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Approximation of L_2-Processes by Gaussian Processes

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Abstract. Let T be an ergodic transformation of a nonatomic probability space, f an L_2-function, and $K \geq 1$ an integer. It is shown that there is another L_2-function g, such that the joint distribution of $T^i g$, $1 \leq i \leq K$, is nearly normal, and such that the corresponding inner products $(T^i f, T^j f)$ and $(T^i g, T^j g)$ are nearly the same for $1 \leq i, j \leq K$. This result can be used to give a simpler and more transparent proof of an important special case of an earlier theorem [3], which was a refinement of Bourgain's entropy theorem [9].

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1. The Main Result

If a random vector $\mathbf{Y} = (Y_1, Y_2, \ldots, Y_K)$ has the multivariate normal distribution, then the special structure allows one to make very precise statements about the set where $\sup_{1 \leq i \leq K} Y_i > \lambda$. Consequently, if one can reduce a general situation to the case of the multivariate normal, then the knowledge of the multivariate normal case can be used to obtain information about the general case. Thus, it is useful to find situations and techniques that allow us to make a reduction to the multivariate normal situation. In this paper we consider a family of operators that arise in ergodic theory, and show that such a reduction of the general case to the multivariate normal is possible by using a Rohlin tower argument.

Let T be an ergodic transformation of a nonatomic probability space, f an L_2-function, and $K \geq 1$ an integer. We will show below that there is another L_2-function g, such that the joint distribution of $T^i g$, $1 \leq i \leq K$, is nearly normal, and such that the corresponding inner products $(T^i f, T^j f)$ and $(T^i g, T^j g)$ are nearly the same for $1 \leq i, j \leq K$. Using this approximation to the multivariate normal,

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and its well understood properties, we will be able to obtain information about the sequence $T^i f$, $1 \le i \le K$.

The particular application we have in mind is Bourgain's celebrated entropy theorem [9], which establishes a connection between the pointwise and L_2 behaviors of sequences of L_2 contractions T_n, applied to L_2 functions. Such a connection does exist, of course, if $f \in L_2$ is such that the joint distribution of $T_i f$, $1 \le i \le K$, is normal for each K. In this case all pointwise relations between these functions are determined by the L_2 behavior of the sequence $T_n f$. The extension of this connection from the normal case to the general case is carried out in an existential way in Bourgain's original proof, and also in the proof of a refined version of this theorem given in [3]. However, in many cases the passage from the normal case to the general case can be made constructively, in a simple and more transparent way, by approximating in the appropriate sense, the sequence $T_i f$, by a sequence with the multivariate normal distribution. Our main result, Theorem 1.6 below, has been obtained with this application in mind. In Section 2 we give a brief sketch of this application.

Let $(X, \mathcal{F}, \mu)$ be a nonatomic probability space, T an ergodic transformation on X, f a function in $L_2(\mu)$, and $K \ge 1$ an integer. We will show below (Theorem 1.6) that there exists another function $g \in L_2(\mu)$, such that the joint distribution of $T^i g$, $1 \le i \le K$, is nearly normal, and such that the corresponding inner products $(T^i f, T^j f)$ and $(T^i g, T^j g)$ are nearly the same for $1 \le i, j \le K$. Theorem 1.6 implies the following, stated formally as Corollary 1.11 below. Let $A_1, \ldots, A_K$ be K operators which are linear combinations of the iterates of T, or more generally are strong limits of such linear combinations. Then, given an $f \in L_2$, there is another $g \in L_2$ such that the joint distribution of $A_i g$, $1 \le i \le K$, is nearly normal, and such that the corresponding inner products $(A_i f, A_j f)$ and $(A_i g, A_j g)$ are nearly the same for $1 \le i, j \le K$.

A special case of the main result, where T is an irrational rotation of the circle, has been proved in [2]. The general result has been stated in [3] without proof, observing that the special case in [2] would imply the general case with routine approximations, via Rohlin's Lemma. As the proof in [2] is rather involved in details, however, a generalization of it turns out to be very complicated, even though rather routine in principle. In the present note we give a very simple and short direct proof of the general result, using essentially only Rohlin's Lemma. This proof could have been made even shorter by replacing Lemmas 1.7 and 1.8 below by a reference to a general fact about Gauss processes. Indeed, the ergodicity of a Gauss process is equivalent to the continuity of its spectral measure (See, for example, pages 191 and 368 in [10]). This implies directly that a Gauss process can be approximated by an ergodic Gauss process, in the sense required for our proof. Lemmas 1.7 and 1.8 show that any L_2-process can be approximated, in the same sense, by an aperiodic L_2-process, which is enough for our purpose. To keep our presentation as elementary and self contained as possible, we give the simple proof of this fact.

Notation 1.1. By a space we mean a probability space and by a measure we mean a probability measure. All functions considered are measurable, either by assumption or construction. Let f_n and g_n be two finite or infinite sequences of real valued functions. Each sequence may be defined on a different space. These sequences will be called isomorphic if the joint distributions of finite sets of functions from one sequence is the same as the corresponding distributions from the other sequence. A transformation T on a space $X = (X, \mathcal{F}, \mu)$ is a measure preserving point transformation, not necessarily invertible. A transformation T on X induces a transformation of functions on X, denoted by the same letter and defined by $(Tf)(x) = f(Tx)$ for $x \in X$, where f is a function on X. Since T is not assumed to be invertible, given $g : X \to \mathsf{R}$, there need not be (measurable) $f : X \to \mathsf{R}$ such that $Tf = g$. If g is measurable with respect to $T^{-1}\mathcal{F}$, however, then there is an f such that $g = Tf$. If $T^i x \neq T^j x$ whenever $i \neq j$, for almost all x, then T is called aperiodic. By a process (T, f) we mean a transformation T together with a function $f : X \to \mathsf{R}$. We identify a process (T, f) by the corresponding sequence $f_n = T^n f$. For each integer $K \geq 1$ a process induces a measure $\xi = \xi_K$ on R^K, as the distribution measure of the function $(T^1 f, \ldots, T^K f) : X \to \mathsf{R}^K$, which will be called the K-distribution measure of the process. Two processes (T, f) and (S, g) are called isomorphic if their K-distribution measures are the same for each $K \geq 1$; or, equivalently, if the corresponding sequences $T^n f$ and $S^n g$ are isomorphic. If f is an L_2 function, then (T, f) is called an L_2-process.

Definition 1.2 (L_2-equivalence)**.** Let f_n and g_n be two (finite or infinite) sequences of L_2-functions, $n \geq 0$. These sequences will be called L_2-equivalent if the corresponding inner products (f_i, f_j) and (g_i, g_j) are equal for all $i, j \geq 0$. Let (T, f) and (S, g) be two L_2 processes, where the associated transformations may or may not be on the same space. Then (T, f) and (S, g) will be called L_2-equivalent if the corresponding sequences $f_n = T^n f$ and $g_n = S^n g$ are L_2-equivalent. A sequence of L_2 functions is called a Gauss (or normal) sequence if the joint distribution of any finite subsequence is normal. A process (T, f) is called a Gauss process if the corresponding sequence $f_n = T^n f$ is a Gauss sequence. Although the following lemma is well known we will recall the simple proof. In this lemma the uniqueness is understood to be up to an isomorphism, as defined in 1.1.

Lemma 1.3. *Any L_2 sequence is L_2-equivalent to a unique Gauss sequence. In particular, any L_2 process is L_2-equivalent to a unique Gauss process.*

Proof. Let $\Omega = \mathsf{R}^{\mathsf{Z}}$ be the shift space. Points in Ω are denoted by $\omega = (\omega_i)_{i \in \mathsf{Z}}$, with coordinate functions $\omega_i : \Omega \to \mathsf{R}$. The shift transformation $S : \Omega \to \Omega$ is defined as $(S\omega)_i = \omega_{i-1}$. If f_i, $1 \leq i \leq K$, is a finite L_2 sequence, then there is a unique Gauss measure on R^K such that the inner product of the coordinate functions ω_i and ω_j with respect to this measure is equal to the corresponding inner product of f_i and f_j, for all $1 \leq i, j \leq K$. If f_n is an infinite sequence then the Gauss measures corresponding to finite segments are compatible and define a unique measure on R^{N}. If $f_n = T^n f$, then this measure on R^{N} is invariant under the restriction of the shift transformation to R^{N}, defined in an obvious way. Hence it has a unique extension to a shift invariant measure on Ω. Then we see that (T, f) is L_2-equivalent to the Gauss process (S^{-1}, ω_0). $\square$

We will need two concepts of "closeness". In the first case, say that two sequences of functions are L^2 *close* if their corresponding inner products are close. To be precise we give the following definition.

Definition 1.4 (L_2-closeness)**.** Let $K \geq 1$ be an integer and $\varepsilon > 0$. Two L_2 sequences f_n and g_n are called L_2-close within (K, ε) if

$$|(f_i,\, f_j) - (g_i,\, g_j)| < \varepsilon$$

for $0 \leq i, j \leq K-1$. Two L_2 processes (T, f) and (S, g) will be called L_2-close within (K, ε), if the corresponding sequences $f_n = T^n f$ and $g_n = S^n g$ are L_2-close within (K, ε).

We will also need a second measure of closeness. The idea is that two measures are *weakly close* if the results of integration against continuous functions are close. Two finite sequences of functions will be called *weakly close* if their distribution measures are weakly close. The formal definition is as follows.

Definition 1.5 (Weak-closeness)**.** Let $K \geq 1$ and $U \geq 1$ be integers, $\varphi_1, \ldots, \varphi_U$ bounded continuous functions $\mathsf{R}^K \to \mathsf{R}$, and $\varepsilon > 0$. Two sequences of measurable functions f_n and g_n, $n \geq 0$, will be called weakly close within $(K, \varepsilon, \varphi_1, \ldots, \varphi_U)$ if

$$\left| \int_{\mathsf{R}^K} \varphi_u d\xi - \int_{\mathsf{R}^K} \varphi_u d\eta \right| < \varepsilon$$

for $1 \leq u \leq U$, where ξ and η denote the joint distribution measures of the R^K-valued functions $(f_0, \ldots, f_{K-1})$ and $(g_0, \ldots, g_{K-1})$. To simplify the notation we will also say that these sequences are weakly close within (K, ε), the choice of a finite number of bounded continuous functions $\mathsf{R}^K \to \mathsf{R}$ being understood implicitly. Two processes (T, f) and (S, g) will be called weakly close within (K, ε), if the corresponding sequences are weakly close within (K, ε).

Theorem 1.6. *Let T be an aperiodic transformation and $f \in L_2(X)$. Let (V, h) be the Gauss process which is L_2-equivalent to (T, f). Let*

$$(K, \varepsilon) = (K, \varepsilon, \varphi_1, \ldots, \varphi_U)$$

be as in the definition 1.5 of weak-closeness. Then there is a function $g \in L_2(X)$, such that (V, h) and (T, g) are both weakly and L_2-close within (K, ε).

Proof. The passage from (V, h) to (T, g) will be accomplished in two steps, through an intermediary process (V', h'). These two passages will be justified by the Lemmas 1.8 and 1.9 to be obtained below. In the first step, the Gauss process (V, h) is replaced by a process (V', h') such that V' is aperiodic and (V, h) and (V', h') are both weakly and L_2-close within $(K, \varepsilon/2)$. This step is justified by Lemma 1.8, which states that any L_2-process can be approximated by an aperiodic L_2-process, both in weak- and L_2-closeness sense. Next, we find an L_2-function g on X so that (V', h') and (T, g) are both weakly and L_2-close within $(K, \varepsilon/2)$. This step is justified by Lemma 1.9, which is a consequence of Rohlin's Lemma for aperiodic transformations. It is clear that (V, h) and (T, g) are both weakly and L_2-close within (K, ε). □

We now give the details of these lemmas. In what follows

$$(K, \varepsilon) = (K, \varepsilon, \varphi_1, \ldots, \varphi_U)$$

is as specified before.

Lemma 1.7. *Let $X = [0, 1)$ be the unit circle with Lebesgue measure. Let α_j be a sequence in X converging to α, with the corresponding rotations T_j and T. Then, for any $f \in L_2$, and for sufficiently large j, the processes (T_j, f) and (T, f) are both weakly and L_2-close within (K, ε).*

Proof. For any fixed n the sequence $T_j^n f$ converges to $T^n f$ in L_2 norm. Hence, (T_j, f) and (T, f) are L_2-close within (K, ε) for all sufficiently large j. Let $Q_j : X \to \mathsf{R}^K$ be defined as

$$Q_j(x) = (f(x), T_j f(x), \dots, T_j^{K-1} f(x))$$

and let $Q : X \to \mathsf{R}^K$ be the similar mapping defined in terms of T instead of T_j. We see that Q_j converges to Q in measure. This shows that (T_j, f) and (T, f) are also weakly close within (K, ε), for all sufficiently large j. □

Lemma 1.8. *For any L_2 process (T, f) there is an aperiodic process (S, g) such that these two processes are both weakly and L_2-close within (K, ε).*

Proof. If T is a periodic transformation of period n, then (T, f) is isomorphic (as defined in 1.1) to a process for which the underlying space is the unit circle and the transformation is the rotation by $1/n$. Approximating $1/n$ by irrational numbers and applying the previous lemma, we see that in this case there is, in fact, an ergodic process (S, g) satisfying our requirements. In the general case, partition the underlying space X for T into the T-invariant sets $X_1, X_2, \dots, X_\infty$, where, for $1 \le k < \infty$, X_k is the set of all $x \in X$ such that $T^k x = x$ but $T^i x \ne x$ if $1 \le i < k$, and $X_\infty = X - \cup_{1 \le k < \infty} X_k$. The restriction of f to $\cup_{N<n<\infty} X_n$ goes to zero both in L_1 and L_2 norms, as $N \to \infty$. Hence we will assume, without loss of generality, that f vanishes on $\cup_{N<n<\infty} X_n$, for some N. Change X to Y, by replacing each X_n by a circle Y_n, with the same measure as X_n, and leaving $X_\infty = Y_\infty$ unchanged. Then (T, f) is isomorphic to a process (R, g), where the restriction of R to Y_n is the rotation by $1/n$, and the restriction to Y_∞ is equal to the restriction of T to X_∞. Then the required aperiodic transformation S will be obtained by replacing each rotation by an irrational rotation. By choosing the rotation α_n on Y_n, for $1 \le n \le N$, sufficiently close to $1/n$, we see that this process (S, g) satisfies our requirements. □

Lemma 1.9. *Let T and S be two aperiodic transformations on X and Y, respectively. Then, given an $f \in L_2(X)$, there is a $g \in L_2(Y)$ such that (T, f) and (S, g) are both weakly and L_2 close within (K, ε).*

Proof. We will show that, given any $\delta > 0$ there is a function g, with the same distribution as f, and two sets $X_0 \subset X$ and $Y_0 \subset Y$, with measures greater than $1 - 2\delta$, such that the joint distributions of $(f, Tf, \dots, T^{K-1} f)$ on X_0 is the same as the joint distribution of $(S^0 g, \dots, S^{K-1} g)$ on Y_0. This will complete the proof. In fact, note that all the functions we are considering, $T^i f$ and $S^j g$, have the same distribution. Hence, if δ is sufficiently small, we see that the processes (T, f) and (S, g) are both weakly and L_2 close within (K, ε).

To construct such a g, find a nonnegative integer R such that $K/(R+1) < \delta$. In what follows r ranges over the integers $\{0, 1, \dots, R\}$. Use Rohlin's Lemma to find $F \subset X$ and $G \subset Y$, with measures equal to $(1-\delta)/(R+1)$, such that both of the families of sets $T^{-r} F$ and $S^{-r} G$ are pairwise disjoint in their respective spaces.

Let $X_1 = \cup_{r=0}^{R} T^{-r}F$ and $Y_1 = \cup_{r=0}^{R} S^{-r}G$. Let $B = T^{-R}F$ and $C = S^{-R}G$. Let $f_r : B \to \mathsf{R}$ be the restriction of $T^r f$ to B. If $Y = (Y, \mathcal{G}, \nu)$, note that $C \in S^{-R}\mathcal{G}$. Consider C as a measure space with the restriction of ν to $C \cap S^{-R}\mathcal{G}$. Then C is a nonatomic measure space with the total measure equal to $\mu(B)$. Hence there are $R+1$ functions $g_r : C \to \mathsf{R}$ with the same joint distribution as the functions f_r. Furthermore, since these functions are $S^{-R}\mathcal{G}$-measurable, there are $w_r : S^{-R+r}G \to \mathsf{R}$ such that $S^r w_r = g_r$. We then define g on Y_1 as the function whose restriction to $S^{-R+r}G$ is equal to w_r. Then g restricted to Y_1 has the same distribution as f restricted to X_1. Define g on $Y - Y_1$ in such a way that g and f have the same distributions. We then let $X_0 = \cup_{r=K}^{R} T^{-r}F$ and $Y_0 = \cup_{r=K}^{R} S^{-r}G$ and see that all the requirements are satisfied. $\square$

Notation 1.10. Let T be a transformation. Let $\mathcal{L} = \mathcal{L}(T)$ be the class of operators on functions that are linear combinations of the iterates of T. Let $\mathcal{A} = \mathcal{A}(T)$ be the class of bounded L_2 operators A for which the following is true. For each $f \in L_2$ and for each $\varepsilon > 0$ there is a $B \in \mathcal{L}$ such that $\|Af - Bf\|_2 < \varepsilon$.

Corollary 1.11. *Let T be an aperiodic transformation on a nonatomic probability space X. Let $(K, \varepsilon, \varphi_1, \dots, \varphi_U)$ be as in the definition of weak-closeness. Given K operators A_i, $1 \le i \le K$, in $\mathcal{A}$ and $f \in L_2(X)$, let q_i be the Gauss sequence which is L_2-equivalent to the sequence $A_i f$. Then there is a $g \in L_2(X)$, such that the sequences q_i and $A_i g$ are both weakly and L_2-close within (K, ε).*

Proof. We will assume that each A_i belongs to $\mathcal{L}$. This is not a loss of generality, since we can find $B_i \in \mathcal{L}$ such that, if q_i' is the Gauss sequence which is L_2-equivalent to the sequence $B_i f$, then the sequences q_i and q_i' are both weakly and L_2-close within $(K, \varepsilon/2)$. Hence we assume that each A_i is of the form $A_i = \sum_{j=0}^{N} \alpha_{ij} T^j$, $1 \le i \le K$, with real coefficients α_{ij}. Let $\psi : \mathsf{R}^{N+1} \to \mathsf{R}^K$ be defined as $(\psi(x))_i = \sum_{j=0}^{N} \alpha_{ij} x_j$, where $1 \le i \le K$, $0 \le j \le N$, and $x = (x_j) \in \mathsf{R}^{N+1}$. Let $M = \sum |\alpha_{ij}\alpha_{kl}|$, where the summation is over $1 \le i, k \le K$ and $0 \le j, l \le N$. Let (V, h) be the Gauss process which is L_2-equivalent to (T, f). Use Theorem 1.6 to find a $g \in L_2(X)$ such that the processes (V, h) and (T, g) are both weakly and L_2-close within $(N+1, \varepsilon/M, \varphi_1 \circ \psi, \dots, \varphi_U \circ \psi)$. This g satisfies the required condition. $\square$

2. An Application

As mentioned at the beginning of this note, we will now apply Corollary 1.11 to give a simple proof of a result in [3], in an important special case. The following two definitions, taken from [3], give a type of L_2 behaviour and a type of pointwise behaviour for a finite sequence of functions. The theorem to be proved establishes a connection between these behaviours.

Definition 2.1 (δ-Spanning sequences)**.** Let $0 < \delta \le 1$. A sequence of L_2 functions $(f_1, \dots, , f_K)$ will be called a δ-spanning sequence if $\|f_k\|_2 \le 1$ and $\|f_k - Q_{k-1} f_k\|_2 \ge \delta$ for each $k = 1, \dots, K$, where $Q_0 = 0$ and Q_k is the orthogonal projection on the subspace spanned by $(f_1, \dots, f_k)$. Let $(T_1, \dots, T_K)$ be finitely many L_2 operators and f be an L_2 function. Then f is called a δ-spanning function (for $(T_1, \dots, T_K)$) if $\|f\|_2 = 1$ and if $(T_1 f, \dots, T_K f)$ is a δ-spanning sequence.

Definition 2.2 ((δ, ε)-sweeping)**.** Let $0 < \varepsilon < \delta \leq 1$. A sequence of functions $(h_1, \ldots, h_K)$ will be called a (δ, ε)-sweeping sequence if $\|h_k\|_1 < \varepsilon$ for each $k = 1, \ldots, K$, and if $\max_{1 \leq k \leq K} |h_k| > \delta - \varepsilon$ on a set of measure greater than $1 - \varepsilon$. Let $(T_1, \ldots, T_K)$ be finitely many operators and h be a function. Then h is called a (δ, ε)-sweeping function (for $(T_1, \ldots, T_K)$) if $\|h\|_\infty \leq 1$, $\|h\|_1 < \varepsilon$, and if $(T_1 h, \ldots, T_K h)$ is a (δ, ε)-sweeping sequence.

Remark 2.3. The significance of (δ, ε)-sweeping is as follows. Let T_n be a sequence of L_∞ contractions and $\delta > 0$ fixed. If for each ε, $0 < \varepsilon < \delta$ there are arbitrarily large Ks for which the segment $(T_1, \ldots, T_K)$ has a (δ, ε)-sweeping function, then one can show that there are functions h for which the sequence $T_n h$ diverges a.e. Also, its degree of divergence can be characterized by "δ-sweeping", as defined in [4]. Also, see [1], [5], [6], [7], [8], and [11] for more details.

Theorem 2.4. *Let $0 < \varepsilon < \delta \leq 1$. Then there is an integer $K = K(\delta, \varepsilon) \geq 1$ with the following property. Let $(T_1, \ldots, T_K)$ be K contractions in L_2. If there is a δ-spanning function f for $(T_1, \ldots, T_K)$ such that the joint distribution of $(f, T_1 f, \ldots, T_K f)$ is normal, then there is an $M > 0$ such that*

$$h = (1/M)((f \wedge M) \vee (-M))$$

is a (δ, ε)-sweeping function for $(T_1, \ldots, T_K)$.

This is proved in [3]. As mentioned earlier, it relates a certain type of pointwise behaviour of K functions with a joint normal distribution to the L_2 behaviour of these functions. The following theorem removes the normality assumption for certain types of contractions. A more general version of it was also proved in [3], in a longer and nonconstructive way. An application of Corollary 1.11 gives a shorter and more transparent proof for the important special case below. Recall the definition of $\mathcal{A}(T)$ given in 1.10.

Theorem 2.5. *Let T be an ergodic transformation in a nonatomic probability space. Let $0 < \varepsilon < \delta \leq 1$. Then there is an integer $K = K(\delta, \varepsilon) \geq 1$ with the following property. Let $(A_1, \ldots, A_K)$ be K L_2 contractions in $\mathcal{A}(T)$. If there is a δ-spanning function f for $(A_1, \ldots, A_K)$, then there is a function g, and an $M > 0$ such that $h = (1/M)((g \wedge M) \vee (-M))$ is a (δ, ε)-sweeping function for $(A_1, \ldots, A_K)$.*

Sketch of the Proof. The δ-spanning property is preserved under L_2 closeness and the (δ, ε)-sweeping property is preserved under weak closeness. If h_i is the Gauss sequence which is L_2-equivalent to $A_i f$, there is $g \in L_2(X)$ such that $A_i g$ which is as close to h_i as we want, both in the weak and in the L_2 sense. This follows from the Corollary 1.11. Using Theorem 2.4, we see that this g satisfies the desired condition. □

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