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An Integrable Flow on a Family of Hilbert Grassmannians

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This article is dedicated to my daughter Rebeka

Abstract. Various researchers have studied examples of infinite-dimensional dynamical systems. In most of the cases, the phase space consisted of a Hilbert or Banach space or a Frechet space of functions. In this article we propose to study a dynamical system, namely the geodesic flow, over more structurally complex manifolds, the tangent bundles of a family of Hilbert Grassmannians. Using the high degree of symmetry of the spaces and the methods of Thimm [9] and Ii and Watanabe [3] we prove that the geodesic flow is integrable. In the process we determine a spectral invariant á la Moser [5] which completely describes the behavior of the geodesics of the Hilbert Grassmannians. As a result we demonstrate the difference in complexity between the various ranked Hilbert Grassmannians.

Contents

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1. Introduction

When attempting to understand the behavior of dynamical systems one can look for invariants of the dynamical system in order to understand for example the topology of the solution set. As a basic example consider the behavior of a point particle P in a Newtonian central gravitational field in a plane. The differential equation describing its behavior would be

$$\frac{d^2\overrightarrow{x}}{dt^2} = -\frac{k\overrightarrow{x}}{||\overrightarrow{x}||^3} \tag{1}$$

where k is a constant and $\overrightarrow{x}$ denotes the vector from the origin of the gravitational well to the point particle P. This system can be realized as a Hamiltonian system with Hamiltonian the total "energy" of the system, namely, $H(\overrightarrow{x}, \dot{\overrightarrow{x}}) = ||\dot{\overrightarrow{x}}||^2 - k/||\overrightarrow{x}||$. The "total energy" is trivially an invariant function of the system. Another invariant function is the "angular momentum" of the system, $A(\overrightarrow{x}, \dot{\overrightarrow{x}}) = x_1\dot{x}_2 - x_2\dot{x}_1$. The intersection of the level sets of $H = c < 0$ and $A \neq 0$ are compact. Arnold's Theorem (see [1]) states that if $(\overrightarrow{x}, \dot{\overrightarrow{x}})$ belongs to such a compact intersection of level sets then the solution of (1) with initial conditions $(\overrightarrow{x}, \dot{\overrightarrow{x}})$ corresponds to a linear trajectory mapped onto a 2-torus.

For a dynamical system which can be realized as a Hamiltonian system over a symplectic manifold M with sufficient invariant independent functions, Arnold's Theorem is a very powerful tool for describing the behavior of the dynamical system. However, Arnold's Theorem does not apply to Hamiltonian systems with infinite degrees of freedom. A more delicate analysis is required for studying such systems.

Some well-known and deeply studied dynamical systems such as the Korteweg-de Vries equation, the Sine-Gordon equation, Schrödinger's equation can be realized as arising from a Hamiltonian for an appropriate Hilbert/Banach/Frechet space of functions over $\mathbb{R}$. In each case, the base manifold is a vector space of functions. For our case our base manifolds will more complex manifolds which will in some sense be the simplest non-trivial manifolds which have an extensive list of properties such as complex structures, Riemannian metrics, well-defined curvature forms, exponential maps, etc.

Few examples of non–trivial integrable Hamiltonian systems existed until Thimm [9] determined an algorithm for demonstrating for a large class of symmetric spaces that their geodesic flow is integrable. Thimm's method is to consider a manifold M for which there exists a so–called momentum map $\mu : TM \to \mathcal{G}$, where $\mathcal{G}$ is a Lie algebra for a Lie group G. Thimm explicitly demonstrates once a Lie algebra is equipped with an inner product, the space of functions $\mathcal{C}^\infty(\mathcal{G})$ admits Poisson structure. Moreover, Thimm demonstrated that his momentum map is a Poisson map for which the flow generated by the pullback of any function commutes with the geodesic flow. Thus, Thimm had reduced the problem of finding sufficient invariant functions of the geodesic flow to finding functions of $\mathcal{G}$ which Poisson commute, a much easier problem to consider.

We demonstrate in Section 4 that Thimm's method can be extended to our case, but at the price of weakening our definition of complete integrability. As in Arnold's Theorem, Thimm's method does not in general work on Hilbert symmetric spaces without some careful consideration of the essential difference between finite dimensional manifolds and Hilbert manifolds. The main problem is in demonstrating how a set of linearly independent vectors span a Hilbert space. The proof is trivial when the Hilbert space is of dimension $n < \infty$ and the set consists of n vectors by the pigeonhole principle. The pigeonhole principle will not work, however, when dealing with sets of the same infinite cardinality.

Another useful set of invariants (when they exist) for studying dynamical systems is Moser's iso-spectral deformations. Basically, for certain Hamiltonian systems, one can construct an associated matrix-valued function whose eigenvalues are invariants of the Hamilton flow. In our particular case, because our spaces are symmetric spaces, we can represent an arbitrary geodesic in the form

$$\gamma(t) := exp(t\xi)$$

where ξ is a skew-Hermetian Hilbert-Schmidt matrix and exp is the classical matrix exponential function. In Section 5, we exhibit the spectrum of ξ as an invariant of the geodesic flow.

In the process of demonstrating the integrability of the geodesic flow and the existence of a spectral invariant, we will also prove in Section 6 that every function which Poisson-commutes with the Energy Hamiltonian can be factored through the moment map. In this fashion we determine that the only method of demonstrating complete integrability of the geodesic flow for our manifolds is Thimm's method.

Lastly, as a method of exhibiting the fundamental difference between the finite rank and the infinite rank Hilbert Grassmannians, we determine in Section 7 the conjugate points along geodesics of the two types of Hilbert Grassmannians. We determine that the topology of the infinite rank Grassmannians, while locally diffeomorphic to that of the the finite rank Grassmannians, is fundamentally more complex in a global sense than the topology of the finite rank Grassmannians.

For background information on Hilbert/Banach manifolds we recommend [7]. For Thimm's and Ii and Watanabe's algorithms we recommend [9] and [3].

We will define our family of Hilbert Grassmannians as coset spaces for a Hilbert Lie group and a Banach/Hilbert Lie group.

Definition 1.1. For $(H, \langle\, , \,\rangle)$ a separable infinite-dimensional complex Hilbert space, we denote the group of unitary operators of the form $I + X$, X Hilbert-Schmidt, as $U_{HS}(H)$.

Let $H = H_+ \oplus H_-$ denote a polar decomposition of H, where both H_- and H_+ are closed, infinite-dimensional subspaces of H. With respect to the polar decomposition any linear operator $A : H \to H$ can be decomposed into 4 maps,

$$A_{--} : H_- \to H_- \quad A_{-+} : H_- \to H_+ \quad A_{+-} : H_+ \to H_- \quad A_{++} : H_+ \to H_+.$$

Our second group, denoted in [8] as U_{res}, consists of the set of unitary operators $\{g|\ g_{++}, g_{--}$ Fredholm and g_{-+}, g_{+-} Hilbert-Schmidt$\}$.

We mention in passing that $U_{HS}(H)$ is a Hilbert manifold modelled on its Lie algebra, the set $\mathcal{U}_{HS}(H)$ of skew-Hermetian Hilbert-Schmidt operators, while U_{res} is a Lie group modelled on its Lie algebra, a direct sum of a closed subalgebra $\mathcal{L}$ of the Banach space of skew-Hermetian bounded linear operators $\{A : H \to H, \quad A^* = -A\}$ and a closed subspace $\mathcal{M}$ of $\mathcal{U}_{HS}(H)$. $\mathcal{L}$ and $\mathcal{M}$ satisfy the commutator relations

$$[\mathcal{L}, \mathcal{M}] \subseteq \mathcal{M} \qquad [\mathcal{M}, \mathcal{M}] \subseteq \mathcal{L}.$$

Definition 1.2. The family of Grassmannians will consist of the set of r-planes in H denoted as $Gr(r, H)$, and the Grassmannian $Gr(H)$ will be the image of $U_{HS}(H)$ as $U_{HS}(H) \cdot H_+$ in the set of planes in H isomorphic to the base plane $o = H_+$.

2. Symplectic Geometry

In this section we will define and state the necessary properties for symplectic structures and integrability conditions for Hilbert manifolds.

Definition 2.1. A *symplectic* Hilbert space (H, ω) is a real separable Hilbert space $(H, \langle\ ,\ \rangle)$ together with a continuous, bilinear, non-degenerate, skew-symmetric form ω. By non-degenerate we mean that if $0 \neq x \in H$, then $\exists y \in H$ such that $\omega(x, y) \neq 0$.

Remark 2.1. This definition is a natural generalization of finite dimensional symplectic spaces which takes into account the topology of the Hilbert space. Moreover, every infinite-dimensional separable Hilbert space can be given a (non-canonical) symplectic structure by choosing a conset basis denoted by $\{e_i, f_i\}$ and setting $\omega(e_i, f_j) = \delta_{ij}$, $\omega(e_i, e_j) = 0$, $\omega(f_i, f_j) = 0$.

Observe also that with this definition that the symplectic structure ω can be represented as $\langle A\_, \_\rangle$ where A is a continuous skew-symmetric operator with dense range.

Definition 2.2. A set $\bigcup_{i=1}^{\infty}\{e_i, f_i\}$ is called a *symplectic basis* for H if

1. The set is a Schauder basis for H and
2. $\omega(e_i, f_j) = \delta_{ij}$, $\omega(e_i, e_j) = \omega(f_i, f_j) = 0$

Remark 2.2. For a symplectic Hilbert space we can always construct a symplectic basis from a conset basis by means of a modified Gram-Schmidt algorithm.

Now that we have defined a symplectic Hilbert space and its attendant properties we can define a symplectic Hilbert manifold.

Definition 2.3. A *symplectic Hilbert manifold* is a pair (M, ω) where M is a Hilbert manifold and ω is a smooth, closed, non-degenerate, bilinear skew symmetric 2-form.

Since so many of the theorems of finite-dimensional manifolds such as the existence of a Riemannian metric are not automatically guaranteed to exist for Hilbert manifolds, one can ask if there are any examples of symplectic Hilbert manifolds other than (H, ω).

The answer surprisingly is yes, there are plenty of examples. Choose any Hilbert manifold N modelled on a Hilbert space H. Let T^*N denote the cotangent bundle of N, which is a well-defined Hilbert manifold in its own right. For $p \in N$ let U denote a coordinate neighborhood with coordinates (x_i). Let (y_i) denote a coordinate system for H. Since $T^*U \cong U \times H$ then (x_i, y_i) is a coordinate system for T^*U. We can now define the so-called canonical 1-form $\theta \in T^*T^*N$ locally by $\theta := \sum y_i dx_i$.

Because N is modelled on a Hilbert space we have that θ is a well-defined smooth 1-form. By taking the exterior derivative we have that $\omega := d\theta$ is a closed 2-form locally represented by $\omega = \sum dy_i \wedge dx_i$ which is clearly non-degenerate.

Remark 2.3. The 1-form θ and the 2-form $\omega := d\theta$ could be defined as well for the cotangent bundle of a Banach manifold modelled on a separable Banach space X which admits a Schauder basis $\{e_i\}$. The only difference would be that the coordinates (x_i) would be with respect to the the Schauder basis and the coordinates (y_i) would be with respect to the dual of the Schauder basis, $\{e_i'\}$.

Remark 2.4. If the skew-symmetric, continuous operator A which represents the symplectic structure ω for a symplectic Hilbert space (H, ω) is invertible then we can define an isomorphism between H and its dual H' via

$$i_\omega : H \to H', \quad i_\omega(X) := \omega(X, \ ),\ X \in H.$$

For a symplectic Hilbert manifold (M, ω) if the tensor A which represents the symplectic structure ω is invertible then TM and T^*M are isomorphic similarly by the isomorphism

$$i_\omega : TM \to T^*M,\ i_\omega(X) := \omega(X, \ ),\ X \in TM.$$

In the case when $M = T^*N$ for a Hilbert manifold N, from the above construction of ω, we see that $TM \cong T^*M$.

For a symplectic manifold (M, ω) if $TM \cong T^*M$ through i_ω then we can define a Hamiltonian field as follows:

Definition 2.4. Let $f \in \mathcal{C}^\infty(M)$. We call $\xi_f := i_\omega^{-1}(df)$ a *Hamiltonian field*, with *Hamiltonian* f.

Remark 2.5. Just as we can define an isomorphism between the tangent and cotangent bundles of a symplectic Hilbert manifold M of the form $M = T^*N$ we can define an isomorphism between TN and T^*N if N admits a Riemannian metric g compatible with its topology.

For our purposes, our symplectic Hilbert manifolds will be the tangent bundles of our Grassmannians which will be given a symplectic structure through the isomorphism of the tangent and cotangent bundles of our Grassmannians via their Riemannian metrics.

For the properties of the flows generated by Hamiltonian fields as well as the Poisson structure generated by a symplectic structure we refer the reader to [4].

Definition 2.5. Let M, N be manifolds with Poisson-structures $\{\, ,\, \}_M$ and $\{\, ,\, \}_N$. A map $\mu : M \to N$ is a said to be a *Poisson-map* if

$$\{\mu^* f, \mu^* g\}_M = \mu^*\{f, g\}_N \quad \forall f, g \in \mathcal{C}^\infty(N)$$

Remark 2.6. It well-known that every real Lie algebra $\mathcal{G}$ with inner product $\langle\, ,\, \rangle$ admits a Poisson structure by the construction:

$$\{f, g\}(x) := \langle x, [\operatorname{grad} f|_x, \operatorname{grad} g|_x]\rangle, \quad f, g \in \mathcal{C}^\infty(\mathcal{G}).$$

Definition 2.6. Let (M, ω) be a symplectic Hilbert manifold and let

$$\{\, ,\, \} : \mathcal{C}^\infty(M) \times \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M)$$

denote the Poisson structure generated by the symplectic structure ω. If M admits a Riemannian metric g compatible with its Hilbert structure, then we will define the Hamiltonian H_0 to be *completely integrable* if there exists a set of functions $\{H_i\}_1^\infty$ so that:

1. $\{H_i, H_j\} = 0 \ \forall i, j$ and
2. $\{\xi_{H_i}\}_{i=1}^{\infty} \cup \{\operatorname{grad} H_i\}_{i=1}^{\infty}$ forms a spanning set of TM on a dense subset E of M.

Definition 2.7. Let (N, g) be a Riemannian Hilbert manifold. For the symplectic Hilbert manifold TN we can define the so-called *kinetic energy* Hamiltonian $H \in \mathcal{C}^{\infty}(TN)$ by

$$H(x, v) := g_x(v, v), \quad v \in T_x N$$

It is well-known (see for example [4]) that the flow generated by the Hamiltonian field for the kinetic energy Hamiltonian is the image of the sets of geodesics of N in TN of the form $(\gamma(t), \gamma'(t))$. For this reason the flow generated by this Hamiltonian field is called the *geodesic flow*.

3. Symmetric Spaces and Moment Maps

Since the action of $U_{HS}(H)$ on $Gr(r, H)$ is transitive we can give $Gr(r, H)$ the structure of a symmetric Hilbert manifold. We can give our Hilbert Grassmannians the structure of symmetric spaces as follows: Let H_r denote a fixed plane of dimension r and $H_r^{\perp}$ its orthogonal complement in H. Since $U_{HS}(H)$ acts transitively on $Gr(r, H)$ and the isotropy group for H_r is isomorphic to $U(r) \times U_{HS}(H_r^{\perp})$ then $Gr(r, H) \cong U_{HS}(H)/U(r) \times U_{HS}(H_r^{\perp})$. For notational convenience we will denote $U(r) \times U_{HS}(H_r^{\perp})$ simply as L_r.

We can define a bi-invariant Hilbert metric $\langle x, y \rangle := -tr(xy) \quad x, y \in \mathcal{U}_{HS}(H)$ for $\mathcal{U}_{HS}(H)$ for which we can decompose $\mathcal{U}_{HS}(H)$ into the orthogonal sum

$$\mathcal{U}_{HS}(H) = \mathcal{M}_r \oplus \mathcal{L}_r$$

where $\mathcal{L}_r$ is the Lie algebra of L_r and $\mathcal{M}_r$ is its orthogonal complement. Note that $\mathcal{M}_r$ and $\langle\, , \rangle_{\mathcal{M}_r}$ are $Ad(L_r)$-invariant.

With respect to a conset basis $\{e_i\}_{i=1}^{\infty}$ for H with $\{e_i\}_{i=1}^{r} \in H_r$ and $\{e_i\}_{i=r+1}^{\infty} \in H_r^{\perp}$ an element $v \in \mathcal{M}_r$ is represented by a skew-Hermetian Hilbert-Schmidt operator of the form:

$$\begin{pmatrix} 0 & X \\ -X^* & 0 \end{pmatrix}.$$

An element $v \in \mathcal{L}_r$ is represented by a skew-Hermetian Hilbert-Schmidt operator of the form:

$$\begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix}.$$

For an arbitrary point $x = [gL_r]$, $T_x Gr(r, H)$ can be identified with the horizontal component

$$T_{gl}^{hor} U_{HS}(H) = \mathcal{M}_r \quad \text{of} \quad T_{gl} U_{HS}(H)$$

for arbitrary $l \in L$. For $u \in T_x Gr(r, H)$ any 2 elements $u_1 \in T_{gl_1}^{hor} U_{HS}(H)$, $u_2 \in T_{gl_2}^{hor} U_{HS}(H)$ which correspond to u are related by $u_2 = Ad(l_2^{-1} l_1) u_1$.

For $(p, z) \in T_p Gr(r, H)$ let $p = [gL_r]$ denote the coset corresponding to p. Let $(gl, u) \in TU_{HS}(H)$ denote a representation of (p, z). Then

$$\mu_r(p, z) = Ad(gl)u.$$

If (gl_1, u_1) and (gl_2, u_2) both correspond to (p, z) then a quick computation

$$Ad(gl_2)u_2 = Ad(gl_2)Ad(l_2^{-1}l_1)u_1 = Ad(gl_1)u_1$$

shows that μ_r is a well-defined map.

We can similarly give the Hilbert Grassmannian $Gr(H)$ a symmetric space structure. We can consider the isotropy group of H_+, namely $U(H_+) \times U(H_-) \subset U_{res}$. We denote the Lie subalgebra of $U(H_+) \times U(H_-)$ as $\mathcal{L}$. Even though $\mathcal{U}_{res}$ is not a Hilbert-Schmidt Lie algebra we can define a Hilbert inner product for any two elements of $\mathcal{U}_{res}$ which are Hilbert-Schmidt as in $\mathcal{U}_{HS}(H)$. We define $\mathcal{M}$ to be the "orthogonal" complement of $\mathcal{L}$ in the sense that with respect to the polar decomposition $H = H_+ \oplus H_-$ any element of $v \in \mathcal{M}$ is a skew-Hermetian Hilbert-Schmidt operator of the form:

$$\begin{pmatrix} 0 & X \\ -X^* & 0 \end{pmatrix}$$

and any element of $v \in \mathcal{L}$ is a skew-Hermetian bounded operator of the form:

$$\begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix}.$$

Similarly, we define a moment map $\mu : TGr(H) \to \mathcal{U}_{HS}(H)$

Because it will help elucidate why we had to weaken our definition of integrability for our Hilbert Grassmannians we compute for $f \in \mathcal{C}^\infty(\mathcal{U}_{HS}(H))$, $\operatorname{grad} \mu^* f, \xi_{\mu^* f} \in \mathcal{M}_r \times \mathcal{M}_r$.

With respect to a representation $(g, u) \in T^{hor}U_{HS}(H)$ for $(p, v) \in TGr(r, H)$:

$$\operatorname{grad} \mu^* f|_{(p,v)} = ([u, Ad(g^{-1})\operatorname{grad} f|_{\mu(p,v)}]_{\mathcal{M}_r}, (Ad(g^{-1})\operatorname{grad} f|_{\mu(p,v)})_{\mathcal{M}_r})$$
$$\xi_{\mu^* f}|_{(p,v)} = ((Ad(g^{-1})\operatorname{grad} f|_{\mu(p,v)})_{\mathcal{M}_r}, -[u, Ad(g^{-1})\operatorname{grad} f|_{\mu(p,v)}]_{\mathcal{M}_r})$$

4. Integrability of the Geodesic Flow

It's well-known that there exists no measure similar to Lebesgue measure for infinite-dimensional vector spaces. However, by appealing to our intuitions it is true that the union of a countable collection of submanifolds of codimension at least one is at best dense in a Hilbert manifold M and that the complement is at least dense in M. This will be the key idea in our proof of integrability. The collection of points where the Hamiltonian fields will fail to be linearly independent is essentially the countable union of the tangent bundles of sub-Grassmannians.

By choosing a conset basis $\{e_i\}$ for H we can construct as in Thimm's algorithm an ascending chain of Lie algebras

$$\mathcal{U}(1) \hookrightarrow \mathcal{U}(2) \hookrightarrow \ldots \hookrightarrow \mathcal{U}_{HS}(H)$$

with corresponding orthogonal projection operators $P_s : \mathcal{U}_{HS}(H) \to \mathcal{U}(s)$, and Poisson-commuting functions $\{f'_{st} : \mathcal{U}_{HS}(H) \to \mathbb{R}\}$,

$$f'_{st}(x) = i^t tr(P_s(x)^t), \text{ with } \begin{cases} 1 \le t \le \min\{s, 2r\}, & s < \infty \\ t = 2, 4, \ldots, 2r & \text{when } s = \infty \end{cases}$$

For our theorem we need an explicit computation of $\operatorname{grad} f'_s t$:

$$\operatorname{grad} f'_s t = t i^t P_s(x)^{t-1}$$

Let f_{st} denote the pullback under the map μ_r of f'_{st}. We define for $n > r$ $A_n : TGr(r,H) \to \hom(\mathbb{R}^{2rn}, TTGr(r,H))$ to be the matrix-valued function whose columns are the Hamiltonian fields and gradient fields for the subset

$$\{f_{st} \mid s \leq n-1\} \cup \{f_{nt} \mid t = 2,4,\ldots,2r\}.$$

Let V_n denote the range of A_n. Let A_∞ denote the formal limit $\lim_{n\to\infty} A_n$ and V the range of A. Then we have the following theorem:

Theorem 4.1. *There exists a dense subset S of $TGr(r,H)$ for which*

$$\mathit{span}\overline{\{\xi_{f_{st}}|_{(x,v)}, \mathit{grad}\, f_{st}|_{(x,v)}\}} = T_{(x,v)}TGr(r,H)$$

for $(x,v) \in S$.

Proof. In proving the kinetic energy Hamiltonian H_0 for $TGr(r,H)$ is integrable, the second step is to show that

$$\operatorname{span}\overline{\{\xi_{f_{st}}(x,v)\}_{i=0}^{\infty} \cup \{\operatorname{grad} f_{st}(x,v)\}_{i=0}^{\infty}} = T_{(x,v)}TGr(r,H)$$

or equivalently that V is dense for a dense subset of $TGr(r,H)$.

Unfortunately a direct proof for the second step as in Thimm's proof by using the analyticity of A_∞ and demonstrating for one point that A_∞ is invertible will not work for our case. The difficulty lies in the fact that for every point $z \in \mathcal{U}(n)$ we have $\operatorname{grad} f'_{st}(z) \in \mathcal{U}(n)$. For any point $(x,v) \in TGr(r,H)$ such that $\mu(p,v) \in \mathcal{U}(n)$ for some n we have necessarily that

$$\operatorname{span}\overline{\{\xi_{f_{st}}(x,v)\}_{i=0}^{\infty} \cup \{\operatorname{grad} f_{st}(x,v)\}_{i=0}^{\infty}}$$

is finite-dimensional. Among the points where A_∞ fails to have dense range are points in the tangent bundles of sub-Grassmannians $Gr(r,q)$. Unfortunately, the set $\bigcup_{q=1}^{\infty} TGr(r,q)$ is dense in $TGr(r,H)$.

Paradoxically, we can use the fact that the set $\bigcup_{q=1}^{\infty} TGr(r,q)$ is dense in $TGr(r,H)$ to generate the dense set of points where A_∞ has dense range as follows: From the inclusion $\mathcal{U}(1) \hookrightarrow \mathcal{U}(2) \hookrightarrow \ldots \hookrightarrow \mathcal{U}_{HS}(H)$ we can define the inclusions

$$Gr(r,1) \hookrightarrow Gr(r,2) \hookrightarrow \ldots \hookrightarrow Gr(r,H) \text{ and } \mathcal{M}_{r1} \hookrightarrow \mathcal{M}_{r2} \hookrightarrow \ldots \hookrightarrow \mathcal{M}_r$$

Let $\{b_i\}_{i=1}^{\infty}$ denote a conset basis for $\mathcal{M}_r$ such that $\{b_i\}_{i=1}^{2rn}$ is a basis for $\mathcal{M}_{rn}$. Let $p \in TGr(r,H)$ and $\epsilon > 0$ be arbitrary. Since $\bigcup_{q=1}^{\infty} TGr(r,q)$ is dense in $TGr(r,H)$ we know there exists a point $p_0 \in TGr(r,n_1)$ within an ϵ-ball of p for some $n_1 \geq r$.

From Thimm's result we know there exists a point $p_1 \in TGr(r,n_1)$ arbitrarily close to p_0, say $d(p_1,p_0) < (\epsilon - d(p,p_0))/4$ such that $A_{n_1}(p_1)$ is invertible.

Unlike A_∞, A_{n_1} is a smooth-valued function. Hence there exists a δ_1-ball U_1 around p_1 within $TGr(r,H)$ such that for $q \in U_1$, A_{n_1} is invertible and the range of A_{n_1} is close to $\mathcal{M}_{rn_1} \times \mathcal{M}_{rn_1}$ in the sense that

$$\epsilon_1 := \sup\{||(b_i,0) - \mathit{proj}_{V_{n_1}}(b_i,0)||, ||(0,b_j) - \mathit{proj}_{V_{n_1}}(0,b_j)|| \quad | \, i,j \leq 2rn_1\}$$

is less than ϵ/n_1.

We proceed by induction to generate a sequence of δ-balls $U_1 \supset U_2 \supset \ldots$ and a Cauchy sequence $(p_i) \to p_\infty$ with the properties:

- $\delta_{n+1} < \delta_n/4$
- $p_i \in U_i$
- $d(p_{i+1},p_i) < \delta_n/2$
- $d(p_\infty,p) < \epsilon$

- For $q \in U_n$, $A_n(q)$ is invertible, and
- For $q \in U_n$,

$$\epsilon_n := \sup\{||(b_j,0) - proj_{V_n}(b_j,0)||, ||(0,b_k) - proj_{V_n}(0,b_k)|| \quad | \, j,k \leq 2rn\}$$

  is less than ϵ/n.

With these properties we have $p_\infty \in \cap_{i=1}^{\infty} U_i \neq \emptyset$, $d(p_\infty, p) < \epsilon$, and $A_n(p_\infty)$ is invertible for all $n \geq n_1$.

We claim that V is dense in $T_{p_\infty}TGr(r,H)$ for p_∞. Suppose not. Then there exists an $x \in T_{p_\infty}TGr(r,H)$ orthogonal to V with $||x|| = 1$. Now $x = \sum \alpha_i(b_i,0) + \sum \beta_j(0,b_j)$ for some ℓ_2 sequences $(\alpha_i),(\beta_j)$. Let $x_n := \mathrm{proj}_{\mathcal{M}_{rn}\times\mathcal{M}_{rn}} x$. We know that $||x_n|| \nearrow 1$ and $||x - x_n|| \searrow 0$. So there exists an n such that

$$||x - x_n|| < \min\{1/2, ||x_n|| \, ||x - x_n||\}.$$

We choose an i sufficiently large such that $\epsilon_i < ||x - x_n||/\sqrt{4rn}$ and $n_i > n$. Let $u' := \mathrm{proj}_{V_{n_i}} u$ Let $\hat{x} := \sum_{j=1}^{j=2rn} \alpha_j(b_j,0)' + \sum_{k=1}^{k=2rn} \beta_k(0,b_k)'$. We have

$$\begin{aligned} ||\hat{x} - x_n||^2 &\leq \sum_{j=1}^{j=2rn} \alpha_j^2 ||x - x_n||^2/4rn + \sum_{k=1}^{k=2rn} \beta_k^2 ||x - x_n||^2/4rn \\ &= ||x_n||^2 ||x - x_n||^2. \end{aligned}$$

If x were orthogonal to V then

$$\begin{aligned} ||x - \hat{x}|| &= \sqrt{||x||^2 + ||\hat{x}||^2} \\ &= \sqrt{1 + ||\hat{x}||^2} \\ &\geq 1. \end{aligned}$$

But,

$$\begin{aligned} ||x - \hat{x}|| &\leq ||x - x_n|| + ||x_n - \hat{x}|| \\ &\leq ||x - x_n|| + ||x_n|| \, ||x_n - x|| \\ &< 2||x - x_n|| \\ &< 1. \end{aligned}$$

Hence x could not be orthogonal to V. Therefore V has dense range for p_∞. Since p and ϵ were arbitrary, there exists a dense set $S \subset TGr(r,H)$ for which $\mathrm{span}\overline{\{\xi_{f_{st}}|_q, \mathrm{grad}\, f_{st}|_q\}} = T_qTGr(r,H)$, $q \in S$. □

REMARK 4.1. For $TGr(H)$ the proof is similar except

- we compose μ with an isometry $\phi : \mathcal{U}_{HS}(H) \to \mathcal{U}_{HS}(H)$ in order to demonstrate points where A has dense range.
- The set of Poisson-commuting functions consist of

$$f_{st} = f'_{st} \circ \phi \circ \mu, \quad \begin{cases} t \leq 2s & s < \infty \\ t = 2,4,\ldots & s = \infty \end{cases}$$

5. A Spectral Invariant of Geodesics

We begin by recalling two properties of symmetric spaces we need:

1. if $\mathcal{K}$ is a closed subalgebra of $\mathcal{U}_{HS}(H) \subset \mathcal{U}_{res}$ then $\exp(\mathcal{K})$ is a totally geodesic subgroup of $U_{HS}(H) \subset U_{res}$ and
2. if $\mathcal{K} \subset \mathcal{M}$ then $\pi \circ \exp(\mathcal{K})$ is a totally geodesic submanifold of $Gr(H)$.

Let $\{f_i\}_{i=1}^n \subset \mathcal{C}^\infty(M)$ be a maximal set of non-degenerate Poisson-commuting functions for a symplectic manifold M. It is well-known ([1]) that the Hamiltonian fields $\{\xi_{f_i}\}$ generate a locally-injective action $\alpha : \mathbb{R}^n \times M \to M$ with $\alpha(0, \_) = id_M$ which preserves the level sets of $\{f_i\}$.

Let $p \in M$ and U a connected component of $\bigcap {f_i}^{-1}(f_i(p))$ containing p. If U is compact then the map $\alpha_p := \alpha(\_, p) : \mathbb{R}^n \to U$ has a discrete kernel lattice K with a basis we denote as $\{e_i\}_{i=1}^n$. By defining an equivalence relation $\sim$ on $\mathbb{R}^n$

$$x, y \in \mathbb{R}^n \quad x \sim y \Leftrightarrow x - y \in K$$

we can define a projection $\pi : \mathbb{R}^n \to T^n$ and factor α_p through a diffeomorphism $\alpha_p^* : T^n \to U$.

Moreover, the flows $\{\phi_i\}$ generated by the $\{f_i\}$ correspond to the images of lines in $\mathbb{R}^n$ projected onto T^n. Let L_i denote the line in $\mathbb{R}^n$ which corresponds to ϕ_i with $\phi_i(0) = p$. Let $(x_i) = (a_i t)$ denote a parameterization of L_i with respect to the basis $\{e_i\}$. Let m_i denote the size of the largest subset of $\{a_i\}$ which is rationally independent. Then the closure of the image of ϕ_i beginning at p is an m_i torus.

Under certain conditions we can generalize the above results to symplectic Hilbert manifolds.

Definition 5.1. Let H be a separable real Hilbert space. Let $\{e_i\}$ be a topological basis for H. We define an equivalence relation $\sim$ via

$$x, y \in H, \quad x - y = \sum n_i e_i \quad n_i \in \mathbb{Z}$$

We define a *Hilbert torus* T_H as $T_H = H/\sim$.

Suppose $\{f_i\}$ is a maximal set of non-degenerate Poisson-commuting functions of a symplectic Hilbert manifold M. Let $p \in M$ and U the path component of $\bigcap {f_i}^{-1}(f_i(p))$ which contains p. Suppose as well that the flows generated by the Hamiltonian fields of the $\{f_i\}$ generates an action $\alpha : H \times U \to U$ for some real separable Hilbert space H. If the action has a kernel K whose basis $\{e_i\}$ is a topological basis for H then we can factor $\alpha_p := \alpha(\_, p) : H \to U$ through a diffeomorphism ${\alpha_p}^* : T_H \to U$.

For the symplectic Hilbert manifold $Gr(H)$ we can define an analogous action by a Hilbert torus. We begin by first defining a real separable Hilbert space:

Lemma 5.1. *For every $x \in \mathcal{M}$ there exists a maximal abelian subalgebra of $\mathcal{M}$ which contains x.*

Proof. The proof consists of an application of Zorn's lemma: Let $\prec$ denote a partial ordering on the set A of abelian subalgebras of $\mathcal{M}$ which contain x where $U \prec V \Leftrightarrow U \subseteq V$. Let $A_1 \prec A_2 \prec \dots$, be a chain. Let $A_\infty := \overline{\bigcup A_i}$. Then A_∞ is a maximal element such that $A_i \prec A_\infty$ for all i. For $x \neq 0$, defining $A_1 = \mathbb{R}x$ demonstrates the non-emptiness of the set A. For $x = 0$, 0 trivially belongs to every maximal abelian subalgebra of $\mathcal{M}$. $\square$

Remark 5.1. For $x \in \mathcal{M}$ it is clear that $x^{2n+1} \in \mathcal{M}$. Moreover, every polynomial of x consisting of odd powers of x commutes with x. If the spectrum of x has no multiplicities then every element of $\mathcal{M}$ which commutes with x has to be a limit of polynomials of odd powers of x.

Remark 5.2. For every $x \in \mathcal{M}$ there exists a $y \in \mathcal{M}$ such that

- $[x, y] = 0$
- The spectrum of y has no multiplicities.

For $x \in \mathcal{M}$ let $y \in \mathcal{M}$ be as in the preceding remark. Let A_∞ be the maximal abelian algebra which is the closure of the span $\{y^{2k+1}\}$. Let (v_i) be a set of eigenvectors for y such that with respect to (v_i), $y = diag(i\lambda_1, -i\lambda_1, i\lambda_2, -i\lambda_2, \dots)$ where $0 < \lambda_{i+1} < \lambda_i$ and $spec(y) = \{0\} \cup \{\pm i\lambda_j\}_{j=1}^{\infty}$. Since all elements belonging to A_∞ commute, then with respect to the basis (v_i) all elements of A_∞ are diagonal matrices. A_∞ will be our real, separable Hilbert space.

We now define an equivalence relation: We consider now the group homomorphism $exp : (A_\infty, +) \to (U_{res}, \cdot)$. The kernel K of this map consists precisely of those elements of A_∞ whose eigenvalues are integer multiples of $2\pi i$. Our equivalence relation is defined to be $x \sim y \Leftrightarrow x - y \in K$ for $x, y \in A_\infty$. We define a basis for the kernel to be elements of the form $e_j = diag(0, \dots, 0, \overbrace{2\pi i, -2\pi i}^{\text{jth pair}}, 0, \dots)$. This basis is also a topological basis for A_∞. Thus we can define a Hilbert torus T_H.

Now we define how our Hilbert torus acts on $Gr(H)$: Since the group U_{res} acts by isometries on $Gr(H)$ then T_H acts as a subgroup of isometries on the $Gr(H)$ such that the action of T_H on the base point o is injective. Moreover, if $\gamma(t)$ is a geodesic of $Gr(H)$ such that $\gamma(0) = o$ and $\gamma'(t)$ corresponds to an $x \in A_\infty$ then $\gamma(t)$ is the just the image of a line $L \subset A_\infty$ projected onto T_H. Let $x = \sum x_i e_i$ denote a decomposition of x with respect to the basis $\{e_i\}$. Then the behavior of $\gamma(t)$ is determined entirely by the ℓ_2-sequence (x_i).

Remark 5.3. The spectrum of a sum $\sum \alpha_j e_j$ is equal to $\overline{\{\pm 2\pi i \alpha_j\}}$. Let $\{e_i\}, \{f_i\}$ denote the respective bases for the maximal abelian subalgebras A'_∞, A''_∞ generated by y_1 and y_2. Suppose $x \in A'_\infty \cap A''_\infty$. Let $x = \sum \alpha_i e_i = \sum \beta_j f_j$ denote the respective decompositions of x. Then (β_j) is at most a permutation of (α_i).

Since the spectrum of $x \in \mathcal{M}$ determines its coefficients in a maximal Abelian algebra we can define an ℓ_2-valued function for $TGr(H)$ which determines completely the behavior of the geodesics of $Gr(H)$. For $TGr(H)$ we define the function $F : TGr(H) \to \ell_2$ as

$$F(p, v) := (\lambda_1, \lambda_2, \dots) \in \ell_2$$

where $(i\lambda_j)$ is the set of eigenvalues of $\mu(p, v)$ counted with multiplicities and

$$(\lambda_{2i-1}) \geq 0, \quad (\lambda_{2i}) = -(\lambda_{2i-1}), \quad |\lambda_{2i+1}| \leq |\lambda_{2i-1}|$$

For $TGr(r, H)$ we can define $F_r : TGr(r, H) \to \ell_2$ similarly. However, since $Gr(r, H)$ has rank r then for $i > 2r, \quad \lambda_i = 0$. Under the inclusion map $i_r : Gr(r, H) \hookrightarrow Gr(H)$, $Gr(r, H)$ is embedded isometrically as a geodesic submanifold of $Gr(H)$ with $F \circ i_r = F_r$.

Theorem 5.1. *F determines the behavior of all geodesics of $Gr(r, H)$ and $Gr(H)$.*

Proof. The only point left to observe is that if $(p, v) = \phi(q, w)$ for (p, v), (q, w) in $Gr(H)$ and ϕ an isometry, then v and w are related by the adjoint action of some element $h \in L$. But the adjoint action preserves the spectrum of the operators in $\mathcal{M}$ which correspond to v and w. Hence F is an invariant of the geodesics of $Gr(H)$. $\square$

For a geodesic γ of $Gr(r, H)$ we have that $\overline{\gamma}$ is a torus of rank $1, \ldots, r$. The interesting cases are for geodesics of $Gr(H)$ which are not geodesics of any $Gr(r, H)$. Three cases have been determined:

- $\overline{\gamma}$ is an Euclidean line.
- $\overline{\gamma}$ is a Hilbert torus.
- $\overline{\gamma}$ is a "dna" group.

For the following theorems A_∞ will denote a maximal abelian subalgebra for $\mathcal{M}$ generated by $y \in \mathcal{M}$ with basis $\{e_i\}$ which is also a basis for the kernel K of the map $\exp : A_\infty \to U_{res}$.

Theorem 5.2. *Let $x = \sum \alpha_i e_i \in A_\infty$. Then $\overline{\exp(\mathbb{R}x)} = T^n \subset T_H$ for some $n < \infty$ if and only if all but a finite number of the α_i are zero.*

Proof. The "if" part of the proof is clear. For the "only if" part of the proof let $v_1, \ldots, v_n \in A_\infty$ denote generators of T^n. Then v_i is of the form $v_i = \sum_{j=1}^{n_i} \alpha_{ij} e_j$. Let $n' = \max\{n_i\}$. Since $x \in span\{v_1, \ldots, v_n\}$ then $x = \sum_{j=1}^{n'} \alpha_j e_j$. $\square$

For the theorems which classify the three interesting types of geodesics we require a metric for T_H which we define as follows: For $x, y \in A_\infty$ we define

$$d(\exp(x), \exp(y)) := \inf\{|x - y + k| k \in K\}.$$

Theorem 5.3. *If $x = \sum \frac{1}{j^p} e_j$, $p > \frac{1}{2}$ then $\exp(\mathbb{R}x)$ is a Euclidean line embedded in T_H.*

Proof. It suffices to show there exists an ϵ neighborhood U of $\exp(0)$ in T_H and a $\delta > 0$ such that $\exp(tx) \subset U \Leftrightarrow |t| < \delta$.

Suppose there does not exist a $\delta > 0$ for an ϵ neighborhood of $\exp(0)$ with $0 < \epsilon \ll 1/3$.

Then there exists a sequence (t_i), $t_i \nearrow \infty$ such that $d(\exp(t_i x), \exp(0)) < \epsilon$.

We define $n_i := [[(2t_i)^{1/p}]]$, $\quad \sigma_i := (2t_i)^{1/p} - n_i$.

For $t_i \gg 1$ we have then

$$\frac{1}{2} \le t_i \frac{1}{{j_i}^p} = \frac{t_i}{((2t_i)^{1/p} - \sigma_i)^p} < \frac{t_i}{2t_i(1 - O(\frac{1}{(2t_i)^{1/p}}))} = \frac{1}{2(1 - O(\frac{1}{(2t_i)^{1/p}}))} < \frac{2}{3}$$

Hence for $t_i \gg 2$, $d(\exp(t_i x), \exp(0)) \ge (t_i \frac{1}{{j_i}^p}) \mod 1 > \frac{1}{3} > \epsilon$.

Hence, $\exp(\mathbb{R}x)$ is an embedded Euclidean line in T_H. $\square$

Corollary 5.1. *Let $x = \sum \alpha_j e_j \in A_\infty$, If there exists a $p > \frac{1}{2}$, and a subsequence $(\alpha_{j_i}) \subseteq (\alpha_j)$ such that $\lim_i |\alpha_{j_i} j_i^p| = c > 0$ then $\exp(\mathbb{R}x)$ is an embedded Euclidean line.*

Theorem 5.4. *If $x = \sum r^j e_j$, $0 < r < 1$ then $\exp(\mathbb{R}x)$ is an embedded Euclidean line.*

Proof. As in the previous theorem the proof is by contradiction. We begin again with an ϵ-neighborhood of $\exp(0)$ with $\epsilon \ll 1$. Again we suppose that there exists a sequence (t_i), $t_i \nearrow \infty$ such that $d(\exp(t_i x), \exp(0)) < \epsilon$.

If $d(\exp(t_i x), \exp(0)) < \epsilon$ then $t_i r^j \mod 1 < \epsilon$ for all $j \geq 1$. By decreasing the size of ϵ we can create a sequence (t_i') such that $d(\exp(t_i' x), \exp(0)) \to 0$. In particular, $t_i' r^j \mod 1 \to 0$ for all $j \geq 1$. Now for $t_i' \gg 1$ we have a decreasing sequence $t_i', t_i' r, t_i' r^2, \ldots, \to 0$. $\mathbb{R}^+ = (0, +\infty)$ can be partioned into intervals of the form $(r^{s+1}, r^s]$. Under the action of multiplication by r the interval $(r^{j+1}, r^j]$ is mapped to $(r^{j+2}, r^{j+1}]$. For $t_i' > 1$ there exists a $j > 0$ such that $t_i' r^j \in (r^2, r]$. Hence $t_i' r^j \mod 1 \geq \min\{1-r, r^2\}$. Hence $d(\exp(t_i' x), \exp(0)) \geq \min\{1-r, r^2\}$. $\square$

Corollary 5.2. *Let $x = \sum \alpha_j e_j \in A_\infty$. If there exists a subsequence $(\alpha_{j_i}) \subseteq (\alpha_j)$ and an $r > 0$ such that $\lim_i |\alpha_{j_i} r^{j_i}| = c > 0$ then $\overline{\exp(\mathbb{R}x)}$ is an embedded Euclidean line.*

Theorem 5.5. *There exist $x \in A_\infty$ such that $\overline{\exp(\mathbb{R}x)} = T_H$.*

Proof. The proof is motivated by a problem from symbolic dynamics. Let X be the space of of sequences of the numbers 1 and 0. Let d be a metric defined for X as follows: for two sequences $(a_n), (b_n)$,

$$d((a_n), (b_n)) := \sum_{n=1}^{\infty} \frac{|a_n - b_n|}{2^n}$$

Let S be the shift map which maps a sequence $a_1, a_2, \ldots$ to the sequence $0, a_1, a_2, \ldots$. The problem is whether or not there exists a sequence (a_n) whose orbit under S is dense in X. The answer is yes, and the example is given by

$$(a_n) = 0, 1, 0, 0, 0, 1, 1, 0, 1, 1, 0, 0, 0, 0, 0, 1 \ldots,$$

The sequence consists of listing all sequences of length 1 then listing all sequences of length 2, etc.

For our theorem we will consider the number:

$$a = .1010010001000010000010000001\ldots$$

This number has the property that the $\lim 10^{\frac{n(n+1)}{2} - r} a = 10^{-r} \mod 1$.

By taking subsequences of the digits of a we generate an ℓ_2 sequence:

$$a_1 = .100001000000000000001\ldots \tag{2}$$

$$a_2 = .0010000001000000000000000001\ldots \tag{3}$$

$$a_3 = .00000000000000100000000000000000001\ldots \tag{4}$$

$$a_4 = .000000000000000000000000000000000000000000001\ldots \tag{5}$$

$$\vdots \tag{6}$$

The a_i's are chosen so that the digits of 1 make a snake–like pattern. Mod 1, multiplying this sequence by 10 is equivalent to the left–shift operator on the digits. By choosing the pattern of 1s and 0s in the sequence (a_i) as above we will be able to demonstrate that the image of $p(t) = \sum_i t a_i e_i$ is dense in A_∞. To be more precise, let $x = \sum a_i e_i$. Then x has the property that

$$d(\exp(10^{n_{ir}} x), \exp(10^{-r} e_i)) \to 0$$

where n_{ir} is a sequence of the form $n_{ir} = \frac{i(i+1)}{2} - r$.

For any element of the form $q = .d_1d_2 \dots d_m e_j$ we have that

$$d(\exp(d_1d_2 \dots d_m * 10^{n_{jm}}x), \exp(q)) \to 0.$$

If $q = \sum_{j=1}^{m} q_j e_j$ with each the decimal expansion of each q_j terminating at the m^{th} digit then

$$d\left(\exp\left(\left(\sum(q_j) * 10^{n_{jm}}\right)x\right), \exp(q)\right) \to 0$$

where (q_j) denotes the integer whose digits consists of the digits of the decimal expansion of q_j.

Since the set of elements of the form $\sum_{i=1}^{<\infty} \alpha_i e_i$ are dense in A_∞ and among those elements, the elements whose coefficients have terminating decimal expansions are also dense in A_∞, we have that $\exp(\mathbb{R}x)$ is dense in T_H. □

Corollary 5.3. *There exists a dense subset $\{x\}$ of A_∞ whose orbits $\{\exp(\mathbb{R}x)\}$ are dense in T_H.*

Proof. It's enough to observe that if $z \in A_\infty$ and $\epsilon > 0$ there exists an a $z' = \sum_{i=1}^{m} \alpha_i e_i + \sum_{i=m+1}^{\infty} a_{i-m} e_i$ for some m such that $|z - z'| < \epsilon$ and the set $\{\alpha_i\} \cup \{a_i\}$ is rationally independent. □

Remark 5.4. The closure of any 1-parameter group for T^n is necessarily a Lie group. In the case of T_H however, this is not true. In the following theorem we prove for particular $x \in A_\infty$ that $\overline{\exp(\mathbb{R}x)}$ is a topological group with an uncountable number of components. Groups of this type we denote as "dna" groups simply because a mental picture of a double helix is the easiest picture we can think of which suggests the form of these groups.

Theorem 5.6. *Let (p_i) denote a strictly increasing sequence of prime numbers. Let (r_i) denote a sequence of positive integers with the properties:*

- $r_i \geq 2 \, \forall i$
- $\lim_{n\to\infty} ({p_1}^{r_1} \dots {p_n}^{r_n})^2 \sum_{i=n+1}^{\infty} (\frac{1}{{p_i}^{r_i}})^2 = 0$

If $x = \sum \frac{1}{{p_i}^{r_i}} e_i$ then $\overline{\exp(\mathbb{R}x)}$ is a dna group.

Proof. From the hypothesis we have $d(\exp(({p_1}^{r_1} \dots {p_n}^{r_n})^2 x), \exp(0)) \to 0$. Hence $\overline{\exp(\mathbb{R}x)}$ cannot be a Euclidean line embedded in T_H. And since none of the coefficients of the expansion of x are zero, $\overline{\exp(\mathbb{R}x)}$ cannot be a torus.

We determine some of the cluster points of $\exp(\mathbb{R}x)$. Let $t_n := {p_2}^{r_2} \dots {p_n}^{r_n}$. Then the cluster points of $\{\exp(t_n x)\}$ are of the form: $\exp(\frac{a}{{p_1}^{r_1}} e_1)$, $0 < a < {p_1}^{r_1}$.

Similarly, it's not too hard to see that points of the form $\exp(\frac{a}{{p_n}^{r_n}} e_n)$, $0 < a < {p_n}^{r_n}$ are cluster points as well of $\exp(\mathbb{R}x)$.

Let g_n denote the element $\exp(\frac{1}{{p_n}^{r_n}}e_n)$ and G_n the abelian group of order ${p_n}^{r_n}$ generated by g_n. We claim

$$\exp(\mathbb{R}x) \oplus \bigoplus_{n=1}^{\infty} G_n = \overline{\exp(\mathbb{R}x)}$$

where it's understood that elements of $\bigoplus_{n=1}^{\infty} G_n$ must be ℓ_2 summable in T_H. It's clear that

$$\exp(\mathbb{R}x) \oplus \bigoplus_{n=1}^{\infty} G_n \subseteq \overline{\exp(\mathbb{R}x)}$$

To show inclusion in the other direction we sketch an induction proof. Let $y \in \overline{\exp(\mathbb{R}x)}$. Let $T_n := \exp(\mathbb{R}e_1 + \mathbb{R}e_2 + ... + \mathbb{R}e_n)$ and let $\pi_n : T_H \to T_n$ denote the projection operator that maps elements of the of T_H onto the n-torus. Note that for any $y \in T_H$ the limit of the projections $\{\pi_n(y)\}$ has to converge to y. Since the set $\{{p_1}^{r_1}, \dots, {p_n}^{r_n}\}$ is rationally dependent, $\pi_n(\mathbb{R}x)$ is necessarily a circle with period ${p_1}^{r_1} \dots {p_n}^{r_n}$.

Since the map $\pi_1 : T_H \to T_1$ is continuous, and $\pi_1(\exp(\mathbb{R}x))$ is a circle there is a $t \in \mathbb{R}$ such that

$$\pi_1(\exp((t + n_1 {p_1}^{r_1})x)) = \pi_1(y) \quad n_1 \in \mathbb{Z}$$

Let $y_1 = \exp((t + n_1 {p_1}^{r_1})x)$. Now consider $\pi_2(y)$. For some $n_1 \in \mathbb{Z}$ we have

$$\pi_2(\exp((t + n_1 {p_1}^{r-1} + n_2 {p_1}^{r_1} {p_2}^{r_2})x)) = \pi_2(y) \quad n_2 \in \mathbb{Z}$$

Let $y_2 = \exp((t + n_1 {p_1}^{r-1} + n_2 {p_1}^{r_1} {p_2}^{r_2})x)$. We can proceed in this fashion such that at the m^{th} induction step we have

$$\pi_m \left(\exp \left(\left(t + \sum_{i=1}^{m+1} n_i {p_1}^{r_1} \cdots {p_i}^{r_i} \right) x \right) \right) = \pi_m(y) \quad n_1, \dots, n_{m+1} \in \mathbb{Z}$$

and define the m^{th} in a sequence as $y_m = \exp((t + \sum_{i=1}^{m+1} n_i {p_1}^{r_1} \cdots {p_i}^{r_i})x)$. In this fashion by choosing the $\{n_i\}$ carefully we generate a Cauchy sequence $\{y_m\}$ which converges to y. Observe that $\exp(tx) \in \exp(\mathbb{R}x)$ and that the sequence of points $y_i \circ \exp(-tx)$ converges to an element of $\bigoplus_{n=1}^{\infty} G_n$. Hence $y \in \exp(\mathbb{R}x) \oplus \bigoplus_{n=1}^{\infty} G_n$. Hence $\overline{\exp(\mathbb{R}x)} = \exp(\mathbb{R}x) \oplus \bigoplus_{n=1}^{\infty} G_n$.

Since $\bigoplus_{n=1}^{\infty} G_n$ has uncountably many components, $\overline{\exp(\mathbb{R}x)}$ is a "dna" group. □

REMARK 5.5. Just as for the first two type of geodesics, geodesics of the above form are also dense in T_H.

6. Functions which Poisson-commute with the Energy Hamiltonian

From [9] we know that the pullback of any function $\tilde{f} \in \mathcal{C}^\infty(\mathcal{U}_{HS}(H))$ Poisson-commutes with the kinetic energy Hamiltonian $H \in \mathcal{C}^\infty(TGr(H))$ (resp. $H_r \in \mathcal{C}^\infty(TGr(r, H))$). We would like to conclude the converse as well, that is, if $f \in \mathcal{C}^\infty(TGr(H))$ (resp. $f \in \mathcal{C}^\infty(TGr(r, H))$ Poisson-commutes with H (resp. H_r) then f must be the pullback of some function $\tilde{f}$ of $\mathcal{U}_{HS}(H)$. In fact, we can.

Theorem 6.1. *If $f \in \mathcal{C}^\infty(TGr(H))$ Poisson-commutes with H then f is the pull-back by μ of some function $\tilde{f} : \mathcal{U}_{HS}(H) \to \mathbb{R}$.*

Proof. From the previous section there exists a dense set of Hilbert tori in $TGr(H)$ generated by the geodesic flow which are contained in the level surfaces of f. For any 2 points $(p_1, v_1), (p_2, v_2) \in TGr(H)$ such that $\mu(p_1, v_1) = \mu(p_2, v_2)$ there is a path connecting the points along one of the mentioned Hilbert tori.
Hence $f(p_1, v_1) = f(p_2, v_2)$. Hence f is determined by μ. □

Remark 6.1. The proof is the same if we replace $TGr(H)$ by $TGr(r, H)$ and Hilbert tori by tori of finite rank r.

7. Conjugate points

Since the Grassmannians are symmetric we can concentrate on the geodesics passing through the point o. Let $\gamma(t)$ denote a geodesic of $Gr(H)$ with unit velocity and $o = \gamma(0)$. Let $T \in \mathcal{M}$ represent $\gamma'(0)$. We have then:

Theorem 7.1. *Conjugate points of o along the geodesic γ occur at*

$$t = \frac{\pi}{(|\lambda_i| + |\lambda_j|)^2} \quad \text{and at} \quad t = \frac{\pi}{(|\lambda_i| - |\lambda_j|)^2} \quad (|\lambda_i| \neq |\lambda_j|)$$

where $F(e, T) = (\lambda_i)$.

Proof. By [6] the symmetric operator which determines the behavior of Jacobi fields is given by

$$[T, [\underline{\quad}, T]] : T_oGr(H) \to T_oGr(H).$$

The eigenvalues of $[T, [\underline{\quad}, T]]$ are explicitly computed to be $(|\lambda_i| - |\lambda_j|)^2$ and $(|\lambda_i| + |\lambda_j|)^2$ where $F(e, T) = (\lambda_i)$. □

Remark 7.1. The above theorem is the same for the finite-rank Grassmannians.

This theorem demonstrates again the fundamental difference between the finite-rank Grassmannians and the Grassmannian $Gr(H)$. For $Gr(r, H)$ if we consider any unit velocity geodesic γ passing through an arbitrary point o then a conjugate point of p along the geodesic γ has to occur by $t = r\pi/4$. Moreover, the conjugate points along any geodesic will be discretely spaced with no cluster points. On the other hand, for $Gr(H)$ if we again consider an arbitrary point p no such upper bound exists for the first conjugate point of p along every every unit velocity geodesic. There is also a dense set S of geodesics passing through p for which the conjugate points of p along any geodesic in S have cluster points.

8. Conclusion

While the Hilbert Grassmannians are all locally diffeomorphic, we have seen that based on the behavior of the geodesics $Gr(H)$ is globally fundamentally more complex than the finite rank Grassmannians. In spite of the complexity of the Grassmannians we've also been able to extend the results of [9] and [3] to our Grassmannians at the price of weakening the definition of integrability. We've also determined that for geodesic flow there is no other method of generating a complete set of Poisson-commuting functions other than using Thimm's algorithm.

This paper has been a summary of the author's dissertation [2].

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