eigenvalues, say $\lambda_6$, of the matrix $\mathbf{M}_{\mathcal{A}}(x)$ defined in (8.93) equals to 1, implying $\delta_6 = 0$. Consequently, we have the following estimates

$$-\frac{1}{2} < a_3 \leqslant 0 \leqslant a_4 < \frac{1}{2}\,. \tag{8.97}$$

**Lemma 8.5.** *The operators*

$$r_{S_C}\mathcal{A} : [\widetilde{H}_p^s(S_C)]^6 \longrightarrow [H_p^{s+1}(S_C)]^6, \tag{8.98}$$

$$r_{S_C}\mathcal{A} : [\widetilde{B}_{p,q}^s(S_C)]^6 \longrightarrow [B_{p,q}^{s+1}(S_C)]^6, \ \ q \geq 1, \tag{8.99}$$

*are invertible if*

$$\frac{1}{p} - \frac{3}{2} + a_4 < s < \frac{1}{p} - \frac{1}{2} + a_3, \tag{8.100}$$

*where $a_3$ and $a_4$ are given by* (8.95).

*Proof.* To prove the invertibility of the operators (8.98) and (8.99), we first consider the particular values of the parameters $s=-1/2$ and $p=q=2$, which fall into the region defined by the inequalities (8.100), and show that the null space of the operator

$$r_{S_C}\mathcal{A}:[\widetilde{H}_2^{-\frac{1}{2}}(S_C)]^6\longrightarrow[H_2^{\frac{1}{2}}(S_C)]^6 \tag{8.101}$$

is trivial, i.e., the equation

$$r_{S_C}\mathcal{A}\widetilde{g}=0 \ \text{ on } \ S_C \tag{8.102}$$

admits only the trivial solution in the space $[\widetilde{H}_2^{-\frac{1}{2}}(S_C)]^6$. Recall that $\widetilde{H}_2^s(S_C)=\widetilde{B}_{2,2}^s(S_C)$ and $H_2^s(S_C)=B_{2,2}^s(S_C)$ for $s\in\mathbb{R}$.

Let $\widetilde{g}\in[\widetilde{B}_{2,2}^{-\frac{1}{2}}(S_C)]^6=[\widetilde{H}_2^{-\frac{1}{2}}(S_C)]^6$ be a solution to the homogeneous equation (8.102) and construct the vectors

$$\widetilde{U}^{(1)}(x)=V^{(1)}\big([\mathcal{H}^{(1)}]^{-1}\mathcal{A}\widetilde{g}\big)(x) \ \text{ in } \ \Omega^{(1)}, \quad \widetilde{U}^{(2)}(x)=V^{(2)}\big([\mathcal{H}^{(2)}]^{-1}\mathcal{A}\widetilde{g}\big)(x) \ \text{ in } \ \Omega^{(2)}.$$

Evidently, $U^{(1)}\in[W_2^1(\Omega^{(1)})]^6$ and $U^{(2)}\in[W_{2,loc}^1(\Omega^{(2)})]^6\cap\mathbf{Z}_\tau(\Omega^{(2)})$. Further, taking into consideration that the following relations hold on $S$:

$$\{\widetilde{U}^{(1)}\}^+=\mathcal{A}\widetilde{g}, \quad \big\{\mathcal{T}^{(1)}\widetilde{U}^{(1)}\big\}^+=\mathcal{A}^{(1)}\mathcal{A}\widetilde{g}, \quad \{U^{(2)}\}^-=\mathcal{A}\widetilde{g}, \quad \big\{\mathcal{T}^{(2)}\widetilde{U}^{(2)}\big\}^-=\mathcal{A}^{(2)}\mathcal{A}\widetilde{g},$$

and using (8.102) and (8.90), and keeping in mind that $r_{S_T}\widetilde{g}=0$, we find

$$r_{S_C}\{\widetilde{U}^{(1)}\}^+=0, \quad r_{S_C}\{\widetilde{U}^{(2)}\}^-=0 \ \text{ on } \ S_C,$$
$$r_{S_T}\big[\{\widetilde{U}^{(1)}\}^+-\{\widetilde{U}^{(2)}\}^-\big]=0, \quad r_{S_T}\big[\{\mathcal{T}^{(1)}U^{(1)}\}^+-\{\mathcal{T}^{(2)}U^{(2)}\}^-\big]=0 \quad \text{on} \quad S_T.$$

Therefore, $\widetilde{U}^{(1)}$ and $\widetilde{U}^{(2)}$ solve the homogeneous interfacial crack problem (MTC-D)$_\tau$ with $p=2$ and by Theorem 8.1 they vanish in the corresponding regions, $\Omega^{(1)}$ and $\Omega^{(2)}$, respectively, which implies that $\widetilde{g}=\{\mathcal{T}^{(1)}\widetilde{U}^{(1)}\}^+-\{\mathcal{T}^{(2)}\widetilde{U}^{(2)}\}^-=0$ on $S$. Consequently, the null space of the operator (8.101) is trivial. Since the principal homogeneous symbol matrix of the operator $\mathcal{A}$ is strongly elliptic, we conclude by Theorem B.1 (see Appendix B) that the operators (8.98) and (8.99) are Fredholm operators with zero indices and with the trivial null spaces for all values of the parameters satisfying the inequalities (8.100). Thus they are invertible. □

This lemma leads to the following existence theorem.

**Theorem 8.6.** *Let the conditions* (8.77) *be fulfilled and*

$$\frac{4}{3-2a_4}<p<\frac{4}{1-2a_3}, \tag{8.103}$$

*where $a_3$ and $a_4$ are defined by* (8.95)*. Then the interfacial crack problem* (MTC-D)$_\tau$ *has a unique solution $U^{(1)}\in[W_p^1(\Omega^{(1)})]^6$ and $U^{(2)}\in[W_{p,loc}^1(\Omega^{(2)})]^6\cap\mathbf{Z}_\tau(\Omega^{(2)})$, and the solution vectors are representable in the form of single layer potentials* (8.87) *and* (8.88)*, where the unknown vector $g$ is defined from the uniquely solvable pseudodifferential equation* (8.89)*.*

*Proof.* In accordance with Lemma 8.5, equation (8.89) is uniquely solvable for $s=-\frac{1}{p}$ with $p$ satisfying inequality (8.103), since the inequalities (8.100) are fulfilled. Therefore the vectors (8.87) and (8.88) represent a solution to the interfacial crack problem (MTC-D)$_\tau$ in the space $[W_p^1(\Omega^{(1)})]^6\times([W_{p,loc}^1(\Omega^{(2)})]^6\cap\mathbf{Z}_\tau(\Omega^{(2)}))$ with $p$ as in (8.103).

It remains to show that the problem (MTC-D)$_\tau$ is uniquely solvable in the space $[W_p^1(\Omega^{(1)})]^6\times([W_{p,loc}^1(\Omega^{(2)})]^6\cap\mathbf{Z}_\tau(\Omega^{(2)}))$ for arbitrary $p$ satisfying (8.103).

Let a pair of vectors $(\widetilde{U}^{(1)},\widetilde{U}^{(2)})\in[W_p^1(\Omega^{(1)})]^6\times([W_{p,loc}^1(\Omega^{(2)})]^6\cap\mathbf{Z}_\tau(\Omega^{(2)}))$ be a solution to the homogeneous problem (MTC-D)$_\tau$. Then by Theorem 8.3 they are representable in the form (8.65), (8.66) with

$$F=\{\mathcal{T}^{(1)}\widetilde{U}^{(1)}\}^+-\{\mathcal{T}^{(2)}\widetilde{U}^{(2)}\}^-, \quad f=\{\widetilde{U}^{(1)}\}^+-\{\widetilde{U}^{(2)}\}^- \ \text{ on } \ S.$$

Due to the homogeneous conditions of the problem (MTC-D)$_\tau$, (8.9)–(8.12) with $f^{(1)}=F^{(1)}=0$ on $S_T$ and $f^{(\pm)}=0$ on $S_C$, we have

$$F=\{\mathcal{T}^{(1)}\widetilde{U}^{(1)}\}^+-\{\mathcal{T}^{(2)}\widetilde{U}^{(2)}\}^-\in[\widetilde{B}_{p,p}^{-\frac{1}{p}}(S_C)]^6, \ \ f=0 \ \text{ on } \ S, \tag{8.104}$$

and, using the notation (8.90), we get the representations

$$\widetilde{U}^{(1)} = V^{(1)}\big([\mathcal{H}^{(1)}]^{-1}\mathcal{A}F\big) \ \text{ in } \ \Omega^{(1)}, \quad \widetilde{U}^{(2)} = V^{(2)}\big([\mathcal{H}^{(2)}]^{-1}\mathcal{A}F\big) \ \text{ in } \ \Omega^{(2)}. \tag{8.105}$$

Since, $r_{S_C}\{\widetilde{U}^{(1)}\}^+ = r_{S_C}\{\widetilde{U}^{(2)}\}^- = 0$ on $S_C$ we arrive at the equation

$$r_{S_C}\mathcal{A}F = 0 \ \text{ on } \ S_C.$$

Therefore, the inclusion (8.104) and Lemma 8.5 imply that $F = 0$ on $S_C$ which completes the proof in view of (8.105). □

Further, we establish almost the best regularity Hölder continuity results for solutions to the crack problem $(\text{MTC-D})_\tau$.

**Theorem 8.7.** *Let inclusions* (8.77) *hold and let*

$$\frac{4}{3-2a_4} < p < \frac{4}{1-2a_3}\,, \ \ 1<r<\infty, \ \ 1\leq q\leq\infty, \ \ \frac{1}{r}-\frac{1}{2}+a_4 < s < \frac{1}{r}+\frac{1}{2}+a_3, \tag{8.106}$$

*with $a_3$ and $a_4$ defined by* (8.95).

*Further, let $U^{(1)} \in [W^1_p(\Omega^{(1)})]^6$ and $U^{(2)} \in [W^1_{p,loc}(\Omega^{(2)})]^6 \cap \mathbf{Z}_\tau(\Omega^{(2)})$, be a unique solution pair to the interfacial crack problem* $(\text{MTC-D})_\tau$. *Then the following hold:*

(i) *if*

$$f^{(1)} \in [B^s_{r,r}(S_T)]^6, \quad F^{(1)} \in [B^{s-1}_{r,r}(S_T)]^6, \quad f^{(\pm)} \in [B^s_{r,r}(S_C)]^6, \tag{8.107}$$

*and $\widetilde{f} \in [B^s_{r,r}(S)]^6$ with $\widetilde{f}$ defined in* (8.82), *then*

$$U^{(1)} \in [H^{s+\frac{1}{r}}_r(\Omega^{(1)})]^6, \quad U^{(2)} \in [H^{s+\frac{1}{r}}_{r,loc}(\Omega^{(2)})]^6 \cap \mathbf{Z}_\tau(\Omega^{(2)}); \tag{8.108}$$

(ii) *if*

$$f^{(1)} \in [B^s_{r,q}(S_T)]^6, \quad F^{(1)} \in [B^{s-1}_{r,q}(S_T)]^6, \quad f^{(\pm)} \in [B^s_{r,q}(S_C)]^6, \tag{8.109}$$

*and $\widetilde{f} \in [B^s_{r,q}(S)]^6$ with $\widetilde{f}$ defined in* (8.82), *then*

$$U^{(1)} \in [B^{s+\frac{1}{r}}_{r,q}(\Omega^{(1)})]^6, \quad U^{(2)} \in [B^{s+\frac{1}{r}}_{r,q,loc}(\Omega^{(2)})]^6 \cap \mathbf{Z}_\tau(\Omega^{(2)}); \tag{8.110}$$

(iii) *if $\alpha > 0$ is not integer and*

$$f^{(1)} \in [C^\alpha(\overline{S_T})]^6, \quad F^{(1)} \in [B^{\alpha-1}_{\infty,\infty}(S_T)]^6, \quad f^{(\pm)} \in [C^\alpha(\overline{S_C})]^6, \tag{8.111}$$

*and $\widetilde{f} \in [C^\alpha(S)]^6$ with $\widetilde{f}$ defined in* (8.82), *then*

$$U^{(1)} \in \bigcap_{\alpha'<\kappa}[C^{\alpha'}(\overline{\Omega^{(1)}})]^6, \quad U^{(2)} \in \bigcap_{\alpha'<\kappa}[C^{\alpha'}(\overline{\Omega^{(2)}})]^6 \cap \mathbf{Z}_\tau(\Omega^{(2)}), \tag{8.112}$$

*where $\kappa = \min\{\alpha, \ a_3+\frac{1}{2}\} > 0$.*

*Proof.* It is word for word of the proof of Theorem 5.22. □

8.2.3. *Asymptotic expansion of solutions to the problem* $(\text{MTC-D})_\tau$. In this subsection we investigate the asymptotic behaviour of the solution to the problem $(\text{MTC-D})_\tau$ near the interfacial crack edge $\ell = \partial S_c$. For simplicity of description of the method applied below, we assume that the crack and transmission boundary data of the problem under consideration are infinitely smooth, namely,

$$f^{(+)}, f^{(-)} \in [C^\infty(\overline{S}_C)]^6, \quad f^{(1)}, F^{(1)} \in [C^\infty(\overline{S}_T)]^6, \quad \widetilde{f} \in [C^\infty(S)]^6.$$

In Subsection 8.2.2 we have shown that the problem $(\text{MTC-D})_\tau$ is uniquely solvable and the solution $(U^{(1)}, U^{(2)})$ is represented by (8.87), (8.88) with the density defined by the pseudodifferential equation (8.89).

To establish the asymptotic expansion of solution $(U^{(1)}, U^{(2)})$ near the crack edge $\ell = \partial S_c$ we preserve the notation of Subsection 8.2.2 and rewrite the representation (8.87), (8.88) in the form

$$U^{(1)} = V^{(1)}(\mathcal{C}_1 g) + R^{(1)} \ \text{ in } \ \Omega^{(1)}, \quad U^{(2)} = V^{(2)}(\mathcal{C}_2 g) + R^{(2)} \ \text{ in } \ \Omega^{(2)},$$

where $\mathcal{C}_1 = [\mathcal{H}^{(1)}]^{-1}\mathcal{A}$, $\mathcal{C}_2 = [\mathcal{H}^{(2)}]^{-1}\mathcal{A}$,

$$R^{(1)} = V^{(1)}(\mathcal{C}_1\widetilde{F}) - V^{(1)}(\mathcal{C}_1\mathcal{A}^{(2)}\widetilde{f}) \in [C^\infty(\overline{\Omega^{(1)}})]^6, \ \ R^{(2)} = V^{(2)}(\mathcal{C}_2\widetilde{F}) - V^{(2)}(\mathcal{C}_2\mathcal{A}^{(1)}\widetilde{f}) \in [C^\infty(\overline{\Omega^{(2)}})]^6,$$

and $g$ solves the pseudodifferential equation

$$r_{S_C}\mathcal{A}g = G^{(D)} \ \text{ on } \ S_C, \tag{8.113}$$

where $\mathcal{A}=(\mathcal{A}^{(2)}-\mathcal{A}^{(1)})^{-1}$ (see (8.90)) and

$$G^{(D)} := \frac{1}{2}\,(f^{+}+f^{-}) + \frac{1}{2}\,r_{S_C}\mathcal{A}(\mathcal{A}^{(1)}+\mathcal{A}^{(2)})\widetilde{f} - r_{S_C}\mathcal{A}\widetilde{F} \in [C^{\infty}(\overline{S}_C)]^6.$$

Now, we can choose a partition of unity and natural local coordinate systems to perform a standard rectifying procedure for $\ell=\partial S_C=\partial S_T$ and $S_C$ based on canonical diffeomorphisms. For simplicity, let us denote the local rectified images of $\ell$ and $S_C$ under this diffeomorphisms by the same symbols. Then we identify a one-sided neighbourhood on $S_C$ of an arbitrary point $\widetilde{x}\in\ell=\partial S_C$ as a part of the half-plane $x_2>0$. Thus we assume that $(x_1,0)=\widetilde{x}\in\ell$ and $(x_1,x_{2,+})\in\partial S_C$ for $0<x_{2,+}<\varepsilon$ with some positive $\varepsilon$.

Consider the $6\times 6$ matrix $\mathbf{M}_{\mathcal{A}}$ related to the principal homogeneous symbol $\mathfrak{S}(\mathcal{A};x,\xi)$ of the operator $\mathcal{A}$ (see (8.93))

$$M_{\mathcal{A}}(x_1) := \big[\mathfrak{S}(\mathcal{A};x_1,0,0,+1)\big]^{-1}\mathfrak{S}(\mathcal{A};x_1,0,0,-1), \ \ (x_1,0)\in\ell=\partial S_C.$$

Denote by $\lambda_1(x_1),\dots,\lambda_6(x_1)$ the eigenvalues of the matrix $\mathbf{M}_{\mathcal{A}}(x_1)$ and by $m_j$ the algebraic multiplicities of $\lambda_j(x_1)$. Let $\mu_1(x_1),\dots,\mu_l(x_1)$ be the distinct eigenvalues. Evidently, $m_j$ and $l$ depend on $x_1$ in general and $m_1+\dots+m_l=6$.

It is well known that the matrix $\mathbf{M}_{\mathcal{A}}(x_1)$ admits the following decomposition (see, e.g. [42, Ch. 7, Section 7])

$$M_{\mathcal{A}}(x_1) = \mathcal{D}(x_1)\mathcal{J}_{\mathbf{M}_{\mathcal{A}}}(x_1)\mathcal{D}^{-1}(x_1), \ \ (x_1,0)\in\partial S_C,$$

where $\mathcal{D}$ is $6\times 6$ non-degenerate matrix with infinitely smooth entries and $\mathcal{J}_{\mathbf{M}_{\mathcal{A}}}$ is block diagonal

$$\mathcal{J}_{\mathbf{M}_{\mathcal{A}}}(x_1) := \operatorname{diag}\big\{\mu_1(x_1)B^{m_1}(1),\dots,\mu_l(x_1)B^{m_l}(1)\big\}.$$

Here $B^{(\nu)}(t)$ with $\nu\in\{m_1,\dots,m_l\}$ are upper triangular matrices:

$$B^{(\nu)}(t) = \|b^{(\nu)}_{jk}(t)\|_{\nu\times\nu}, \quad b^{(\nu)}_{jk}(t) = \begin{cases} \dfrac{t^{k-j}}{(k-j)!}\,, & j<k,\\ 1, & j=k,\\ 0, & j>k, \end{cases}$$

i.e.,

$$B^{(\nu)}(t) = \begin{pmatrix} 1 & t & \dfrac{t^2}{2!} & \cdots & \dfrac{t^{\nu-2}}{(\nu-2)!} & \dfrac{t^{\nu-1}}{(\nu-1)!}\\ 0 & 1 & t & \cdots & \dfrac{t^{\nu-3}}{(\nu-3)!} & \dfrac{t^{\nu-2}}{(\nu-2)!}\\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\ 0 & 0 & 0 & \cdots & 1 & t\\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}_{\nu\times\nu}.$$

Denote

$$B_0(t) := \operatorname{diag}\big\{B^{(m_1)}(t),\dots,B^{(m_l)}(t)\big\}. \tag{8.114}$$

Applying the results from the reference [22] we arrive at the following asymptotic expansion for the solution of the pseudodifferential equation (8.113)

$$\begin{aligned} g(x_1,x_{2,+}) = {} & \mathcal{D}(x_1)x_{2,+}^{-\frac{1}{2}+\Delta(x_1)}B_0\Big(-\frac{1}{2\pi i}\,\log x_{2,+}\Big)\mathcal{D}^{-1}(x_1)b_0(x_1)\\ & + \sum_{k=1}^{M}\mathcal{D}(x_1)x_{2,+}^{-\frac{1}{2}+\Delta(x_1)+k}B_k(x_1,\log x_{2,+}) + g_{M+1}(x_1,x_{2,+}), \end{aligned} \tag{8.115}$$

where $b_0\in[C^{\infty}(\ell)]^6$, $g_{M+1}\in[C^{\infty}(\ell^{+}_{\varepsilon})]^6$, $\ell^{+}_{\varepsilon}=\ell\times[0,\varepsilon]$, and

$$B_k(x_1,t) = B_0\Big(-\frac{t}{2\pi i}\Big)\sum_{j=1}^{k(2m_0-1)} t^j d_{kj}(x_1).$$

Here $m_0 = \max\{m_1, \dots, m_6\}$, the coefficients $d_{kj} \in [C^\infty(\ell)]^6$,

$$\Delta(x_1) := (\Delta_1(x_1), \dots, \Delta_6(x_1)), \quad \Delta_j(x_1) = \frac{1}{2\pi i} \log \lambda_j(x_1) = \frac{1}{2\pi} \arg \lambda_j(x_1) + \frac{1}{2\pi i} \log |\lambda_j(x_1)|,$$
$$-\pi < \arg\lambda_j(x_1) < \pi, \ \ (x_1, 0) \in \ell = \partial S_C, \ \ j = 1, \dots, 6.$$

Furthermore,

$$x_{2,+}^{-\frac{1}{2}+\Delta(x_1)} := \operatorname{diag}\Big\{x_{2,+}^{-\frac{1}{2}+\Delta_1(x_1)}, \dots, x_{2,+}^{-\frac{1}{2}+\Delta_6(x_1)}\Big\}.$$

Now, having at hand the above asymptotic expansion for the density vector function $g$, we can apply Theorem 2.2 in the reference [23], and write the following spatial asymptotic expansions of the solution $(U^{(1)}, U^{(2)})$

$$U^{(\beta)}(x) = \sum_{\mu=\pm 1} \Bigg\{ \sum_{s=1}^{l_0^{(\beta)}} \sum_{j=0}^{n_s^{(\beta)}-1} x_3^j \Big[ d_{sj}^{(\beta)}(x_1, \mu) (z_{s,\mu}^{(\beta)})^{\frac{1}{2}+\Delta(x_1)-j} B_0\Big( -\frac{1}{2\pi i} \log z_{s,\mu}^{(\beta)} \Big) \Big] c_j(x_1) +$$
$$+ \sum_{k,l=0}^{M+2} \sum_{j+p=0}^{M+2-l} x_2^l x_3^j d_{sljp}^{(\beta)}(x_1, \mu) (z_{s,\mu}^{(\beta)})^{\frac{1}{2}+\Delta(x_1)+p+k} B_{skjp}^{(\beta)}\big(x_1, \log z_{s,\mu}^{(\beta)}\big) \Bigg\} +$$
$$+ U_{M+1}^{(\beta)}(x), \ \ x_3 > 0, \ \ \beta = 1, 2. \tag{8.116}$$

The coefficients $d_{sj}^{(\beta)}(\,\cdot\,, \mu)$ and $d_{sljp}^{(\beta)}(\,\cdot\,, \mu)$ are $6 \times 6$ matrices with entries from the space $C^\infty(\ell)$, while $B_{skjp}^{(\beta)}(x_1, t)$ are polynomials in $t$ with vector coefficients which depend on the variable $x_1$ and have the order $\nu_{kjp} = k(2m_0 - 1) + m_0 - 1 + p + j$ in general, where $m_0 = \max\{m_1, \dots, m_l\}$ and $m_1 + \dots + m_l = 6$,

$$c_j \in [C^\infty(\ell)]^6, \ \ U_{M+1}^{(\beta)} \in [C^{M+1}(\overline{\Omega^{(\beta)}})]^6, \ \ (z_{s,\mu}^{(\beta)})^{\kappa+\Delta(x_1)} := \operatorname{diag}\big\{ (z_{s,\mu}^{(\beta)})^{\kappa+\Delta_1(x_1)}, \dots, (z_{s,\mu}^{(\beta)})^{\kappa+\Delta_6(x_1)} \big\},$$
$$\kappa \in \mathbb{R}, \ \ \mu = \pm 1, \ \ \beta = 1, 2, \ \ (x_1, 0) \in \ell = \partial S_C,$$
$$z_{s,+1}^{(\beta)} := -x_2 - x_3 \zeta_{s,+1}^{(\beta)}, \quad z_{s,-1}^{(\beta)} := x_2 - x_3 \zeta_{s,-1}^{(\beta)}, \ \ -\pi < \arg z_{s,\pm 1}^{(\beta)} < \pi, \ \ \zeta_{s,\pm 1}^{(\beta)} \in C^\infty(\ell).$$

Here $\{\zeta_{s,\pm 1}^{(\beta)}\}_{s=1}^{l_0^{(\beta)}}$ are the different roots of multiplicity $n_s^{(\beta)}$, $s = 1, \dots, l_0^{(\beta)}$, of the polynomial in $\zeta$, $\det A^{(\beta,0)}([J_{\varkappa_\beta}^\top(x, 0, 0)]^{-1} \eta_\pm)$ with $\eta_\pm = (0, \pm 1, \zeta)^\top$, satisfying the conditions $\operatorname{Re} \zeta_{s,\pm 1} < 0$. The matrix $J_{\varkappa_\beta}$ stands for the Jacobian matrix corresponding to the canonical diffeomorphism $\varkappa_\beta$ related to the chosen local coordinate system. Under this diffeomorphism the curve $\ell = \partial S_C$ and $S_C$ are locally rectified and we assume that $(x_1, 0, 0) \in \ell = \partial S_C$, $x_2 = \operatorname{dist}(x_C^{(\beta)}, \ell)$, $x_3 = \operatorname{dist}(x, S_C)$, where $x_C^{(\beta)}$ is the projection of the reference point $x \in \Omega^{(\beta)}$, $\beta = 1, 2$, on the plane corresponding to the image of $S_C$ under the diffeomorphism $\varkappa_\beta$.

Note that, the coefficients $d_{sj}^{(\beta)}(\,\cdot\,, \mu)$ can be calculated explicitly, whereas the coefficients $c_j$ can be expressed by means of the first coefficient $b_0$ in the asymptotic expansion (8.115) (see [23]),

$$d_{sj}^{(\beta)}(x_1, +1) = \frac{1}{2\pi} \mathcal{G}_{\varkappa_\beta}(x_1, 0) P_{sj}^{\beta,+}(x_1) \mathcal{D}(x_1), \ \ d_{sj}^{(\beta)}(x_1, -1) = \frac{1}{2\pi} \mathcal{G}_{\varkappa_\beta}(x_1, 0) P_{sj}^{\beta,-}(x_1) \mathcal{D}(x_1) e^{i\pi(\frac{1}{2}-\Delta(x_1))},$$
$$s = 1, \dots, l_0^{(\beta)}, \ \ j = 0, \dots, n_s^{(\beta)} - 1, \ \ \beta = 1, 2,$$

where

$$P_{sj}^{\pm,\beta}(x_1) := V_{-1,j}^{\beta,s}(x_1, 0, 0, \pm 1) \mathfrak{S}(\mathcal{C}_\beta; 0, 0, \pm 1),$$
$$V_{-1,j}^{\beta,s}(x_1, 0, 0, \pm 1) := \frac{i^{j+1}}{j!(n_s^{(\beta)} - 1 - j)!} \frac{d^{n_s^{(\beta)}-1-j}}{d\zeta^{n_s^{(\beta)}-1-j}} (\zeta - \zeta_{s,\pm 1}^{(\beta)})^{n_s^{(\beta)}} \times$$
$$\times \Big( A^{(\beta,0)}\big( (J_{\varkappa_\beta}^\top(x_1, 0) v)^{-1} (0, \pm 1, \zeta)^\top \big) \Big)^{-1} \Big|_{\zeta=\zeta_{s,\pm 1}^{(\beta)}},$$
$$e^{i\pi(\frac{1}{2}-\Delta(x_1))} := \operatorname{diag}\Big\{ e^{i\pi(\frac{1}{2}-\Delta_1(x_1))}, \dots, e^{i\pi(\frac{1}{2}-\Delta_6(x_1))} \Big\},$$

$\mathcal{G}_{\varkappa_\beta}$ are the square roots of Gram's determinant of the diffeomorphisms $\varkappa_\beta$, $\beta = 1, 2$, and

$$c_j(x_1) = a_j(x_1) B_0^-\Big( -\frac{1}{2} + \Delta(x_1) \Big) \mathcal{D}^{-1}(x_1) b_0(x_1), \ \ j = 0, , \dots, n_s^{(\beta)} - 1,$$

$$B_0^-\Big(-\frac{1}{2}+\Delta(x_1)\Big)=\operatorname{diag}\Big\{B_-^{m_1}\Big(-\frac{1}{2}+\Delta(x_1)\Big),\dots,B_-^{m_l}\Big(-\frac{1}{2}+\Delta(x_1)\Big)\Big\},$$

$$B_-^{m_r}(t)=\|\widetilde{b}_{kp}^{m_r}(t)\|_{m_r\times m_r},\ \ r=1,\dots,l,$$

$$\widetilde{b}_{kp}^{m_r}(t)=\begin{cases}\Big(\dfrac{1}{2\pi i}\Big)^{p-k}\dfrac{(-1)^{p-k}}{(p-k)!}\,\dfrac{d^{p-k}}{dt^{p-k}}\,\Gamma(t+1)e^{\frac{i\pi(t+1)}{2}} & \text{for } \ k\le p,\\ 0 & \text{for } k>p,\end{cases}$$

$\Gamma(t+1)$ is the Euler function,

$$a_j(x_1)=\operatorname{diag}\big\{a^{m_1}(\alpha_1^{(j)}),\dots,a^{m_l}(\alpha_l^{(j)})\big\},$$

$$\alpha_r^{(j)}=-\frac{3}{2}-\Delta_r(x_1)+j,\ \ r=1,\dots,l,\ \ j=0,\dots,r_s^{(\beta)}-1,\ \ \ a^{m_r}(\alpha_r^{(j)})=\big\|a_{kp}^{m_r}(\alpha_r^{(j)})\big\|_{m_r\times m_r},$$

$$a_{kp}^{m_r}(\alpha_r^{(j)})=\begin{cases}-i\displaystyle\sum_{l=k}^{p}\frac{(-1)^{p-k}(2\pi i)^{l-p}\widetilde{b}_{kl}^{mr}(\mu_r^{(0)})}{(\alpha_r^{(0)}+1)^{p-l+1}}\,, & j=0,\ \ k\le p,\\ (-1)^{p-k}\widetilde{b}_{kp}^{m_r}(\alpha_r^{(j)}), & j=1,\dots,n_s^{(r)}-1,\ \ k\le p,\\ 0, & k>p,\end{cases}$$

with $\alpha_r^{(j)}=-1+j+\mu_r^{(j)}$, $-1<\operatorname{Re}\mu_r^{(j)}<0$.

*Remark* 8.8. The above asymptotic expansions (8.116) of solutions imply that for sufficiently smooth boundary data (e.g., $C^\infty$-smooth data say) the solution vectors $(U^{(1)},U^{(2)})$ to the interfacial problem $(\text{MTC-D})_\tau$ belong to the class of semi-regular functions described in Definition 2.2:

$$U^{(1)}\in[\mathbf{C}(\widetilde{\Omega}_\ell^{(1)};\alpha)]^6,\quad U^{(2)}\in[\mathbf{C}(\widetilde{\Omega}_\ell^{(2)};\alpha)]^6,\quad \widetilde{\Omega}_\ell^{(\beta)}=\overline{\Omega^{(\beta)}}\setminus\ell,\ \ \ell=\partial S_C,\ \ \beta=1,2,$$

where $\alpha=\frac{1}{2}-a_3+\varepsilon$ with $a_3$ defined in (8.95) and $\varepsilon$ being an arbitrarily small positive number. Due to the relations (8.97), it is evident that $\frac{1}{2}<\alpha<1$ if $0<\varepsilon<\frac{1}{4}+\frac{1}{2}a_3$.

Moreover, the dominant terms of the solution vectors $U^{(\beta)}$ near the curves $\ell=\partial S_C$ can be represented as the product of $C^\infty$-smooth vector-functions and factors of the form $[\ln\varrho(x)]^{m_j-1}[\varrho(x)]^{\kappa_j+i\nu_j}$, where $\varrho(x)$ is the distance from a reference point $x$ to the curve $\ell$. Therefore, near the curve $\ell$ the dominant singular terms of the corresponding generalized stress vectors $\mathcal{T}^{(\beta)}U^{(\beta)}$ are represented as the product of $C^\infty$-smooth vector-functions and the singular factors $[\ln\varrho(x)]^{m_j-1}[\varrho(x)]^{-1+\kappa_j+i\nu_j}$. The numbers $\nu_j$ are different from zero, in general, and describe the oscillating character of the stress singularities.

The exponents $\kappa_j+i\nu_j$ are related to the corresponding eigenvalues $\lambda_j$ of the matrix (8.93) by the equalities

$$\kappa_j=\frac{1}{2}+\frac{\arg\lambda_j}{2\pi}\,,\quad \nu_j=-\frac{\ln|\lambda_j|}{2\pi}\,,\ \ j=1,2,\dots,6. \tag{8.117}$$

Recall that in the above expressions the parameter $m_j$ denotes the multiplicity of the eigenvalue $\lambda_j$.

It is evident that at the curve $\ell$ the components of the generalized stress vector $\mathcal{T}^{(\beta)}U^{(\beta)}$ behave like $\mathcal{O}([\ln\varrho(x)]^{m_0-1}[\varrho(x)]^{-\frac{1}{2}+a_3})$, where $m_0$ denotes the maximal multiplicity of the eigenvalues. This is a global singularity effect for the first order derivatives of the vectors $U^{(\beta)}$, $\beta=1,2$. In contrast to the classical pure elasticity case (where $a_3=0$), here $a_3$ depends on the material parameters and is different from zero, in general. Since $a_3\leqslant0$, we see that the stress singularity exponents may become less than $-\frac{1}{2}$, in general.

8.2.4. *Existence and regularity of solutions to the interfacial crack problem* $(\text{MTC-N})_\tau$. Let us assume that the conditions (cf. (8.23))

$$f^{(1)}\in[B_{p,p}^{1-\frac{1}{p}}(S_T)]^6,\quad F^{(1)}\in[B_{p,p}^{-\frac{1}{p}}(S_T)]^6,\quad F^{(\pm)}\in[B_{p,p}^{-\frac{1}{p}}(S_C)]^6, \tag{8.118}$$

are satisfied and reformulate equivalently the crack conditions (8.13) and (8.14) in the setting of the crack problem $(\text{MTC-N})_\tau$ in the following form:

$$\{U^{(1)}\}^+-\{U^{(2)}\}^-=f^{(1)}\ \text{ on }\ S_T, \tag{8.119}$$

$$\big\{\mathcal{T}^{(1)}(\partial_x,n,\tau)U^{(1)}\big\}^+ - \big\{\mathcal{T}^{(2)}(\partial_x,n,\tau)U^{(2)}\big\}^- = F^{(1)} \ \text{ on } \ S_T, \tag{8.120}$$

$$\big\{\mathcal{T}^{(1)}(\partial_x,n,\tau)U^{(1)}\big\}^+ - \big\{\mathcal{T}^{(2)}(\partial_x,n,\tau)U^{(2)}\big\}^- = F^{(+)} - F^{(-)} \ \text{ on } \ S_C, \tag{8.121}$$

$$\big\{\mathcal{T}^{(1)}(\partial_x,n,\tau)U^{(1)}\big\}^+ + \big\{\mathcal{T}^{(2)}(\partial_x,n,\tau)U^{(2)}\big\}^- = F^{(+)} + F^{(-)} \ \text{ on } \ S_C. \tag{8.122}$$

Define the vector function

$$\widetilde{F} := \begin{cases} F^{(1)} & \text{on } \ S_T, \\ F^{(+)} - F^{(-)} & \text{on } \ S_C. \end{cases} \tag{8.123}$$

We assume that the following natural compatibility condition is satisfied

$$\widetilde{F} \in [B_{p,p}^{-\frac{1}{p}}(S)]^6. \tag{8.124}$$

We see that the weak formulation of the crack problem $(\text{MTC-N})_\tau$ now reads as follows: Find vector functions $U^{(1)} \in [W_p^1(\Omega^{(1)})]^6$ and $U^{(2)} \in [W_{p,loc}^1(\Omega^{(2)})]^6 \cap \mathbf{Z}_\tau(\Omega^{(2)})$ satisfying the homogeneous differential equations (8.2) of pseudo-oscillations in the distributional sense with $\Phi^{(\beta)} = 0$, $\beta = 1,2$, and

$$\{U^{(1)}\}^+ - \{U^{(2)}\}^- = f^{(1)} \ \text{ on } \ S_T, \tag{8.125}$$

$$\big\{\mathcal{T}^{(1)}(\partial_x,n,\tau)U^{(1)}\big\}^+ - \big\{\mathcal{T}^{(2)}(\partial_x,n,\tau)U^{(2)}\big\}^- = \widetilde{F} \ \text{ on } \ S, \tag{8.126}$$

$$\big\{\mathcal{T}^{(1)}(\partial_x,n,\tau)U^{(1)}\big\}^+ + \big\{\mathcal{T}^{(2)}(\partial_x,n,\tau)U^{(2)}\big\}^- = F^{(+)} + F^{(-)} \ \text{ on } \ S_C. \tag{8.127}$$

Let $\widetilde{f} \in [B_{p,p}^{1-\frac{1}{p}}(S)]^6$ be a fixed extension of the vector function $f^{(1)}$ from $S_T$ onto the whole of $S$. Then an arbitrary extension preserving the space has the form $f = \widetilde{f} + g$, where $g \in [\widetilde{B}_{p,p}^{1-\frac{1}{p}}(S_C)]^6$. Evidently, $r_{S_T} f = f^{(1)}$ on $S_T$.

Motivated by the results obtained in Subsection 8.2.1 (see Theorem 8.3) and applying the notation (8.90), we look for a weak solution of the problem $(\text{MTC-N})_\tau$ again in the form

$$U^{(1)} = V^{(1)}\big([\mathcal{H}^{(1)}]^{-1}\mathcal{A}\widetilde{F} - [\mathcal{H}^{(1)}]^{-1}\mathcal{A}\mathcal{A}^{(2)}(\widetilde{f}+g)\big) \ \text{ in } \ \Omega^{(1)}, \tag{8.128}$$

$$U^{(2)} = V^{(2)}\big([\mathcal{H}^{(2)}]^{-1}\mathcal{A}\widetilde{F} - [\mathcal{H}^{(2)}]^{-1}\mathcal{A}\mathcal{A}^{(1)}(\widetilde{f}+g)\big) \ \text{ in } \ \Omega^{(2)}. \tag{8.129}$$

It can easily be verified that all the conditions of the problem $(\text{MTC-N})_\tau$ are satisfied automatically, except the condition (8.127), which leads to the following pseudodifferential equation on $S_C$ with respect to the unknown vector $g \in [\widetilde{B}_{p,p}^{1-\frac{1}{p}}(S_C)]^6$,

$$(\mathcal{A}^{(1)} + \mathcal{A}^{(2)})\mathcal{A}\widetilde{F} - \big(\mathcal{A}^{(1)}\mathcal{A}\mathcal{A}^{(2)} + \mathcal{A}^{(2)}\mathcal{A}\mathcal{A}^{(1)}\big)(\widetilde{f}+g) = F^{(+)} + F^{(-)} \ \text{ on } \ S_C.$$

In view of the equality (8.67), the later equation can be rewritten as

$$r_{S_C}\mathcal{A}^{(N)} g = G^{(N)} \ \text{ on } \ S_C, \tag{8.130}$$

where

$$\mathcal{A}^{(N)} := -\mathcal{A}^{(1)}\mathcal{A}\mathcal{A}^{(2)} \equiv -\mathcal{A}^{(2)}\mathcal{A}\mathcal{A}^{(1)}, \tag{8.131}$$

$$G^{(N)} := \frac{1}{2}\,(F^{(+)} + F^{(-)}) - \frac{1}{2} r_{S_C}(\mathcal{A}^{(1)} + \mathcal{A}^{(2)})\mathcal{A}\widetilde{F} + r_{S_C}\mathcal{A}^{(N)}\widetilde{f} \in [B_{p,p}^{-\frac{1}{p}}(S_C)]^6. \tag{8.132}$$

Now, we show that the principal homogenous symbol matrix of the first order pseudodifferential operator $\mathcal{A}^{(N)}$ is strongly elliptic. Indeed, keeping in mind that the symbol matrices of the operators $\mathcal{A}^{(1)}$, $-\mathcal{A}^{(2)}$, and $\mathcal{A} = [\mathcal{A}^{(1)} - \mathcal{A}^{(2)}]^{-1}$ are strongly elliptic, we can transform the symbol matrix $\mathfrak{S}(\mathcal{A}^{(N)})$ as follows

$$\begin{aligned}
\mathfrak{S}(\mathcal{A}^{(N)}) &= -\mathfrak{S}(\mathcal{A}^{(1)})\mathfrak{S}(\mathcal{A})\mathfrak{S}(\mathcal{A}^{(2)}) = -\mathfrak{S}(\mathcal{A}^{(1)})\mathfrak{S}\big([\mathcal{A}^{(1)} - \mathcal{A}^{(2)}]^{-1}\big)\mathfrak{S}(\mathcal{A}^{(2)}) \\
&= -\mathfrak{S}(\mathcal{A}^{(1)})\big[\mathfrak{S}(\mathcal{A}^{(1)} - \mathcal{A}^{(2)})\big]^{-1}\mathfrak{S}(\mathcal{A}^{(2)}) = -\Big[[\mathfrak{S}(\mathcal{A}^{(2)})]^{-1}\mathfrak{S}(\mathcal{A}^{(1)} - \mathcal{A}^{(2)})[\mathfrak{S}(\mathcal{A}^{(1)})]^{-1}\Big]^{-1} \\
&= -\Big[[\mathfrak{S}(\mathcal{A}^{(2)})]^{-1}\big[\mathfrak{S}(\mathcal{A}^{(1)}) - \mathfrak{S}(\mathcal{A}^{(2)})\big]\,[\mathfrak{S}(\mathcal{A}^{(1)})]^{-1}\Big]^{-1} = \Big[[\mathfrak{S}(\mathcal{A}^{(1)})]^{-1} - [\mathfrak{S}(\mathcal{A}^{(2)})]^{-1}\Big]^{-1}.
\end{aligned} \tag{8.133}$$

Whence we conclude that $\mathfrak{S}(\mathcal{A}^{(N)})$ is a strongly elliptic symbol matrix, i.e.,

$$\text{Re}\,\big[\mathfrak{S}(\mathcal{A}^{(N)}; x,\xi_1,\xi_2)\eta\cdot\eta\big] \geq c|\xi||\eta|^2 \ \text{ for all } \ x \in S, \ (\xi_1,\xi_2) \in \mathbb{R}^2 \setminus \{0\}, \ \ \eta \in \mathbb{C}^6. \tag{8.134}$$

Consequently, the operators

$$r_{S_C}\mathcal{A}^{(N)}:[\widetilde{H}_p^{1-\frac{1}{p}}(S_C)]^6\longrightarrow[H_p^{-\frac{1}{p}}(S_C)]^6,\quad r_{S_C}\mathcal{A}^{(N)}:[\widetilde{B}_{p,p}^{1-\frac{1}{p}}(S_C)]^6\longrightarrow[B_{p,p}^{-\frac{1}{p}}(S_C)]^6$$

are bounded. To analyse the invertibility of these operators, in accordance with Theorem B.1 in Appendix B, let us consider the matrix

$$\mathbf{M}_{\mathcal{A}^{(N)}}(x):=\big[\mathfrak{S}(\mathcal{A}^{(N)};x,0,+1)\big]^{-1}\big[\mathfrak{S}(\mathcal{A}^{(N)};x,0,-1)\big],\ \ x\in\partial S_C, \tag{8.135}$$

constructed by the principal homogeneous symbol matrix $\mathfrak{S}(\mathcal{A}^{(N)};x,\xi_1,\xi_2)$ of the pseudodifferential operator $\mathcal{A}^{(N)}$. Let $\lambda_1'(x),\dots,\lambda_6'(x)$ be the eigenvalues of the matrix (8.135) and

$$\delta_j'(x)=\operatorname{Re}\big[(2\pi i)^{-1}\ln\lambda_j'(x)\big],\ \ j=1,\dots,6, \tag{8.136}$$

$$a_5=\inf_{x\in\ell,1\le j\le 6}\delta_j'(x),\quad a_6=\sup_{x\in\ell,1\le j\le 6}\delta_j'(x); \tag{8.137}$$

here $\ln\zeta$ denotes the branch of the logarithm analytic in the complex plane cut along $(-\infty,0]$. Due to the strong ellipticity of the operator $\mathcal{A}$ we have the strict inequalities $-\frac12<\delta_j'(x)<\frac12$ for $x\in\overline{S_C}$, $j=1,\dots,6$. Therefore

$$-\frac{1}{2}<a_5\le a_6<\frac{1}{2}\,. \tag{8.138}$$

Moreover, by the same arguments as in Subsection 5.7, we can show that one of the eigenvalues, say $\lambda_6$, of the matrix $M_{\mathcal{A}^{(N)}}(x)$ defined in (8.135) equals to 1, implying $\delta_6=0$. Consequently, we have the following estimates

$$-\frac{1}{2}<a_5\leqslant 0\leqslant a_6<\frac{1}{2}\,. \tag{8.139}$$

**Lemma 8.9.** *The operators*

$$r_{S_C}\mathcal{A}^{(N)}:[\widetilde{H}_p^{s}(S_C)]^6\longrightarrow[H_p^{s-1}(S_C)]^6, \tag{8.140}$$

$$r_{S_C}\mathcal{A}^{(N)}:[\widetilde{B}_{p,q}^{s}(S_C)]^6\longrightarrow[B_{p,q}^{s-1}(S_C)]^6,\ \ q\ge 1, \tag{8.141}$$

*are invertible if*

$$\frac{1}{p}-\frac{1}{2}+a_6<s<\frac{1}{p}+\frac{1}{2}+a_5, \tag{8.142}$$

*where $a_5$ and $a_6$ are given by* (8.137).

*Proof.* To prove the invertibility of the operators (8.140) and (8.141), we first consider the particular values of the parameters $s=1/2$ and $p=q=2$, which fall into the region defined by the inequalities (8.142), and show that the null space of the operator

$$r_{S_C}\mathcal{A}^{(N)}:[\widetilde{H}_2^{\frac12}(S_C)]^6\longrightarrow[H_2^{-\frac12}(S_C)]^6 \tag{8.143}$$

is trivial, i.e., the equation

$$r_{S_C}\mathcal{A}^{(N)}\widetilde{g}=0\ \text{ on }\ S_C \tag{8.144}$$

admits only the trivial solution in the space $[\widetilde{H}_2^{\frac12}(S_C)]^6$. Recall that $\widetilde{H}_2^s(S_C)=\widetilde{B}_{2,2}^s(S_C)$ and $H_2^s(S_C)=B_{2,2}^s(S_C)$ for $s\in\mathbb{R}$.

Let $\widetilde{g}\in[\widetilde{B}_{2,2}^{\frac12}(S)]^6=[\widetilde{H}_2^{\frac12}(S)]^6$ be a solution to the homogeneous equation (8.144) and construct the vectors

$$\widetilde{U}^{(1)}(x)=-V^{(2)}\big([\mathcal{H}^{(1)}]^{-1}\mathcal{A}\mathcal{A}^{(2)}\widetilde{g}\big)(x)\ \text{ in }\ \Omega^{(1)},$$

$$\widetilde{U}^{(2)}(x)=-V^{(1)}\big([\mathcal{H}^{(2)}]^{-1}\mathcal{A}\mathcal{A}^{(1)}\widetilde{g}\big)(x)\ \text{ in }\ \Omega^{(2)}.$$

Evidently, $U^{(1)}\in[W_2^1(\Omega^{(1)})]^6$ and $U^{(2)}\in[W_{2,loc}^1(\Omega^{(2)})]^6\cap\mathbf{Z}_\tau(\Omega^{(2)})$. Further, taking into consideration that the following relations hold on $S$:

$$\{\widetilde{U}^{(1)}\}^+=-\mathcal{A}\mathcal{A}^{(2)}\widetilde{g},\quad\{\mathcal{T}^{(1)}\widetilde{U}^{(1)}\}^+=-\mathcal{A}^{(1)}\mathcal{A}\mathcal{A}^{(2)}\widetilde{g},$$

$$\{U^{(2)}\}^-=-\mathcal{A}\mathcal{A}^{(1)}\widetilde{g},\quad\{\mathcal{T}^{(2)}\widetilde{U}^{(2)}\}^-=-\mathcal{A}^{(2)}\mathcal{A}\mathcal{A}^{(1)}\widetilde{g},$$

and using (8.131) and (8.144), and keeping in mind that $r_{S_T}\widetilde{g}=0$, we find

$$r_{S_C}\{\mathcal{T}^{(1)}\widetilde{U}^{(1)}\}^+=0,\quad r_{S_C}\{\mathcal{T}^{(2)}\widetilde{U}^{(2)}\}^-=0\ \text{ on }\ S_C,$$

$$r_{S_T}\big[\{\widetilde{U}^{(1)}\}^+ - \{\widetilde{U}^{(2)}\}^-\big] = 0, \quad r_{S_T}\big[\{\mathcal{T}^{(1)}U^{(1)}\}^+ - \{\mathcal{T}^{(2)}U^{(2)}\}^-\big] = 0 \ \text{ on } \ S_T.$$

Therefore, $\widetilde{U}^{(1)}$ and $\widetilde{U}^{(2)}$ solve the homogeneous interfacial crack problem $(\text{MTC-N})_\tau$ with $p=2$ and by Theorem 8.1 they vanish in the corresponding regions, $\Omega^{(1)}$ and $\Omega^{(2)}$, respectively, which implies that $\widetilde{g} = \{\widetilde{U}^{(1)}\}^+ - \{\widetilde{U}^{(2)}\}^- = 0$ on $S$. Consequently, the null space of the operator (8.143) is trivial. Since the principal homogeneous symbol matrix of the operator $\mathcal{A}^{(N)}$ is strongly elliptic, we conclude by Theorem B.1 (see Appendix B) that the operators (8.140) and (8.141) are Fredholm operators with zero indices and with the trivial null spaces for all values of the parameters satisfying the inequalities (8.142). Thus they are invertible. □

This lemma leads to the following existence theorem.

**Theorem 8.10.** *Let the conditions* (8.118) *be fulfilled and*

$$\frac{4}{3-2a_6} < p < \frac{4}{1-2a_5}\,, \tag{8.145}$$

*where $a_5$ and $a_6$ are defined by* (8.137). *Then the interfacial crack problem* $(\text{MTC-N})_\tau$ *has a unique solution $U^{(1)} \in [W^1_p(\Omega^{(1)})]^6$ and $U^{(2)} \in [W^1_{p,loc}(\Omega^{(2)})]^6 \cap \mathbf{Z}_\tau(\Omega^{(2)})$, and the solution vectors are representable in the form of single layer potentials* (8.128) *and* (8.129)*, where the unknown vector $g$ is defined from the uniquely solvable pseudodifferential equation* (8.130).

*Proof.* In accordance with Lemma 8.9, equation (8.130) is uniquely solvable for $s = 1 - \frac{1}{p}$ with $p$ satisfying inequality (8.145), since the inequalities (8.142) are fulfilled. Therefore the vectors (8.128) and (8.129) represent a solution to the interfacial crack problem $(\text{MTC-N})_\tau$ in the space $[W^1_p(\Omega^{(1)})]^6 \times ([W^1_{p,loc}(\Omega^{(2)})]^6 \cap \mathbf{Z}_\tau(\Omega^{(2)}))$ with $p$ as in (8.145).

It remains to show that the problem $(\text{MTC-N})_\tau$ is uniquely solvable in the space $[W^1_p(\Omega^{(1)})]^6 \times ([W^1_{p,loc}(\Omega^{(2)})]^6 \cap \mathbf{Z}_\tau(\Omega^{(2)}))$ for arbitrary $p$ satisfying (8.145).

Let a pair of vectors $(\widetilde{U}^{(1)}, \widetilde{U}^{(2)}) \in [W^1_p(\Omega^{(1)})]^6 \times ([W^1_{p,loc}(\Omega^{(2)})]^6 \cap \mathbf{Z}_\tau(\Omega^{(2)}))$ be a solution to the homogeneous problem $(\text{MTC-N})_\tau$. Then by Theorem 8.3 they are representable in the form (8.65), (8.66) with

$$F = \{\mathcal{T}^{(1)}\widetilde{U}^{(1)}\}^+ - \{\mathcal{T}^{(2)}\widetilde{U}^{(2)}\}^-, \quad f = \{\widetilde{U}^{(1)}\}^+ - \{\widetilde{U}^{(2)}\}^- \ \text{ on } \ S.$$

Due to the homogeneous conditions of the problem $(\text{MTC-N})_\tau$, (8.9), (8.10) and (8.13), (8.14) with $f^{(1)} = F^{(1)} = 0$ on $S_T$ and $F^{(\pm)} = 0$ on $S_C$, we have

$$f = \{\widetilde{U}^{(1)}\}^+ - \{\widetilde{U}^{(2)}\}^- \in [\widetilde{B}^{1-\frac{1}{p}}_{p,p}(S_C)]^6, \ \ F = 0 \ \text{ on } \ S, \tag{8.146}$$

and, using the notation (8.90), we get the representations

$$\widetilde{U}^{(1)} = -V^{(1)}\big([\mathcal{H}^{(1)}]^{-1}\mathcal{A}\mathcal{A}^{(2)}f\big) \ \text{ in } \ \Omega^{(1)}, \quad \widetilde{U}^{(2)} = -V^{(2)}\big([\mathcal{H}^{(2)}]^{-1}\mathcal{A}\mathcal{A}^{(1)}f\big) \ \text{ in } \ \Omega^{(2)}.$$

Since, $r_{S_C}\{\mathcal{T}^{(1)}\widetilde{U}^{(1)}\}^+ = r_{S_C}\{\mathcal{T}^{(2)}\widetilde{U}^{(2)}\}^- = 0$ on $S_C$ we arrive at the equation $r_{S_C}\mathcal{A}^{(N)}f = 0$ on $S_C$. Therefore the inclusion (8.146) and Lemma 8.9 imply that $f = 0$ on $S_C$ which completes the proof. □

Further, we deduce almost the best regularity Hölder continuity results for solutions to the crack problem $(\text{MTC-N})_\tau$.

**Theorem 8.11.** *Let inclusions* (8.118) *hold and let*

$$\frac{4}{3-2a_6} < p < \frac{4}{1-2a_5}, \ \ 1 < r < \infty, \ \ 1 \le q \le \infty, \ \ \frac{1}{r} - \frac{1}{2} + a_6 < s < \frac{1}{r} + \frac{1}{2} + a_5, \tag{8.147}$$

*with $a_5$ and $a_6$ defined by* (8.137).

*Further, let $U^{(1)} \in [W^1_p(\Omega^{(1)})]^6$ and $U^{(2)} \in [W^1_{p,loc}(\Omega^{(2)})]^6 \cap \mathbf{Z}_\tau(\Omega^{(2)})$, be a unique solution pair to the interfacial crack problem (MTC-N)$_\tau$. Then the following hold:*

(i) *if*

$$f^{(1)} \in [B^s_{r,r}(S_T)]^6, \quad F^{(1)} \in [B^{s-1}_{r,r}(S_T)]^6, \quad F^{(\pm)} \in [B^{s-1}_{r,r}(S_C)]^6, \tag{8.148}$$

*and* $\widetilde{F} \in [B^{s-1}_{r,r}(S)]^6$ *with* $\widetilde{F}$ *defined in* (8.123)*, then*

$$U^{(1)} \in [H_r^{s+\frac{1}{r}}(\Omega^{(1)})]^6, \quad U^{(2)} \in [H^{s+\frac{1}{r}}_{r,loc}(\Omega^{(2)})]^6 \cap \mathbf{Z}_\tau(\Omega^{(2)}); \tag{8.149}$$

(ii) *if*

$$f^{(1)} \in [B^s_{r,q}(S_T)]^6, \quad F^{(1)} \in [B^{s-1}_{r,q}(S_T)]^6, \quad F^{(\pm)} \in [B^{s-1}_{r,q}(S_C)]^6, \tag{8.150}$$

*and* $\widetilde{F} \in [B^{s-1}_{r,q}(S)]^6$ *with* $\widetilde{F}$ *defined in* (8.123)*, then*

$$U^{(1)} \in [B_{r,q}^{s+\frac{1}{r}}(\Omega^{(1)})]^6, \quad U^{(2)} \in [B^{s+\frac{1}{r}}_{r,q,loc}(\Omega^{(2)})]^6 \cap \mathbf{Z}_\tau(\Omega^{(2)}); \tag{8.151}$$

(iii) *if* $\alpha > 0$ *is not integer and*

$$f^{(1)} \in [C^\alpha(\overline{S_T})]^6, \quad F^{(1)} \in [B^{\alpha-1}_{\infty,\infty}(S_T)]^6, \quad F^{(\pm)} \in [B^{\alpha-1}_{\infty,\infty}(S_C)]^6, \tag{8.152}$$

*and* $\widetilde{F} \in [B^{\alpha-1}_{\infty,\infty}(S)]^6$ *with* $\widetilde{F}$ *defined in* (8.123)*, then*

$$U^{(1)} \in \bigcap_{\alpha'<\kappa} [C^{\alpha'}(\overline{\Omega^{(1)}})]^6, \quad U^{(2)} \in \bigcap_{\alpha'<\kappa} [C^{\alpha'}(\overline{\Omega^{(2)}})]^6 \cap \mathbf{Z}_\tau(\Omega^{(2)}), \tag{8.153}$$

*where* $\kappa = \min\{\alpha,\ a_5 + \frac{1}{2}\} > 0$.

*Proof.* It is word for word of the proof of Theorem 5.22. □

8.2.5. *Asymptotic expansion of solutions to the problem* $(\text{MTC-N})_\tau$. In this subsection we investigate the asymptotic behaviour of the solution to the problem $(\text{MTC-N})_\tau$ near the interfacial crack edge $\ell = \partial S_C$. For simplicity of description of the method applied below, we again assume that the crack and transmission boundary data of the problem are infinitely smooth, namely,

$$F^{(+)}, F^{(-)} \in [C^\infty(\overline{S}_C)]^6, \quad f^{(1)}, F^{(1)} \in [C^\infty(\overline{S}_T)]^6, \quad \widetilde{F} \in [C^\infty(S)]^6.$$

In Subsection 8.2.4 we have shown that the problem $(\text{MTC-N})_\tau$ is uniquely solvable and the solution $(U^{(1)}, U^{(2)})$ is represented by (8.128), (8.129) with the density defined by the pseudodifferential equation (8.130).

To establish the asymptotic expansion of solution $(U^{(1)}, U^{(2)})$ near the crack edge $\partial S_C$ we again employ the notation introduced in Subsection 8.2.4 and rewrite the representation (8.128), (8.129) in the form

$$U^{(1)} = V^{(1)}\mathcal{B}_1 g + R^{(1)} \ \text{ in } \ \Omega^{(1)}, \quad U^{(2)} = V^{(2)}\mathcal{B}_2 g + R^{(2)} \ \text{ in } \ \Omega^{(2)},$$

where $\mathcal{B}_1 = -(\mathcal{H}^{(1)})^{-1}\mathcal{A}\mathcal{A}^{(2)}$, $\mathcal{B}_2 = -(\mathcal{H}^{(2)})^{-1}\mathcal{A}\mathcal{A}^{(1)}$,

$$R^{(1)} = V^{(1)}(\mathcal{H}^{(1)})^{-1}\mathcal{A}\widetilde{F} - V^{(1)}\mathcal{B}_1\widetilde{f} \in [C^\infty(\overline{\Omega^{(1)}})]^6,$$

$$R^{(2)} = V^{(2)}(\mathcal{H}^{(2)})^{-1}\mathcal{A}\widetilde{F} - V^{(2)}\mathcal{B}_2\widetilde{f} \in [C^\infty(\overline{\Omega^{(2)}})]^6,$$

and $g$ solves the pseudodifferential equation

$$r_{_{S_C}}\mathcal{A}^{(N)} g = G^{(N)} \ \text{ on } \ S_C \tag{8.154}$$

with $G^{(N)} \in [C^\infty(\overline{S}_C)]^6$.

Applying again a partition of unity and natural local coordinate systems we perform a standard rectifying procedure for $\ell = \partial S_C = \partial S_T$ and $S_C$ based on canonical diffeomorphisms. For simplicity, we again denote the local rectified images of $\ell$ and $S_C$ under this diffeomorphisms by the same symbols. Then we identify a one-sided neighbourhood on $S_C$ of an arbitrary point $\widetilde{x} \in \ell = \partial S_C$ as a part of the half-plane $x_2 > 0$. Thus we assume that $(x_1, 0) = \widetilde{x} \in \ell$ and $(x_1, x_{2,+}) \in \partial S_C$ for $0 < x_{2,+} < \varepsilon$ with some positive $\varepsilon$.

Consider the $6\times 6$ matrix $M_{\mathcal{A}^{(N)}}$ related to the principal homogeneous symbol $\mathfrak{S}(\mathcal{A}^{(N)}; x, \xi)$ of the operator $\mathcal{A}^{(N)}$

$$M_{\mathcal{A}^{(N)}}(x_1) := \big[\mathfrak{S}(\mathcal{A}^{(N)}; x_1, 0, 0, +1)\big]^{-1}\mathfrak{S}(\mathcal{A}^{(N)}; x_1, 0, 0, -1), \ \ (x_1, 0) \in \ell = \partial S_C.$$

Denote by $\lambda_1(x_1), \dots, \lambda_6(x_1)$ the eigenvalues of the matrix $M_{\mathcal{A}^{(N)}}(x_1)$ and by $m_j$ the algebraic multiplicity of $\lambda_j(x_1)$. Let $\mu_1(x_1), \dots, \mu_l(x_1)$ be the distinct eigenvalues. Evidently, $m_j$ and $l$ depend on $x_1$ in general and $m_1 + \dots + m_l = 6$.

It is well known that the matrix $M_{\mathcal{A}^{(N)}}(x_1)$ admits the following decomposition (see, e.g. [42, Ch. 7, Section 7])

$$M_{\mathcal{A}^{(N)}}(x_1) = \mathcal{N}(x_1)\mathcal{J}_{M_{\mathcal{A}^{(N)}}}(x_1)\mathcal{N}^{-1}(x_1), \ \ (x_1,0)\in\ell=\partial S_C,$$

where $\mathcal{N}$ is $6\times 6$ non-degenerate matrix with infinitely smooth entries and $\mathcal{J}_{M_{\mathcal{A}^{(N)}}}$ is block diagonal

$$\mathcal{J}_{M_{\mathcal{A}^{(N)}}}(x_1) := \operatorname{diag}\big\{\mu_1(x_1)B^{m_1}(1),\dots,\mu_l(x_1)B^{m_l}(1)\big\}.$$

Here $B^{(\nu)}(t)$ with $\nu\in\{m_1,\dots,m_l\}$ are upper triangular matrices introduced in Subsection 8.2.3.

Denote

$$B_0(t) := \operatorname{diag}\big\{B^{(m_1)}(t),\dots,B^{(m_l)}(t)\big\}. \tag{8.155}$$

Applying the results from the reference [22] we derive the following asymptotic expansion for the solution of the pseudodifferential equation (8.154)

$$\begin{aligned} g(x_1,x_{2,+}) &= \mathcal{N}(x_1)x_{2,+}^{\frac12+\Delta(x_1)}B_0\Big(-\frac{1}{2\pi i}\log x_{2,+}\Big)\mathcal{N}^{-1}(x_1)b_0(x_1)\\ &\quad+\sum_{k=1}^{M}\mathcal{N}(x_1)x_{2,+}^{\frac12+\Delta(x_1)+k}B_k(x_1,\log x_{2,+})+g_{M+1}(x_1,x_{2,+}), \end{aligned} \tag{8.156}$$

where $b_0\in[C^\infty(\ell)]^6$, $g_{M+1}\in[C^\infty(\ell^+_{c,\varepsilon})]^6$, $\ell^+_{c,\varepsilon}=\ell\times[0,\varepsilon]$, and

$$B_k(x_1,t) = B_0\Big(-\frac{t}{2\pi i}\Big)\sum_{j=1}^{k(2m_0-1)} t^j d_{kj}(x_1).$$

Here $m_0=\max\{m_1,\dots,m_6\}$, the coefficients $d_{kj}\in[C^\infty(\ell)]^6$,

$$\Delta(x_1) := (\Delta_1(x_1),\dots,\Delta_6(x_1)), \quad \Delta_j(x_1) = \frac{1}{2\pi i}\,\log\lambda_j'(x_1) = \frac{1}{2\pi}\,\arg\lambda_j'(x_1)+\frac{1}{2\pi i}\,\log|\lambda_j'(x_1)|,$$
$$-\pi<\arg\lambda_j'(x_1)<\pi, \ \ (x_1,0)\in\ell=\partial S_C, \ \ j=1,\dots,6.$$

Furthermore,

$$x_{2,+}^{\frac12+\Delta(x_1)} := \operatorname{diag}\Big\{x_{2,+}^{\frac12+\Delta_1(x_1)},\dots,x_{2,+}^{\frac12+\Delta_6(x_1)}\Big\}.$$

Now, having at hand the above asymptotic expansion for the density vector function $g$, we can apply the results of the reference [23] and write the following spatial asymptotic expansions of the solution $(U^{(1)},U^{(2)})$

$$\begin{aligned} U^{(\beta)}(x) = \sum_{\mu=\pm1}&\Bigg\{\sum_{s=1}^{l_0^{(\beta)}}\sum_{j=0}^{n_s^{(\beta)}-1} x_3^j\Big[d_{sj}^{(\beta)}(x_1,\mu)\Big(z_{s,\mu}^{(\beta)}\Big)^{\frac12+\Delta(x_1)-j}B_0\Big(-\frac{1}{2\pi i}\,\log z_{s,\mu}^{(\beta)}\Big)\Big]c_j(x_1)\\ &+\sum_{\substack{k,l=0\\k+l+j+p\ge1}}^{M+2}\sum_{j+p=0}^{M+2-l} x_2^l x_3^j d_{sljp}^{(\beta)}(x_1,\mu)(z_{s,\mu}^{(\beta)})^{\frac12+\Delta(x_1)+p+k}B_{skjp}^{(\beta)}(x_1,\log z_{s,\mu}^{(\beta)})\Bigg\}\\ &+U_{M+1}^{(\beta)}(x), \ \ x_3>0, \ \ \beta=1,2. \end{aligned} \tag{8.157}$$

The coefficients $d_{sj}^{(\beta)}(\,\cdot\,,\mu)$ and $d_{sljp}^{(\beta)}(\,\cdot\,,\mu)$ are matrices with entries from the space $C^\infty(\ell)$, while $B_{skjp}^{(\beta)}(x_1,t)$ are polynomials in $t$ with vector coefficients which depend on the variable $x_1$ and have the order $\nu_{kjp}=k(2m_0-1)+m_0-1+p+j$ in general, where $m_0=\max\{m_1,\dots,m_l\}$ and $m_1+\dots+m_l=6$,

$$c_j\in[C^\infty(\ell)]^6, \ \ U_{M+1}^{(\beta)}\in[C^{M+1}(\overline{\Omega}^{(\beta)})]^6, \ \ (z_{s,\mu}^{(\beta)})^{\kappa+\Delta(x_1)} := \operatorname{diag}\big\{(z_{s,\mu}^{(\beta)})^{\kappa+\Delta_1(x_1)},\dots,(z_{s,\mu}^{(\beta)})^{\kappa+\Delta_6(x_1)}\big\},$$
$$\kappa\in\mathbb{R}, \ \ \mu=\pm1, \ \ \beta=1,2, \ (x_1,0)\in\ell,$$
$$z_{s,+1}^{(\beta)} := -x_2-x_3\zeta_{s,+1}^{(\beta)}, \quad z_{s,-1}^{(\beta)} := x_2-x_3\zeta_{s,-1}^{(\beta)}, \ \ -\pi<\arg z_{s,\pm1}^{(\beta)}<\pi, \ \ \zeta_{s,\pm1}^{(\beta)}\in C^\infty(\ell).$$

Note that, the coefficients $d_{sj}^{(\beta)}(\,\cdot\,,\mu)$ can be calculated explicitly, whereas the coefficients $c_j$ can be expressed by means of the first coefficient $b_0$ in the asymptotic expansion of (8.156) (for details see [23]),

$$
\begin{aligned}
d_{sj}^{(\beta)}(x_1,+1) &= \frac{1}{2\pi}\,\mathcal{G}_{\varkappa_\beta}(x_1,0)P_{sj}^{+,\beta}(x_1)\mathcal{N}(x_1),\\
d_{sj}^{(\beta)}(x_1,-1) &= \frac{1}{2\pi}\mathcal{G}_{\varkappa_\beta}(x_1,0)P_{sj}^{-,\beta}(x_1)\mathcal{N}(x_1)e^{i\pi(-\frac12-\Delta(x_1))},\\
&\qquad s=1,\dots,l_0^{(\beta)},\ \ j=0,\dots,n_s^{(\beta)}-1,\ \ \beta=1,2,
\end{aligned}
$$

where

$$
\begin{aligned}
P_{sj}^{\pm,\beta}(x_1) &:= V_{-1,j}^{\beta,s}(x_1,0,0,\pm1)\mathfrak{S}(\mathcal{B}_\beta;0,0,\pm1),\\
V_{-1,j}^{\beta,s}(x_1,0,0,\pm1) &:= \frac{i^{j+1}}{j!(n_s^{(\beta)}-1-j)!}\,\frac{d^{n_s^{(\beta)}-1-j}}{d\zeta^{n_s^{(\beta)}-1-j}}\,(\zeta-\zeta_{s,\pm1}^{(\beta)})^{n_s^{(\beta)}}\\
&\qquad\times\Big(A^{(\beta,0)}\big((J_{\varkappa_\beta}^\top(x_1,0))^{-1}(0,\pm1,\zeta)^\top\big)\Big)^{-1}\bigg|_{\zeta=\zeta_{s,\pm1}^{(\beta)}},\\
e^{i\pi(-\frac12-\Delta(x_1))} &:= \operatorname{diag}\big\{e^{i\pi(-\frac12-\Delta_1(x_1))},\dots,e^{i\pi(-\frac12-\Delta_6(x_1))}\big\},
\end{aligned}
$$

$\mathcal{G}_{\varkappa_\beta}$ are the square roots of Gram's determinant of the diffeomorphisms $\varkappa_\beta$, $\beta=1,2$, and

$$
\begin{aligned}
c_j(x_1) &= a_j(x_1)B_0^-\Big(\frac12+\Delta(x_1)\Big)\mathcal{N}^{-1}(x_1)b_0(x_1),\ \ j=0,\dots,n_s^{(\beta)}-1,\\
B_0^-\Big(\frac12+\Delta(x_1)\Big) &= \operatorname{diag}\Big\{B_-^{m_1}\Big(\frac12+\Delta(x_1)\Big),\dots,B_-^{m_l}\Big(\frac12+\Delta(x_1)\Big)\Big\},\\
B_-^{m_r}(t) &= \|\widetilde{b}_{kp}^{m_r}(t)\|_{m_r\times m_r},\ \ r=1,\dots,l,\\
\widetilde{b}_{kp}^{m_r}(t) &= \begin{cases}\Big(\dfrac{1}{2\pi i}\Big)^{p-k}\dfrac{(-1)^{p-k}}{(p-k)!}\,\dfrac{d^{p-k}}{dt^{p-k}}\,\Gamma(t+1)e^{\frac{i\pi(t+1)}{2}}, & \text{for } k\le p,\\ 0 & \text{for } k>p,\end{cases}
\end{aligned}
$$

$\Gamma(t+1)$ is the Euler function,

$$
a_j(x_1)=\operatorname{diag}\big\{a^{m_1}(\alpha_1^{(j)}),\dots,a^{m_l}(\alpha_l^{(j)})\big\},
$$

$$
\alpha_r^{(j)}=-\frac32-\Delta_r(x_1)+j,\ \ r=1,\dots,l,\ \ j=0,\dots,r_s^{(\beta)}-1,\quad a^{m_r}(\alpha_r^{(j)})=\big\|a_{kp}^{m_r}(\alpha_r^{(j)})\big\|_{m_r\times m_r},
$$

$$
a_{kp}^{m_r}(\alpha_r^{(j)})=\begin{cases}-i\displaystyle\sum_{l=k}^{p}\frac{(-1)^{p-k}(2\pi i)^{l-p}\widetilde{b}_{kl}^{mr}(\mu_r^{(0)})}{(\alpha_r^{(0)}+1)^{p-l+1}}, & j=0,\ \ k\le p,\\ (-1)^{p-k}\widetilde{b}_{kp}^{m_r}(\alpha_r^{(j)}), & j=1,\dots,n_s^{(r)}-1,\ \ k\le p,\\ 0, & k>p,\end{cases}
$$

with $\alpha_r^{(j)}=-1+j+\mu_r^{(j)}$, $-1<\operatorname{Re}\mu_r^{(j)}<0$.

*Remark* 8.12. The above asymptotic expansions (8.157) of solutions imply that for sufficiently smooth boundary data (e.g., $C^\infty$-smooth data say) the solution vectors $(U^{(1)},U^{(2)})$ to the interfacial problem $(\text{MTC-N})_\tau$ belong to the class of semi-regular functions described in Definition 2.2:

$$
U^{(1)}\in[\mathbf{C}(\widetilde{\Omega}_\ell^{(1)};\alpha)]^6,\quad U^{(2)}\in[\mathbf{C}(\widetilde{\Omega}_\ell^{(2)};\alpha)]^6,\quad \widetilde{\Omega}_\ell^{(\beta)}=\overline{\Omega^{(\beta)}}\setminus\ell,\ \ \ell=\partial S_C,\ \ \beta=1,2,
$$

where $\alpha=\frac12-a_5+\varepsilon$ with $a_5$ defined in (8.137) and $\varepsilon$ being an arbitrarily small positive number. Due to the relations (8.139), it is evident that $\frac12<\alpha<1$ if $0<\varepsilon<\frac14+\frac12\,a_5$.

Moreover, the dominant terms of the solution vectors $U^{(\beta)}$ near the curves $\ell=\partial S_C$ can be represented as the product of $C^\infty$-smooth vector-functions and factors of the form $[\ln\varrho(x)]^{m_j-1}[\varrho(x)]^{\kappa_j'+i\nu_j'}$, where $\varrho(x)$ is the distance from a reference point $x$ to the curve $\ell$. Therefore, near the curve $\ell$ the dominant singular terms of the corresponding generalized stress vectors $\mathcal{T}^{(\beta)}U^{(\beta)}$ are represented as the product of $C^\infty$-smooth vector-functions and the singular factors $[\ln\varrho(x)]^{m_j-1}[\varrho(x)]^{-1+\kappa_j'+i\nu_j'}$.

The numbers $\nu_j$ are different from zero, in general, and describe the oscillating character of the stress singularities.

The exponents $\kappa'_j + i\nu'_j$ are related to the corresponding eigenvalues $\lambda'_j$ of the matrix (8.135) by the equalities

$$\kappa'_j = \frac{1}{2} + \frac{\arg \lambda'_j}{2\pi}, \quad \nu'_j = -\frac{\ln|\lambda'_j|}{2\pi}, \quad j = 1, 2, \dots, 6. \tag{8.158}$$

Recall that in the above expressions the parameter $m_j$ denotes the multiplicity of the eigenvalue $\lambda'_j$.

It is evident that at the curve $\ell$ the components of the generalized stress vector $\mathcal{T}^{(\beta)}U^{(\beta)}$ behave like $\mathcal{O}([\ln \varrho(x)]^{m_0-1}[\varrho(x)]^{-\frac{1}{2}+a_5})$, where $m_0$ denotes the maximal multiplicity of the eigenvalues. This is a global singularity effect for the first order derivatives of the vectors $U^{(\beta)}$, $\beta = 1, 2$. In contrast to the classical pure elasticity case (where $a_5 = 0$), here $a_5$ depends on the material parameters and is different from zero, in general. Since $a_5 \leqslant 0$, we see that the stress singularity exponents may become less than $-\frac{1}{2}$, in general.

8.2.6. *Existence and regularity of solutions to the interfacial crack problem* $(\text{MTC-M})_\tau$. Here we assume that

$$\begin{gathered} f_k^{(1)} \in B_{p,p}^{1-\frac{1}{p}}(S_T), \quad F_k^{(1)} \in B_{p,p}^{-\frac{1}{p}}(S_T), \;\; k = 1, \dots, 6, \\ F_r^{(\pm)} \in B_{p,p}^{-\frac{1}{p}}(S_C), \quad f_j^* \in B_{p,p}^{1-\frac{1}{p}}(S_C), \quad F_j^* \in B_{p,p}^{-\frac{1}{p}}(S_C), \;\; r = 1, 2, 3, \;\; j = 4, 5, 6. \end{gathered} \tag{8.159}$$

and reformulate equivalently the conditions of the problem $(\text{MTC-M})_\tau$ as follows

$$\{U^{(1)}\}^+ - \{U^{(2)}\}^- = f^{(1)} \;\text{ on }\; S_T, \tag{8.160}$$

$$\big\{\mathcal{T}^{(1)}(\partial_x, n, \tau)U^{(1)}\big\}^+ - \big\{\mathcal{T}^{(2)}(\partial_x, n, \tau)U^{(2)}\big\}^- = F^{(1)} \;\text{ on }\; S_T, \tag{8.161}$$

$$\big\{\big[\mathcal{T}^{(1)}(\partial_x, n, \tau)U^{(1)}\big]_r\big\}^+ - \big\{\big[\mathcal{T}^{(2)}(\partial_x, n, \tau)U^{(2)}\big]_r\big\}^- = F_r^{(+)} - F_r^{(-)} \;\text{ on }\; S_C, \;\; r = 1, 2, 3, \tag{8.162}$$

$$\big\{\big[\mathcal{T}^{(1)}(\partial_x, n, \tau)U^{(1)}\big]_r\big\}^+ + \big\{\big[\mathcal{T}^{(2)}(\partial_x, n, \tau)U^{(2)}\big]_r\big\}^- = F_r^{(+)} + F_r^{(-)} \;\text{ on }\; S_C, \;\; r = 1, 2, 3, \tag{8.163}$$

$$\{\varphi^{(1)}\}^+ - \{\varphi^{(2)}\}^- = f_4^* \;\text{ on }\; S_C, \tag{8.164}$$

$$\big\{\big[\mathcal{T}^{(1)}(\partial_x, n, \tau)U^{(1)}\big]_4\big\}^+ - \big\{\big[\mathcal{T}^{(2)}(\partial_x, n, \tau)U^{(2)}\big]_4\big\}^- = F_4^* \;\text{ on }\; S_C, \tag{8.165}$$

$$\{\psi^{(1)}\}^+ - \{\psi^{(2)}\}^- = f_5^* \;\text{ on }\; S_C, \tag{8.166}$$

$$\big\{\big[\mathcal{T}^{(1)}(\partial_x, n, \tau)U^{(1)}\big]_5\big\}^+ - \big\{\big[\mathcal{T}^{(2)}(\partial_x, n, \tau)U^{(2)}\big]_5\big\}^- = F_5^* \;\text{ on }\; S_C, \tag{8.167}$$

$$\{\vartheta^{(1)}\}^+ - \{\vartheta^{(2)}\}^- = f_6^* \;\text{ on }\; S_C, \tag{8.168}$$

$$\big\{\big[\mathcal{T}^{(1)}(\partial_x, n, \tau)U^{(1)}\big]_6\big\}^+ - \big\{\big[\mathcal{T}^{(2)}(\partial_x, n, \tau)U^{(2)}\big]_6\big\}^- = F_6^* \;\text{ on }\; S_C. \tag{8.169}$$

Denote by $\widetilde{f}_r^{(1)}$, $r = 1, 2, 3$, a fixed extension, preserving the space, of the functions $f_r^{(1)}$ from $S_T$ to the whole of $S$. An arbitrary extension can be written then as $\widetilde{f}_r^{(1)} + g_r$, where $g_r \in \widetilde{B}_{p,p}^{1-\frac{1}{p}}(S_C)$, $r = 1, 2, 3$. Evidently, $r_{S_T}[\widetilde{f}_r^{(1)} + g_r] = f_r^{(1)}$ on $S_T$.

Now, let us define the functions:

$$\widetilde{f}_r = \begin{cases} f_r^{(1)} & \text{on } S_T, \\ \widetilde{f}_r^{(1)} & \text{on } S_C, \end{cases} \quad r = 1, 2, 3, \qquad \widetilde{f}_j = \begin{cases} f_j^{(1)} & \text{on } S_T, \\ f_j^* & \text{on } S_C, \end{cases} \quad j = 4, 5, 6, \tag{8.170}$$

$$\widetilde{F}_r = \begin{cases} F_r^{(1)} & \text{on } S_T, \\ F_r^{(+)} - F_r^{(-)} & \text{on } S_C, \end{cases} \quad r = 1, 2, 3, \qquad \widetilde{F}_j = \begin{cases} F_j^{(1)} & \text{on } S_T, \\ F_j^* & \text{on } S_C, \end{cases} \quad j = 4, 5, 6. \tag{8.171}$$

We require that the following natural compatibility conditions are satisfied:

$$\widetilde{f}_j \in B_{p,p}^{1-\frac{1}{p}}(S), \;\; j = 4, 5, 6, \quad \widetilde{F}_k \in B_{p,p}^{-\frac{1}{p}}(S), \;\; k = 1, \dots, 6.$$

Therefore, we have

$$\widetilde{f} = (\widetilde{f}_1, \dots, \widetilde{f}_6)^\top \in [B_{p,p}^{1-\frac{1}{p}}(S)]^6, \quad \widetilde{F} = (\widetilde{F}_1, \dots, \widetilde{F}_6)^\top \in [B_{p,p}^{-\frac{1}{p}}(S)]^6. \tag{8.172}$$

As in the previous subsections, motivated by the results stated in Theorem 8.3, we employ the same notation and look for a weak solution of the problem $(\text{MTC-M})_\tau$ in the form

$$U^{(1)} = V^{(1)}\big([\mathcal{H}^{(1)}]^{-1}\mathcal{A}\widetilde{F} - [\mathcal{H}^{(1)}]^{-1}\mathcal{A}\mathcal{A}^{(2)}(\widetilde{f}+g)\big) \ \text{ in } \ \Omega^{(1)}, \tag{8.173}$$

$$U^{(2)} = V^{(2)}\big([\mathcal{H}^{(2)}]^{-1}\mathcal{A}\widetilde{F} - [\mathcal{H}^{(2)}]^{-1}\mathcal{A}\mathcal{A}^{(1)}(\widetilde{f}+g)\big) \ \text{ in } \ \Omega^{(2)}, \tag{8.174}$$

where

$$g = (g_1, g_2, g_3, 0, 0, 0)^\top \ \text{ with } \ g_r \in \widetilde{B}^{1-\frac{1}{p}}_{p,p}(S_C), \ \ r = 1, 2, 3, \tag{8.175}$$

is an unknown vector. From (8.173), (8.174) it follows that all the above stated conditions of the problem $(\text{MTC-M})_\tau$ are satisfied automatically, except the conditions (8.163), which lead to the following pseudodifferential equation on $S_C$ with respect to the components $g_1$, $g_2$, $g_3$, of the unknown vector $g$,

$$\Big[(\mathcal{A}^{(1)} + \mathcal{A}^{(2)})\mathcal{A}\widetilde{F} - \big(\mathcal{A}^{(1)}\mathcal{A}\mathcal{A}^{(2)} + \mathcal{A}^{(2)}\mathcal{A}\mathcal{A}^{(1)}\big)(\widetilde{f}+g)\Big]_r = F_r^{(+)} + F_r^{(-)} \ \text{ on } \ S_C, \ \ r = 1, 2, 3.$$

In view of the equality (8.67), the later equation can be rewritten as

$$r_{S_C}[\mathcal{A}^{(M)} g]_r = G_r^{(M)} \ \text{ on } \ S_C, \ \ r = 1, 2, 3, \tag{8.176}$$

where the pseudodifferential operator $\mathcal{A}^{(M)}$ coincides with $\mathcal{A}^{(N)}$ defined in (8.131),

$$\mathcal{A}^{(M)} = \mathcal{A}^{(N)} = -\mathcal{A}^{(1)}\mathcal{A}\mathcal{A}^{(2)} \equiv -\mathcal{A}^{(2)}\mathcal{A}\mathcal{A}^{(1)}, \tag{8.177}$$

while the known functions $G_r^{(M)}$, $r = 1, 2, 3$, are given by the relations

$$G_r^{(M)} := \frac{1}{2}\,(F_r^{(+)} + F_r^{(-)}) - \frac{1}{2} r_{S_C}\big[(\mathcal{A}^{(1)} + \mathcal{A}^{(2)})\mathcal{A}\widetilde{F}\big]_r + r_{S_C}[\mathcal{A}^{(M)}\widetilde{f}]_r \in B^{-\frac{1}{p}}_{p,p}(S_C) \tag{8.178}$$

with $\widetilde{F}_k$ and $\widetilde{f}_k$ defined in (8.170), (8.171).

Denote the upper left hand side $3\times 3$ matrix block of the pseudodifferential operator $\mathcal{A}^{(M)}$ by $\widetilde{\mathcal{A}}^{(M)}$,

$$\widetilde{\mathcal{A}}^{(M)} = [\widetilde{\mathcal{A}}^{(M)}_{lj}]_{3\times 3} := [\mathcal{A}^{(M)}_{lj}]^3_{l,j=1} = [\mathcal{A}^{(N)}_{lj}]^3_{l,j=1}. \tag{8.179}$$

Equation (8.176) can be written then as

$$r_{S_C}\widetilde{\mathcal{A}}^{(M)}\widetilde{g} = \widetilde{G}^{(M)} \ \text{ on } \ S_C, \tag{8.180}$$

where $\widetilde{g} = (g_1, g_2, g_3)^\top \in [\widetilde{B}^{1-\frac{1}{p}}_{p,p}(S_C)]^3$ is the unknown vector and the right hand side $\widetilde{G}^{(M)}$ is a known vector with components defined in (8.178), $\widetilde{G}^{(M)} = (G_1^{(M)}, G_2^{(M)}, G_3^{(M)})^\top \in [B^{-\frac{1}{p}}_{p,p}(S_C)]^3$.

As we have shown in the previous Section 8.2.4 the principal homogenous $6\times 6$ symbol matrix of the first order pseudodifferential operator $\mathcal{A}^{(N)}$ is strongly elliptic. In view of (8.177) and (8.179), this implies that the principal homogenous $3\times 3$ symbol matrix $\mathfrak{S}(\widetilde{\mathcal{A}}^{(M)})$ of the pseudodifferential operator $\widetilde{\mathcal{A}}^{(M)}$ is strongly elliptic as well, i.e.,

$$\operatorname{Re}\big[\mathfrak{S}(\widetilde{\mathcal{A}}^{(M)}; x, \xi_1, \xi_2)\eta\cdot\eta\big] \geq c|\xi|\,|\eta|^2 \ \text{ for all } \ x \in S, \ \ (\xi_1, \xi_2) \in \mathbb{R}^2\setminus\{0\}, \ \ \eta \in \mathbb{C}^3. \tag{8.181}$$

Consequently the operators

$$r_{S_C}\widetilde{\mathcal{A}}^{(M)} : [\widetilde{H}^{1-\frac{1}{p}}_p(S_C)]^3 \longrightarrow [H^{-\frac{1}{p}}_p(S_C)]^3, \quad r_{S_C}\widetilde{\mathcal{A}}^{(N)} : [\widetilde{B}^{1-\frac{1}{p}}_{p,p}(S_C)]^3 \longrightarrow [B^{-\frac{1}{p}}_{p,p}(S_C)]^3,$$

are bounded. To analyse the invertibility of these operators, in accordance with Theorem B.1 in Appendix B, let us consider the $3\times 3$ matrix

$$\mathbf{M}_{\widetilde{\mathcal{A}}^{(M)}}(x) := \big[\mathfrak{S}(\widetilde{\mathcal{A}}^{(M)}; x, 0, +1)\big]^{-1}\big[\mathfrak{S}(\widetilde{\mathcal{A}}^{(M)}; x, 0, -1)\big], \ \ x \in \ell = \partial S_C, \tag{8.182}$$

constructed by the principal homogeneous symbol matrix $\mathfrak{S}(\widetilde{\mathcal{A}}^{(M)}; x, \xi_1, \xi_2)$. Let $\lambda_1''(x)$, $\lambda_2''(x)$, and $\lambda_3''(x)$ be the eigenvalues of the matrix (8.182) and

$$\delta_j''(x) = \operatorname{Re}\big[(2\pi i)^{-1}\ln\lambda_j''(x)\big], \ \ j = 1, 2, 3, \tag{8.183}$$

$$a_7 = \inf_{x\in\ell,\, 1\leq j\leq 3}\delta_j''(x), \quad a_8 = \sup_{x\in\ell,\, 1\leq j\leq 3}\delta_j''(x). \tag{8.184}$$

Here $\ln\zeta$ denotes the branch of the logarithm analytic in the complex plane cut along $(-\infty,0]$. Due to the strong ellipticity of the operator $\mathcal{A}$ we have the strict inequalities $-\frac{1}{2}<\delta_j''(x)<\frac{1}{2}$ for $x\in\overline{S_C}$, $j=1,2,3$. Therefore

$$-\frac{1}{2}<a_7\le a_8<\frac{1}{2}\,. \tag{8.185}$$

**Lemma 8.13.** *The operators*

$$r_{S_C}\widetilde{\mathcal{A}}^{(M)}:[\widetilde{H}_p^s(S_C)]^3\longrightarrow[H_p^{s-1}(S_C)]^3, \tag{8.186}$$

$$r_{S_C}\widetilde{\mathcal{A}}^{(M)}:[\widetilde{B}_{p,q}^s(S_C)]^3\longrightarrow[B_{p,q}^{s-1}(S_C)]^3,\ \ q\ge1, \tag{8.187}$$

*are invertible if*

$$\frac{1}{p}-\frac{1}{2}+a_8<s<\frac{1}{p}+\frac{1}{2}+a_7, \tag{8.188}$$

*where $a_7$ and $a_8$ are defined in* (8.184).

*Proof.* To prove the invertibility of the operators (8.186) and (8.187), we again first consider the particular values of the parameters $s=1/2$ and $p=q=2$, which fall into the region defined by the inequalities (8.188), and show that the null space of the operator

$$r_{S_C}\widetilde{\mathcal{A}}^{(M)}:[\widetilde{H}_2^{\frac{1}{2}}(S_C)]^3\longrightarrow[H_2^{-\frac{1}{2}}(S_C)]^3 \tag{8.189}$$

is trivial, i.e., the equation

$$r_{S_C}\widetilde{\mathcal{A}}^{(M)}\widetilde{g}=0\ \text{ on }\ S_C \tag{8.190}$$

admits only the trivial solution in the space $[\widetilde{H}_2^{\frac{1}{2}}(S_C)]^3$. Recall again that $\widetilde{H}_2^s(S_C)=\widetilde{B}_{2,2}^s(S_C)$ and $H_2^s(S_C)=B_{2,2}^s(S_C)$ for $s\in\mathbb{R}$.

Let $\widetilde{g}=(\widetilde{g}_1,\widetilde{g}_2,\widetilde{g}_3)^\top\in[B_{2,2}^{\frac{1}{2}}(S)]^3=[H_2^{\frac{1}{2}}(S)]^3$ be a solution to the homogeneous equation (8.190) and construct the vectors

$$\widetilde{U}^{(1)}(x)=-V^{(1)}\big([\mathcal{H}^{(1)}]^{-1}\mathcal{A}\mathcal{A}^{(2)}g^*\big)(x)\ \text{ in }\ \Omega^{(1)},\quad \widetilde{U}^{(2)}(x)=-V^{(2)}\big([\mathcal{H}^{(2)}]^{-1}\mathcal{A}\mathcal{A}^{(1)}g^*\big)(x)\ \text{ in }\ \Omega^{(2)},$$

where $g^*=(\widetilde{g}_1,\widetilde{g}_2,\widetilde{g}_3,0,0,0)^\top$. Evidently, $U^{(1)}\in[W_2^1(\Omega^{(1)})]^6$ and $U^{(2)}\in[W_{2,loc}^1(\Omega^{(2)})]^6\cap\mathbf{Z}_\tau(\Omega^{(2)})$. Further, taking into consideration that the following relations hold on $S$:

$$\{\widetilde{U}^{(1)}\}^+=-\mathcal{A}\mathcal{A}^{(2)}g^*,\quad \big\{\mathcal{T}^{(1)}\widetilde{U}^{(1)}\big\}^+=-\mathcal{A}^{(1)}\mathcal{A}\mathcal{A}^{(2)}g^*,$$
$$\{U^{(2)}\}^-=-\mathcal{A}\mathcal{A}^{(1)}g^*,\quad \big\{\mathcal{T}^{(2)}\widetilde{U}^{(2)}\big\}^-=-\mathcal{A}^{(2)}\mathcal{A}\mathcal{A}^{(1)}g^*,$$

and using (8.177) and (8.190), and keeping in mind that $r_{S_T}g_r^*=r_{S_T}\widetilde{g}_r=0$ for $r=1,2,3$, and $g_j^*=0$ on $S$ for $j=4,5,6$, we find

$$r_{S_C}\big\{[\mathcal{T}^{(1)}\widetilde{U}^{(1)}]_r\big\}^+=0,\quad r_{S_C}\big\{[\mathcal{T}^{(2)}\widetilde{U}^{(2)}]_r\big\}^-=0\ \text{ on }\ S_C,\ \ r=1,2,3,$$
$$r_{S_C}\big[\{\widetilde{U}_j^{(1)}\}^+-\{\widetilde{U}_j^{(2)}\}^-\big]=0,\quad r_{S_C}\big[\big\{[\mathcal{T}^{(1)}\widetilde{U}^{(1)}]_j\big\}^+-\big\{[\mathcal{T}^{(2)}\widetilde{U}^{(2)}]_j\big\}^-\big]=0\ \text{ on }\ S_C,\ \ j=4,5,6,$$
$$r_{S_T}\big[\{\widetilde{U}^{(1)}\}^+-\{\widetilde{U}^{(2)}\}^-\big]=0,\quad r_{S_T}\big[\{\mathcal{T}^{(1)}U^{(1)}\}^+-\{\mathcal{T}^{(2)}U^{(2)}\}^-\big]=0\ \text{ on }\ S_T.$$

Therefore, $\widetilde{U}^{(2)}$ and $\widetilde{U}^{(1)}$ solve the homogeneous interfacial crack problem $(\text{MTC-M})_\tau$ with $p=2$ and by Theorem 8.1 they vanish in the corresponding regions, $\Omega^{(1)}$ and $\Omega^{(2)}$, respectively, which implies that $g_r^*=\widetilde{g}_r=\{\widetilde{U}_r^{(1)}\}^+-\{\widetilde{U}_r^{(2)}\}^-=0$ on $S$, $r=1,2,3$. Consequently, the null space of the operator (8.189) is trivial. Since the operator $\widetilde{\mathcal{A}}^{(M)}$ is strongly elliptic, we conclude by Theorem B.1 (see Appendix B) that the operators (8.186) and (8.187) are Fredholm operators with zero indices and with the trivial null spaces for all values of the parameters satisfying the inequalities (8.188). Thus they are invertible. □

This lemma leads to the following existence theorem.

**Theorem 8.14.** *Let the conditions* (8.159) *be fulfilled and*

$$\frac{4}{3-2a_8}<p<\frac{4}{1-2a_7}\,, \tag{8.191}$$

*where $a_7$ and $a_8$ are defined by* (8.184). *Then the interfacial crack problem* $(\text{MTC-M})_\tau$ *has a unique solution* $U^{(1)} \in [W^1_p(\Omega^{(1)})]^6$ *and* $U^{(2)} \in [W^1_{p,loc}(\Omega^{(2)})]^6 \cap \mathbf{Z}_\tau(\Omega^{(2)})$, *and the solution vectors are representable in the form of single layer potentials* (8.173) *and* (8.174), *where the unknown vector* $\widetilde{g}$ *is defined from the uniquely solvable pseudodifferential equation* (8.180).

*Proof.* In accordance with Lemma 8.13, equation (8.180) is uniquely solvable for $s = 1 - \frac{1}{p}$ with $p$ satisfying inequality (8.191), since the inequalities (8.188) are fulfilled. Therefore the vectors (8.173) and (8.174) represent a solution to the interfacial crack problem $(\text{MTC-M})_\tau$ in the space $[W^1_p(\Omega^{(1)})]^6 \times ([W^1_{p,loc}(\Omega^{(2)})]^6 \cap \mathbf{Z}_\tau(\Omega^{(2)}))$ with $p$ as in (8.191).

It remains to show that the problem $(\text{MTC-M})_\tau$ is uniquely solvable in the space $[W^1_p(\Omega^{(1)})]^6 \times ([W^1_{p,loc}(\Omega^{(2)})]^6 \cap \mathbf{Z}_\tau(\Omega^{(2)}))$ for arbitrary $p$ satisfying (8.191).

Let a pair of vectors $(\widetilde{U}^{(1)}, \widetilde{U}^{(2)}) \in [W^1_p(\Omega^{(1)})]^6 \times ([W^1_{p,loc}(\Omega^{(2)})]^6 \cap \mathbf{Z}_\tau(\Omega^{(2)}))$ be a solution to the homogeneous problem $(\text{MTC-M})_\tau$. Then by Theorem 8.3 they are representable in the form (8.65), (8.66) with

$$F = \{\mathcal{T}^{(1)}\widetilde{U}^{(1)}\}^+ - \{\mathcal{T}^{(2)}\widetilde{U}^{(2)}\}^-, \quad f = \{\widetilde{U}^{(1)}\}^+ - \{\widetilde{U}^{(2)}\}^- \ \text{ on } \ S.$$

Due to the homogeneous conditions of the problem $(\text{MTC-M})_\tau$, (8.9), (8.10) and (8.15), (8.22) with $f^{(1)} = F^{(1)} = 0$ on $S_T$ and $F^{(\pm)}_r = 0$ on $S_C$ for $r = 1,2,3$, $f^*_j = F^*_j = 0$ on $S_C$ for $j = 4,5,6$, we have

$$f_r \in \widetilde{B}^{1-\frac{1}{p}}_{p,p}(S_C) \ \text{ for } \ r = 1,2,3, \quad f_j = 0 \ \text{ on } \ S \ \text{ for } j = 4,5,6, \quad F = 0 \ \text{ on } \ S, \tag{8.192}$$

and, using the notation (8.90), we get the representations

$$\widetilde{U}^{(1)} = -V^{(1)}\big([\mathcal{H}^{(1)}]^{-1}\mathcal{A}\mathcal{A}^{(2)}f\big) \ \text{ in } \ \Omega^{(1)}, \quad \widetilde{U}^{(2)} = -V^{(2)}\big([\mathcal{H}^{(2)}]^{-1}\mathcal{A}\mathcal{A}^{(1)}f\big) \ \text{ in } \ \Omega^{(2)},$$

where $f = (\widetilde{f}, 0, 0, 0)^\top$ with $\widetilde{f} = (f_1, f_2, f_3)^\top$.

Since, $r_{S_C}\{[\mathcal{T}^{(1)}\widetilde{U}^{(1)}]_r\}^+ = r_{S_C}\{[\mathcal{T}^{(2)}\widetilde{U}^{(2)}]_r\}^- = 0$ on $S_C$ for $r = 1,2,3$, we arrive at the equation $r_{S_C}\widetilde{\mathcal{A}}^{(M)}\widetilde{f} = 0$ on $S_C$, where $\widetilde{f} = (f_1, f_2, f_3)^\top \in [\widetilde{B}^{1-\frac{1}{p}}_{p,p}(S_C)]^3$. Therefore Lemma 8.13 implies that $\widetilde{f} = 0$ on $S_C$, i.e., $f = 0$ on $S$, which completes the proof. □

Finally, we establish almost the best regularity Hölder continuity results for solutions to the crack problem $(\text{MTC-M})_\tau$.

**Theorem 8.15.** *Let inclusions* (8.159) *hold and let*

$$\frac{4}{3 - 2a_8} < p < \frac{4}{1 - 2a_7}\,, \ \ 1 < r < \infty, \ \ 1 \le q \le \infty, \ \ \frac{1}{r} - \frac{1}{2} + a_8 < s < \frac{1}{r} + \frac{1}{2} + a_7, \tag{8.193}$$

*with $a_7$ and $a_8$ defined by* (8.184).

*Further, let* $U^{(1)} \in [W^1_p(\Omega^{(1)})]^6$ *and* $U^{(2)} \in [W^1_{p,loc}(\Omega^{(2)})]^6 \cap \mathbf{Z}_\tau(\Omega^{(2)})$, *be a unique solution pair to the interfacial crack problem* $(\text{MTC-M})_\tau$. *Then the following hold:*

(i) *if*

$$\begin{gathered} f^{(1)}_k \in B^s_{r,r}(S_T), \quad F^{(1)}_k \in B^{s-1}_{r,r}(S_T), \ \ k = 1,\dots,6, \\ F^{(\pm)}_l \in B^{s-1}_{r,r}(S_C), \quad f^*_j \in B^s_{r,r}(S_C), \quad F^*_j \in B^{s-1}_{r,r}(S_C), \ \ l = 1,2,3, \ \ j = 4,5,6. \end{gathered} \tag{8.194}$$

*and* $\widetilde{f} = (\widetilde{f}_1, \dots, \widetilde{f}_6)^\top \in [B^s_{r,r}(S)]^6$ *and* $\widetilde{F} = (\widetilde{F}_1, \dots, \widetilde{F}_6)^\top \in [B^{s-1}_{r,r}(S)]^6$ *with* $\widetilde{f}$ *and* $\widetilde{F}$ *defined by* (8.170), (8.171), *then*

$$U^{(1)} \in [H^{s+\frac{1}{r}}_r(\Omega^{(1)})]^6, \quad U^{(2)} \in [H^{s+\frac{1}{r}}_{r,loc}(\Omega^{(2)})]^6 \cap \mathbf{Z}_\tau(\Omega^{(2)}); \tag{8.195}$$

(ii) *if*

$$\begin{gathered} f^{(1)}_k \in B^s_{r,q}(S_T), \quad F^{(1)}_k \in B^{s-1}_{r,q}(S_T), \ \ k = 1,\dots,6, \\ F^{(\pm)}_l \in B^{s-1}_{r,q}(S_C), \quad f^*_j \in B^s_{r,q}(S_C), \quad F^*_j \in B^{s-1}_{r,q}(S_C), \ \ l = 1,2,3, \ \ j = 4,5,6. \end{gathered} \tag{8.196}$$

*and* $\widetilde{f} = (\widetilde{f}_1, \dots, \widetilde{f}_6)^\top \in [B^s_{r,q}(S)]^6$ *and* $\widetilde{F} = (\widetilde{F}_1, \dots, \widetilde{F}_6)^\top \in [B^{s-1}_{r,q}(S)]^6$ *with* $\widetilde{f}$ *and* $\widetilde{F}$ *defined by* (8.170), (8.171), *then*

$$U^{(1)} \in [B^{s+\frac{1}{r}}_{r,q}(\Omega^{(1)})]^6, \quad U^{(2)} \in [B^{s+\frac{1}{r}}_{r,q,loc}(\Omega^{(2)})]^6 \cap \mathbf{Z}_\tau(\Omega^{(2)}); \tag{8.197}$$

(iii) *if* $\alpha > 0$ *is not integer and*

$$\begin{gathered} f^{(1)}_k \in C^\alpha(\overline{S_T}), \quad F^{(1)}_k \in B^{\alpha-1}_{\infty,\infty}(S_T), \ \ k = 1, \dots, 6, \\ F^{(\pm)}_l \in B^{\alpha-1}_{\infty,\infty}(S_C), \quad f^*_j \in C^\alpha(\overline{S_C}), \quad F^*_j \in B^{\alpha-1}_{\infty,\infty}(S_C), \ \ l = 1,2,3, \ \ j = 4,5,6. \end{gathered} \tag{8.198}$$

*and* $\widetilde{f} = (\widetilde{f}_1, \dots, \widetilde{f}_6)^\top \in [C^\alpha(S)]^6$ *and* $\widetilde{F} = (\widetilde{F}_1, \dots, \widetilde{F}_6)^\top \in [B^{\alpha-1}_{\infty,\infty}(S)]^6$ *with* $\widetilde{f}$ *and* $\widetilde{F}$ *defined by* (8.170), (8.171), *then*

$$U^{(1)} \in \bigcap_{\alpha'<\kappa} [C^{\alpha'}(\overline{\Omega^{(1)}})]^6, \quad U^{(2)} \in \bigcap_{\alpha'<\kappa} [C^{\alpha'}(\overline{\Omega^{(2)}})]^6 \cap \mathbf{Z}_\tau(\Omega^{(2)}), \tag{8.199}$$

*where* $\kappa = \min\{\alpha, \ a_7 + \frac{1}{2}\} > 0$.

*Proof.* It is word for word of the proof of Theorem 5.22. □

8.2.7. *Asymptotic expansion of solutions to the problem* $(\text{MTC-M})_\tau$. In this subsection we investigate the asymptotic behaviour of the solution to the problem $(\text{MTC-M})_\tau$ near the interfacial crack edge $\ell = \partial S_C$. For simplicity of description of the method applied below, we assume that the crack and transmission boundary data of the problem are infinitely smooth, namely,

$$\begin{gathered} F^{(+)}_r, F^{(-)}_r \in C^\infty(\overline{S}_C), \ \ r = 1,2,3, \quad f^*_j, F^*_j \in C^\infty(\overline{S}_C), \ \ j = 4,5,6, \\ f^{(1)}, F^{(2)} \in [C^\infty(\overline{S}_T)]^6, \quad \widetilde{f}, \widetilde{F} \in [C^\infty(S)]^6. \end{gathered}$$

In Subsection 8.2.6 we have shown that the problem $(\text{MTC-M})_\tau$ is uniquely solvable and the solution $(U^{(1)}, U^{(2)})$ is represented by (8.173), (8.174) with the density defined by the pseudodifferential equation (8.180).

To establish the asymptotic expansion of solution $(U^{(1)}, U^{(2)})$ near the crack edge $\ell = \partial S_C$ we again preserve the notation introduced in Subsection 8.2.6 and rewrite the representation (8.173), (8.174) in the form

$$U^{(1)} = V^{(1)}\mathcal{B}_1 g + R^{(1)} \ \text{ in } \ \Omega^{(1)}, \quad U^{(2)} = V^{(2)}\mathcal{B}_2 g + R^{(2)} \ \text{ in } \ \Omega^{(2)},$$

where $\mathcal{B}_1 = -(\mathcal{H}^{(1)})^{-1}\mathcal{A}\mathcal{A}^{(2)}$, $\mathcal{B}_2 = -(\mathcal{H}^{(2)})^{-1}\mathcal{A}\mathcal{A}^{(1)}$,

$$R^{(1)} = V^{(1)}(\mathcal{H}^{(1)})^{-1}\mathcal{A}F - V^{(1)}\mathcal{B}_1\widetilde{f} \in [C^\infty(\overline{\Omega^{(1)}})]^6, \ \ R^{(2)} = V^{(2)}(\mathcal{H}^{(2)})^{-1}\mathcal{A}F - V^{(2)}\mathcal{B}_2\widetilde{f} \in [C^\infty(\overline{\Omega^{(2)}})]^6,$$

and $g = (\widetilde{g}, 0, 0, 0)^\top$, $\widetilde{g} = (g_1, g_2, g_3)^\top$ solves the pseudodifferential equation

$$r_{S_C}\widetilde{\mathcal{A}}^{(M)} g = \widetilde{G}^{(M)} \ \text{ on } \ S_C \tag{8.200}$$

with $\widetilde{G}^{(M)} \in [C^\infty(\overline{S}_C)]^3$.

Consider the $3 \times 3$ matrix $M_{\widetilde{\mathcal{A}}^{(M)}}$ related to the principal homogeneous symbol matrix $\mathfrak{S}(\widetilde{\mathcal{A}}^{(M)}; x, \xi)$ of the operator $\widetilde{\mathcal{A}}^{(M)}$

$$M_{\widetilde{\mathcal{A}}^{(M)}}(x_1) := \big[\mathfrak{S}(\widetilde{\mathcal{A}}^{(M)}; x_1, 0, 0, +1)\big]^{-1}\mathfrak{S}(\widetilde{\mathcal{A}}^{(M)}; x_1, 0, 0, -1), \ \ (x_1, 0) \in \ell = \partial S_C.$$

Denote by $\lambda_1(x_1), \lambda_2(x_1), \lambda_3(x_1)$ the eigenvalues of the matrix $M_{\widetilde{\mathcal{A}}^{(M)}}(x_1)$ and by $m_j$ the algebraic multiplicity of $\lambda_j(x_1)$. Let $\mu_1(x_1), \dots, \mu_l(x_1)$ $(1 \le l \le 3)$ be the distinct eigenvalues. Evidently, $m_j$ and $l$ depend on $x_1$ in general and $m_1 + \dots + m_l = 3$.

It is well known that the matrix $M_{\mathcal{A}^{(N)}}(x_1)$ admits the following decomposition (see, e.g. [42, Ch. 7, Section 7])

$$M_{\widetilde{\mathcal{A}}^{(M)}}(x_1) = \mathcal{K}(x_1)\mathcal{J}_{M_{\widetilde{\mathcal{A}}^{(M)}}}(x_1)\mathcal{K}^{-1}(x_1), \ \ (x_1, 0) \in \ell = \partial S_C,$$

where $\mathcal{K}$ is $3 \times 3$ non-degenerate matrix with infinitely smooth entries and $\mathcal{J}_{M_{\widetilde{\mathcal{A}}^{(M)}}}$ is block diagonal

$$\mathcal{J}_{M_{\widetilde{\mathcal{A}}^{(M)}}}(x_1) := \operatorname{diag}\big\{\mu_1(x_1)B^{m_1}(1), \dots, \mu_l(x_1)B^{m_l}(1)\big\}.$$

Here $B^{(\nu)}(t)$, $\nu \in \{m_1, \dots, m_l\}$ are upper triangular matrices (see Subsection 5.2.3).

Introduce a $3\times 3$ matrix

$$B_0(t):=\operatorname{diag}\big\{B^{(m_1)}(t),\dots,B^{(m_l)}(t)\big\}. \tag{8.201}$$

Applying the results from the reference [22] we derive the following asymptotic expansion for the solution of the pseudodifferential equation (8.200)

$$\begin{aligned}\widetilde{g}(x_1,x_{2,+})&=\mathcal{K}(x_1)x_{2,+}^{\frac12+\Delta(x_1)}B_0\Big(-\frac{1}{2\pi i}\log x_{2,+}\Big)\mathcal{K}^{-1}(x_1)b_0(x_1)\\&\quad+\sum_{k=1}^{M}\mathcal{K}(x_1)x_{2,+}^{\frac12+\Delta(x_1)+k}B_k(x_1,\log x_{2,+})+\widetilde{g}_{M+1}(x_1,x_{2,+}),\end{aligned} \tag{8.202}$$

where $b_0\in[C^\infty(\ell)]^3$, $\widetilde{g}_{M+1}\in[C^\infty(\ell^+_{c,\varepsilon})]^3$, $\ell^+_{c,\varepsilon}=\ell\times[0,\varepsilon]$, and

$$B_k(x_1,t)=B_0\Big(-\frac{t}{2\pi i}\Big)\sum_{j=1}^{k(2m_0-1)}t^j d_{kj}(x_1).$$

Here $m_0=\max\{m_1,\dots,m_l\}$, the coefficients $d_{kj}\in[C^\infty(\ell)]^3$,

$$\begin{gathered}\Delta(x_1):=(\Delta_1(x_1),\Delta_2(x_1),\Delta_3(x_1)),\quad \Delta_j(x_1)=\frac{1}{2\pi i}\log\lambda_j''(x_1)=\frac{1}{2\pi}\arg\lambda_j''(x_1)+\frac{1}{2\pi i}\log|\lambda_j''(x_1)|,\\ -\pi<\arg\lambda_j''(x_1)<\pi,\ \ (x_1,0)\in\ell,\ \ j=1,2,3.\end{gathered}$$

Furthermore,

$$x_{2,+}^{\frac12+\Delta(x_1)}:=\operatorname{diag}\Big\{x_{2,+}^{\frac12+\Delta_1(x_1)},x_{2,+}^{\frac12+\Delta_2(x_1)},x_{2,+}^{\frac12+\Delta_3(x_1)}\Big\}.$$

Now, having at hand the above asymptotic expansion for the density vector function $\widetilde{g}$, we can apply the results of the reference [23] and write the following spatial asymptotic expansions of the solution $(U^{(1)},U^{(2)})$

$$\begin{aligned}U^{(\beta)}(x)=\sum_{\mu=\pm1}&\Bigg\{\sum_{s=1}^{l_0^{(\beta)}}\sum_{j=0}^{n_s^{(\beta)}-1}x_3^j\Big[d_{sj}^{(\beta)}(x_1,\mu)\Big(\widetilde{z}_{s,\mu}^{(\beta)}\Big)^{\frac12+\widetilde{\Delta}(x_1)-j}\widetilde{B}_0\Big(-\frac{1}{2\pi i}\log z_{s,\mu}^{(\beta)}\Big)\Big]\widetilde{c}_j(x_1)\\&+\sum_{\substack{k,l=0\\k+l+j+p\geq1}}^{M+2}\sum_{j+p=0}^{M+2-l}x_2^l x_3^j d_{sljp}^{(\beta)}(x_1,\mu)(\widetilde{z}_{s,\mu}^{(\beta)})^{\frac12+\widetilde{\Delta}(x_1)+p+k}B_{skjp}^{(\beta)}(x_1,\log\widetilde{z}_{s,\mu}^{(\beta)})\Bigg\}\\&+U_{M+1}^{(\beta)}(x),\ \ x_3>0,\ \ \beta=1,2.\end{aligned} \tag{8.203}$$

The coefficients $d_{sj}^{(\beta)}(\,\cdot\,,\mu)$ and $d_{sljp}^{(\beta)}(\,\cdot\,,\mu)$ are $6\times6$ matrices with entries from the space $C^\infty(\ell)$, while $B_{skjp}^{(\beta)}(x_1,t)$ are polynomials in $t$ with vector coefficients which depend on the variable $x_1$ and have the order $\nu_{kjp}=k(2m_0-1)+m_0-1+p+j$ in general, where $m_0=\max\{m_1,\dots,m_l\}$ and $m_1+\cdots+m_l=3$,

$$\begin{gathered}\widetilde{\Delta}(x_1):=(\Delta(x_1),\Delta(x_1)),\ \ \widetilde{c}_j\in[C^\infty(\ell)]^6,\quad U_{M+1}^{(\beta)}\in[C^{M+1}(\overline{\Omega}^{(\beta)})]^6,\\ (\widetilde{z}_{s,\mu}^{(\beta)})^{\kappa+\widetilde{\Delta}(x_1)}:=\operatorname{diag}\big\{(z_{s,\mu}^{(\beta)})^{\kappa+\Delta(x_1)},(z_{s,\mu}^{(\beta)})^{\kappa+\Delta(x_1)}\big\},\end{gathered}$$

where

$$\begin{gathered}(z_{s,\mu}^{(\beta)})^{\kappa+\Delta(x_1)}:=\operatorname{diag}\big\{(z_{s,\mu}^{(\beta)})^{\kappa+\Delta_1(x_1)},(z_{s,\mu}^{(\beta)})^{\kappa+\Delta_2(x_1)},(z_{s,\mu}^{(\beta)})^{\kappa+\Delta_3(x_1)}\big\},\\ \kappa\in\mathbb{R},\ \ \mu=\pm1,\ \ \beta=1,2,\ \ (x_1,0)\in\ell,\\ z_{s,+1}^{(\beta)}:=-x_2-x_3\zeta_{s,+1}^{(\beta)},\quad z_{s,-1}^{(\beta)}:=x_2-x_3\zeta_{s,-1}^{(\beta)},\\ -\pi<\arg z_{s,\pm1}^{(\beta)}<\pi,\ \ \zeta_{s,\pm1}^{(\beta)}\in C^\infty(\ell),\ \ \widetilde{B}_0(t):=\operatorname{diag}\{B_0(t),B_0(t)\}.\end{gathered}$$

Note that, the coefficients $d_{sj}^{(\beta)}(\,\cdot\,,\mu)$ can be calculated explicitly, whereas the coefficients $c_j$ can be expressed by means of the first coefficient $b_0$ in the asymptotic expansion of (3.65) (see [23, Theorem 2.3])

$$\begin{gathered}
d_{sj}^{(\beta)}(x_1,+1)=\frac{1}{2\pi}\,\mathcal{G}_{\varkappa_\beta}(x_1,0)P_{sj}^{+,\beta}(x_1)\widetilde{\mathcal{K}}(x_1),\\
d_{sj}^{(\beta)}(x_1,-1)=\frac{1}{2\pi}\mathcal{G}_{\varkappa_\beta}(x_1,0)P_{sj}^{-,\beta}(x_1)\widetilde{\mathcal{K}}(x_1)e^{i\pi(-\frac{1}{2}-\widetilde{\Delta}(x_1))},\\
s=1,\dots,l_0^{(\beta)},\ \ j=0,\dots,n_s^{(\beta)}-1,\ \ \beta=1,2,\\
\widetilde{\mathcal{K}}(x_1):=\operatorname{diag}\{\mathcal{K}(x_1),\mathcal{K}(x_1)\},
\end{gathered}$$

where

$$P_{sj}^{\pm,\beta}(x_1):=V_{-1,j}^{\beta,s}(x_1,0,0,\pm1)\mathfrak{S}\big([\mathcal{B}_\beta^{jk}]_{6\times3};0,0,\pm1\big),\ \ j=1,\dots,6,\ \ k=1,\dots,3,$$

$$\begin{aligned}
V_{-1,j}^{\beta,s}(x_1,0,0,\pm1):=\ &\frac{i^{j+1}}{j!(n_s^{(\beta)}-1-j)!}\,\frac{d^{n_s^{(\beta)}-1-j}}{d\zeta^{n_s^{(\beta)}-1-j}}\,(\zeta-\zeta_{s,\pm1}^{(\beta)})^{n_s^{(\beta)}}\\
&\times\Big(A^{(\beta,0)}\big((J_{\varkappa_\beta}^\top(x_1,0))^{-1}(0,\pm1,\zeta)^\top\big)\Big)^{-1}\bigg|_{\zeta=\zeta_{s,\pm1}^{(\beta)}},
\end{aligned}$$

$\mathcal{G}_{\varkappa_\beta}$ are the square roots of Gram's determinant of the diffeomorphisms $\varkappa_\beta$, $\beta=1,2$, and

$$\begin{gathered}
\widetilde{c}_j(x_1):=(c_j(x_1),c_j(x_1)),\\
c_j(x_1)=a_j(x_1)B_0^-\Big(\frac{1}{2}+\Delta(x_1)\Big)\mathcal{K}^{-1}(x_1)b_0(x_1),\ \ j=0,\dots,n_s^{(\beta)}-1,
\end{gathered}$$

where $a_j$ and $B_0^-$ have the same structure and properties as in Subsection 8.2.5.

*Remark* 8.16. The above asymptotic expansions (8.203) of solutions imply that for sufficiently smooth boundary data (e.g., $C^\infty$-smooth data say) the solution vectors $(U^{(1)},U^{(2)})$ to the interfacial problem $(\text{MTC-M})_\tau$ belong to the class of semi-regular functions described in Definition 2.2:

$$U^{(1)}\in[\mathbf{C}(\widetilde{\Omega}_\ell^{(1)};\alpha)]^6,\quad U^{(2)}\in[\mathbf{C}(\widetilde{\Omega}_\ell^{(2)};\alpha)]^6,\quad \widetilde{\Omega}_\ell^{(\beta)}=\overline{\Omega^{(\beta)}}\setminus\ell,\ \ \ell=\partial S_C,\ \ \beta=1,2,$$

where $\alpha=\frac{1}{2}-a_7+\varepsilon$ with $a_7$ defined in (8.184) and $\varepsilon$ being an arbitrarily small positive number. Due to the relations (8.185), it is evident that $0<\alpha<1$ if $0<\varepsilon<\frac{1}{4}+\frac{1}{2}\,a_7$.

Moreover, the dominant terms of the vectors $U^{(\beta)}$ near the curves $\ell=\partial S_C$ can be represented as the product of $C^\infty$-smooth vector-functions and factors of the following form $[\ln\varrho(x)]^{m_j-1}[\varrho(x)]^{\kappa_j''+i\nu_j''}$, where $\varrho(x)$ is the distance from a reference point $x$ to the curves $\ell$. Therefore, near the curve $\ell$ the dominant singular terms of the corresponding generalized stress vectors $\mathcal{T}^{(\beta)}U^{(\beta)}$ are represented as the product of $C^\infty$-smooth vector-functions and the singular factors $[\ln\varrho(x)]^{m_j-1}[\varrho(x)]^{-1+\kappa_j''+i\nu_j''}$. The numbers $\nu_j$ are different from zero, in general, and describe the oscillating character of the stress singularities.

The exponents $\kappa_j''+i\nu_j''$ are related to the corresponding eigenvalues $\lambda_j''$ of the matrix (8.182) by the equalities

$$\kappa_j''=\frac{1}{2}+\frac{\arg\lambda_j''}{2\pi},\quad \nu_j''=-\frac{\ln|\lambda_j''|}{2\pi},\ \ j=1,2,3. \tag{8.204}$$

Recall that in the above expressions the parameter $m_j$ denotes the multiplicity of the eigenvalue $\lambda_j''$.

It is evident that at the curve $\ell$ the components of the generalized stress vector $\mathcal{T}^{(\beta)}U^{(\beta)}$ behave like $\mathcal{O}([\ln\varrho(x)]^{m_0-1}[\varrho(x)]^{-\frac{1}{2}+a_7})$, where $m_0$ denotes the maximal multiplicity of the eigenvalues. This is a global singularity effect for the first order derivatives of the vectors $U^{(\beta)}$, $\beta=1,2$.

8.3. **General transmission problems of pseudo-oscillations for multi-layered composite structures.** First of all we describe the geometrical structure of layered elastic three-dimensional composites treated in this subsection. Let us consider a nested set of closed disjoint surfaces: $S_1,S_2,\dots,S_K$, assuming that the surface $S_k$ is located inside the surface $S_{k+1}$, $k=1,\dots,K-1$, and $S_j\cap S_l=\varnothing$ for all $j, l = 1, 2, \dots, K$. Denote by $\Omega^{(1)}$ the domain surrounded by the surface $S_1$, by $\Omega^{(k)}$ the region between the surfaces $S_{k-1}$ and $S_k$, $k = 2, \dots, K$, and, finally, by $\Omega^{(K+1)}$ the unbounded region exterior to $S_K$. Evidently,

$$\partial\Omega^{(1)} = S_1, \quad \partial\Omega^{(k)} = S_{k-1} \cup S_k, \quad k = 2, \dots, K, \quad \partial\Omega^{(K+1)} = S_K,$$
$$\overline{\Omega^{(1)}} = \Omega^{(1)} \cup S_1, \quad \overline{\Omega^{(k)}} = \Omega^{(k)} \cup S_{k-1} \cup S_k, \quad k = 2, \dots, K, \quad \overline{\Omega^{(K+1)}} = \Omega^{(K+1)} \cup S_K.$$

For $x \in S_l$, by $n(x)$ we again denote the outward unit normal vector to the surface $S_l$, $l = 1, 2, \dots, K$. For simplicity, we again assume that the surfaces $S_l$ are sufficiently smooth ($C^\infty$-smooth say) if not otherwise stated.

Further, as in Subsection 8.1, we assume that the domains $\Omega^{(j)}$, $j = 1, 2, \dots, K+1$, are occupied by anisotropic homogeneous materials possessing different thermo-electro-magneto-elastic properties described in Section 2. Again, the material parameters, thermo-mechanical and electro-magnetic characteristics (*displacement vectors, strain and stress tensors*, *electric and magnetic potentials, electric displacements and magnetic inductions*, *temperature functions and heat fluxes*) associated with the domain $\Omega^{(j)}$ for $j = 1, \dots, K+1$, we equip with the superscript $(j)$ and employ the notation introduced in Subsection 8.1.

For the region $\Omega^{(j)}$, we again use the notation introduced in the previous sections for the basic field equations, differential and boundary operators, as well as the fundamental solutions, single layer, double layer and volume potentials, and the corresponding boundary integral operators, but now equipped with the superscript $(j)$, e.g., $A^{(j)}(\partial, \tau)$, $\mathcal{T}^{(j)}(\partial, n, \tau)$, $\mathcal{P}^{(j)}(\partial, n, \tau)$, $V^{(j)}_{S_l}$, $W^{(j)}_{S_l}$, $N^{(j)}_{\Omega^{(j)}}$, $\mathcal{H}^{(j)}_{S_l}$, $\mathcal{K}^{(j)}_{S_l}$, $\mathcal{N}^{(j)}_{S_l}$, $\mathcal{L}^{(j)}_{S_l}$ etc. Recall that the subindex $S_l$ in the notation of payer potentials and the corresponding boundary integral operators denotes the integration surface, e.g., $W^{(j)}_{S_l}$ stands for the double layer potential where the integrand kernel is constructed by the fundamental solution matrix $\Gamma^{(j)}$ and by the boundary operator $\mathcal{P}^{(j)}$ associated with the field equations in the region $\Omega^{(j)}$ and $S_l$ is the integration surface.

Further, depending on the structure of the composed body, the following four main cases can be considered:

(i) *multi-layered composite space:*

$$\overline{\Omega^{(1)}} \cup \overline{\Omega^{(2)}} \cup \dots \cup \overline{\Omega^{(K)}} \cup \overline{\Omega^{(K+1)}} = \mathbb{R}^3 \ \ \textit{with interfaces } S_1, \dots, S_K;$$

(ii) *multi-layered bounded composite structure:*

$$\overline{\Omega^{(1)}} \cup \overline{\Omega^{(2)}} \cup \dots \cup \overline{\Omega^{(K)}} = \mathbb{R}^3 \setminus \Omega^{(K+1)} \ \ \textit{with interfaces } S_1, \dots, S_{K-1}, \textit{ and exterior boundary } S_K;$$

(iii) *multi-layered unbounded composite structure with interior cavity:*

$$\overline{\Omega^{(2)}} \cup \dots \cup \overline{\Omega^{(K+1)}} = \mathbb{R}^3 \setminus \Omega^{(1)} \ \ \textit{with interfaces } S_2, \dots, S_K, \textit{ and interior boundary } S_1;$$

(iv) *multi-layered bounded composite structure with interior cavity:*

$$\overline{\Omega^{(2)}} \cup \dots \cup \overline{\Omega^{(K)}} = \mathbb{R}^3 \setminus [\Omega^{(1)} \cup \Omega^{(K+1)}]$$

*with interfaces $S_2, \dots, S_{K-1}$, interior boundary $S_1$, and exterior boundary $S_K$.*

In the case of *General transmission problems of pseudo-oscillations* of the GTEME theory, the sought for vectors should satisfy the corresponding differential equations in each subdomain $\Omega^{(j)}$, the transmission conditions on the interfaces, and the boundary conditions on the boundary surface of the composite body under consideration. If the composite body contains interior or interfacial cracks, then the corresponding crack conditions should be prescribed on the crack faces.

Among the huge number of possible general mathematical transmission problems for the above listed composite structures, for illustration of our approach, here we formulate and analyse only one general transmission problem for multi-layered bounded composite structure with interior cavity. In particular, as a model problem, we consider a composed body

$$\mathbf{\Omega} := \overline{\Omega^{(2)}} \cup \dots \cup \overline{\Omega^{(K)}} = \mathbb{R}^3 \setminus [\Omega^{(1)} \cup \Omega^{(K+1)}] \tag{8.205}$$

with interfaces $S_2, \dots, S_{K-1}$, interior boundary $S_1$ and exterior boundary $S_K$, and formulate the following transmission problem.

**Problem (GTP-Ω-DN)$_\tau$:** *Find vector functions*

$$U^{(j)} = \big(u_1^{(j)}, u_2^{(j)}, u_3^{(j)}, \varphi^{(j)}, \psi^{(j)}, \vartheta^{(j)}\big)^\top \in [W_p^1(\Omega^{(j)})]^6, \ \ j = 2, \dots, K, \ \ p > 1, \tag{8.206}$$

*satisfying the corresponding differential equations of pseudo-oscillations in each region* $\Omega^{(j)}$,

$$A^{(j)}(\partial_x, \tau)U^{(j)}(x) = 0, \ \ x \in \Omega^{(j)}, \ \ j = 2, \dots, K, \tag{8.207}$$

*the transmission conditions on the interfaces* $S_l$, $l = 2, \dots, K-1$,

$$\{U^{(l)}(x)\}^+ - \{U^{(l+1)}(x)\}^- = f^{(l)}(x), \ \ x \in S_l, \tag{8.208}$$

$$\big\{\mathcal{T}^{(l)}(\partial_x, n, \tau)U^{(l)}(x)\big\}^+ - \big\{\mathcal{T}^{(l+1)}(\partial_x, n, \tau)U^{(l+1)}(x)\big\}^- = F^{(l)}(x), \ \ x \in S_l, \tag{8.209}$$

*and the Dirichlet boundary conditions on the interior boundary* $S_1$ *and the Neumann boundary conditions on the exterior boundaries* $S_K$,

$$\{U^{(2)}(x)\}^- = f^{(1)}(x), \ \ x \in S_1, \tag{8.210}$$

$$\big\{\mathcal{T}^{(K)}(\partial_x, n, \tau)U^{(K)}(x)\big\}^+ = F^{(K)}(x), \ \ x \in S_K, \tag{8.211}$$

*where*

$$f^{(l)} \in [B_{p,p}^{1-\frac{1}{p}}(S_l)]^6, \ \ l = 1, \dots, K-1, \quad F^{(j)} \in [B_{p,p}^{-\frac{1}{p}}(S_l)]^6, \ \ j = 2, \dots, K. \tag{8.212}$$

**Theorem 8.17.** *Let the surfaces* $S_k$, $k = 1, \dots, K$, *be Lipschitz and* $\tau = \sigma + i\omega$ *with* $\sigma > \sigma_0 \geqslant 0$ *and* $\omega \in \mathbb{R}$. *The homogeneous transmission problem* (GTP-**Ω**-DN)$_\tau$ *for* $p = 2$ *possesses only the trivial weak solution assuming that the time relaxation parameters* $\nu_0^{(j)}$ *are the same for all regions* $\Omega^{(j)}$, $j = 2, \dots, K$,

$$\nu_0^{(2)} = \cdots = \nu_0^{(K)} =: \nu_0. \tag{8.213}$$

*Proof.* By the corresponding Green's formulas for the regions $\Omega^{(j)}$ and by the word-for-word arguments applied in the proof of Theorem 8.1, we deduce the relations

$$u^{(j)} = 0, \ \ \varphi^{(j)} = b_1^{(j)} = const, \ \ \psi^{(j)} = b_2^{(j)} = const, \ \ \vartheta^{(j)} = 0 \text{ in } \Omega^{(j)}, \ \ j = 2, \dots, K, \tag{8.214}$$

for all $\tau = \sigma + i\omega$ with $\sigma > \sigma_0 \geqslant 0$ and $\omega \in \mathbb{R}$. Since the vector $U^{(2)}$ satisfies the homogeneous Dirichlet condition (8.210) with $f^{(1)} = 0$ on $S_1$ we conclude that $b_1^{(2)} = b_2^{(2)} = 0$. Therefore due to the homogeneous transmission conditions (8.208) on $S_l$ with $f^{(l)} = 0$ for $l = 2, \dots, K-1$, we get that all the constants $b_1^{(j)}$ and $b_2^{(j)}$ vanish for $j = 2, \dots, K$, which completes the proof. □

In what follows we assume that the condition (8.213) is satisfied.

To prove the existence of solutions to the nonhomogeneous problem (GTP-**Ω**-DN)$_\tau$, (8.206)–(8.212), we look for solution vectors (8.206) in the form of single layer potentials

$$U^{(j)}(x) = V_{S_{j-1}}^{(j)}(g_{j-1}^{(j)})(x) + V_{S_j}^{(j)}(g_j^{(j)})(x), \ \ x \in \Omega^{(j)}, \ \ j = 2, \dots, K, \tag{8.215}$$

where

$$g_{j-1}^{(j)} \in [B_{p,p}^{-\frac{1}{p}}(S_{j-1})]^6, \quad g_j^{(j)} \in [B_{p,p}^{-\frac{1}{p}}(S_j)]^6, \ \ j = 2, \dots, K, \tag{8.216}$$

are $2(K-1)$ unknown density vector functions (the superscript corresponds to the number of the layer region and the subscript corresponds to the number of integration surface). The transmission and boundary conditions (8.210), (8.208), (8.209), and (8.211) lead then to the following pseudodifferential equations for the unknown densities $g_1^{(2)}, g_2^{(2)}, g_2^{(3)}, g_3^{(3)}, \dots, g_{K-1}^{(K)}, g_K^{(K)}$:

$$\mathcal{H}_{S_1}^{(2)} g_1^{(2)} + \gamma_{S_1}^- V_{S_2}^{(2)}(g_2^{(2)}) = f^{(1)} \ \text{ on } \ S_1, \tag{8.217}$$

$$\gamma_{S_2}^+ V_{S_1}^{(2)}(g_1^{(2)}) + \mathcal{H}_{S_2}^{(2)} g_2^{(2)} - \mathcal{H}_{S_2}^{(3)} g_2^{(3)} - \gamma_{S_2}^- V_{S_3}^{(3)}(g_3^{(3)}) = f^{(2)} \ \text{ on } \ S_2, \tag{8.218}$$

$$\begin{aligned} &\gamma_{S_2}^+ \big\{\mathcal{T}^{(2)} V_{S_1}^{(2)}(g_1^{(2)})\big\} + \Big(-\frac{1}{2} I_6 + \mathcal{K}_{S_2}^{(2)}\Big) g_2^{(2)} \\ &\qquad - \Big(\frac{1}{2} I_6 + \mathcal{K}_{S_2}^{(3)}\Big) g_2^{(3)} - \gamma_{S_2}^- \big\{\mathcal{T}^{(3)} V_{S_3}^{(3)}(g_3^{(3)})\big\} = F^{(2)} \ \text{ on } \ S_2, \end{aligned} \tag{8.219}$$

$$\gamma^+_{S_3} V^{(3)}_{S_2}(g^{(3)}_2) + \mathcal{H}^{(3)}_{S_3} g^{(3)}_3 - \mathcal{H}^{(4)}_{S_3} g^{(4)}_3 - \gamma^-_{S_3} V^{(4)}_{S_4}(g^{(4)}_4) = f^{(3)} \ \text{ on } \ S_3, \tag{8.220}$$

$$\gamma^+_{S_3}\big\{\mathcal{T}^{(3)} V^{(3)}_{S_2}(g^{(3)}_2)\big\} + \Big(-\frac{1}{2}\, I_6 + \mathcal{K}^{(3)}_{S_3}\Big) g^{(3)}_3 - \Big(\frac{1}{2}\, I_6 + \mathcal{K}^{(4)}_{S_3}\Big) g^{(4)}_3 \\ -\gamma^-_{S_3}\big\{\mathcal{T}^{(4)} V^{(4)}_{S_4}(g^{(4)}_4)\big\} = F^{(3)} \ \text{ on } \ S_3, \tag{8.221}$$

. . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . .

$$\gamma^+_{S_{K-1}} V^{(K-1)}_{S_{K-2}}(g^{(K-1)}_{K-2}) + \mathcal{H}^{(K-1)}_{S_{K-1}} g^{(K-1)}_{K-1} - \mathcal{H}^{(K)}_{S_{K-1}} g^{(K)}_{K-1} \\ -\gamma^-_{S_{K-1}} V^{(K)}_{S_K}(g^{(K)}_K) = f^{(K-1)} \ \text{ on } \ S_{K-1}, \tag{8.222}$$

$$\gamma^+_{S_{K-1}}\big\{\mathcal{T}^{(K-1)} V^{(K-1)}_{S_{K-2}}(g^{(K-1)}_{K-2})\big\} + \Big(-\frac{1}{2} I_6 + \mathcal{K}^{(K-1)}_{S_{K-1}}\Big) g^{(K-1)}_{K-1} \\ -\Big(\frac{1}{2} I_6 + \mathcal{K}^{(K)}_{S_{K-1}}\Big) g^{(K)}_{K-1} - \gamma^-_{S_{K-1}}\big\{\mathcal{T}^{(K)} V^{(K)}_{S_K}(g^{(K)}_K)\big\} = F^{(K-1)} \ \text{ on } \ S_{K-1}, \tag{8.223}$$

$$\gamma^+_{S_K}\big\{\mathcal{T}^{(K)} V^{(K)}_{S_{K-1}}(g^{(K)}_{K-1})\big\}^+ + \Big(-\frac{1}{2} I_6 + \mathcal{K}^{(K)}_{S_K}\Big) g^{(K)}_K = F^{(K-1)} \ \text{ on } \ S_K, \tag{8.224}$$

where $\gamma^\pm_{S_l}$ denote one-sided traces on $S_l$, the integral operators $\mathcal{H}^{(l)}_{S_l}$ and $\mathcal{K}^{(l)}_{S_l}$ are associated with the single layer potential and are defined in (4.6) and (4.7), respectively.

Let us denote by $\mathbb{T} = [\mathbb{T}_{pq}]_{12(K-1)\times 12(K-1)}$ the matrix pseudodifferential operator generated by the left hand side expressions in equations (8.217)–(8.224), and introduce the notation for $12(K-1)$ dimensional vectors constructed by the sought for unknowns and given right hand sides in the simultaneous equations (8.217)–(8.224):

$$\mathbf{G} := \big(g^{(2)}_1, g^{(2)}_2, g^{(3)}_2, g^{(3)}_3, \dots, g^{(K)}_{K-1}, g^{(K)}_K\big)^\top \in \mathbb{X}_p, \tag{8.225}$$

$$\mathbf{F} := \big(f^{(1)}, f^{(2)}, F^{(2)}, f^{(3)}, F^{(3)}, \dots, f^{(K-1)}, F^{(K-1)}, F^{(K)}\big)^\top \in \mathbb{Y}_p, \tag{8.226}$$

where the structures of the function spaces $\mathbb{X}$ and $\mathbb{Y}$ are determined bythe conditions (8.216) and (8.212), respectively,

$$\mathbb{X}_p := \big[B^{-\frac{1}{p}}_{p,p}(S_1)\big]^6 \times \big[B^{-\frac{1}{p}}_{p,p}(S_2)\big]^6 \times \big[B^{-\frac{1}{p}}_{p,p}(S_2)\big]^6 \times \big[B^{-\frac{1}{p}}_{p,p}(S_3)\big]^6 \times \big[B^{-\frac{1}{p}}_{p,p}(S_3)\big]^6 \times \cdots \\ \times \big[B^{-\frac{1}{p}}_{p,p}(S_{K-1})\big]^6 \times \big[B^{-\frac{1}{p}}_{p,p}(S_{K-1})\big]^6 \times \big[B^{-\frac{1}{p}}_{p,p}(S_K)\big]^6, \tag{8.227}$$

$$\mathbb{Y}_p := \big[B^{1-\frac{1}{p}}_{p,p}(S_1)\big]^6 \times \big[B^{1-\frac{1}{p}}_{p,p}(S_2)\big]^6 \times \big[B^{-\frac{1}{p}}_{p,p}(S_2)\big]^6 \times \big[B^{1-\frac{1}{p}}_{p,p}(S_3)\big]^6 \times \big[B^{-\frac{1}{p}}_{p,p}(S_3)\big]^6 \times \cdots \\ \times \big[B^{1-\frac{1}{p}}_{p,p}(S_{K-1})\big]^6 \times \big[B^{-\frac{1}{p}}_{p,p}(S_{K-1})\big]^6 \times \big[B^{-\frac{1}{p}}_{p,p}(S_K)\big]^6. \tag{8.228}$$

The system of equations (8.217)–(8.224) can be rewritten then as

$$\mathbb{T}\mathbf{G} = \mathbf{F}, \tag{8.229}$$

where the operator

$$\mathbb{T} : \mathbb{X}_p \longrightarrow \mathbb{Y}_p \tag{8.230}$$

is continuous due to the mapping properties of the operators $\mathcal{H}^{(l)}_{S_l}$ and $\mathcal{K}^{(l)}_{S_l}$ (see Theorem 4.4).

Let us remark that the integral operators generated by the summands containing the trace operators $\gamma^\pm_{S_l}$ are compact. Therefore from (8.217)–(8.224) it follows that the "principal part" of the operator $\mathbb{T}$ is blockwise diagonal matrix operator and reads as follows:

$$\widetilde{\mathbb{T}} := \operatorname{diag}\big[\widetilde{\mathbb{T}}^{(1)}, \widetilde{\mathbb{T}}^{(2)}, \dots, \widetilde{\mathbb{T}}^{(K-1)}, \widetilde{\mathbb{T}}^{(K)}\big], \tag{8.231}$$

where

$$\widetilde{\mathbb{T}}^{(1)} = \big[\widetilde{\mathbb{T}}^{(1)}_{pq}\big]_{6\times 6} := \mathcal{H}^{(2)}_{S_1}, \tag{8.232}$$

$$\widetilde{\mathbb{T}}^{(2)} = \big[\widetilde{\mathbb{T}}^{(2)}_{pq}\big]_{12\times 12} := \begin{bmatrix} \mathcal{H}^{(2)}_{S_2} & -\mathcal{H}^{(3)}_{S_2} \\ -\frac{1}{2}\, I_6 + \mathcal{K}^{(2)}_{S_2} & -\frac{1}{2}\, I_6 - \mathcal{K}^{(2)}_{S_3} \end{bmatrix}, \tag{8.233}$$

$$\widetilde{\mathbb{T}}^{(3)} = \big[\widetilde{\mathbb{T}}^{(3)}_{pq}\big]_{12\times 12} := \begin{bmatrix} \mathcal{H}^{(3)}_{S_3} & -\mathcal{H}^{(4)}_{S_3} \\ -\dfrac{1}{2}\,I_6 + \mathcal{K}^{(3)}_{S_3} & -\dfrac{1}{2}\,I_6 - \mathcal{K}^{(4)}_{S_3} \end{bmatrix}, \tag{8.234}$$

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$$\widetilde{\mathbb{T}}^{(K-1)} = \big[\widetilde{\mathbb{T}}^{(K-1)}_{pq}\big]_{12\times 12} := \begin{bmatrix} \mathcal{H}^{(K-1)}_{S_{K-1}} & -\mathcal{H}^{(K)}_{S_{K-1}} \\ -\dfrac{1}{2}\,I_6 + \mathcal{K}^{(K-1)}_{S_{K-1}} & -\dfrac{1}{2}\,I_6 - \mathcal{K}^{(K)}_{S_{K-1}} \end{bmatrix}, \tag{8.235}$$

$$\widetilde{\mathbb{T}}^{(K)} = \big[\widetilde{\mathbb{T}}^{(K)}_{pq}\big]_{6\times 6} := -\frac{1}{2}\,I_6 + \mathcal{K}^{(K)}_{S_K}. \tag{8.236}$$

Note that, the operators $\widetilde{\mathbb{T}}^{(j)}$ for $j = 2, 3, , \dots, K-1$, are counterparts of the elliptic pseudodifferential operator $\mathbb{A}$ defined in (8.58) and their principal homogeneous symbol matrices are non-degenerate. In view of the ellipticity of the symbol matrices $\mathfrak{S}(\mathcal{H}^{(2)}_{S_1})$ and $\mathfrak{S}(\frac{1}{2}I_6 + \mathcal{K}^{(K)}_{S_K})$ we deduce that the principal homogeneous symbol matrix $\mathfrak{S}(\widetilde{\mathbb{T}})$ is elliptic, implying in turn that the pseudodifferential operator $\mathbb{T}$ is elliptic. Evidently, the operators

$$\widetilde{\mathbb{T}}^{(1)} : [B^{-\frac{1}{p}}_{p,p}(S_1)]^6 \longrightarrow [B^{1-\frac{1}{p}}_{p,p}(S_1)]^6, \tag{8.237}$$

$$\widetilde{\mathbb{T}}^{(j)} : [B^{-\frac{1}{p}}_{p,p}(S_j)]^6 \times [B^{-\frac{1}{p}}_{p,p}(S_j)]^6 \longrightarrow [B^{1-\frac{1}{p}}_{p,p}(S_j)]^6 \times [B^{-\frac{1}{p}}_{p,p}(S_j)]^6, \quad j = 2, 3, \dots, K-1, \tag{8.238}$$

$$\widetilde{\mathbb{T}}^{(K)} : [B^{-\frac{1}{p}}_{p,p}(S_K)]^6 \longrightarrow [B^{-\frac{1}{p}}_{p,p}(S_K)]^6, \tag{8.239}$$

are bounded and, consequently, the operator

$$\widetilde{\mathbb{T}} : \mathbb{X}_p \longrightarrow \mathbb{Y}_p \tag{8.240}$$

is continuous. Moreover, the operator

$$\mathbb{T} - \widetilde{\mathbb{T}} : \mathbb{X}_p \longrightarrow \mathbb{Y}_p \tag{8.241}$$

is compact.

By Remark 5.17 and Theorem 8.2 the operators (8.237), (8.238) are invertible, while (8.239) is a Fredholm operator with zero index. Therefore, (8.240) and (8.230) are Fredholm operators with zero index. Now we show that the null space of the operator (8.230) for $p = 2$ is trivial. Indeed, let

$$\widetilde{\mathbf{G}} := \big(\widetilde{g}^{(2)}_1, \widetilde{g}^{(2)}_2, \widetilde{g}^{(3)}_2, \widetilde{g}^{(3)}_3, \dots, \widetilde{g}^{(K)}_{K-1}, \widetilde{g}^{(K)}_K\big)^\top \in \mathbb{X}_2,$$

be a solution to the homogeneous equation $\mathbb{T}\widetilde{\mathbf{G}} = 0$ and construct the vectors

$$\widetilde{U}^{(j)}(x) = V^{(j)}_{S_{j-1}}(\widetilde{g}^{(j)}_{j-1})(x) + V^{(j)}_{S_j}(\widetilde{g}^{(j)}_j)(x), \quad x \in \Omega^{(j)}, \quad j = 2, \dots, K. \tag{8.242}$$

In view of the homogeneous equation $\mathbb{T}\widetilde{\mathbf{G}} = 0$, it follows that the vectors $\widetilde{U}^{(j)}$, $j = 2, \dots, K$, solve the homogeneous transmission problem (GTP-$\mathbf{\Omega}$-DN)$_\tau$, and by Theorem 8.17 they vanish in the corresponding domains. By continuity of the single layer potential and the uniqueness theorem for solutions to the Dirichlet problem, we deduce that all the vectors $\widetilde{U}^{(j)}$ vanish in $\mathbb{R}^3$, implying that $\widetilde{g}^{(j)}_j = 0$ on $S_j$ and $\widetilde{g}^{(j)}_{j-1} = 0$ on $S_{j-1}$ for $j = 2, \dots, K$. Thus the null space of the operator (8.230) for $p = 2$ is trivial which implies invertibility of (8.230) for $p = 2$. But then, due to the general theory of pseudodifferential equations on manifolds without boundary, we conclude the invertibility of (8.230) for arbitrary $p > 1$. Thus we have proved the following lemma.

**Lemma 8.18.** *The pseudodifferential operator* (8.230) *is invertible for arbitrary* $p > 1$.

Now, we can prove the following existence results.

**Theorem 8.19.** *Let conditions* (8.212) *be satisfied with* $p > 1$. *Then the transmission problem* (GTP-$\mathbf{\Omega}$-DN)$_\tau$, (8.206)–(8.212), *is uniquely solvable in the space*

$$\mathbb{W}^1_p := [W^1_p(\Omega^{(2)})]^6 \times [W^1_p(\Omega^{(3)})]^6 \times \cdots \times [W^1_p(\Omega^{(K)})]^6$$

*and the solution vectors* $U^{(j)}$, $j = 2, 3, , \dots, K$, *are representable in the form of single layer potentials* (8.215), *where the density vectors* $g_{j-1}^{(j)} \in [B_{p,p}^{-\frac{1}{p}}(S_{j-1})]^6$ *and* $g_j^{(j)} \in \big[B_{p,p}^{-\frac{1}{p}}(S_j)\big]^6$ *are defined from the uniquely solvable system of pseudodifferential equations* (8.217)–(8.224).

*Proof.* Existence of a solution in the space $\mathbb{W}_p^1$ follows from Lemma 8.18, while the uniqueness of solution to the problem (GTP-**Ω**-DN)$_\tau$ for arbitrary $p > 1$ can be shown by using exactly the same arguments as in the proof of Theorem 8.3. □

For the regular setting and smooth boundary-transmission data, due to the imbedding theorems and the invertibility of the corresponding integral operators in Hölder continuous spaces, we have the following counterpart of Theorem 8.4.

**Theorem 8.20.** *Let* $S_q \in C^{k+2,\kappa}$ *for* $q = 1, \dots, K$, $f^{(l)} \in [C^{k+1,\kappa'}(S_l)]^6$ *for* $l = 1, \dots, K-1$, *and* $F^{(j)} \in [C^{k,\kappa'}(S)]^6$ *for* $j = 2, \dots, K$, *with a nonnegative integer* $k$ *and* $0 < \kappa' < \kappa \leqslant 1$. *Then the basic transmission problem* (GTP-**Ω**-DN)$_\tau$ *possesses a unique regular solution in the space*

$$[C^{k+1,\kappa'}(\overline{\Omega^{(2)}})]^6 \times [C^{k+1,\kappa'}(\overline{\Omega^{(3)}})]^6 \times \dots \times [C^{k+1,\kappa'}(\overline{\Omega^{(K)}})]^6$$

*representable in the form of single layer potentials* (8.215), *where the density vector-functions* $g_{j-1}^{(j)} \in [C^{k,\kappa'}(S_{j-1})]^6$ *and* $g_j^{(j)} \in [C^{k,\kappa'}(S_j)]^6$ *are defined from the uniquely solvable system of pseudodifferential equations* (8.217)–(8.224).

*Proof.* It is word for word of the proof of Theorem 8.4. □

## 9. Appendix A: Structural Properties of Bounded Solutions in Exterior Domains

Here we prove several technical lemmas.

**Lemma A.1.** *Let* $U = (u_1, u_2, \dots, u_N)^\top$ *be a bounded solution to the homogeneous differential equation*

$$L(\partial)U(x) = 0, \;\; x \in \Omega^-, \tag{A.1}$$

*where* $\Omega^- \subset \mathbb{R}^3$ *is a complement of a bounded region* $\overline{\Omega^+}$ *with a compact boundary* $S = \partial\Omega^+$ *and* $L(\partial) = [L_{kj}(\partial)]_{N\times N}$ *is a strongly elliptic second order matrix differential operator with constant coefficients,*

$$L_{kj}(\partial) = \sum_{p,q=1}^{3} a_{pq}^{kj}\partial_p\partial_q, \;\; k, j = 1, \dots, N.$$

*Then*

$$U(x) = C + \mathcal{O}(|x|^{-1}) \;\; as \;\; |x| \to +\infty, \tag{A.2}$$

*where* $C = (C_1, \dots, C_N)^\top$ *is a constant vector.*

*Proof.* Let $U$ be a bounded solution to equation (A.1) and $B(O,R)$ be a ball centered at the origin and radius $R$, such that $\overline{\Omega^+} \subset B(O,R)$. Clearly, $U \in [C^\infty(\Omega^-)]^N$ due to the ellipticity of the operator $L(\partial)$. Let $V = (v_1, \dots, v_N)^\top \in [C^\infty(\mathbb{R}^3)]^N$ be a vector whose restriction on $\Omega_R^- := \Omega^- \setminus \overline{B(O,R)}$ coincides with $U$, i.e,

$$V(x) = U(x) \;\text{ for }\; x \in \Omega_R^-. \tag{A.3}$$

Due to (A.1) and (A.3) the vector $V$ solves the nonhomogeneous differential equation

$$L(\partial)V(x) = \Phi(x), \;\; x \in \mathbb{R}^3, \tag{A.4}$$

with $\Phi = (\Phi_1, \dots, \Phi_N)^\top \in [C_{comp}^\infty(\mathbb{R}^3)]^N$ having a compact support, $\operatorname{supp}\Phi \subset \overline{B(O,R)}$. Keeping in mind that $V$ is bounded, we can apply the generalized Fourier transform to equation (A.4) to obatin

$$L(-i\xi)\widehat{V}(\xi) = \widehat{\Phi}(\xi), \;\; \xi \in \mathbb{R}^3, \tag{A.5}$$

where $\widehat{V} = \mathcal{F}[V]$ and $\widehat{\Phi} = \mathcal{F}[\Phi] \in C^\infty(\mathbb{R}^3)$. This equation is understood in the sense of tempered distributions. Since $\det L(-i\xi) \neq 0$ for $\xi \neq 0$ and the entries of the inverse matrix $[L(-i\xi)]^{-1}$ are $C^\infty$-smooth homogeneous functions of order $-2$ in $\mathbb{R}^3 \setminus \{0\}$, from (A.5) we conclude

$$\widehat{V}(\xi) = [L(-i\xi)]^{-1}\widehat{\Phi}(\xi) + \sum_{|\alpha|\leq M} C_\alpha \delta^{(\alpha)}(\xi), \tag{A.6}$$

where $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ is a multi-index, $C_\alpha = (C_{\alpha,1}, \dots, C_{\alpha,N})^\top$ are arbitrary constant vectors, $M$ is a nonnegative integer, $\delta(\,\cdot\,)$ is Dirac's distribution and $\delta^{(\alpha)} = \partial^\alpha \delta$.

By applying the inverse Fourier transform to (A.6) we get

$$V(x) = \mathcal{F}^{-1}_{\xi\to x}\big([L(-i\xi)]^{-1}\widehat{\Phi}(\xi)\big) + \sum_{|\alpha|\le M} C_\alpha x^\alpha. \tag{A.7}$$

Denote by $\Gamma_L(x)$ the fundamental matrix of the operator $L(\partial)$ whose entries are homogeneous functions of order $-1$,

$$\Gamma_L(x) := \mathcal{F}^{-1}_{\xi\to x}\big([L(-i\xi)]^{-1}\big), \ \ \Gamma_L \in C^\infty(\mathbb{R}^3 \setminus \{0\}), \ \ L(\partial)\Gamma_L(x) = \delta(x) I_N. \tag{A.8}$$

Then (A.7) can be rewritten as follows

$$V(x) = \mathcal{F}^{-1}_{\xi\to x}\mathcal{F}(\Gamma_L * \Phi) + \sum_{|\alpha|\le M} C_\alpha x^\alpha = (\Gamma_L * \Phi)(x) + \sum_{|\alpha|\le M} C_\alpha x^\alpha, \tag{A.9}$$

where $*$ denotes the convolution operator. Therefore,

$$V(x) = \int_{\mathbb{R}^3} \Gamma_L(x-y)\Phi(y)\,dy + \sum_{|\alpha|\le M} C_\alpha x^\alpha. \tag{A.10}$$

Since $\operatorname{supp}\Phi \subset \overline{B(O,R)}$ is compact, the first summand in the right hand side in (A.10) decays at infinity as $\mathcal{O}(|x|^{-1})$. Then it follows that $C_\alpha = 0$ for $|\alpha| \ge 1$ due to boundedness of $V$ at infinity. Finally, we get

$$V(x) = \int_{B(O,R)} \Gamma_L(x-y)\Phi(y)\,dy + C = C + \mathcal{O}(|x|^{-1}), \tag{A.11}$$

where $C = C_{(0,\dots,0)} =: (C_1, \dots, C_N)^\top$ is an arbitrary constant vector. □

**Lemma A.2.** *Let $L(\partial)$ be as in Lemma* A.1 *and $P = (P_1, P_2, \dots, P_N)^\top \in [C^\infty(\mathbb{R}^3 \setminus \{0\})]^N$ be an odd homogeneous vector function of order $-2$. Then the equation*

$$L(\partial)U(x) = P(x), \ \ x \in \mathbb{R}^3 \setminus \{0\}, \tag{A.12}$$

*has a unique homogeneous solution $U^{(0)} \in [C^\infty(\mathbb{R}^3 \setminus \{0\})]^N$ of zero order satisfying the condition*

$$\int_{|x|=1} U^{(0)}(x)\,dS = 0. \tag{A.13}$$

*Proof.* From (A.12) by the Fourier transform we get

$$L(-i\xi)\widehat{U}(\xi) = \widehat{P}(\xi), \ \ x \in \mathbb{R}^3, \tag{A.14}$$

where $\widehat{P}(\xi)$ is an odd homogeneous vector function of order $-1$, $\det L(-i\xi) \ne 0$ for $\xi \ne 0$ and the entries of the inverse matrix $[L(-i\xi)]^{-1}$ are even, $C^\infty$-smooth homogeneous functions of order $-2$.

The equation (A.14) is understood in the sense of the space of tempered distributions and applying the same arguments as in the proof of Lemma A.1 we arrive at the relation

$$\widehat{U}(\xi) = [L(-i\xi)]^{-1}\widehat{P}(\xi) + \sum_{|\alpha|\le M} C_\alpha \delta^{(\alpha)}(\xi), \tag{A.15}$$

where again $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ is a multi-index, $C_\alpha = (C_{\alpha,1}, \dots, C_{\alpha,N})^\top$ are arbitrary constant vectors, and $M$ is a nonnegative integer.

Note that, the first summand in the right hand side is an odd homogeneous function of order $-3$ satisfying the condition

$$\int_{|\xi|=1} [L(-i\xi)]^{-1}\widehat{P}(\xi)\,dS = 0. \tag{A.16}$$

Therefore, we can regularize this summand and consider it in the Cauchy Principal Value (v.p.) sense. Then the corresponding inverse Fourier transform

$$V(x) := \mathcal{F}^{-1}_{\xi\to x}\big(\text{v.p.}[L(-i\xi)]^{-1}\widehat{P}(\xi)\big) \tag{A.17}$$

is a homogeneous vector function of order zero satisfying the condition

$$\int_{|x|=1} V(x)\,dS = 0. \tag{A.18}$$

Moreover, $U \in [C^\infty(\mathbb{R}^3 \setminus \{0\})]^N$ (see, e.g., [80], Assertion 2.13 and Theorem 2.16).

Now, from (A.15) by the inverse Fourier transform we get

$$U(x) = V(x) + \sum_{|\alpha|\le M} C_\alpha x^\alpha.$$

Since $U$ should be a homogeneous vector function of order zero satisfying condition (A.13), we conclude that $C_\alpha = 0$ for all $\alpha$ in view of (A.18), and, consequently,

$$U(x) = V(x), \tag{A.19}$$

which completes the proof. □

**Lemma A.3.** *Let $L(\partial)$ be as in Lemma* A.1*, $\Gamma_L(x)$ be the fundamental solution of the operator $L(\partial)$ defined by* (A.8)*, and $Q = (Q_1, Q_2, \dots, Q_N)^\top \in [C^\infty(\mathbb{R}^3)]^N$ with*

$$\partial^\alpha Q_j(x) = \mathcal{O}(|x|^{-3-|\alpha|}) \ \ \text{as} \ \ |x| \to \infty, \ \ j = 1, \dots, N, \tag{A.20}$$

*for any multi-index $\alpha = (\alpha_1, \alpha_2, \alpha_3)$.*

*Then the Newtonian volume potential*

$$V(x) = \int_{\mathbb{R}^3} \Gamma_L(x-y) Q(y)\,dy = (\Gamma_L * Q)(x) \tag{A.21}$$

*is a particular solution of the equation*

$$L(\partial)U(x) = Q(x), \ \ x \in \mathbb{R}^3. \tag{A.22}$$

*Moreover, the embedding $V \in [C^\infty(\mathbb{R}^3)]^N$ holds and*

$$\partial^\alpha V(x) = \mathcal{O}\big(|x|^{-1-|\alpha|} \ln |x|\big) \ \ \text{as} \ \ |x| \to \infty \tag{A.23}$$

*for any multi-index $\alpha = (\alpha_1, \alpha_2, \alpha_3)$.*

*Proof.* The equation $L(\partial)V = Q$ in $\mathbb{R}^3$ and the inclusion $V \in [C^\infty(\mathbb{R}^3)]^N \cap [C^2(\overline{\Omega^-})]^N$ follow from the properties of the Newtonian volume potential and can be shown by standard arguments.

To derive the estimate (A.23) we proceed as follows. Assume that the origin of the coordinate system belongs to the domain $\Omega^+$ and $r = |x|$ is sufficiently large, such that $\overline{\Omega^+} \subset B(O, \frac{r}{2})$. Then $V$ can be represented as

$$\partial_x^\alpha V(x) = \int_{\mathbb{R}^3} \Gamma_L(x-y) \partial_y^\alpha Q(y)\,dy = \sum_{k=1}^{4} \mathcal{I}_k(x) \tag{A.24}$$

with

$$\mathcal{I}_k(x) := \int_{\Omega_k} \Gamma_L(x-y) \partial_y^\alpha Q(y)\,dy, \ \ k = 1, 2, 3, 4, \tag{A.25}$$

where $\mathbb{R}^3 = \Omega_1 \cup \Omega_2 \cup \Omega_3 \cup \Omega_4$ and

$$\Omega_1 = B(O, \frac{r}{2}), \quad \Omega_2 = B(x, \frac{r}{2}), \quad \Omega_3 = \mathbb{R}^3 \setminus B(O, \frac{3r}{2}), \quad \Omega_4 = B\Big(O, \frac{3r}{2}\Big) \setminus \Big[B\Big(x, \frac{r}{2}\Big) \cup B\Big(x, \frac{r}{2}\Big)\Big].$$

From the assumptions of the lemma and relation (A.20) it follows that for arbitrary multi-index $\alpha$ there is a positive constant $C_1$ such that

$$|\partial^\alpha Q(x)| \leqslant \frac{C_1}{(1+|x|)^{3+|\alpha|}} \ \ \text{for all} \ \ x \in \mathbb{R}^3. \tag{A.26}$$

For arbitrary multi-index $\alpha$ we have also the following relation

$$|\partial^\alpha \Gamma_L(x-y)| \leqslant \frac{C_2}{|x-y|^{-1-|\alpha|}} \ \ \text{for all} \ \ x, y \in \mathbb{R}^3, \ \ x \neq y, \tag{A.27}$$

with a positive constant $C_2$.

First, we estimate $\mathcal{I}_1(x)$. Note that if $y\in\Omega_1$, then $|x-y|\geqslant\frac{|x|}{2}$. Applying the Gauss divergence theorem $|\alpha|$ times and relation (A.26), we easily derive

$$\mathcal{I}_1(x)=\int_{\Omega_1}\Gamma_L(x-y)\partial_y^\alpha Q(y)\,dy=\int_{\Omega_1}\partial_y^\alpha\Gamma_L(x-y)Q(y)\,dy+\mathcal{O}(r^{-1-|\alpha|}). \tag{A.28}$$

Further, we have

$$\begin{aligned}\Big|\int_{\Omega_1}\partial_y^\alpha\Gamma_L(x-y)Q(y)\,dy\Big|&\leqslant C_3\int_{\Omega_1}\frac{1}{|x-y|^{1+|\alpha|}}\,\frac{1}{(1+|y|)^3}\,dy\\&\leqslant C_4\,\frac{1}{|x|^{1+|\alpha|}}\int_{\Omega_1}\frac{1}{(1+|y|)^3}\,dy\leqslant C_5\,\frac{\ln|x|}{|x|^{1+|\alpha|}}\,,\end{aligned}$$

where $C_j$, $j=3,4,5$, are some positive constants. Consequently,

$$\mathcal{I}_1(x)=\mathcal{O}\big(|x|^{-1-|\alpha|}\ln|x|\big). \tag{A.29}$$

If $y\in\Omega_2$, then $|y|\geqslant\frac{|x|}{2}$, and we have

$$|\mathcal{I}_2(x)|=\Big|\int_{\Omega_2}\Gamma_L(x-y)\partial_y^\alpha Q(y)\,dy\Big|\leqslant C_6\int_{\Omega_2}\frac{1}{|x-y|}\,\frac{1}{(1+|y|)^{3+|\alpha|}}\,dy=\mathcal{O}(|x|^{-1-|\alpha|}). \tag{A.30}$$

Now, if $y\in\Omega_3$, then $|y|\geqslant\frac{3|x|}{2}$ and $|x-y|\geqslant\frac{|y|}{3}$, and we get

$$|\mathcal{I}_3(x)|=\Big|\int_{\Omega_3}\Gamma_L(x-y)\partial_y^\alpha Q(y)\,dy\Big|\leqslant C_7\int_{\Omega_2}\frac{1}{|x-y|}\,\frac{1}{(1+|y|)^{3+|\alpha|}}\,dy=\mathcal{O}(|x|^{-1-|\alpha|}). \tag{A.31}$$

And finally, if $y\in\Omega_4$, then $|x-y|\geqslant\frac{|x|}{2}$ and $|y|\geqslant\frac{|x|}{2}$, and we obtain

$$|\mathcal{I}_4(x)|=\Big|\int_{\Omega_4}\Gamma_L(x-y)\partial_y^\alpha Q(y)\,dy\Big|\leqslant C_7\int_{\Omega_2}\frac{1}{|x-y|}\,\frac{1}{(1+|y|)^{3+|\alpha|}}\,dy=\mathcal{O}(|x|^{-1-|\alpha|}), \tag{A.32}$$

which completes the proof. □

From the above lemma the following assertions follow directly.

**Corollary A.4.** *Let $L(\partial)$ be as in Lemma* A.1, *$\Gamma_L(x)$ be the fundamental solution of the operator $L(\partial)$ defined by* (A.8), *and $Q=(Q_1,Q_2,\dots,Q_N)^\top\in[C^\infty(\Omega^-)]^N$ with*

$$\partial^\alpha Q_j(x)=\mathcal{O}(|x|^{-3-|\alpha|})\ \text{ as }\ |x|\to\infty,\ \ j=1,\dots,N,$$

*for any multi-index $\alpha=(\alpha_1,\alpha_2,\alpha_3)$.*

*Then the Newtonian volume potential*

$$V(x)=\int_{\Omega^-}\Gamma_L(x-y)Q(y)\,dy \tag{A.33}$$

*is a particular solution of the equation*

$$L(\partial)U(x)=Q(x),\ \ x\in\Omega^-. \tag{A.34}$$

*Moreover, the embedding $V\in[C^\infty(\Omega^-)]^N$ holds and*

$$\partial^\alpha V(x)=\mathcal{O}\big(|x|^{-1-|\alpha|}\ln|x|\big)\ \text{ as }\ |x|\to\infty \tag{A.35}$$

*for any multi-index $\alpha=(\alpha_1,\alpha_2,\alpha_3)$.*

*Proof.* Let $R_0$ be a positive number such that $\overline{\Omega^+}:=\mathbb{R}^3\setminus\Omega^-\subset B(O,R_0)$ and $\chi(x)$ be a $C^\infty$ regular function with properties $\chi(x)=1$ for $|x|\geqslant 2R_0$ and $\chi(x)=0$ for $|x|\leqslant R_0$. Represent $Q(x)$ as

$$Q(x)=Q_1(x)+Q_2(x)\ \text{ with }\ Q_1(x):=[1-\chi(x)]Q(x),\ \ Q_2(x):=\chi(x)Q(x).$$

Evidently, $Q_1$ has a compact support and $Q_2$, extended by zero onto $\overline{\Omega^+}$, is $C^\infty$ regular in $\mathbb{R}^3$ and satisfies the condition (A.26). Then the proof follows from the representation

$$V(x)=\int_{\Omega^-\cap\,\mathrm{supp}(1-\chi)}\Gamma_L(x-y)Q_1(y)\,dy+\int_{\mathbb{R}^3}\Gamma_L(x-y)Q_2(y)\,dy \tag{A.36}$$

and Lemma A.3. □

**Corollary A.5.** *Let $L(\partial)$, $\Omega^-$, $P$, and $Q$ be as in Lemmas* A.1–A.3 *and Corollary* A.4 *and let $\Phi \in [L_{2,comp}(\Omega^-)]^N$. Further, let $U \in [W^1_{2,loc}(\Omega^-)]^N$ be a solution of the equation*

$$L(\partial)U(x) = P(x) + Q(x) + \Phi(x), \quad x \in \Omega^- \tag{A.37}$$

*satisfying the condition $U(x) = \mathcal{O}(1)$ as $|x| \to \infty$.*

*Then $U$ can be represented as*

$$U(x) = C + U^{(0)}(x) + V^{(0)}(x),$$

*where $C = (C_1, \dots, C_N)^\top$ is a constant vector, $U^{(0)}$ is given by* (A.17) *and*

$$V^{(0)} \in [W^1_{2,loc}(\Omega^-)]^N \cap [C^\infty(\mathbb{R}^3 \setminus \operatorname{supp}\Phi)]^N$$

*possesses the following asymptotic property at infinity*

$$\partial^\alpha V^{(0)}(x) = \mathcal{O}\big(|x|^{-1-|\alpha|} \ln|x|\big) \;\; as \;\; |x| \to \infty$$

*for arbitrary multi-index $\alpha = (\alpha_1, \alpha_2, \alpha_3)$.*

*Proof.* Let $\Gamma_L(x)$ be the fundamental matrix of the operator $L(\partial)$ defined by (A.8). Note that, the Newtonian potential

$$N_{\Omega^-}(\Phi)(x) := \int_{\Omega^-} \Gamma_L(x-y)\Phi(y)\,dy = \int_{\Omega^- \cap \operatorname{supp}\Phi} \Gamma_L(x-y)\Phi(y)\,dy$$

belongs to $[W^2_{2,loc}(\Omega^-)]^N \cap [C^\infty(\mathbb{R}^3 \setminus \operatorname{supp}\Phi)]^N$, solves the equation $L(\partial)N_{\Omega^-}(\Phi) = \Phi$ in $\Omega^-$, and at infinity has the property $\partial^\alpha N_{\Omega^-}(\Phi)(x) = \mathcal{O}(|x|^{-1-|\alpha|})$ as $|x| \to \infty$ for arbitrary multi-index $\alpha = (\alpha_1, \alpha_2, \alpha_3)$. Then it is evident that the vector

$$U^*(x) := U^{(0)}(x) + N_{\Omega^-}(Q)(x) + N_{\Omega^-}(\Phi)(x)$$

is bounded at infinity and solves the nonhomogeneous equation (A.37) due to Lemmas A.2–A.3. Now, the proof follows from Lemma A.1. □

## 10. Appendix B: Fredholm Properties of Strongly Elliptic Pseudodifferential Operators on Manifolds with Boundary

Here we recall some results from the theory of strongly elliptic pseudodifferential equations on manifolds with boundary, in both Bessel potential and Besov spaces. These are the main tools for proving existence theorems for mixed boundary value, boundary–transmission and crack type problems using the potential method. They can be found e.g. in [35], [44], [101].

Let $\overline{\mathcal{M}} \in C^\infty$ be a compact, $n$–dimensional, non-self-intersecting manifold with boundary $\partial\mathcal{M} \in C^\infty$, and let $\mathbf{A}$ be a strongly elliptic $N \times N$ matrix pseudodifferential operator of order $\nu \in \mathbb{R}$ on $\overline{\mathcal{M}}$. Denote by $\mathfrak{S}(\mathbf{A}; x, \xi)$ the principal homogeneous symbol matrix of the operator $\mathbf{A}$ in some local coordinate system ($x \in \overline{\mathcal{M}}, \xi \in \mathbb{R}^n \setminus \{0\}$).

Let $\lambda_1(x), \dots, \lambda_N(x)$ be the eigenvalues of the matrix

$$\big[\mathfrak{S}(\mathbf{A}; x, 0, \dots, 0, +1)\big]^{-1}\big[\mathfrak{S}(\mathbf{A}; x, 0, \dots, 0, -1)\big], \quad x \in \partial\mathcal{M}.$$

Introduce the notation

$$\delta_j(x) = \operatorname{Re}\big[(2\pi i)^{-1} \ln \lambda_j(x)\big], \quad j = 1, \dots, N,$$

where $\ln\zeta$ denotes the branch of the logarithm analytic in the complex plane cut along $(-\infty, 0]$. Due to the strong ellipticity of $\mathbf{A}$ we have the strict inequality

$$-1/2 < \delta_j(x) < 1/2 \;\text{ for }\; x \in \overline{\mathcal{M}}, \;\; j = 1, \dots, N.$$

The numbers $\delta_j(x)$ do not depend on the choice of the local coordinate system. Note that, in particular cases, when $\mathfrak{S}(\mathbf{A}; x, \xi)$ is an even matrix function in $\xi$ or $\mathfrak{S}(\mathbf{A}; x, \xi)$ is a positive definite matrix for every $x \in \overline{\mathcal{M}}$ and $\xi \in \mathbb{R}^n \setminus \{0\}$, we have $\delta_j(x) = 0$ for $j = 1, \dots, N$, since all the eigenvalues $\lambda_j(x)$ ($j = 1, \dots, N$) are positive numbers for any $x \in \overline{\mathcal{M}}$.

The Fredholm properties of strongly elliptic pseudodifferential operators on manifolds with boundary are characterized by the following theorem.

**Theorem B.1.** *Let $s \in \mathbb{R}$, $1 < p < \infty$, $1 \le t \le \infty$, and let $\mathbf{A}$ be a strongly elliptic pseudodifferential operator of order $\nu \in \mathbb{R}$, that is, there is a positive constant $c_0$such that*

$$\operatorname{Re}\mathfrak{S}(\mathcal{A}; x, \xi)\eta \cdot \eta \ge c_0|\eta|^2$$

*for $x \in \overline{\mathcal{M}}$, $\xi \in \mathbb{R}^n$ with $|\xi| = 1$, and $\eta \in \mathbb{C}^N$.*

*Then the operators*

$$\mathbf{A} : [\widetilde{H}^s_p(\mathcal{M})]^N \longrightarrow [H^{s-\nu}_p(\mathcal{M})]^N, \tag{B.1}$$

$$\mathbf{A} : [\widetilde{B}^s_{p,t}(\mathcal{M})]^N \longrightarrow [B^{s-\nu}_{p,t}(\mathcal{M})]^N, \tag{B.2}$$

*are Fredholm with zero index if*

$$\frac{1}{p} - 1 + \sup_{x\in\partial\mathcal{M},\, 1\le j\le N} \delta_j(x) < s - \frac{\nu}{2} < \frac{1}{p} + \inf_{x\in\partial\mathcal{M},\, 1\le j\le N} \delta_j(x). \tag{B.3}$$

*Moreover, the null–spaces and indices of the operators* (B.1) *and* (B.2) *are the same* (*for all values of the parameter $t \in [1, +\infty]$*) *provided $p$ and $s$ satisfy the inequality*(B.3)*.*

## 11. Appendix C: Explicit Expressions for Symbol Matrices

Here we present the explicit expressions for the homogeneous principal symbol matrices of the pseudodifferential operators introduced in the main body of the monograph, in Section 4. With the help of the results presented in Subsection 3.1 and, in particular, using formulas (3.7) and (3.8), we can derive the following formulas for the principal homogeneous symbol matrices of the pseudodifferential operators generated by the layer potentials:

$$\mathfrak{S}(\mathcal{H}; x, \xi') = H(x, \xi') = [H_{pq}(x, \xi')]_{6\times 6} = \begin{bmatrix} [H_{kj}(x, \xi')]_{5\times 5} & [0]_{5\times 1} \\ [0]_{1\times 5} & H_{66}(x, \xi') \end{bmatrix}_{6\times 6}$$

$$:= -\frac{1}{2\pi}\int_{\ell_\pm} [A^{(0)}(B\xi)]^{-1}\, d\xi_3 = -\frac{1}{2\pi}\int_{-\infty}^{+\infty} [A^{(0)}(B\xi)]^{-1}\, d\xi_3, \tag{C.1}$$

$$\mathfrak{S}(\pm 2^{-1} I_6 + \mathcal{K}; x, \xi') = K^{(\pm)}(x, \xi') = [K^{(\pm)}_{pq}(x, \xi')]_{6\times 6} = \begin{bmatrix} [K^{(\pm)}_{kj}(x, \xi')]_{5\times 5} & [0]_{5\times 1} \\ [0]_{1\times 5} & \pm 2^{-1} \end{bmatrix}_{6\times 6}$$

$$:= \frac{i}{2\pi}\int_{\ell_\mp} \mathcal{T}^{(0)}(B\xi, n)[A^{(0)}(B\xi)]^{-1}\, d\xi_3, \tag{C.2}$$

$$\mathfrak{S}(\pm 2^{-1} I_6 + \mathcal{N}; x, \xi') = N^{(\pm)} x, \xi') = [N^{(\pm)}_{pq}(x, \xi')]_{6\times 6} = \begin{bmatrix} [N^{(\pm)}_{kj}(x, \xi')]_{5\times 5} & [0]_{5\times 1} \\ [0]_{1\times 5} & \pm 2^{-1} \end{bmatrix}_{6\times 6}$$

$$:= -\frac{i}{2\pi}\int_{\ell_\pm} [A^{(0)}(B\xi)]^{-1}[\mathcal{P}^{(0)}(B\xi, n)]^\top\, d\xi_3, \tag{C.3}$$

$$\mathfrak{S}(\mathcal{L}; x, \xi') = L(x, \xi') = [L_{pq}(x, \xi')]_{6\times 6} = \begin{bmatrix} [L_{kj}(x, \xi_1, \xi_2)]_{5\times 5} & [0]_{5\times 1} \\ [0]_{1\times 5} & L_{66}(x, \xi_1, \xi_2) \end{bmatrix}_{6\times 6}$$

$$:= -\frac{1}{2\pi}\int_{\ell_\pm} \mathcal{T}^{(0)}(B\xi, n)[A^{(0)}(B\xi)]^{-1}[\mathcal{P}^{(0)}(B\xi, n)]^\top\, d\xi_3, \tag{C.4}$$

where $\xi' = (\xi_1, \xi_2) \in \mathbb{R}^2 \setminus \{0\}$, $\xi = (\xi_1, \xi_2, \xi_3)^\top$, the matrices $A^{(0)}(\,\cdot\,)$, $\mathcal{T}^{(0)}(\,\cdot\,,\,\cdot\,)$ and $\mathcal{P}^{(0)}(\,\cdot\,,\,\cdot\,)$ are defined by (2.47), (2.59), and (2.60), respectively,

$$B(x) = \begin{bmatrix} l_1(x) & m_1(x) & n_1(x) \\ l_2(x) & m_2(x) & n_2(x) \\ l_3(x) & m_3(x) & n_3(x) \end{bmatrix} \tag{C.5}$$

is an orthogonal matrix with $\det B(x) = 1$ for $x \in \partial\Omega^\pm = S$; here $n(x)$ is the exterior unit normal vector to $S$, while $l(x)$ and $m(x)$ are orthogonal unit vectors in the tangential plane associated with some local chart; $\ell_-$ ($\ell_+$) is a closed contours in the lower (upper) complex $\xi_3 = \xi_3' + i\xi_3''$ half-plane, orientated clockwise (counterclockwise) and enclosing all roots with negative (positive) imaginary parts of the equation $\det A^{(0)}(B\xi) = 0$ with respect to $\xi_3$, while $\xi' = (\xi_1, \xi_2) \in \mathbb{R}^2 \setminus \{0\}$ is to be considered as a parameter. Let $R > 0$ be sufficiently large positive number, such that the circle $C_R$ centered at the origin and radius $R$, encloses all the roots. Then without loss of generality we can take

$$\ell_- = [-R, +R] \cup C_R^{(-)}, \quad \ell_+ = [-R, +R] \cup C_R^{(+)}, \tag{C.6}$$

where $C_R^{(-)} \subset C_R$ is a semi-circle in the lower half-plane orientated clockwise and $C_R^{(+)} \subset C_R$ is a semi-circle in the upper half-plane orientated counterclockwise.

In (C.2) and (C.3) we employed the fact that $\mathcal{N}_{66}$ and $\mathcal{K}_{66}$ are weakly singular integral operators since their kernel functions, the co-normal derivatives $\eta_{jl} n_j(y) \partial_l \Gamma_{66}(x-y)$ and $\eta_{jl} n_j(x) \partial_l \Gamma_{66}(x-y)$, are weakly singular functions of type $\mathcal{O}(|x-y|^{-2+\kappa})$ on a $C^{1,\kappa}$ smooth surface $S$ with $0 < \kappa \leqslant 1$.

The principal homogeneous symbol matrices (C.1)–(C.4) are elliptic (see Remark 4.13).

The entries of the matrices $H(x, \xi')$ and $L(x, \xi')$ are homogeneous functions in $\xi'$ of order $-1$ and $+1$, respectively, while the entries of the matrices $K^{(\pm)}(x, \xi')$ and $N^{(\pm)}(x, \xi')$ are homogeneous functions in $\xi' = (\xi_1, \xi_2)$ of order 0.

Moreover, the matrices $-\mathfrak{S}(\mathcal{H}; x, \xi_1, \xi_2)$ and $\mathfrak{S}(\mathcal{L}; x, \xi_1, \xi_2)$ are strongly elliptic, i.e., there is a positive constant $c$ depending on the material parameters such that (see Subsections 4.2, formula (4.67), and Subsections 5.5, formula (5.67))

$$\operatorname{Re}\big[-\mathfrak{S}(\mathcal{H}; x, \xi_1, \xi_2)\eta \cdot \eta\big] \geq c|\xi|^{-1}|\eta|^2 \ \text{ for all } \ x \in S, \ \ (\xi_1, \xi_2) \in \mathbb{R}^2 \setminus \{0\}, \ \ \eta \in \mathbb{C}^6, \tag{C.7}$$

$$\operatorname{Re}\big[\mathfrak{S}(\mathcal{L}; x, \xi_1, \xi_2)\eta \cdot \eta\big] \geq c|\xi||\eta|^2 \ \text{ for all } \ x \in S, \ \ (\xi_1, \xi_2) \in \mathbb{R}^2 \setminus \{0\}, \ \ \eta \in \mathbb{C}^6. \tag{C.8}$$

Now, we prove some structural properties of the above introduced symbol matrices.

**Lemma C.1.** *Entries of the symbol matrices $\mathfrak{S}(\mathcal{K}; x, \xi_1, \xi_2)$ and $\mathfrak{S}(\mathcal{N}; x, \xi_1, \xi_2)$ are pure imaginary complex functions, that is,* $\operatorname{Re}\mathfrak{S}(\mathcal{K}; x, \xi_1, \xi_2) = \operatorname{Re}\mathfrak{S}(\mathcal{N}; x, \xi_1, \xi_2) = 0$ *implying*

$$K^{(\pm)}(x, \xi') = \pm\frac{1}{2} I_6 + i\widetilde{K}(x, \xi') \ \text{ and } \ N^{(\pm)}(x, \xi') = \pm\frac{1}{2} I_6 + i\widetilde{N}(x, \xi'), \ \ \xi' = (\xi_1, \xi_2),$$

*where $\widetilde{K}(x, \xi')$ and $\widetilde{N}(x, \xi')$ are matrices with real entries and they read as*

$$\widetilde{K}(x, \xi') := -i\mathfrak{S}(\mathcal{K}; x, \xi') = \begin{bmatrix} [\widetilde{K}_{kj}(x, \xi')]_{5\times 5} & [0]_{5\times 1} \\ [0]_{1\times 5} & 0 \end{bmatrix}_{6\times 6}$$
$$:= \pm\frac{i}{2} I_6 + \frac{1}{2\pi}\int_{\ell_\mp} \mathcal{T}^{(0)}(B\xi, n)[A^{(0)}(B\xi)]^{-1}\, d\xi_3, \tag{C.9}$$

*and*

$$\widetilde{N}(x, \xi') := -i\mathfrak{S}(\mathcal{N}; x, \xi') = \begin{bmatrix} [\widetilde{N}_{kj}(x, \xi')]_{5\times 5} & [0]_{5\times 1} \\ [0]_{1\times 5} & 0 \end{bmatrix}_{6\times 6}$$
$$:= \pm\frac{i}{2} I_6 - \frac{1}{2\pi}\int_{\ell_\pm} [A^{(0)}(B\xi)]^{-1}[\mathcal{P}^{(0)}(B\xi, n)]^\top\, d\xi_3. \tag{C.10}$$

*Proof.* Formulas (C.9) and (C.10) follow from (C.2) and (C.3). From the same formulas (C.2) and (C.3), we get

$$\begin{aligned} 2\mathfrak{S}(\mathcal{K}; x, \xi') &= \mathfrak{S}(2^{-1} I_6 + \mathcal{K}; x, \xi') + \mathfrak{S}(-2^{-1} I_6 + \mathcal{K}; x, \xi') \\ &= \frac{i}{2\pi}\bigg[\int_{\ell_-} \mathcal{T}^{(0)}(B\xi, n)[A^{(0)}(B\xi)]^{-1}\, d\xi_3 + \int_{\ell_+} \mathcal{T}^{(0)}(B\xi, n)[A^{(0)}(B\xi)]^{-1}\, d\xi_3\bigg], \end{aligned} \tag{C.11}$$

$$\begin{aligned} 2\mathfrak{S}(\mathcal{N}; x, \xi') &= \mathfrak{S}(2^{-1} I_6 + \mathcal{N}; x, \xi') + \mathfrak{S}(-2^{-1} I_6 + \mathcal{N}; x, \xi') \\ &= -\frac{i}{2\pi}\bigg[\int_{\ell_+} [A^{(0)}(B\xi)]^{-1}[\mathcal{P}^{(0)}(B\xi, n)]^\top\, d\xi_3 + \int_{\ell_-} [A^{(0)}(B\xi)]^{-1}[\mathcal{P}^{(0)}(B\xi, n)]^\top\, d\xi_3\bigg], \end{aligned} \tag{C.12}$$

implying

$$2\widetilde{K}(x, \xi') = \frac{1}{2\pi}\int_{\ell_-} \mathcal{T}^{(0)}(B\xi, n)[A^{(0)}(B\xi)]^{-1}\, d\xi_3 + \frac{1}{2\pi}\int_{\ell_+} \mathcal{T}^{(0)}(B\xi, n)[A^{(0)}(B\xi)]^{-1}\, d\xi_3, \tag{C.13}$$

$$2\widetilde{N}(x,\xi') = -\frac{1}{2\pi}\int_{\ell_-}[A^{(0)}(B\xi)]^{-1}[\mathcal{P}^{(0)}(B\xi,n)]^\top\,d\xi_3 - \frac{1}{2\pi}\int_{\ell_+}[A^{(0)}(B\xi)]^{-1}[\mathcal{P}^{(0)}(B\xi,n)]^\top\,d\xi_3, \quad (C.14)$$

The first and the second summands in both formulas (C.13) and (C.14) are mutually complex conjugate quantities,

$$\int_{\ell_-}\mathcal{T}^{(0)}(B\xi,n)[A^{(0)}(B\xi)]^{-1}\,d\xi_3 = \overline{\int_{\ell_+}\mathcal{T}^{(0)}(B\xi,n)[A^{(0)}(B\xi)]^{-1}\,d\xi_3},$$

$$\int_{\ell_-}[A^{(0)}(B\xi)]^{-1}[\mathcal{P}^{(0)}(B\xi,n)]^\top\,d\xi_3 = \overline{\int_{\ell_+}[A^{(0)}(B\xi)]^{-1}[\mathcal{P}^{(0)}(B\xi,n)]^\top\,d\xi_3}.$$

Therefore $\widetilde{K}(x,\xi')$ and $\widetilde{N}(x,\xi')$ are matrices with real-valued entries. □

**Lemma C.2.** *Entries of the matrices $\mathfrak{S}(\mathcal{K};x,\xi_1,\xi_2)$ and $\mathfrak{S}(\mathcal{N};x,\xi_1,\xi_2)$ are odd functions in $\xi'=(\xi_1,\xi_2)$, while the entries of $\mathfrak{S}(\mathcal{H};x,\xi_1,\xi_2)$ and $\mathfrak{S}(\mathcal{L};x,\xi_1,\xi_2)$ are real-valued even functions in $\xi'=(\xi_1,\xi_2)$.*

*Proof.* The claim of the lemma for the matrix $\mathfrak{S}(\mathcal{H};x,\xi_1,\xi_2)$ follows from (C.1) and (2.47).

To prove the claim of the lemma for the matrices $\mathfrak{S}(\mathcal{K};x,\xi_1,\xi_2)$ and $\mathfrak{S}(\mathcal{N};x,\xi_1,\xi_2)$, let us write formulas (C.11) and (C.12) for $-\xi'=(-\xi_1,-\xi_2)$ and perform the transformation of the integration variable $\xi_3=-\eta_3$. In view of (C.6), by this substitution the integration line $\ell_-$ will be transformed into the line $\ell_+$ preserving the clockwise orientation, and the line $\ell_+$ will be transformed into the line $\ell_-$ preserving the counterclockwise orientation. Therefore the expressions for the symbol matrices $\mathfrak{S}(\mathcal{K};x,-\xi')$ and $\mathfrak{S}(\mathcal{K};x,-\xi')$ have the same form (C.11) and (C.12), respectively, but with opposite orientation for the integration lines. Consequently,

$$\mathfrak{S}(\mathcal{K};x,-\xi') = -\mathfrak{S}(\mathcal{K};x,\xi'), \quad \mathfrak{S}(\mathcal{N};x,-\xi') = -\mathfrak{S}(\mathcal{N};x,\xi').$$

Further, we apply the first relation in (4.30),

$$\mathcal{H}\mathcal{L} = -\frac{1}{4}\,I_6 + \mathcal{N}^2, \quad (C.15)$$

which leads to the following equality for the corresponding principal homogeneous symbol matrices

$$\mathfrak{S}(\mathcal{H};x,\xi')\mathfrak{S}(\mathcal{L};x,\xi') = -\frac{1}{4}\,I_6 + \big[\mathfrak{S}(\mathcal{N};x,\xi')\big]^2, \quad (C.16)$$

implying that the entries of the symbol matrix $\mathfrak{S}(\mathcal{L};x,\xi')$ are real-valued even functions in $\xi'$. This completes the proof. □

*Remark* C.3. From the relations (C.1), (C.4), (C.7), and (C.15) it follows that for all $x\in S$ and $(\xi_1,\xi_2)\in\mathbb{R}^2\setminus\{0\}$,

$$-H_{66}(x,\xi_1,\xi_2)>0, \quad \mathfrak{S}_{66}(\mathcal{L};x,\xi') = L_{66}(x,\xi_1,\xi_2) = -\frac{1}{4H_{66}(x,\xi_1,\xi_2)}>0.$$

Further, we present some auxiliary formulas associated with the above symbol matrices.

Due to (C.5), we have

$$B(x)\mathbf{e}^{(3)} = n(x) = \big(n_1(x),n_2(x),n_3(x)\big)^\top, \quad (C.17)$$

with $\mathbf{e}^{(3)}=(0,0,1)^\top$ and the same normal vector $n(x)$ as in (C.5). Consequently, for arbitrary $\xi=(\xi_1,\xi_2,\xi_3)^\top$, we get the representation,

$$B(x)\xi = B(x)\widetilde{\xi} + B(x)\mathbf{e}^{(3)}\xi_3 = B(x)\widetilde{\xi} + \xi_3 n(x) \text{ with } \widetilde{\xi}=(\xi',0)^\top,\ \xi'=(\xi_1,\xi_2)\in\mathbb{R}^2\setminus\{0\}. \quad (C.18)$$

Note also that

$$\mathcal{T}^{(0)}(n,n) = \big[\mathcal{P}^{(0)}(n,n)\big]^\top = A^{(0)}(n), \quad (C.19)$$

due to (2.47), (2.59), and (2.60).

Using these relations we derive

$$\mathcal{T}^{(0)}(B\xi,n) = \xi_3\mathcal{T}^{(0)}(n,n) + \mathcal{T}^{(0)}(B\widetilde{\xi},n) = \xi_3 A^{(0)}(n) + \mathcal{T}^{(0)}(B\widetilde{\xi},n), \quad (C.20)$$

$$\mathcal{P}^{(0)}(B\xi,n) = \xi_3\mathcal{P}^{(0)}(n,n) + \mathcal{P}^{(0)}(B\widetilde{\xi},n) = \xi_3[A^{(0)}(n)]^\top + \mathcal{P}^{(0)}(B\widetilde{\xi},n), \quad (C.21)$$

$$A^{(0)}(B\xi) = \xi_3^2 A^{(0)}(n) + \xi_3 A^{(1)}(x,\xi') + A^{(0)}(B\widetilde{\xi}), \quad (C.22)$$

where the entries of the matrix $A^{(1)}(x,\xi')$ are linear homogeneous functions in $\xi'=(\xi_1,\xi_2)$, as well as the entries of the matrices $\mathcal{T}^{(0)}(B\widetilde{\xi},n)$ and $\mathcal{P}^{(0)}(B\widetilde{\xi},n)$, while the entries of the matrix $A^{(0)}(B\widetilde{\xi})$ are second order homogeneous functions in $\xi'=(\xi_1,\xi_2)$.

From these relations it follows that for sufficiently large $|\xi_3|$ and $\xi'\in\mathbb{R}^2\setminus\{0\}$ the following asymptotic relations hold true:

$$[A^{(0)}(B\xi)]^{-1}=\xi_3^{-2}[A^{(0)}(n)]^{-1}+\mathcal{O}(|\xi_3|^{-3}), \tag{C.23}$$

$$\mathcal{T}^{(0)}(B\xi,n)[A^{(0)}(B\xi)]^{-1}=\xi_3^{-1}I_6+\mathcal{O}(|\xi_3|^{-2}), \tag{C.24}$$

$$[A^{(0)}(B\xi)]^{-1}\big[\mathcal{P}^{(0)}(B\xi,n)\big]^\top=\xi_3^{-1}I_6+\mathcal{O}(|\xi_3|^{-2}). \tag{C.25}$$

These relations imply that

$$\lim_{R\to+\infty}\int_{C_R^{(\pm)}}\mathcal{T}^{(0)}(B\xi,n)\big[A^{(0)}(B\xi)\big]^{-1}=\pm i\pi I_6, \tag{C.26}$$

$$\lim_{R\to+\infty}\int_{C_R^{(\pm)}}\big[A^{(0)}(B\xi)\big]^{-1}\big[\mathcal{P}^{(0)}(B\xi,n)\big]^\top=\pm i\pi I_6. \tag{C.27}$$

Now, from (C.2) and (C.3) with the help of (C.6), (C.26), and (C.26) we deduce

$$\begin{aligned}
&\mathfrak{S}(\pm2^{-1}I_6+\mathcal{K};x,\xi')\\
&=\frac{i}{2\pi}\lim_{R\to+\infty}\bigg[\int_{C^{(\mp)}}\mathcal{T}^{(0)}(B\xi,n)[A^{(0)}(B\xi)]^{-1}\,d\xi_3+\int_{-R}^{R}\mathcal{T}^{(0)}(B\xi,n)[A^{(0)}(B\xi)]^{-1}\,d\xi_3\bigg]\\
&=\pm\frac12\,I_6+\frac{i}{2\pi}\lim_{R\to+\infty}\int_{-R}^{R}\mathcal{T}^{(0)}(B\xi,n)[A^{(0)}(B\xi)]^{-1}\,d\xi_3,
\end{aligned} \tag{C.28}$$

$$\begin{aligned}
&\mathfrak{S}(\pm2^{-1}I_6+\mathcal{N};x,\xi')\\
&=-\frac{i}{2\pi}\lim_{R\to+\infty}\bigg[\int_{C^{(\pm)}}[A^{(0)}(B\xi)]^{-1}[\mathcal{P}^{(0)}(B\xi,n)]^\top\,d\xi_3+\int_{R}^{R}[A^{(0)}(B\xi)]^{-1}[\mathcal{P}^{(0)}(B\xi,n)]^\top\,d\xi_3\bigg]\\
&=\pm\frac12\,I_6-\frac{i}{2\pi}\lim_{R\to+\infty}\int_{R}^{R}[A^{(0)}(B\xi)]^{-1}[\mathcal{P}^{(0)}(B\xi,n)]^\top\,d\xi_3.
\end{aligned} \tag{C.29}$$

These equalities give alternative formulas for calculation of the symbol matrices of the operators $\mathcal{K}$ and $\mathcal{N}$,

$$\mathfrak{S}(\mathcal{K};x,\xi')=i\widetilde{K}(x,\xi'),\quad \mathfrak{S}(\mathcal{N};x,\xi')=i\widetilde{N}(x,\xi'), \tag{C.30}$$

where $\widetilde{K}(x,\xi')$ and $\widetilde{N}(x,\xi')$ are matrices with real entries defined by the relations

$$\widetilde{K}(x,\xi')=\frac{1}{2\pi}\lim_{R\to+\infty}\int_{-R}^{R}\mathcal{T}^{(0)}(B\xi,n)[A^{(0)}(B\xi)]^{-1}\,d\xi_3, \tag{C.31}$$

$$\widetilde{N}(x,\xi')=-\frac{1}{2\pi}\lim_{R\to+\infty}\int_{R}^{R}[A^{(0)}(B\xi)]^{-1}[\mathcal{P}^{(0)}(B\xi,n)]^\top\,d\xi_3. \tag{C.32}$$

Finally, we formulate the following technical lemma.

**Lemma C.4.** *Let* $\mathbf{Q}$ *be the set of all non-singular* $k\times k$ *square matrices with complex-valued entries and having the structure*

$$\begin{bmatrix}[Q_{lj}]_{(k-1)\times(k-1)} & \{0\}_{(k-1)\times1}\\ \{0\}_{1\times(k-1)} & Q_{kk}\end{bmatrix}_{k\times k},\quad k\in\mathbb{N}. \tag{C.33}$$

*If* $X,Y\in\mathbf{Q}$*, then* $XY\in\mathbf{Q}$ *and* $X^{-1}\in\mathbf{Q}$*. Moreover, if in addition* $X=[X_{jl}]_{k\times k}$ *and* $Y=[Y_{jl}]_{k\times k}$ *are strongly elliptic, i.e.*

$$\operatorname{Re}(X\zeta\cdot\zeta)>0,\ \ \operatorname{Re}(Y\zeta\cdot\zeta)>0\ \ \text{for all }\ \zeta\in\mathbb{C}^k\setminus\{0\}, \tag{C.34}$$

*and* $X_{kk}$ *and* $Y_{kk}$ *are real numbers, then at least one eigenvalue of the matrix* $XY$ *is positive.*

*In particular, if* $X_{kk}Y_{kk}=1$*, then* $\lambda=1$ *is an eigenvalue of the matrix* $XY$*.*

*Proof.* The first part of the lemma follows from the structure (C.33) and can be verified trivially.

The second part of the lemma is also trivial and follows from the strong ellipticity property (C.34), since it implies that if $X \in \mathbf{Q}$ and $Y \in \mathbf{Q}$ are strongly elliptic with $X_{kk}$ and $Y_{kk}$ being real numbers, and $B = XY$, then evidently $X_{kk} > 0$, $Y_{kk} > 0$, and, consequently, the number

$$\lambda = B_{kk} = X_{kk} Y_{kk} > 0$$

is a solution of the equation $\det[B - \lambda I] = 0$ and is a positive eigenvalue of the matrix $B = XY$. □

*Remark* C.5. Note that, the principal homogeneous symbol matrices (C.1)–(C.4) are non-singular and have the structure (C.33) and therefore along with their inverse matrices they belong to the class $\mathbf{Q}$.

## 12. Appendix D: Calculation of Some Specific Integrals

**D.1. Proof of formula (3.63).** Here we prove the equality

$$\lim_{\varepsilon\to 0} \int_{\Sigma(x,\varepsilon)} \mathcal{P}(\partial_y, n(y), \overline{\tau}) \Gamma^\top(x-y,\tau)\, dS = I_6, \tag{D.1}$$

where $\Sigma(x,\varepsilon) = \partial B(x,\varepsilon)$ is a sphere of radius $\varepsilon$ centered at the point $x \in \mathbb{R}^3$, $n(y) = (y-x)/\varepsilon$ is the exterior normal vector to $\Sigma(x,\varepsilon)$ at the point $y \in \Sigma(x,\varepsilon)$, boundary operator $\mathcal{P}(\partial_y, n(y), \overline{\tau})$ is given by (2.58), and $\Gamma(x-y,\tau)$ is the fundamental matrix constructed in Subsection 3.2.

Due to Lemma 3.3 we easily deduce that

$$\lim_{\varepsilon\to 0} \int_{\Sigma(x,\varepsilon)} \mathcal{P}(\partial_y, n(y), \overline{\tau}) \Gamma^\top(x-y,\tau)\, dS = \lim_{\varepsilon\to 0} \int_{\Sigma(x,\varepsilon)} \mathcal{P}^{(0)}(\partial_y, n(y)) [\Gamma^{(0)}(x-y)]^\top\, dS \tag{D.2}$$

where the boundary operator $\mathcal{P}^{(0)}(\partial_y, n(y))$ is given by (2.60) and is the principal part of the operator $\mathcal{P}(\partial_y, n(y), \overline{\tau})$, and the matrix $\Gamma^{(0)}(x-y)$, constructed in Subsection 3.1, is the fundamental solution of the differential operator $A^{(0)}(\partial)$ defined in (2.46). Evidently, $\Gamma^{(0)*}(y-x) := [\Gamma^{(0)}(x-y)]^\top$ is then a fundamental matrix of the adjoint operator $A^{(0)*}(\partial) = [\overline{A^{(0)}(-\partial)}]^\top = [A^{(0)}(\partial)]^\top$,

$$A^{(0)*}(\partial_y)\Gamma^{(0)*}(y-x) = \delta(x) I_6, \tag{D.3}$$

where $\delta(\,\cdot\,)$ is Dirac's distribution. For arbitrary test function $U = (U_1, U_2, \dots, U_6)^\top \in [\mathcal{D}(\mathbb{R}^3)]^6$ we then have

$$\big\langle A^{(0)*}(\partial_y)\Gamma^{(0)*}_{(j)}(\,\cdot\,-x), U\big\rangle_{\mathbb{R}^3} = \int_{\mathbb{R}^3} \Gamma^{(0)*}_{(j)}(y-x)\cdot A^{(0)}(\partial_y)U(y)\, dy = U_j(x), \quad \forall\, x \in \mathbb{R}^3, \tag{D.4}$$

where $\Gamma^{(0)*}_{(j)}(y-x)$ is the $j$-th column of the matrix $\Gamma^{(0)*}(y-x)$. From (D.4) we have

$$U_j(x) = \lim_{\varepsilon\to 0} \int_{\mathbb{R}^3_\varepsilon} A^{(0)}(\partial_y)U(y)\cdot \Gamma^{(0)*}_{(j)}(y-x)\, dy, \ \ \forall\, x \in \mathbb{R}^3, \ \ j = 1, 2, \dots, 6, \tag{D.5}$$

where $\mathbb{R}^3_\varepsilon = \mathbb{R}^3 \setminus \overline{B(x,\varepsilon)}$. Taking into account that the support of $U$ is compact and $n(y) = (y-x)/\varepsilon$ is the inward normal vector for the domain $\mathbb{R}^3_\varepsilon$, by Green's formula (2.198) with $\Gamma^{(0)*}_{(j)}(y-x)$ for $U'(y)$, we can rewrite (D.5) as follows

$$\begin{aligned} U_j(x) &= -\lim_{\varepsilon\to 0} \Big[ \int_{\mathbb{R}^3_\varepsilon} \mathcal{E}^{(0)}\big(U, \Gamma^{(0)*}_{(j)}(y-x)\big)\, dy + \int_{\Sigma(x,\varepsilon)} \mathcal{T}(\partial, n, \tau)U \cdot \Gamma^{(0)*}_{(j)}(y-x)\, dS_y \Big] \\ &= -\lim_{\varepsilon\to 0} \int_{\mathbb{R}^3_\varepsilon} \mathcal{E}^{(0)}\big(U, \Gamma^{(0)*}_{(j)}(y-x)\big)\, dy, \end{aligned} \tag{D.6}$$

where $\mathcal{E}^{(0)}(\,\cdot\,,\,\cdot\,)$ is defined in (4.50). Now, applying Green's formula (2.199) and keeping in mind that $A^{(0)*}(\partial_y)\Gamma^{(0)*}_{(j)}(y-x) = 0$ for $x \neq y$, we find from (D.6)

$$\begin{aligned} U_j(x) &= \lim_{\varepsilon\to 0} \int\limits_{\Sigma(x,\varepsilon)} U(y)\cdot \mathcal{P}(\partial, n, \tau)\Gamma^{(0)*}_{(j)}(y-x)\, dS_y \\ &= \lim_{\varepsilon\to 0} \int\limits_{\Sigma(x,\varepsilon)} [U(y) - U(x)]\cdot \mathcal{P}(\partial, n, \tau)\Gamma^{(0)*}_{(j)}(y-x)\, dS_y \end{aligned}$$

$$+ U(x)\cdot \lim_{\varepsilon\to 0}\int_{\Sigma(x,\varepsilon)} \mathcal{P}(\partial,n,\tau)\Gamma^{(0)*}_{(j)}(y-x)\,dS_y$$
$$= U(x)\cdot \lim_{\varepsilon\to 0}\int\limits_{\Sigma(x,\varepsilon)} \mathcal{P}(\partial,n,\tau)\Gamma^{(0)*}_{(j)}(y-x)\,dS_y,\quad j=1,2,\dots,6. \tag{D.7}$$

Evidently, the last formula is equivalent to the relation

$$U(x) = U(x)\lim_{\varepsilon\to 0}\int\limits_{\Sigma(x,\varepsilon)} \mathcal{P}(\partial,n,\tau)\Gamma^{(0)*}(y-x)\,dS_y \ \ \forall\, x\in\mathbb{R}^3 \tag{D.8}$$

implying

$$\lim_{\varepsilon\to 0}\int\limits_{\Sigma(x,\varepsilon)} \mathcal{P}(\partial,n,\tau)\Gamma^{(0)*}(y-x)\,dS_y = I_6 \ \ \forall\, x\in\mathbb{R}^3. \tag{D.9}$$

And finally, since $\Gamma^{(0)*}(y-x) := [\Gamma^{(0)}(x-y)]^\top$, from equations (D.2) and (D.9) we conclude that (D.1) holds true.

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