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# On Classes of $p$-adic Lie Groups

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Abstract. We consider non-contracting $p$-adic Lie groups and we establish equivalence relations and connections among the following classes of $p$-adic Lie groups: (1) non-contracting; (2) type $R$; (3) distal and (4) Tortrat. We also deduce that non-contracting $p$-adic Lie groups are unimodular and IN $p$-adic Lie groups are non-contracting.

In this note we prove $p$-adic analogue of results in [DR2] and [Ro].

Let $G$ be a locally compact group and $e$ denote the identity of $G$. Let $\mathcal{P}(G)$ be the space of all regular Borel probability measures on $G$, equipped with the weak* topology with respect to all bounded continuous functions on $G$: see [H] for more details on probability measures on locally compact groups. A locally compact group $G$ is called *non-contracting* if $e$ is not a limit point of $\{x^n g x^{-n} \mid n \in \mathbb{Z}\}$ for any $g \in G \setminus (e)$ and any $x \in G$.

Let $V$ be a finite-dimensional vector space over $\mathbb{Q}_p$ and $T$ be a group of linear transformations on $V$. Then we say that $V$ is of *type* $R_T$ if all eigenvalues of $T$ are of absolute value one. A $p$-adic Lie group $G$ is called *type* $R$ if $L(G)$ is type $\mathrm{R}_{\mathrm{Ad}(G)}$ where $L(G)$ is the Lie algebra of $G$. Let $\mathrm{Aut}(L(G))$ be the group of all Lie algebra automorphisms of $L(G)$. It should be noted that $\mathrm{Aut}(L(G))$ is an algebraic subgroup of $GL(L(G))$, the general linear group on $L(G)$. We now prove the following using methods in [Wa2].

**Theorem 1.** *Let $G$ be a $p$-adic Lie group and $L(G)$ be the Lie algebra of $G$. Then the following are equivalent.*

1. *$G$ is non-contracting.*
2. *For each $x \in G$ there exists an open subgroup $U(x)$ invariant under the conjucation of $x$ and for any compact subset $C$ of $U(x)$ the orbit $\{x^n C x^{-n} \mid n \in \mathbb{Z}\}$ is relatively compact.*
3. *The closed subgroup generated by $\mathrm{Ad}(x)$ is compact in $\mathrm{Aut}(L(G))$ for any $x \in G$.*
4. *$G$ is of type $R$.*

**Proof.** Let $x \in G$ and $\alpha\colon G \to G$ be $\alpha(g) = xgx^{-1}$ for all $g \in G$. Suppose $G$ is non-contracting. Then Theorem 3.6 of [Wa2] implies (2).

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We now prove (2) $\Rightarrow$ (3). Suppose there is an open subgroup $U$ invariant under $\alpha$ and orbits of the cyclic group generated by $\alpha$ in $U$ are all relatively compact. Then let $\alpha = \alpha_u\alpha_s$ be the Jordan decomposition of $\alpha$ in $\mathrm{Aut}(L(G))$ where $\alpha_s$ and $\alpha_u$ are semisimple and unipotent parts of $\alpha$ respectively. Let $T$ be the torus generated by $\alpha_s$ and $T_a$ and $T_d$ be the anisotropic and split parts of $T$. Then $T_a(\mathbb{Q}_p)T_d(\mathbb{Q}_p)$ is of finite index in $T(\mathbb{Q}_p)$ (see [Wa2]). By considering a power of $\alpha_s$ we may assume that $\alpha_s$ is in $T_a(\mathbb{Q}_p)T_d(\mathbb{Q}_p)$ and let $\alpha_d$ be the split part of $\alpha$. Since the closed subgroup generated by $\alpha_u$ and the subgroup $T_a(\mathbb{Q}_p)$ are compact, to prove the closed subgroup generated by $\alpha$ is compact it is enough to prove that the subgroup generated by $\alpha_d$ is relatively compact. Since orbits in $U$ for the cyclic group generated by $\alpha$ are relatively compact the eigenvalues of $\alpha_d$ are of $p$-adic absolute value one and hence the closed subgroup generated by $\alpha_d$ is compact.

It is easy to see that (3) $\Rightarrow$ (4). We now claim that (4) $\Rightarrow$ (1). Suppose there exits a $g \in G$ and $h \in G$ such that $e$ is a limit point $(g^nhg^{-n})$. Let $V_x = \{v \in L(G) \mid \mathrm{Ad}(x)^n(v) \to 0\}$, for $x \in G$. Suppose $G$ is of type $R$. Then both $V_g$ and $V_{g^{-1}}$ are of dimension zero. Then by Theorem 3.6 of [Wa2], there exists a closed open subgroup $M$ of $G$ such that $gMg^{-1} = M$ and $\{g^nxg^{-n} \mid n \in \mathbb{Z}\}$ is relatively compact for all $x \in M$. Since $e$ is a limit point of $(g^nhg^{-n})$, $h$ belongs to any neighbourhood of $e$ in $G$ that is invariant under the conjugation by $g$. By Corollary 1.4 of [Wa2], $M$ has arbitrarily small open subgroups invariant under the conjugation by $g$ and hence $h = e$. This proves that $G$ is non-contracting. □

**Corollary 1.** *Let $G$ be a $p$-adic Lie group. Suppose $G$ is non-contracting. Then $G$ is unimodular.*

**Proof.** Let $m$ be the left Haar measure on $G$ and $\Delta$ be the unimodular homomorphism on $G$, that is $m(Ex) = \Delta(x)m(E)$ for all $x \in G$ and for all Borel sets $E$ of $G$. Let $x \in G$. Then by Theorem 1, there exists an open subgroup $U$ such that $xUx^{-1} = U$. Since $G$ is totally disconnected, there exists a compact open subgroup $K$ of $U$. Again by Theorem 1, we get that $\cup x^nKx^{-n} = L$, say is a relatively compact open subgroup of $G$ and $xLx^{-1} = L$. Thus, $m(L) = m(xLx^{-1}) = m(Lx^{-1}) = \Delta(x^{-1})m(L)$. Since $L$ is a relatively compact open subgroup, we have $0 < m(L) < \infty$. This implies that $\Delta(x) = 1$. Thus, $G$ is unimodular. □

**Proposition 1.** *Let $G$ be a Zariski-connected $p$-adic algebraic group. Suppose $G$ is non-contracting. Then $G$ is a compact extension of its nilradical.*

**Proof.** Let $G$ be connected algebraic group that is non-contracting. Let us first consider the case when $G$ is semisimple. Let $T$ be a maximal $\mathbb{Q}_p$-split torus of $G$. Then $\mathrm{Ad}T$ is isomorphic to $(\mathbb{Q}_p^*)^n$ for some $n$ where $\mathbb{Q}_p^*$ is the multiplicative group of units in $\mathbb{Q}_p$. By Theorem 1, every element of $\mathrm{Ad}T$ generates a relatively compact subgroup. This implies that $T$ is central and hence since $G$ is semisimple, $T$ is trivial. Now by Theorem 3.1 of [PR], $G$ is compact.

We now consider the case when $G$ is solvable. Let $U$ be the unipotent radical of $G$ and $T$ be a torus such that $G$ is the semidirect product of $U$ and $T$. Let $T_s$ be the $\mathbb{Q}_p$-split part of $T$. Then as in the previous case, we may prove that $T_s$ centralizes $U$. This implies that $G$ is a compact extension of a nilpotent normal subgroup.

Now let $G$ be any connected algebraic group. Let $S$ and $U$ be the solvable and unipotent radicals of $G$ respectively. By a result of G. D. Mostow, there exists a reductive Levi subgroup $L$ of $G$ such that $G$ is the semidirect product of $L$ and $U$

and the connected component of identity in the center of $L$, say $T$ is a maximal torus of $S$ (see 11.22 and Theorem 11.23 of [B]). Also, by Theorem 2.4 of [PR], $L = RT$ where $R$ is a connected semisimple subgroup of $G$ and hence since $S = TU$ (see Theorem 10.6 of [B]), we have $G = LU = RTU = RS$. Since $G$ is non-contracting, $R$ is also non-contracting. This implies that $R$ is compact. Since $S$ is a solvable connected that is non-contracting, we have $S$ is a compact extension of its nilradical. Since the nilradical of $S$ is same as the nilradical of $G$, we get that $G$ is a compact extension of its nilradical. $\square$

A locally compact group $G$ is called *distal* if $e$ is not a limit point of $\{gxg^{-1} \mid g \in G\}$ for any $x \in G \setminus (e)$.

A locally compact group $G$ is said to be a *IN-group* if there exists a compact invariant neighbourhood of $e$. See [GM] and [P] for more details on IN-groups.

A locally compact group $G$ is called *Tortrat* if a sequence of the form $(g_n \lambda g_n^{-1})$, where $\lambda \in \mathcal{P}(G)$ and $(g_n)$ is a sequence in $G$, has an idempotent limit point only if $\lambda$ is an idempotent. See [Ra] for more details on Tortrat groups.

A *local field* $\mathbb{K}$ is a commutative non-discrete locally compact field (see [We]). A locally compact group $G$ is said to be a *linear group* if $G$ is a closed subgroup of $GL(V)$, the general linear group on a finite-dimensional vector space $V$ over a local field $\mathbb{K}$.

It is proved in [Ro], that compact extensions of nilpotent normal subgroups are distal. Here, we prove that compact extensions of (not necessarily normal) unipotent groups are distal.

**Proposition 2.** *Let $G$ be a linear group. Suppose there exist an unipotent algebraic (not necessarily normal) subgroup $U$ of $G$ such that $G/U$ is compact. Then $G$ is distal.*

**Proof.** Let $V$ be a finite-dimensional vector space over a local field such that $G$ is a closed subgrup of $GL(V)$. Let $W$ be the algebra of all linear endomorphisms on $V$. Now, for $g \in GL(V)$, define $\phi_g \colon W \to W$ by $\phi_g(w) = gwg^{-1}$ for all $w \in W$. Let $(g_n)$ be a sequence in $G$ such that $g_n x g_n^{-1} \to e$ for some $x \in G$. Since $G/U$ is compact, by passing to a subsequence of $(g_n)$, we may assume that there exists a sequence $(h_n)$ in $G$ such that $u_n = h_n^{-1} g_n \in U$ and $h_n \to h \in G$. This implies that $u_n x u_n^{-1} \to e$. Let $\phi_n = \phi_{u_n}$. Then by Lemma 2.2 of [DR1], there exist sequences $(a_n)$ and $(b_n)$ in $U$ such that $u_n = a_n b_n$, $a_n \to a$ in $U$ and $b_n w b_n^{-1} = w$ for all $w$ such that $(u_n w u_n^{-1})$ converges. Since $u_n x u_n^{-1} \to e$, we have $b_n x b_n^{-1} \to e$ and hence $x = e$. This proves that $G$ is distal. $\square$

A $p$-adic Lie group $G$ is called *Ad-regular* if $Z(G)$ is the kernel of the adjoint representation of $G$.

**Theorem 2.** *Let $G$ be a Ad-regular $p$-adic Lie group. Then the following are equivalent:*

(1) *$G$ is non-contracting;*
(2) *$G$ is distal;*
(3) *$G$ is of type R.*

*In addition, if $G$ is a $p$-adic linear group, then* (1), (2) *and* (3) *are equivalent to*

(4) *$G$ is Tortrat.*

**Proof.** In view of Theorem 1, it is enough to prove that (1) is equivalent to (2). Let $G$ be a Ad-regular $p$-adic Lie group. Suppose $G$ is non-contracting. Let $H$ be the algebraic closure of $\mathrm{Ad}(G)$. Then any $\mathbb{Q}_p$-split semisimple element occurring in the Jordan decomposition of any element of Inn $(G)$ generates a relatively compact subgroup and hence $\mathrm{Ad}(G)$ is contained in a compact extension of an unipotent subgroup of $H$. By Proposition 2, we get that $\mathrm{Ad}(G)$ is distal. Since the kernel of the adjoint representation is the center of $G$, $G$ is distal. This proves that (1) $\Rightarrow$ (2). That (2) $\Rightarrow$ (1) is obvious. The second part of the theorem is proved in Theorem 2 of [Ra]. □

**Remark 1.** Let $G$ be a $p$-adic Lie group. Suppose for each $g \in G$ there exists a compact neighbourhood $K(g)$ of $e$ such that $gK(g)g^{-1} = K(g)$. Then $G$ is non-contracting, which may be seen as follows: Let $x$ be a point in $G$ and $C(x) = \{g \in G \mid x^n g x^{-n} \to e\}$. Then it is easy to see that $C(x) \subset K(x)$. By Theorem 3.6 of [Wa2], $C(x)$ is a closed subgroup of $G$. Thus, $C(x)$ is compact and hence it is trivial (see Theorem 3.5 of [Wa2]). By Theorem 1, we have $G$ is non-contracting. In particular, IN $p$-adic Lie groups are non-contracting. In fact, a similar argument proves that IN $p$-adic Lie groups are distal without using Theorem 2.

A compactly generated locally compact group is said to be of *polynomial growth* if for every compact neighbourhood $U$ of $e$, $m(U^n) \leq Kn^l$ for all $n$ and for some constant $K$ and an integer $l$ where $m$ is a Haar measure on $G$. See [Gu], [L] and [P] for more details on the theory of polynomial growth. Since only reductive $p$-adic algebraic groups are compactly generated (see Proposition 3.15 of [PR]), we have the following.

**Corollary 2.** *Let $G$ be a Zariski-connected $p$-adic reductive algebraic group. Then* (1), (2), (3) *and* (4) *of Theorem* 2 *are equivalent to either of the following conditions.*

(5) *$G$ has polynomial growth.*

(6) *$G$ is an IN-group.*

**Proof.** Suppose $G$ has polynomial growth. Then by Theorem 2 of [L], there exists a compact normal subgroup $H$ of $G$ such that $G/H$ is a real Lie group. Since $G$ is totally disconnected, $G/H$ is discrete. This implies that $G$ has a compact open normal subgroup $H$. Thus, $G$ is an IN-group. This proves (5) implies (6) and that (6) implies (1) follows from Remark 1.

Suppose $G$ is non-contracting. By Proposition 1, $G$ is a compact extension of its nilradical, say $N$. Since $G$ is a reductive group, the connected component of the identity of the center of $G$, say $Z$, is a torus and $Z$ is the solvable radical of $G$ (see 11.21 of [B]). This implies that $N \subset Z$. Thus, $G$ is a compact extension of its center and hence $G$ has polynomial growth (see [P]). □

**Remark 2.** The results in this note may be proved for any linear algebraic group $G$ defined over a non-Archimedean local field $\mathbb{K}$ provided $G$ is connected and has a Levi-decomposition defined over $\mathbb{K}$. It may be mentioned that results in [Wa1] are used in the place of results in [Wa2] in the arguement.

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## References

[B] A. Borel, *Linear Algebraic Groups*, Graduate Texts in Mathmatics 126, Springer-Verlag, New York, 1991, MR 92d:20001, Zbl 726.20030.

[DR1] S. G. Dani and C. R. E. Raja, *Asymptotics of measures under group automorphisms and an application to factor sets*, Proceedings of the International Colloquium on Lie Groups and Ergodic Theory 1996, Tata Institute of Fundamental Research, Mumbai, Narosa Publishing House, New Delhi, 1998, pp. 59–73.

[DR2] S. G. Dani and C. R. E. Raja, *A note on Tortrat groups*, Journal of Theoretical Probability **11** (1998), 571–576, MR 99g:60012, Zbl 980.57605.

[GM] S. Grosser and M. Moskowitz, *Compactness conditions in topological groups*, J. Reine Angew. Math. **246** (1971), 1–40, MR 44 #1766, Zbl 219.22011.

[Gu] Y. Guivarc'h, *Croisance polynomiale et périodes des fonctions harmoniques*, Bull. Soc. Math. France **101** (1973), 333–379, MR 51 #5841.

[H] H. Heyer, *Probability Measures on Locally Compact Groups*, Ergebnisse der Mathematik und ihrer Grenzgebiete 94, Springer-verlag, Berlin, 1977, MR 58 #18648, Zbl 376.60002.

[L] V. Losert, *On the structure of groups with polynomial growth*, Math. Z. **195** (1987), 109–117, MR 88e:53059 Zbl 633.22002.

[P] T. W. Palmer, *Classes of nonabelian, noncompact, locally compact groups*, Rocky Mountain J. Math. **8** (1978), 683–741, MR 81j:22003, Zbl 396.22001.

[PR] V. Platonov and A. Rapinchuk, *Algebraic Groups and Number Theory*, Academic Press, Boston, 1994, MR 95b:11039, Zbl 841.20046.

[Ra] C. R. E. Raja, *Some class of distal groups are Tortrat*, preprint.

[Ro] J. Rosenblatt, *A distal property of groups and the growth of connected locally compact groups*, Mathematika **26** (1979), 94–98, MR 81c:22014, Zbl 414.22011.

[Wa1] S. P. Wang, *On Mautner phenomonen and groups with property* (T), Amer. J. Math. **104** (1982), 1191–1210, Zbl 507.22011.

[Wa2] J. S. P. Wang, *Mautner phenemenon for p-adic Lie groups*, Math. Z. **185** (1984), 403–412, MR 85i:22031, Zbl 539.22015.

[We] A. Weil, *Basic Number Theory*, Springer-verlag, Berlin, 1967, MR 38 #3244, Zbl 823.11001.

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