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Three Results on Mixing Shapes

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Abstract. Let α be a $\mathbf{Z}^d$-action ($d \geq 2$) by automorphisms of a compact metric abelian group. For any non-linear shape $I \subset \mathbf{Z}^d$, there is an α with the property that I is a minimal mixing shape for α. The only implications of the form "I is a mixing shape for $\alpha \implies J$ is a mixing shape for α" are trivial ones for which I contains a translate of J.

If all shapes are mixing for α, then α is mixing of all orders. In contrast to the algebraic case, if β is a $\mathbf{Z}^d$-action by measure-preserving transformations, then all shapes mixing for β does not preclude rigidity.

Finally, we show that mixing of all orders in cones — a property that coincides with mixing of all orders for $\mathbf{Z}$–actions — holds for algebraic mixing $\mathbf{Z}^2$-actions.

Contents

1. Introduction

Let α be a measure-preserving action of $\mathbb{Z}^d$ on a standard probability space $(X, \mathcal{B}, \mu)$ ($d \geq 2$). If X is a compact metrizable abelian group, μ is Haar measure, and each $\alpha_{\mathbf{n}}$ is a group automorphism, then α is an algebraic dynamical system (as studied in [10], where the notions below are found).

The action α is rigid if there is a sequence $\mathbf{n}_j \to \infty$ (going to infinity means leaving finite sets) with the property that $\mu(A \cap \alpha_{\mathbf{n}_j} A) \to \mu(A)$ as $j \to \infty$ for all $A \in \mathcal{B}$. The action α is mixing of all orders if for all $r \geq 1$ and for all sets $B_1, \dots, B_r$ in $\mathcal{B}$,

$$\lim_{\mathbf{n}_l \in \mathbb{Z}^d \text{ and } \mathbf{n}_l - \mathbf{n}_{l'} \to \infty \text{ for } 1 \leq l' < l \leq r} \mu\left(\bigcap_{l=1}^{r} \alpha_{-\mathbf{n}_l}(B_l)\right) = \prod_{l=1}^{r} \mu(B_l).$$

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The shape $F = \{\mathbf{n}_1, \dots, \mathbf{n}_r\}$ is mixing for α if for all sets $B_1, \dots, B_r$ in $\mathcal{B}$,

$$\lim_{k\to\infty} \mu\left(\bigcap_{l=1}^{r} \alpha_{-k\mathbf{n}_l}(B_l)\right) = \prod_{l=1}^{r} \mu(B_l).$$

The shape F is a minimal non-mixing shape for α if F is non-mixing but any subset of F is mixing. A shape is admissable if it does not lie on a line in $\mathbb{Z}^d$, it contains 0, and for any $k > 1$ the set $\frac{1}{k}S$ contains non-integral points.

For the last mixing property, take $d = 2$ for simplicity and let α be a measure-preserving $\mathbb{Z}^2$-action on $(X, \mathcal{B}, \mu)$ as before. An oriented line through the origin in $\mathbb{Z}^2$ is a half-line starting at the origin. An oriented cone $\mathcal{C} = (\ell_1, \ell_2)$ in $\mathbb{Z}^2$ is the region between an ordered pair (ℓ_1, ℓ_2) of oriented half lines, including the edges. Notice that if $\ell_1 = \ell_2$ then the cone (ℓ_1, ℓ_2) comprises exactly a half-line. The cone defined by no lines is all of $\mathbb{Z}^2$. Given a collection $\{\ell_j\}$ of half-lines, there is an associated collection of oriented cones $\{\mathcal{C}_j\}$ where $\mathcal{C}_j$ is the cone associated to the ordered pair (ℓ_j, ℓ_{j+1}) (if there are n lines, with $j+1$ reduced mod n).

The $\mathbb{Z}^2$-action α is mixing of all orders in the oriented cone $\mathcal{C}$ if for every $r \geq 1$ and all sets $B_1, \dots, B_r$ in $\mathcal{B}$,

$$\lim_{\mathbf{n}_j \in \mathcal{C} \text{ and } \mathbf{n}_j \to \infty \text{ for } 1 \le j \le r} \mu\left(\bigcap_{l=1}^{r} \alpha_{-(\mathbf{n}_1+\mathbf{n}_2+\cdots+\mathbf{n}_l)}(B_l)\right) = \prod_{l=1}^{r} \mu(B_l). \tag{1}$$

Theorem 1.1. *If S is any admissable shape, then there is an algebraic $\mathbb{Z}^d$-action for which S is a minimal non-mixing shape. If S and T are admissable shapes, then there is an algebraic $\mathbb{Z}^d$-action that is mixing on S and not mixing on T unless a translate of T is a subset of S.*

That is, the poset formed by equivalence classes (under translation) of admissable shapes in $\mathbb{Z}^d$, partially ordered by inclusion, embeds in the hierarchy of mixing properties for $\mathbb{Z}^d$-actions.

Theorem 1.2. *If α is an algebraic $\mathbb{Z}^d$-action for which every shape is mixing, then α is mixing of all orders. In general, a measure-preserving $\mathbb{Z}^d$-action for which every shape is mixing can be rigid.*

Notice that the notion of mixing shapes still makes sense for $d = 1$, and there it is not clear whether in general all shapes mixing implies mixing of all orders.

For the next theorem, notice that if an action α is mixing of all orders in the oriented cones associated to a family of lines $\mathcal{L}$, then the same is true of any larger family $\mathcal{L}' \supset \mathcal{L}$. It follows that the object of interest is the smallest set of lines for which the property holds. Examples related to parts (b) and (c) of Theorem 1.3 are given below (Example 3.5).

Theorem 1.3. *Let α be a mixing algebraic $\mathbb{Z}^2$-action on the compact abelian group X. Then there is a collection $\mathcal{L} = \{\ell_j\}$ of half-lines in $\mathbb{Z}^2$ with the property that α is mixing of all orders in the oriented cones associated to the family of lines. Moreover,*

(a) *if X is connected then $\mathcal{L}$ may be taken to be empty;*
(b) *if α is expansive then $\mathcal{L}$ may be taken to be finite;*
(c) *if α is not expansive and X is not connected, then the smallest such set $\mathcal{L}$ may contain a line through every point in $\mathbb{Z}^2$.*

2. Proofs of Theorems 1.1 and 1.2

Let R be any ring; a polynomial $f \in R[u_1^{\pm 1}, \dots, u_d^{\pm 1}]$ may be written $\sum_{\mathbf{n} \in S} c_{\mathbf{n}} \mathbf{u}^{\mathbf{n}}$, where each $c_{\mathbf{n}} \in R \backslash \{0\}$, and $\mathbf{u}^{\mathbf{n}}$ is the monomial $u_1^{n_1} \dots u_d^{n_d}$. The set $S = \mathrm{Supp}(f)$ is the support of f. If R is an integral domain, then the polynomial f is absolutely irreducible if f is irreducible over an algebraic closure of the field of fractions of R. A polynomial is primitive if its support includes the origin and is not an integer dilate of another set.

Let $\mathfrak{R} = \mathbb{Z}[u_1^{\pm 1}, \dots, u_d^{\pm 1}]$ and $\mathfrak{R}_p = \mathbb{F}_p[u_1^{\pm 1}, \dots, u_d^{\pm 1}]$. Following [10], if $\mathfrak{M}$ is a module over $\mathfrak{R}$, then the d commuting automorphisms given by multiplication by $u_1, \dots, u_d$ have as duals d commuting automorphism of $X = \widehat{\mathfrak{M}}$, defining an algebraic $\mathbb{Z}^d$-action $\alpha^{\mathfrak{M}}$ on X. Conversely, any algebraic action is of the form $\alpha^{\mathfrak{M}}$ for some $\mathfrak{R}$-module $\mathfrak{M}$. Notice that any $\mathfrak{R}_p$-module is an $\mathfrak{R}$-module.

Proof of Theorem 1.1. The following result is proved in Section 3 of [4]: if the polynomials

$$f^{(k)}(u_1, \dots, u_d) \;=\; f(u_1^k, \dots, u_d^k)$$

have no primitive irreducible factors for any $k \geq 1$ (apart from k a power of p), and the support of f is the admissable shape S, then S is a minimal non-mixing shape for the $\mathbb{Z}^d$-action $\alpha^{\mathfrak{R}_p/\langle f \rangle}$.

So it is enough to show that for any admissable shape S there is a prime p, and a polynomial f over $\mathbb{F}_p$ whose support is S and with the property that $f^{(k)}$ is absolutely irreducible for all $k \geq 1$. By Lemma 3.10 of [4] (see also Theorem I,II in [3]), if $\mathrm{Supp}(f)$ is admissable, then there is an $N(\mathrm{Supp}(f))$ with the property that if $f^{(k)}$ has no primitive irreducible divisors over $\bar{\mathbb{F}}_p$ for $1 \leq k \leq N(\mathrm{Supp}(f))$, then $f^{(k)}$ has no primitive irreducible divisors for all k not a power of p.

Fix an admissable shape S with $s = |S|$, an integral domain R, and a generic polynomial $h \in R[u_1^{\pm 1}, \dots, u_d^{\pm 1}]$ with support S. Then $h = h(\mathbf{u}) = h(u_1, \dots, u_d)$ is a polynomial $h^*(\mathbf{u}, \mathbf{a}) \in R[\mathbf{u}, a_1, \dots, a_s]$ in which the variables $a_1, \dots, a_s$ all appear with degree one. By the Bertini-Noether Theorem (Proposition 9.29 in [2]), there exist polynomials $R_1, \dots, R_t \in R[\mathbf{a}]$ with the property that $h^*(\mathbf{u}, \mathbf{a}^0)$ is absolutely irreducible if and only if at least one of $R_1(\mathbf{a}^0), \dots, R_t(\mathbf{a}^0)$ is not zero. So, if the polynomial $h(\mathbf{u}, \mathbf{a})$ is absolutely irreducible over $\mathbb{Q}(\mathbf{a})$, then the polynomials $R_1, \dots, R_t$ don't vanish identically. Therefore, in this case there exists $\mathbf{a}^0$ integral such that for all but finitely many primes p, $\bar{h}(\mathbf{u}, \mathbf{a}^0)$ is absolutely irreducible over $\mathbb{F}_p$ and $\mathrm{Supp}(\bar{h}(\mathbf{u}, \mathbf{a}^0) = S$, where $g \mapsto \bar{g}$ is the canonical map $\mathbb{Z} \to \mathbb{F}_p$. Now consider the collection of all the polynomials $h^*(\mathbf{u}, \mathbf{a})$ with support S. By Bertini's Theorem (see Theorem I.11.18 of [9] or Theorem IX.6.17 of [13]), the generic member of this linear system (of dimension greater than or equal to 2) is irreducible if and only if the general member is not composite with a pencil (h^* is composite with a pencil if $h^*(\mathbf{u}, \mathbf{a}) = P(Q(\mathbf{u}))$ with $P \in \mathbb{Q}(\mathbf{a})[\lambda]$). Assume the general member is composite with a pencil, and let $P(\lambda) = \sum_{i=0}^n a_i \lambda^i$ and $Q(\mathbf{u}) = \sum_{\mathbf{n} \in S_0} c_{\mathbf{n}} \mathbf{u}^{\mathbf{n}}$. Then $h^*(\mathbf{u}, \mathbf{a}) = \sum_{i=1}^n a_i \left(\sum_{\mathbf{n} \in S_0} c_{\mathbf{n}} \mathbf{u}^{\mathbf{n}} \right)^i$. Now count the number of coefficients that may be chosen freely in the family: in $h^*(\mathbf{u}, \mathbf{a})$ there are s; in $P(Q(\mathbf{u}))$ there are n, so $s = n$. On the other hand, the support of the family $P(Q(\mathbf{u}))$ has cardinality $|S_0| + |2S_0| + \dots + |nS_0|$ where $2S_0 = \{\mathbf{n} + \mathbf{m} \mid \mathbf{n} \in S_0, \mathbf{m} \in S_0\}$ and so on. If $|S_0| > 1$, then it follows that the cardinality of the support of the family of $P(Q(\mathbf{u}))$ exceeds s, which is impossible. If $|S_0| = 1$, then Q is a monomial, so the shape S

is not admissable, contrary to our assumption. We deduce that the family $h^*(\mathbf{u},\mathbf{a})$ is not composite with a pencil, and therefore is generically absolutely irreducible. Now apply the bound $N(\mathrm{Supp}(f))$ to deduce that the generic specialization $h(\mathbf{u},\mathbf{a}^0)$ has the property that for all but finitely many primes, the reduction mod p is a polynomial f with $\mathrm{Supp}(f)=S$ and with $f^{(k)}$ absolutely irreducible for all $k\geq 1$ not a power of p. By the remarks above, this shows that there is an algebraic $\mathbb{Z}^d$-action for which S is a minimal non-mixing shape.

Now fix two admissable shapes S and T, with the properties that for all $\mathbf{n}\in\mathbb{Z}^d$, $T+\mathbf{n}\not\subset S$, and $0\in S\cap T$. By the construction above, we can find a polynomial f in the ring $\mathfrak{R}$ with the property that, for a generic prime p, the reduction mod p of f gives a polynomial $\bar{f}$ in $\mathfrak{R}_p$ whose support is S and which has the property that $f^{(k)}$ has no primitive irreducible factors for k not a power of p.

It follows from Proposition 28.9 in [10] that for a generic prime p, the $\mathbb{Z}^d$-action $\alpha^{\mathfrak{R}_p/\langle\bar{f}\rangle}$ has S as its unique extremal non-mixing set (see Definition 28.8 in [10]). We now need to show that, for an appropriate choice of the prime p, the shape T is a mixing set for $\alpha^{\mathfrak{R}_p/\langle\bar{f}\rangle}$. This is not guaranteed because of possible cancellations mod p.

The following example (Example 28.10(7) in [10]) illustrates the problem. If $f(u_1,u_2)=1+u_1+u_2$, and p is chosen to be 2, then $\{(0,0),(1,0),(0,1)\}$ is the unique extremal non-mixing set for $\alpha^{\mathfrak{R}_2/\langle\bar{f}\rangle}$, but the identity

$$(1+u_1+u_2)(1+u_1)=1+u_1^2+u_2+u_1u_2 \mod 2$$

shows that the set $\{(0,0),(2,0),(0,1),(1,1)\}$ is also a minimal non-mixing set for $\alpha^{\mathfrak{R}_2/\langle\bar{f}\rangle}$. However, choosing for the fixed shape $T=\{(0,0),(2,0),(0,1),(1,1)\}$ a sufficiently large prime p (in this case, $p>2$ will suffice), this cancellation will not occur mod p and so the shape T will be mixing for $\alpha^{\mathfrak{R}_p/\langle\bar{f}\rangle}$.

Similarly, by Proposition 28.9 in [10] if the prime p is chosen large enough for the given shape T, the shape T will be mixing for the action $\alpha^{\mathfrak{R}_p/\langle\bar{f}\rangle}$. $\square$

Proof of Theorem 1.2. The first part follows from characterisations of higher-order mixing and mixing shapes for algebraic dynamical systems in Sections 27 and 28 of [10].

Before turning to the second part of Theorem 1.2, we assemble some basic facts about Gaussian processes (see for instance [12]). The entropy of a d-dimensional Gaussian process has been computed in [8]. Define a measure space by $(\Omega,\mathcal{F}_0)=\prod_{\mathbf{n}\in\mathbb{Z}^d}(\mathbb{R},\mathcal{B})$ where $\mathcal{B}$ is the Borel σ-algebra on $\mathbb{R}$. Let $\xi_{\mathbf{n}}(\omega)$ be the $\mathbf{n}$th coordinate of $\omega\in\Omega$. Let μ be a probability measure on $(\Omega,\mathcal{F}_0)$ with the property that for any k-tuple of integer vectors $\mathbf{n}_1,\dots,\mathbf{n}_k$ of the k-dimensional random variable $(\xi_{\mathbf{n}_1},\dots,\xi_{\mathbf{n}_k})$ is a k-dimensional Gaussian law, and the joint distribution is stationary in the sense that $\mu^{(\mathbf{n}_1+\mathbf{m},\dots,\mathbf{n}_k+\mathbf{m})}=\mu^{(\mathbf{n}_1,\dots,\mathbf{n}_k)}$ for any $\mathbf{m}\in\mathbb{Z}^d$. Let $\mathcal{F}$ denote the completion of $\mathcal{F}_0$ under μ. Then $(\Omega,\mathcal{F},\mu,\{\xi_{\mathbf{n}}\}_{\mathbf{n}\in\mathbb{Z}^d})$ is a d-dimensional Gaussian stationary sequence. Assume that $E\{\xi_{\mathbf{n}}\}=0$ for each $\mathbf{n}\in\mathbb{Z}^d$. The covariance function $R:\mathbb{Z}^d\to\mathbb{C}$ may be expressed in terms of a (symmetric) spectral measure ρ on $\mathbb{T}^d$ via Khinchine's decomposition, $R(\mathbf{n})=E\{\xi_{\mathbf{n}+\mathbf{m}}\xi_{\mathbf{m}}\}=\int_0^1\cdots\int_0^1 e^{-2\pi i(n_1s_1+\cdots+n_ds_d)}\rho(ds_1\dots ds_d)$. Conversely, if ρ is a symmetric finite measure on $\mathbb{T}^d$, then there is a unique d-dimensional Gaussian stationary sequence whose spectral measure is ρ.

Associated to any Gaussian stationary sequence of the above form there is a measure-preserving $\mathbb{Z}^d$-action α, defined by the shift on Ω. Standard approximation arguments (see [12]) give the following. Let $\mathcal{C}$ denote the class of functions $f:\Omega\to\mathbb{C}$ with the property that $f(\omega)=F(\xi_{\mathbf{m}_1}(\omega),\dots,\xi_{\mathbf{m}_t}(\omega))$ for some $\mathbf{m}_1,\dots,\mathbf{m}_t$ and some bounded continuous function $F:\mathbb{R}^t\to\mathbb{C}$. Let α be a Gaussian $\mathbb{Z}^d$-action. Then, in order to check any mixing property, it is sufficient to check it for functions in the class $\mathcal{C}$.

For each $\mathbf{n}\in\mathbb{Z}^d$, the $\mathbb{Z}$-action generated by the tranformation $\alpha_{\mathbf{n}}$ is again Gaussian, on $(\Omega,\mathcal{F}_{\mathbf{n}})$, where $\mathcal{F}_{\mathbf{n}}$ is the sub-σ-algebra of $\mathcal{F}$ generated by the projections $\{\xi_{k\mathbf{n}}\}_{k\in\mathbb{Z}}$. The spectral measure of $\alpha_{\mathbf{n}}$ is $\rho_{\mathbf{n}}=\rho\psi_{\mathbf{n}}^{-1}$, where $\psi_{\mathbf{n}}:\mathbb{T}^d\to\mathbb{T}$ is given by $\psi_{\mathbf{n}}(s_1,\dots,s_d)=n_1s_1+\cdots+n_ds_d \bmod 1$.

To exhibit an example for the second part of Theorem 1.2, we simply check that a simple modification of the construction of Ferenci and Kaminski in [1] has the stated properties. Choose $\mathbb{Q}$-independent numbers $1,\beta_1,\dots,\beta_d$, and let $f(t)=(\beta_1t,\dots,\beta_dt)\pmod 1$ for $t\in\mathbb{T}$ the additive circle. Let $\imath:\mathbb{T}^d\to\mathbb{T}^d$ be the involution $\imath(t_1,\dots,t_d)=(1-t_1,\dots,1-t_d)$, and let λ be Lebesgue measure on $\mathbb{T}^d$. Define a symmetric, singular, continuous measure ρ on $\mathbb{T}^d$ by $\rho=\frac{1}{2}\left(\lambda f^{-1}+\lambda(\imath\circ f)^{-1}\right)$. Let α be the Gaussian $\mathbb{Z}^d$-action with spectral measure ρ. The covariance function is given by

$$R(\mathbf{n})=\frac{\sin(2\pi(n_1\beta_1+\cdots+n_d\beta_d))}{2\pi(n_1\beta_1+\cdots+n_d\beta_d)}. \tag{2}$$

Choose a sequence $\mathbf{n}_j=(n_1^{(j)},\dots,n_d^{(j)})\to\infty$ for which $n_1^{(j)}\beta_1+\cdots+n_d^{(j)}\beta_d\to 0$ as $j\to\infty$. Then $R(\mathbf{n}_j)\to 1$ as $j\to\infty$. It follows that the $2t$-dimensional random Gaussian vector

$$\Phi_j(\omega)=\left(\xi_{\mathbf{m}_1}(\omega),\dots,\xi_{\mathbf{m}_t}(\omega),\xi_{\mathbf{m}_1-\mathbf{n}_j}(\omega),\dots,\xi_{\mathbf{m}_t-\mathbf{n}_j}(\omega)\right)$$

has covariance matrix $\begin{bmatrix}V_{00}^{(j)} & V_{10}^{(j)}\\ V_{01}^{(j)} & V_{11}^{(j)}\end{bmatrix}$, where $V_{00}^{(j)}=V_{11}^{(j)}$ is the covariance matrix V of $(\xi_{\mathbf{m}_1}(\omega),\dots,\xi_{\mathbf{m}_t}(\omega))$, and $V_{01}^{(j)}$ has (p,q)th entry

$$E\{\xi_{\mathbf{m}_p}\xi_{\mathbf{m}_q-\mathbf{n}_j}\}=R(\mathbf{m}_p-\mathbf{m}_q+\mathbf{n}_j)\to R(\mathbf{m}_p-\mathbf{m}_q)$$

as $j\to\infty$ by our choice of $\mathbf{n}_j$. Thus $V_{01}^{(j)}\to V$; similarly $V_{10}^{(j)}\to V$. By the remark above, this shows that $\mu(\alpha_{\mathbf{n}_j}(A)\cap A))\to\mu(A)$ for all $A\in\mathcal{F}$, so α is rigid.

Let $S=\{\mathbf{n}_1,\dots,\mathbf{n}_r\}$, and define a random vector of dimension $r\times t$ by $\Psi_k(\omega)=\left(\xi_{\mathbf{m}_i-k\mathbf{n}_j}(\omega)\mid i=1,\dots,t;j=1,\dots,r\right)$. This vector is Gaussian with zero mean and covariance matrix

$$V_k=\begin{bmatrix}V_k^{11} & V_k^{12} & \dots & V_k^{1r}\\ \vdots & & & \vdots\\ V_k^{r1} & V_k^{r2} & \dots & V_k^{rr}\end{bmatrix},$$

where V_k^{jl} is the $t\times t$ matrix whose (p,q)th element is

$$v_{(p,q)}^{(j,l)}(k)=E\left(\xi_{\mathbf{m}_p-k\mathbf{n}_j}\xi_{\mathbf{m}_q-k\mathbf{n}_l}\right)=\begin{cases}R(\mathbf{m}_p-\mathbf{m}_q) & \text{if } j=l\\ R(\mathbf{m}_p-\mathbf{m}_q+k\mathbf{n}_l-k\mathbf{n}_j) & \text{if } j\neq l.\end{cases}$$

Notice that $V_0 = V_k^{jj}$ is the covariance matrix of $(\xi_{\mathbf{n}_1}, \dots, \xi_{\mathbf{n}_t})$. For $j \neq l$, it is clear from (2) that

$$\lim_{k\to\infty} v_{(p,q)}^{(j,l)}(k) = 0,$$

so that

$$\lim_{k\to\infty} V_k = \begin{bmatrix} V_0 & 0 & \dots & 0 & \\ 0 & V_0 & \dots & 0 & \\ & & \ddots & & \\ 0 & 0 & \dots & V_0 & \end{bmatrix}.$$

It follows that α is mixing for all shapes. $\square$

3. Proof of Theorem 1.3

As in the proof of Theorem 1.1, the (countable) dual group $\mathfrak{M} = \widehat{X}$ is a module over the ring $\mathfrak{R} = \mathbb{Z}[u_1^{\pm 1}, u_2^{\pm 1}]$.

Following [11], expanding the characteristic functions of the sets appearing in (1) as Fourier series on X shows that property (1) is equivalent to the following: for any non-zero r-tuple $(m_1, \dots, m_r) \in \mathfrak{M}^r$,

$$\mathbf{u}^{\mathbf{n}_1} m_1 + \mathbf{u}^{\mathbf{n}_1+\mathbf{n}_2} m_2 + \dots + \mathbf{u}^{\mathbf{n}_1+\mathbf{n}_2+\dots+\mathbf{n}_r} m_r \neq 0 \tag{3}$$

whenever $\mathbf{n}_1, \dots, \mathbf{n}_r \in \mathcal{C}$ lie outside some sufficiently large finite set in $\mathbb{Z}^2$ (how large depending on the characters $(m_1, \dots, m_r) \in \mathfrak{M}^r$).

Recall that a prime ideal $\mathfrak{p} \subset \mathfrak{R}$ is associated with the module $\mathfrak{M}$ if there is an element $m \in \mathfrak{M}$ for which $\mathfrak{p} = \{f \in \mathfrak{R} \mid f \cdot m = 0 \in \mathfrak{M}\}$. The basic mixing behaviour is governed by the following lemmas.

Lemma 3.1. *The following conditions are equivalent*:

(i) $\alpha^{\mathfrak{M}}$ *is mixing.*
(ii) $\alpha_{\mathbf{n}}^{\mathfrak{M}}$ *is ergodic for every* $\mathbf{n} \neq 0$.
(iii) *No prime ideal associated with the module* $\mathfrak{M}$ *contains a polynomial of the form* $\mathbf{u}^{\mathbf{m}}\phi(\mathbf{u}^{\mathbf{n}})$ *where* ϕ *is cyclotomic.*

Proof. See Proposition 6.6(3) in [10] $\square$

Lemma 3.2. *The following conditions are equivalent*:

(i) $\alpha^{\mathfrak{M}}$ *is mixing of all orders in the cone* $\mathcal{C}$.
(ii) *For every prime ideal* $\mathfrak{p}$ *associated with* $\mathfrak{M}$, $\alpha^{\mathfrak{R}/\mathfrak{p}}$ *is mixing of all orders in the cone* $\mathcal{C}$.

Proof. This follows from the proof of Theorem 2.2 in [11] or Theorem 27.2 in [10] by restricting those proofs to the special sequence of mixing times in the cone. $\square$

Lemma 3.3. *If* $X = X^{\mathfrak{M}}$ *is connected, and* $\alpha^{\mathfrak{M}}$ *is mixing, then* $\alpha^{\mathfrak{M}}$ *is mixing of all orders.*

Proof. This is proved in [11]. $\square$

According to Lemma 3.2, in order to prove Theorem 1.3 it is sufficient to consider mixing actions of the form $\alpha^{\mathfrak{R}/\mathfrak{p}}$ on $X^{\mathfrak{R}/\mathfrak{p}}$. If $X^{\mathfrak{R}/\mathfrak{p}}$ is connected, then by Lemma 3.3 the action $\alpha^{\mathfrak{R}/\mathfrak{p}}$ is mixing of all orders, which proves Theorem 1.3 (a).

Assume therefore that $X^{\mathfrak{R}/\mathfrak{p}}$ is not connected. It follows that $p = \operatorname{char}(\mathfrak{R}/\mathfrak{p})$ is a rational prime. Let $\mathfrak{R}_p = \mathbb{F}_p[u_1^{\pm 1}, u_2^{\pm 2}]$; then $\mathfrak{R}/\mathfrak{p}$ becomes $\mathfrak{R}_p/\mathfrak{q}$ for a prime ideal

$\mathfrak{q} \subset \mathfrak{R}_p$. Notice that the ideal $\mathfrak{q}$ may be $\{0\}$: in this case the original ideal $\mathfrak{p}$ must have been $p \cdot \mathbb{Z}[u_1^{\pm 1}, u_2^{\pm 1}]$. The corresponding $\mathbb{Z}^2$-action is the full two-dimensional shift on p symbols which is mixing of all orders. From now on we therefore assume that $\mathfrak{q}$ is non-zero. By Proposition 25.5 of [10], if α is ergodic then $\mathfrak{q}$ must be principal, so it is enough to look at mixing $\mathbb{Z}^2$-actions of the form $\alpha^{\mathfrak{R}_p/\langle f\rangle}$, where $f \in \mathfrak{R}_p$. For any polynomial $g \in \mathfrak{R}_p$, let $CH(g)$ denote the convex hull of $\mathrm{Supp}(g)$. Choose a finite set of oriented lines through the origin $\mathcal{L}(f)$ with the following properties:

(i) For each extreme point $\mathbf{n}$ of $CH(f)$, there is a line $\ell(\mathbf{n}) \in \mathcal{L}(f)$ such that $CH(f)\backslash\{\mathbf{n}\}$ is entirely contained in one of the open half-planes defined by the line parallel to $\ell(\mathbf{n})$ through $\mathbf{n}$.
(ii) All the cones defined by $\mathcal{L}(f)$ are strictly acute.

The group $X = X^{\mathfrak{R}_p/\langle f\rangle}$ has the following form. If $f = \sum_{\mathbf{n}\in\mathrm{Supp}(f)} f_{\mathbf{n}}\mathbf{u}^{\mathbf{n}}$, then

$$X^{\mathfrak{R}_p/\langle f\rangle} = \{\mathbf{x} \in \mathbb{F}_p^{\mathbb{Z}^2} \mid \sum_{\mathbf{n}\in\mathrm{Supp}(f)} f_{\mathbf{n}} x_{\mathbf{n}+\mathbf{m}} = 0 \in \mathbb{F}_p \text{ for all } \mathbf{m} \in \mathbb{Z}^2\}. \tag{4}$$

When described in this way, the $\mathbb{Z}^2$-action $\alpha^{\mathfrak{R}_p/\langle f\rangle}$ is the shift on the closed shift-invariant subgroup of $\mathbb{F}_p^{\mathbb{Z}^2}$ defined by (4).

Lemma 3.4. *If $\mathcal{C}$ is a cone determined by the lines $\mathcal{L}(f)$ and $\alpha^{\mathfrak{R}_p/\langle f\rangle}$ is mixing, then $\alpha^{\mathfrak{R}_p/\langle f\rangle}$ is mixing of all orders in $\mathcal{C}$.*

Proof. First notice that the set $\mathrm{Supp}(f)$ does not lie on a line — if it did, then f would be a polynomial in a single monomial $t = \mathbf{u}^{\mathbf{n}}$ say. In this case the action of $\alpha_{\mathbf{n}}^{\mathfrak{R}_p/\langle f\rangle}$ is isomorphic to the infinite direct product of one-dimensional systems determined by the $\mathbb{Z}[t^{\pm 1}]$-module $\mathbb{Z}[t^{\pm 1}]/\langle p, f\rangle$. Since the ideal $\langle p, f\rangle$ is non-principal and $\mathbb{Z}[t^{\pm 1}] \otimes \mathbb{Q}$ is a principal ideal domain, the group $\mathbb{Z}[t^{\pm 1}]/\langle p, f\rangle$ is finite (see Examples 6.17(3) in [10]). It follows that $\alpha_{\mathbf{n}}^{\mathfrak{R}_p/\langle f\rangle}$ is periodic and therefore cannot be mixing.

Fix the cone $\mathcal{C}$. With the chosen ordering described in Section 1, the cone $\mathcal{C}$ is defined by a "bottom" half-line ℓ_1 and a "top" half-line ℓ_2. Each polynomial $h \in \mathfrak{R}_p$ defines a character on $X = X^{\mathfrak{R}_p/\langle f\rangle}$. Two polynomials h_1 and h_2 will define the same character if $h_1 - h_2 \in \langle f\rangle$. Denote by $\overline{h}$ a single character on X, and let h denote any polynomial that defines that character. Each character $\overline{h}$ with $\mathrm{Supp}(h) \subset \mathcal{C}$ has a distinguished representative $\widetilde{h}$, defined as follows. Let $B_f(\mathcal{C})$ denote the half-open strip along the bottom ($= \ell_1$) edge of $\mathcal{C}$, with width exactly equal to the width of $CH(f)$ in the direction orthogonal to ℓ_1. The polynomial $\widetilde{h}$ is defined by the following two properties:

(i) $\widetilde{h}$ defines the character $\overline{h}$.
(ii) $\mathrm{Supp}(\widetilde{h}) \subset B_f(\mathcal{C})$.

There is such a representative: by construction there is a line parallel to ℓ_1 that meets $\mathrm{Supp}(f)$ in a singleton and has the property that any other line parallel to ℓ_1 above it does not meet $\mathrm{Supp}(f)$. It follows that if $\mathbf{n} \in \mathrm{Supp}(h)\backslash B_f(\mathcal{C})$, an appropriate multiple (of the form $c\mathbf{u}^{\mathbf{m}}f$ with $c \in \mathbb{F}_p$) of f may be added to h to give h' with $\mathbf{n} \notin \mathrm{Supp}(h')$ and with the top edge of $\mathrm{Supp}(h')$ the same as the top edge of $\mathrm{Supp}(h)$ at all points other than $\mathbf{n}$. After finitely many such additions, we end up with the desired polynomial $\widetilde{h}$.

CLAIM 1: The representative $\widetilde{h}$ is unique. That is, $\overline{h_1} = \overline{h_2}$ if and only if $\widetilde{h_1} = \widetilde{h_2}$.

To see this, first notice that if $\widetilde{h_1} = \widetilde{h_2}$, then $\overline{h_1} = \overline{h_2}$. Now the set $B_f(\mathcal{C})$ has, by construction, the following property: given any element $\mathbf{y} \in \mathbb{F}_p^{B_f(\mathcal{C})}$, there is an element $\mathbf{y}^* \in X$ such that $\mathbf{y}^*$ restricted to $B_f(\mathcal{C})$ coincides with $\mathbf{y}$. This is clear from (8). If then $\widetilde{h_1} \neq \widetilde{h_2}$, there is a point $\mathbf{n} \in B_f(\mathcal{C})$ with $(h_1)_\mathbf{n} \neq (h_2)_\mathbf{n}$; choose $\mathbf{y} \in \mathbb{F}_p^{B_f(\mathcal{C})}$ with the property that the characters defined by h_1 and h_2 differ on this point. Then $\overline{h_1}$ and $\overline{h_2}$ must differ on $\mathbf{y}^*$.

For a character $\overline{h}$ with $\mathrm{Supp}(h) \subset \mathcal{C}$ define a number $r(\overline{h})$ by $r(\overline{h}) = k$ if the line orthogonal to ℓ_2 most distant from the origin that intersects $\mathrm{Supp}(\widetilde{h})$ meets ℓ_1 at distance k from the origin.

CLAIM 2: If $\mathbf{n} \in \mathcal{C}$, then $r(\mathbf{u}^\mathbf{n} h) > r(h)$.

This is clear: the polynomial $\mathbf{u}^\mathbf{n}\widetilde{m}$ has an associated representative $\widetilde{\mathbf{u}^\mathbf{n}\widetilde{m}}$ obtained by adding multiples of monomials times f. There is a face of $CH(f)$ orthogonal to ℓ_2, so the support of the resulting polynomial moves further away from the origin.

Now consider property (3). Let $m_1, \dots, m_r$ be a collection of polynomials, not all zero, with $\mathrm{Supp}(m_i) \in \mathcal{C}$ (if this is not the case, multiply all of them by a monomial $\mathbf{u}^\mathbf{n}$ to ensure their supports move into $\mathcal{C}$). By the second claim, if $\mathbf{n}_2, \dots, \mathbf{n}_r \in \mathcal{C}$ are large enough, then for each $j = 2, \dots, r$ the set $\mathrm{Supp}(\widetilde{\mathbf{u}^{\mathbf{n}_1 + \cdots + \mathbf{n}_j} m_j})$ contains points not in

$$\mathrm{Supp}(\widetilde{\mathbf{u}^{\mathbf{n}_1} m_1} + \widetilde{\mathbf{u}^{\mathbf{n}_1+\mathbf{n}_2} m_2} + \cdots + \widetilde{\mathbf{u}^{\mathbf{n}_1+\cdots+\mathbf{n}_{j-1}} m_{j-1}}).$$

By the first claim, it follows that the character

$$\mathbf{u}^{\mathbf{n}_1} m_1 + \mathbf{u}^{\mathbf{n}_1+\mathbf{n}_2} m_2 + \cdots + \mathbf{u}^{\mathbf{n}_1+\mathbf{n}_2+\cdots+\mathbf{n}_r} m_r$$

is non-trivial, proving Lemma 3.4. □

Proof of Theorem 1.3. Let $\mathfrak{M}$ be the $\mathfrak{R}$-module associated to the action α on X. As pointed out above, (a) follows from Lemma 3.3, so we may assume that X is not connected and α acts expansively. By Corollary 6.13 of [10], it follows that the $\mathfrak{R}$-module $\mathfrak{M}$ is Noetherian, so there are only finitely many prime ideals associated to $\mathfrak{M}$ (see Theorem 6.5, Chapter 2 of [6]). Let $\mathcal{L}$ be the finite set of lines given by the union of the set of lines chosen before Lemma 3.4 for each of the associated prime ideals. Then any cone $\mathcal{C}$ defined by $\mathcal{L}$ is a sub-cone of a cone in Lemma 3.4, so by Lemma 3.2 the action $\alpha = \alpha^{\mathfrak{M}}$ is mixing of all orders in $\mathcal{C}$, proving (b).

Finally, (c) follows from Example 3.5(2) below. □

Example 3.5. (1) An example to illustrate Theorem 1.3(b) is given by Ledrappier's example [5] for which the shape $\{(0,0),(0,1),(1,0)\}$ is non-mixing. In the $\mathfrak{R}$-module description, Ledrappier's example corresponds to the module

$$\frac{\mathfrak{R}}{\langle 2, 1+u_1+u_2\rangle}.$$

In the notation of Section 3, this means that the prime p is 2 and the polynomial g is $1+u_1+u_2$. The convex hull is $CH(g) = \{(s,t) \in \mathbb{R}^2 \mid 0 \leq s,t \leq 1, s+t \leq 1\}$ with extreme points $(0,0),(0,1)$, and $(1,0)$. A suitable set of lines that satisfy properties (i) and (ii) are the five oriented lines through the origin and the points

$(1,0),(-1,1),(-1,-1),(1,-2)$ and $(1,2)$. Notice that there are many other possible choices, though all of them have at least five lines. The statement (b) for this example is then that mixing of all orders in the sense of equation (1) occurs in each of the five associated cones.
(2) Without the assumption that the group be connected or that the action be expansive, there may be no cones in which mixing of all orders can occur. An example to show this starts again with Ledrappier's example [5] for which the shape $\{(0,0),(0,1),(1,0)\}$ is non-mixing, and applies linear maps in $\mathbb{Z}^2$ to produce similar examples for which any given triangle is a non-mixing shape. Since any cone subtending a positive angle contains some triangle, the product of these (countably many) examples gives the required example. Let

$$\mathfrak{M} = \bigoplus_{a\in\mathbb{Z}\setminus\{0\},b\in\mathbb{Z}} \frac{\mathfrak{R}}{\langle 2, 1+u_1^a u_2^b + u_1^a u_2^{b+1}\rangle}.$$

Then the $\mathbb{Z}^2$-action corresponding to the module $\mathfrak{M}$ is not mixing on the shapes $\{(0,0),(a,b),(a,b+1)\}$ for each $a \neq 0$, $b \in \mathbb{Z}$. It follows that $\alpha^{\mathfrak{M}}$ cannot be mixing of all orders in any cone subtending a positive angle.

4. Remarks

I thank Prof. Fried for pointing out [2] and the connection between the Bertini-Noether Theorem and irreducibility. The Gaussian construction above is based on that of Ferenci and Kamiński, who used it to exhibit a rigid $\mathbb{Z}^2$-action each of whose elements is a Bernoulli shift; I thank Prof. Kamiński for showing me a preprint of the paper [1]. Mixing properties in the positive quadrant and their relationship to mixing properties of a complete $\mathbb{Z}^2$-action are discussed in [7].

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