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Quasilinear Elliptic Systems in Divergence Form with Weak Monotonicity

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Abstract. We consider the Dirichlet problem for the quasilinear elliptic system

$$\begin{aligned} -\operatorname{div}\sigma(x,u(x),Du(x)) &= f \qquad \text{on } \Omega \\ u(x) &= 0 \qquad \text{on } \partial\Omega \end{aligned}$$

for a function $u:\Omega\to\mathbb{R}^m$, where Ω is a bounded open domain in $\mathbb{R}^n$. For arbitrary right hand side $f\in W^{-1,p'}(\Omega)$ we prove existence of a weak solution under classical regularity, growth and coercivity conditions, but with only very mild monotonicity assumptions.

Contents

1. Introduction

On a bounded open domain $\Omega\subset\mathbb{R}^n$ we consider the Dirichlet problem for the quasilinear elliptic system

$$-\operatorname{div}\sigma(x,u(x),Du(x)) = f \qquad \text{on } \Omega \tag{1}$$

$$u(x) = 0 \qquad \text{on } \partial\Omega \tag{2}$$

for a function $u:\Omega\to\mathbb{R}^m$. Here, $f\in W^{-1,p'}(\Omega)$ for some $p\in(1,\infty)$, and σ satisfies the conditions (H0)–(H2) below. A feature of this paper is that we treat a class of problems for which the classical monotone operator methods developed by Višik [11], Minty [10], Browder [2], Brézis [1], Lions [9] and others do not apply. The reason for this is that σ does not need to satisfy the strict monotonicity condition of a typical Leray-Lions operator. The tool we use in order to prove the

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needed compactness of approximating solutions is Young measures. The methods are inspired by [3].

To fix some notation, let $\mathbb{M}^{m\times n}$ denote the real vector space of $m\times n$ matrices equipped with the inner product $M : N = M_{ij}N_{ij}$ (with the usual summation convention).

The following notion of monotonicity will play a rôle in part of the exposition: Instead of assuming the usual pointwise monotonicity condition for σ, we will also use a weaker, integrated version of monotonicity which is called quasimonotonicity (see [3]). The definition is phrased in terms of gradient Young measures. Note, however, that although quasimonotonicity is "monotonicity in integrated form", the gradient $D\eta$ of a quasiconvex function η is not necessarily quasimonotone.

Definition. A function $\eta : \mathbb{M}^{m\times n} \to \mathbb{M}^{m\times n}$ is said to be strictly p-quasimonotone, if

$$\int_{\mathbb{M}^{m\times n}} (\eta(\lambda) - \eta(\bar\lambda)) : (\lambda - \bar\lambda) d\nu(\lambda) > 0$$

for all homogeneous $W^{1,p}$-gradient Young measures ν with center of mass $\bar\lambda = \langle \nu, \mathrm{id}\rangle$ which are not a single Dirac mass.

A simple example is the following: Assume that η satisfies the growth condition

$$|\eta(F)| \leqslant C\,|F|^{p-1}$$

with $p > 1$ and the structure condition

$$\int_\Omega (\eta(F + \nabla\varphi) - \eta(F)) : \nabla\varphi\, dx \geqslant c \int_\Omega |\nabla\varphi|^p dx$$

for all $\varphi \in C_0^\infty(\Omega)$ and all $F \in \mathbb{M}^{m\times n}$. Then η is strictly p-quasimonotone. This follows easily from the definition if one uses that for every $W^{1,p}$-gradient Young measure ν there exists a sequence $\{Dv_k\}$ generating ν for which $\{|Dv_k|^p\}$ is equi-integrable (see [4], [6]).

Now, we state our main assumptions.

(H0) (Continuity) $\sigma : \Omega \times \mathbb{R}^m \times \mathbb{M}^{m\times n} \to \mathbb{M}^{m\times n}$ is a Carathéodory function, i.e., $x \mapsto \sigma(x, u, F)$ is measurable for every $(u, F) \in \mathbb{R}^m \times \mathbb{M}^{m\times n}$ and $(u, F) \mapsto \sigma(x, u, F)$ is continuous for for almost every $x \in \Omega$.

(H1) (Growth and coercivity) There exist $c_1 \geqslant 0$, $c_2 > 0$, $\lambda_1 \in L^{p'}(\Omega)$, $\lambda_2 \in L^1(\Omega)$, $\lambda_3 \in L^{(p/\alpha)'}(\Omega)$, $0 < \alpha < p$ and $0 < q \leqslant n\frac{p-1}{n-p}$ such that

$$\begin{aligned} |\sigma(x, u, F)| &\leqslant \lambda_1(x) + c_1(|u|^q + |F|^{p-1}) \\ \sigma(x, u, F) : F &\geqslant -\lambda_2(x) - \lambda_3(x)|u|^\alpha + c_2|F|^p \end{aligned}$$

(H2) (Monotonicity) σ satisfies one of the following conditions:

(a) For all $x \in \Omega$ and all $u \in \mathbb{R}^m$, the map $F \mapsto \sigma(x, u, F)$ is a C^1-function and is monotone, i.e.,

$$(\sigma(x, u, F) - \sigma(x, u, G)) : (F - G) \geqslant 0$$

for all $x \in \Omega$, $u \in \mathbb{R}^m$ and $F, G \in \mathbb{M}^{m\times n}$.

(b) There exists a function $W : \Omega \times \mathbb{R}^m \times \mathbb{M}^{m\times n} \to \mathbb{R}$ such that $\sigma(x, u, F) = \frac{\partial W}{\partial F}(x, u, F)$, and $F \mapsto W(x, u, F)$ is convex and C^1.

(c) σ is strictly monotone, i.e., σ is monotone and

$$(\sigma(x, u, F) - \sigma(x, u, G)) : (F - G) = 0 \quad \text{implies} \quad F = G.$$

(d) $\sigma(x, u, F)$ is strictly p-quasimonotone in F.

The condition (H0) ensures that $\sigma(x, u(x), U(x))$ is measurable on Ω for measurable functions $u : \Omega \to \mathbb{R}^m$ and $U : \Omega \to \mathbb{M}^{m\times n}$. (H1) are standard growth and coercivity conditions. The main point is that we do not require strict monotonicity or monotonicity in the variables (u, F) in (H2) as it is usually assumed in previous work (see, e.g., [7] or [8]). For example, take a potential $W(x, u, F)$, which is only convex but not strictly convex in F, and consider the corresponding elliptic problem (1)–(2) with $\sigma(x, u, F) = \frac{\partial W}{\partial F}(x, u, F)$. Even such a very simple situation cannot be treated by conventional methods: The problem is that the gradients of approximating solutions do not converge pointwise where W is not strictly convex. The idea is now, that in a point where W is not strictly convex, it is locally affine, and therefore, passage to the limit should locally still be possible. Technically, this can indeed be achieved by a suitable blow-up process, or (and this seems to be much more efficient) by considering the Young measure generated by the sequence of gradients.

We prove the following result:

Theorem. *If σ satisfies the conditions* (H0)–(H2)*, then the Dirichlet problem* (1), (2) *has a weak solution $u \in W_0^{1,p}(\Omega)$ for every $f \in W^{-1,p}(\Omega)$.*

2. Galerkin approximation

Let $V_1 \subset V_2 \subset \ldots \subset W_0^{1,p}(\Omega)$ be a sequence of finite dimensional subspaces with the property that $\cup_{i\in\mathbb{N}} V_i$ is dense in $W_0^{1,p}(\Omega)$. We define the operator

$$\begin{aligned} F : W_0^{1,p}(\Omega) &\to W^{-1,p'}(\Omega) \\ u &\mapsto \Big(w \mapsto \int_\Omega \sigma(x, u(x), Du(x)) : Dw\, dx - \langle f, w\rangle\Big), \end{aligned}$$

where $\langle \cdot, \cdot \rangle$ denotes the dual pairing of $W^{-1,p'}(\Omega)$ and $W_0^{1,p}(\Omega)$. Observe that for arbitrary $u \in W_0^{1,p}(\Omega)$, the functional $F(u)$ is well defined by the growth condition in (H1), linear, and bounded (again by the growth condition in (H1)).

By the continuity assumption (H0) and the growth condition in (H1), it is easy to check, that the restriction of F to a finite linear subspace of $W_0^{1,p}(\Omega)$ is continuous.

Let us fix some k and assume that V_k has dimension r and that $\varphi_1, \ldots, \varphi_r$ is a basis of V_k. Then we define the map

$$G : \mathbb{R}^r \to \mathbb{R}^r, \quad \begin{pmatrix} a^1 \\ a^2 \\ \vdots \\ a^r \end{pmatrix} \mapsto \begin{pmatrix} \langle F(a^i\varphi_i), \varphi_1\rangle \\ \langle F(a^i\varphi_i), \varphi_2\rangle \\ \vdots \\ \langle F(a^i\varphi_i), \varphi_r\rangle \end{pmatrix}.$$

G is continuous, since F is continuous on finite dimensional subspaces. Moreover, for $a = (a_1, \ldots, a_r)^t$ and $u = a^i\varphi_i \in V_k$, we have by the coercivity assumption in (H1) that

$$G(a) \cdot a = (F(u), u) \to \infty$$

as $\|a\|_{\mathbb{R}^r} \to \infty$. Hence, there exists $R > 0$ such that for all $a \in \partial B_R(0) \subset \mathbb{R}^r$ we have $G(a) \cdot a > 0$ and the usual topological argument (see, e.g., [10] or [9]) gives that $G(x) = 0$ has a solution in $B_R(0)$. Hence, for all k there exists $u_k \in V_k$ such that

$$\langle F(u_k), v\rangle = 0 \qquad \text{for all } v \in V_k.$$

3. The Young measure generated by the Galerkin approximation

From the coercivity assumption in (H1) it follows that there exists $R > 0$ with the property, that $\langle F(u), u\rangle > 1$ whenever $\|u\|_{W_0^{1,p}(\Omega)} > R$. Thus, for the sequence of Galerkin approximations $u_k \in V_k$ constructed above, there is a uniform bound

$$\|u_k\|_{W_0^{1,p}(\Omega)} \leqslant R \qquad \text{for all } k. \tag{3}$$

Thus, we may extract a subsequence (again denoted by u_k) such that

$$u_k \rightharpoonup u \qquad \text{in } W_0^{1,p}(\Omega)$$

and such that

$$u_k \rightharpoonup u \qquad \text{in measure and in } L^s(\Omega)$$

for all $s < p^*$. The sequence of gradients Du_k generates a Young measure ν_x, and since u_k converges in measure to u, the sequence (u_k, Du_k) generates the Young measure $\delta_{u(x)} \otimes \nu_x$ (see, e.g., [5]). Moreover, for almost all $x \in \Omega$, ν_x

(i) is a probability measure,
(ii) is a homogeneous $W^{1,p}$-gradient Young measure, and
(iii) satisfies $\langle \nu_x, \mathrm{id}\rangle = Du(x)$.

The proofs are standard (see, e.g., [3]).

4. Passage to the limit

Let us consider the sequence

$$\mathrm{I}_k := \big(\sigma(x, u_k, Du_k) - \sigma(x, u, Du)\big) : \big(Du_k - Du\big)$$

and prove, that its negative part I_k^- is equiintegrable: To do this, we write I_k^- in the form

$$\begin{aligned} \mathrm{I}_k &= \sigma(x, u_k, Du_k) : Du_k - \sigma(x, u_k, Du_k) : Du \\ &\quad - \sigma(x, u, Du) : Du_k + \sigma(x, u, Du) : Du =: \mathrm{II}_k + \mathrm{III}_k + \mathrm{IV}_k + \mathrm{V}. \end{aligned}$$

The sequences II_k^- and V^- are easily seen to be equiintegrable by the coercivity condition in (H1). Then, to see equiintegrability of the sequence III_k we take a measurable subset $\Omega' \subset \Omega$ and write

$$\begin{aligned} &\int_{\Omega'} |\sigma(x, u_k, Du_k) : Du| dx \leqslant \\ &\qquad \leqslant \Big(\int_{\Omega'} |\sigma(x, u_k, Du_k)|^{p'} dx\Big)^{1/p'} \Big(\int_{\Omega'} |Du|^p dx\Big)^{1/p} \\ &\qquad \leqslant C\Big(\int_{\Omega'} (|\lambda_1(x)^{p'} + |u_k|^{qp'} + |Du_k|^p) dx\Big)^{1/p'} \Big(\int_{\Omega'} |Du|^p dx\Big)^{1/p}. \end{aligned}$$

The first integral is uniformly bounded in k by (3). The second integral is arbitrarily small if the measure of Ω' is chosen small enough. A similar argument gives the equiintegrability of the sequence IV_k.

Having established the equiintegrability of I_k^-, we may use [3, Lemma 6] which gives that

$$(4)\qquad X := \liminf_{k\to\infty} \int_\Omega I_k \geqslant \int_\Omega \int_{\mathbb{M}^{m\times n}} \sigma(x,u,\lambda) : (\lambda - Du) d\nu_x(\lambda) dx.$$

On the other hand, we will now see that $X \leqslant 0$. To prove this, we fix $\varepsilon > 0$. Then, there exists $k_0 \in \mathbb{N}$ such that $\operatorname{dist}(u, V_k) < \varepsilon$ for all $k > k_0$, or equivalently, that $\operatorname{dist}(u_k - u, V_k) < \varepsilon$ for all $k > k_0$. Then, for $v_k \in V_k$, we may estimate X as follows

$$\begin{aligned} X &= \liminf_{k\to\infty} \int_\Omega \sigma(x,u_k,Du_k) : (Du_k - Du) dx \\ &= \liminf_{k\to\infty} \left(\int_\Omega \sigma(x,u_k,Du_k) : D(u_k - u - v_k) dx + \int_\Omega \sigma(x,u_k,Du_k) : Dv_k dx \right) \\ &\leqslant \liminf_{k\to\infty} \left(\Big(\int_\Omega |\sigma(x,u_k,Du_k)|^{p'} dx\Big)^{1/p'} \Big(\int_\Omega |D(u_k - u - v_k)|^p dx\Big)^{1/p} + \langle f, v_k\rangle \right). \end{aligned}$$

The term $\left(\int_\Omega |\sigma(x,u_k,Du_k)|^{p'} dx\right)^{1/p'}$ is bounded uniformly in k by the growth condition in (H1) and (3). On the other hand, by choosing $v_k \in V_k$ in such a way that $\|u_k - u - v_k\|_{W_0^{1,p}(\Omega)} < 2\varepsilon$ for all $k > k_0$, the term $\left(\int_\Omega |D(u_k - u - v_k)|^p dx\right)^{1/p}$ is bounded by 2ε. Moreover, we have

$$|\langle f, v_k\rangle| \leqslant |\langle f, v_k - (u_k - u)\rangle| + |\langle f, u_k - u\rangle| \leqslant 2\varepsilon \|f\|_{W^{-1,p}(\Omega)} + o(k).$$

Since $\varepsilon > 0$ was arbitrary, this proves $X \leqslant 0$. We conclude from (4), that

$$(5)\qquad \int_\Omega \int_{\mathbb{M}^{m\times n}} \sigma(x,u,\lambda) : \lambda d\nu_x(\lambda) dx \leqslant \int_\Omega \int_{\mathbb{M}^{m\times n}} \sigma(x,u,\lambda) : Du d\nu_x(\lambda) dx.$$

Now, we have to prove the theorem separately in the cases (a), (b), (c) and (d) of (H2). We start with the easiest case:

Case (d): Suppose that ν_x is not a Dirac mass on a set $x \in M$ of positive Lebesgue measure $|M| > 0$. Then, by the strict p-quasimonotonicity of $\sigma(x,u,\cdot)$, we have for a.e. $x \in M$

$$\int_{\mathbb{M}^{m\times n}} \sigma(x,u,\lambda) : \lambda d\nu_x(\lambda) > \int_{\mathbb{M}^{m\times n}} \sigma(x,u,\lambda) d\nu_x(\lambda) : \underbrace{\int_{\mathbb{M}^{m\times n}} \lambda d\nu_x(\lambda)}_{= Du(x)}.$$

Hence, by integrating over Ω, we get

$$\begin{aligned} &\int_\Omega \int_{\mathbb{M}^{m\times n}} \sigma(x,u,\lambda) d\nu_x(\lambda) : Du(x) dx \geqslant \\ &\qquad \int_\Omega \int_{\mathbb{M}^{m\times n}} \sigma(x,u,\lambda) : \lambda d\nu_x(\lambda) dx > \int_\Omega \int_{\mathbb{M}^{m\times n}} \sigma(x,u,\lambda) d\nu_x(\lambda) : Du(x) dx \end{aligned}$$

which is a contradiction. Hence, we have $\nu_x = \delta_{Du(x)}$ for almost every $x \in \Omega$. From this, it follows that $Du_k \to Du$ in measure for $k \to \infty$, and thus, $\sigma(x,u_k,Du_k) \to \sigma(x,u,Du)$ almost everywhere. Since, by the growth condition in (H1), $\sigma(x,u_k,Du_k)$ is equiintegrable, it follows that $\sigma(x,u_k,Du_k) \to \sigma(x,u,Du)$ in $L^1(\Omega)$ by the Vitali convergence theorem. This implies that $\langle F(u), v\rangle = 0$ for all $v \in \cup_{k\in\mathbb{N}} V_k$ and hence $F(u) = 0$, which proves the theorem in this case.

To prepare the proof in the remaining cases (a)–(c), we proceed as follows: From (5), we infer that

$$\int_\Omega \int_{\mathbb{M}^{m\times n}} \big(\sigma(x,u,\lambda)-\sigma(x,u,Du)\big):\big(\lambda-Du\big)d\nu_x(\lambda)dx \leqslant 0. \tag{6}$$

On the other hand, the integrand in (6) is nonnegative by monotonicity. It follows that the integrand must vanish almost everywhere with respect to the product measure $d\nu_x\otimes dx$. Hence, we have that for almost all $x\in\Omega$

$$(\sigma(x,u,\lambda)-\sigma(x,u,Du)):(\lambda-Du)=0 \quad \text{on } \operatorname{spt}\nu_x \tag{7}$$

and thus

$$\operatorname{spt}\nu_x\subset\{\lambda \mid (\sigma(x,u,\lambda)-\sigma(x,u,Du)):(\lambda-Du)=0\}. \tag{8}$$

Now, we proceed with the proof in the single cases.

Case (c): By strict monotonicity, it follows from (7) that $\nu_x=\delta_{Du(x)}$ for almost all $x\in\Omega$, and hence $Du_k\to Du$ in measure. The reminder of the proof in this case is exactly as in case (d).

Case (b): We start by showing that for almost all $x\in\Omega$, the support of ν_x is contained in the set where W agrees with the supporting hyper-plane $L:=\{(\lambda, W(x,u,Du)+\sigma(x,u,Du)(\lambda-Du))\}$ in $Du(x)$, i.e., we want to show that

$$\operatorname{spt}\nu_x\subset K_x=\{\lambda\in\mathbb{M}^{m\times n}:W(x,u,\lambda)=W(x,u,Du)+\sigma(x,u,Du):(\lambda-Du)\}.$$

If $\lambda\in\operatorname{spt}\nu_x$ then by (8)

$$(1-t)(\sigma(x,u,Du)-\sigma(x,u,\lambda)):(Du-\lambda)=0 \quad \text{for all } t\in[0,1]. \tag{9}$$

On the other hand, by monotonicity, we have for $t\in[0,1]$ that

$$0\leqslant(1-t)(\sigma(x,u,Du+t(\lambda-Du))-\sigma(x,u,\lambda)):(Du-\lambda). \tag{10}$$

Subtracting (9) from (10), we get

$$0\leqslant(1-t)(\sigma(x,u,Du+t(\lambda-Du))-\sigma(x,u,Du)):(Du-\lambda) \tag{11}$$

for all $t\in[0,1]$. But by monotonicity, in (11) also the reverse inequality holds and we may conclude, that

$$(\sigma(x,u,Du+t(\lambda-Du))-\sigma(x,u,Du)):(\lambda-Du)=0 \tag{12}$$

for all $t\in[0,1]$, whenever $\lambda\in\operatorname{spt}\nu_x$. Now, it follows from (12) that

$$\begin{aligned} W(x,u,\lambda) &= W(x,u,Du)+\int_0^1\sigma(x,u,Du+t(\lambda-Du)):(\lambda-Du)dt \\ &= W(x,u,Du)+\sigma(x,u,Du):(\lambda-Du) \end{aligned}$$

as claimed.

By the convexity of W we have $W(x,u,\lambda)\geq W(x,u,Du)+\sigma(x,u,Du):(\lambda-Du)$ for all $\lambda\in\mathbb{M}^{m\times n}$ and thus L is a supporting hyper-plane for all $\lambda\in K_x$. Since the mapping $\lambda\mapsto W(x,u,\lambda)$ is by assumption continuously differentiable we obtain

$$\sigma(x,u,\lambda)=\sigma(x,u,Du) \quad \text{for all } \lambda\in K_x\supset\operatorname{spt}\nu_x \tag{13}$$

and thus

$$\bar\sigma:=\int_{\mathbb{M}^{m\times n}}\sigma(x,u,\lambda)\,d\nu_x(\lambda)=\sigma(x,u,Du)\,. \tag{14}$$

Now consider the Carathéodory function

$$g(x,u,p) = |\sigma(x,u,p) - \bar{\sigma}(x)|\,.$$

The sequence $g_k(x) = g(x, u_k(x), Du_k(x))$ is equiintegrable and thus

$$g_k \rightharpoonup \bar{g} \quad \text{weakly in } L^1(\Omega)$$

and the weak limit $\bar{g}$ is given by

$$\begin{aligned}\bar{g}(x) &= \int_{\mathbb{R}^m \times \mathbb{M}^{m\times n}} |\sigma(x,\eta,\lambda) - \bar{\sigma}(x)|\, d\delta_{u(x)}(\eta) \otimes d\nu_x(\lambda)\\ &= \int_{\operatorname{spt}\nu_x} |\sigma(x,u(x),\lambda) - \bar{\sigma}(x)|\, d\nu_x(\lambda) = 0\end{aligned}$$

by (13) and (14). Since $g_k \geqslant 0$ it follows that

$$g_k \to 0 \quad \text{strongly in } L^1(\Omega).$$

This again suffices to pass to the limit in the equation and the proof of the case (b) is finished.

Case (a): We claim that in this case for almost all $x \in \Omega$ the following identity holds for all $\mu \in \mathbb{M}^{m\times n}$ on the support of ν_x:

$$\sigma(x,u,\lambda) : \mu = \sigma(x,u,Du) : \mu + (\nabla\sigma(x,u,Du)\mu) : (Du - \lambda), \tag{15}$$

where ∇ is the derivative with respect to the third variable of σ. Indeed, by the monotonicity of σ we have for all $t \in \mathbb{R}$

$$(\sigma(x,u,\lambda) - \sigma(x,u,Du + t\mu)) : (\lambda - Du - t\mu) \geq 0,$$

whence, by (7),

$$\begin{aligned}-\sigma(x,u,\lambda) : (t\mu) &\geqslant -\sigma(x,u,Du) : (\lambda - Du) + \sigma(x,u,Du + t\mu) : (\lambda - Du - t\mu)\\ &= t\big((\nabla\sigma(x,u,Du)\mu)(\lambda - Du) - \sigma(x,u,Du) : \mu\big) + o(t).\end{aligned}$$

The claim follows from this inequality since the sign of t is arbitrary. Since the sequence $\sigma(x,u_k,Du_k)$ is equiintegrable, its weak L^1-limit $\bar{\sigma}$ is given by

$$\begin{aligned}\bar{\sigma} &= \int_{\operatorname{spt}\nu_x} \sigma(x,u,\lambda) d\nu_x(\lambda)\\ &= \int_{\operatorname{spt}\nu_x} \sigma(x,u,Du) d\nu_x(\lambda) + (\nabla\sigma(x,u,Du))^t \int_{\operatorname{spt}\nu_x} (Du - \lambda) d\nu_x(\lambda)\\ &= \sigma(x,u,Du),\end{aligned}$$

where we used (15) in this calculation. This finishes the proof of the case (a) and hence of the theorem.

Remark. Notice, that in case (b) we have $\sigma(x,u_k,Du_k) \to \sigma(x,u,Du)$ in $L^1(\Omega)$. In the cases (c) and (d), we even have $Du_k \to Du$ in measure as $k \to \infty$.

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