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Reduced Cowen Sets

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Abstract. For $f \in H^2$, let

$$G'_f := \{g \in zH^2 : f + \overline{g} \in L^\infty \text{ and } T_{f+\overline{g}} \text{ is hyponormal}\}.$$

In 1988, C. Cowen posed the following question: If $g \in G'_f$ is such that $\lambda\, g \notin G'_f$ (all $\lambda \in \mathbb{C}$, $|\lambda| > 1$), is g an extreme point of G'_f ? In this note we answer this question in the negative. At the same time, we obtain a general sufficient condition for the answer to be affirmative; that is, when $f \in H^\infty$ is such that rank $H_{\overline{f}} < \infty$.

Contents

1. Introduction

A bounded linear operator A on a Hilbert space is said to be hyponormal if its self-commutator $[A^*, A] := A^*A - AA^*$ is positive (semidefinite). Given $\varphi \in L^\infty(\mathbb{T})$, the Toeplitz operator with symbol φ is the operator T_φ on the Hardy space $H^2(\mathbb{T})$ of the unit circle $\mathbb{T} \equiv \partial\mathbb{D}$ defined by $T_\varphi f := P(\varphi \cdot f)$, where $f \in H^2(\mathbb{T})$ and P denotes the orthogonal projection that maps $L^2(\mathbb{T})$ onto $H^2(\mathbb{T})$. Let $H^\infty(\mathbb{T}) := L^\infty \cap H^2$, that is, H^∞ is the set of bounded analytic functions on $\mathbb{D}$. The problem of determining which symbols induce hyponormal Toeplitz operators was solved by C. Cowen [Co2] in 1988. Cowen's method is to recast the operator-theoretic problem of hyponormality for Toeplitz operators as a functional equation involving the operator's symbol.

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Suppose that $\varphi \in L^\infty(\mathbb{T})$ is arbitrary and consider the following subset of the closed unit ball of $H^\infty(\mathbb{T})$,

$$\mathcal{E}(\varphi) := \{k \in H^\infty(\mathbb{T}) : ||k||_\infty \le 1 \text{ and } \varphi - k\overline{\varphi} \in H^\infty(\mathbb{T})\}.$$

Cowen's Theorem states that T_φ is hyponormal if and only if $\mathcal{E}(\varphi)$ is nonempty [Co2], [NT]. We also recall the connection between Hankel and Toeplitz operators. For φ in L^∞, the *Hankel operator* $H_\varphi : H^2 \to H^2$ is defined by $H_\varphi f := J(I-P)(\varphi f)$, where $J : (H^2)^\perp \to H^2$ is given by $Jz^{-n} = z^{n-1}$ for $n \ge 1$. The following are two basic identities:

$$(1) \qquad T_{\varphi\psi} - T_\varphi T_\psi = H^*_{\overline{\varphi}} H_\psi \quad (\varphi, \psi \in L^\infty) \quad \text{and} \quad H_{\varphi h} = T^*_{\widetilde{h}} H_\varphi \quad (h \in H^\infty),$$

where for $\zeta \in L^\infty$, we define $\widetilde{\zeta}(z) := \overline{\zeta(\overline{z})}$. From this we can see that if $k \in \mathcal{E}(\varphi)$ then

$$[T^*_\varphi, T_\varphi] = H^*_{\overline{\varphi}} H_{\overline{\varphi}} - H^*_\varphi H_\varphi = H^*_{\overline{\varphi}} H_{\overline{\varphi}} - H^*_{k\,\overline{\varphi}} H_{k\,\overline{\varphi}} = H^*_{\overline{\varphi}}(1 - T_{\widetilde{k}} T^*_{\widetilde{k}}) H_{\overline{\varphi}},$$

which implies that $\ker H_{\overline{\varphi}} \subseteq \ker [T^*_\varphi, T_\varphi]$.

To describe the set of g such that $T_{f+\overline{g}}$ is hyponormal for a given f, C. Cowen [Co1] defined the set G'_f as follows. If $H := \{h \in zH^\infty : ||h||_2 \le 1\}$, let

$$G'_f := \left\{ g \in zH^2 : \sup_{h_0 \in H} |\langle hh_0, f\rangle| \ \ge \ \sup_{h_0 \in H} |\langle hh_0, g\rangle| \ \text{ for every } h \in H^2 \right\}.$$

To see how this definition is relevant to hyponormality of Toeplitz operators, we assume that $f + \overline{g} \in L^\infty$. Note that if $f \in H^2$ then $H_{\overline{f}}$ makes sense when f has an L^∞-conjugate $g \in H^2$, that is, $f + \overline{g} \in L^\infty$. For, given $h \in H^2$ we have $H_{\overline{f}+g}(h) = J(I-P)(\overline{f}h + gh) = J(I-P)(\overline{f}h) =: H_{\overline{f}}h$. If $f + \overline{g} \in L^\infty$ $(f \in H^2, g \in zH^2)$ and $h \in H^2$ then

$$\begin{aligned}
\sup_{h_0 \in H} |\langle hh_0, f\rangle| &= \sup_{h_0 \in H} \left| \int_{\mathbb{T}} hh_0 \overline{f}\, d\mu \right| = \sup_{h_0 \in H} \left| \int_{\mathbb{T}} (I-P)(\overline{f}h + gh) h_0\, d\mu \right| \\
&= \sup_{h_0 \in H} |\langle (I-P)\overline{f}h, \overline{h_0}\rangle| = \sup_{h_0 \in H} |\langle J(I-P)\overline{f}h, h_0\rangle| \\
&= ||H_{\overline{f}}h||
\end{aligned}$$

and similarly,

$$\sup_{h_0 \in H} |\langle hh_0, g\rangle| = ||H_{\overline{g}}h||.$$

Recall ([Ab, Lemma 1]) that if $\varphi = f + \overline{g} \in L^\infty$ $(f \in H^2, g \in zH^2)$ then the following are equivalent:

(a) T_φ is hyponormal;
(b) $||H_{\overline{f}}h|| \ge ||H_{\overline{g}}h||$ for every $h \in H^2$.

Therefore we can see that for $f \in H^2$,

$$(2) \qquad G'_f = \left\{ g \in zH^2 : f + \overline{g} \in L^\infty \text{ and } T_{f+\overline{g}} \text{ is hyponormal} \right\}.$$

We call G'_f the *reduced Cowen set* for f. To avoid some technical difficulties using the original definition of G'_f when dealing with hyponormality of $T_{f+\overline{g}}$, hereafter we assume that $f + \overline{g} \in L^\infty$ and adopt (2) as our definition of G'_f; this appears to be natural when studying the set G'_f. We can easily see that G'_f is balanced and convex. Write

$$\nabla G'_f := \{ g \in G'_f : \ \lambda g \notin G'_f \ (\text{all } \lambda \in \mathbb{C}, \ |\lambda| > 1) \}$$

and $\operatorname{ext} G'_f$ for the set of all extreme points of G'_f. In [Co1] the following question was posed:

Question. Is $\nabla G'_f \subseteq \operatorname{ext} G'_f$?

In [CCL] an affirmative answer to the above question was given in case f is an analytic polynomial. In this note we answer the above question in the negative, and give a general sufficient condition for the answer to be affirmative: If $\operatorname{rank} H_{\overline{f}} < \infty$ then $\nabla G'_f \subseteq \operatorname{ext} G'_f$. In [CCL], our ploy was to use the Carathéodory-Schur Interpolation Problem to deal with the case of an analytic polynomial f. By comparison, we here resort to the classical Hermite-Fejér Interpolation Problem.

2. Main results

If $\varphi \in L^\infty$, write $\varphi_+ = P(\varphi) \in H^2$ and $\varphi_- = \overline{(I-P)(\varphi)} \in zH^2$. Thus $\varphi = \varphi_+ + \overline{\varphi_-}$ is the decomposition of φ into its analytic and co-analytic parts. We first reformulate Cowen's Theorem. Suppose that $\varphi \in L^\infty$ is of the form $\varphi(z) = \sum_{n=-\infty}^{\infty} a_n z^n$ and that $k(z) = \sum_{n=0}^{\infty} c_n z^n$ is in H^2. Then $\varphi - k\,\overline{\varphi} \in H^\infty$ if and only if

$$\begin{pmatrix} \overline{a_1} & \overline{a_2} & \overline{a_3} & \dots & \overline{a_n} & \dots \\ \overline{a_2} & \overline{a_3} & \dots & \overline{a_n} & \dots & \\ \overline{a_3} & \dots & \dots & \dots & & \\ \vdots & \overline{a_n} & \dots & & & \\ \overline{a_n} & \dots & & & & \\ \vdots & & & & & \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ \vdots \\ \vdots \end{pmatrix} = \begin{pmatrix} a_{-1} \\ a_{-2} \\ a_{-3} \\ \vdots \\ \vdots \\ \vdots \end{pmatrix}, \tag{3}$$

that is, $H_{\overline{\varphi_+}} k = \overline{z}\widetilde{\varphi_-}$. Thus by Cowen's Theorem we have:

Lemma 1 ([CuL])**.** *If $\varphi \equiv \varphi_+ + \overline{\varphi_-} \in L^\infty$, then $\mathcal{E}(\varphi) \neq \emptyset$ if and only if the equation $H_{\overline{\varphi_+}} k = \overline{z}\widetilde{\varphi_-}$ admits a solution k satisfying $||k||_\infty \leq 1$.*

Recall that a function $\varphi \in L^\infty$ is of bounded type (or in the Nevanlinna class) if it can be written as the quotient of two functions in $H^\infty(\mathbb{D})$, that is, there are functions ψ_1, ψ_2 in $H^\infty(\mathbb{D})$ such that

$$\varphi(z) = \frac{\psi_1(z)}{\psi_2(z)} \quad \text{for almost all } z \in \mathbb{T}.$$

For example, rational functions in L^∞ are of bounded type. By an argument of M. Abrahamse [Ab, Lemma 3], the function φ is of bounded type if and only if $\ker H_{\overline{\varphi}} \neq \{0\}$. Thus if $\varphi \equiv \varphi_+ + \overline{\varphi_-} \in L^\infty$ and $\overline{\varphi}$ is not of bounded type then $\ker H_{\overline{\varphi_+}} = \ker H_{\overline{\varphi}} = \{0\}$, so that the equation $H_{\overline{\varphi_+}} k = \overline{z}\widetilde{\varphi_-}$ has a unique solution whenever it is solvable; in other words, if $\overline{\varphi}$ is not of bounded type, and T_φ is hyponormal, then $\mathcal{E}(\varphi)$ has exactly one element.

We now have:

Theorem 2. *Suppose that $\psi \in H^\infty$ is such that $\overline{\psi}$ is not of bounded type, and let $f := z^3\psi$. Then $\nabla G'_f \not\subseteq \operatorname{ext} G'_f$.*

Proof. By assumption, $f \in H^\infty$ and $\overline{f}$ is not of bounded type; indeed, if $\overline{f}$ were of bounded type then $\overline{f} = \frac{g}{h}$ $(g, h \in H^\infty(\mathbb{D}))$, and so $\overline{\psi} = \frac{z^3 g}{h}$ would be of bounded type. Observe now that by definition and Lemma 1,

$$G'_f = \{g \in zH^2 : f + \bar{g} \in L^\infty \text{ and } H_{\overline{f}}k = \bar{z}\widetilde{g} \text{ for some } k \in H^\infty \text{ with } ||k||_\infty \le 1\}.$$

Since $f \in z^3H^\infty$, we have that $\bar{z}f$, $\bar{z}^2 f$, $\frac{1}{2}(\bar{z}+\bar{z}^2)f$ all are in zH^∞. A straightforward calculation shows that

$$H_{\overline{f}}(q) = \overline{zq}\,\widetilde{f} \quad \text{for } q = z,\ z^2,\ \frac{1}{2}(z+z^2).$$

Since $||q||_\infty \le 1$ and $\bar{q}\widetilde{f} = \widetilde{\bar{q}f} \in zH^\infty$ we have that $\{\bar{z}\,f,\ \bar{z}^2 f,\ \frac{1}{2}(\bar{z}+\bar{z}^2)f\} \subseteq G'_f$. We will now show that $\frac{1}{2}(\bar{z}+\bar{z}^2)f \in \nabla G'_f$, which proves $\nabla G'_f \not\subseteq \operatorname{ext} G'_f$. Since $\overline{f}$ is not of bounded type (so $\ker H_{\overline{f}} = \{0\}$), we know that for $|\lambda| > 1$ and $q := \frac{1}{2}(z+z^2)$, the unique solution of the equation $H_{\overline{f}}k = \overline{\lambda z q}\,\widetilde{f}$ is $k = \bar{\lambda}\,q$. But $||\bar{\lambda}\,q||_\infty > 1$, so $\lambda\,\bar{q}\,f \notin G'_f$ and therefore $\frac{1}{2}(\bar{z}+\bar{z}^2)f \equiv \bar{q}f \in \nabla G'_f$. $\square$

For a concrete example satisfying the hypotheses of Theorem 2, let ψ be a Riemann mapping of the unit disk onto the interior of the ellipse with vertices $\pm i(1-\alpha)^{-1}$ and passing through $\pm(1+\alpha)^{-1}$, where $0 < \alpha < 1$. Then ψ is in H^∞, and $\overline{\psi}$ is not of bounded type ([CoL, Corollary 2]).

In [CCL], an affirmative answer to Cowen's Question was given in case f is an analytic polynomial. We now establish that the answer is also affirmative in the more general instances of $\operatorname{rank} H_{\overline{f}} < \infty$.

To see this we need the following auxiliary lemma.

Lemma 3. *Let q be a finite Blaschke product, let $k \in H^\infty$, and let*

$$G \equiv G(q,k) := \{b \in k + qH^\infty :\ ||b||_\infty \le 1\}.$$

If G contains at least two functions then it contains a function b with $||b||_\infty < 1$.

Proof. Write

$$q \equiv e^{i\theta}\prod_{i=1}^{n} b_i^{n_i}, \quad \text{where } b_i \equiv \frac{z-\alpha_i}{1-\overline{\alpha_i}z}, \quad \theta \in [0, 2\pi),$$

and $\alpha_1, \cdots, \alpha_n$ are distinct points in $\mathbb{D}$. If we define

$$\mathbf{x}_{i,j} := \frac{z^j}{(1-\overline{\alpha_i}z)^{j+1}} \quad \text{for } 1 \le i \le n \text{ and } 0 \le j < n_i,$$

then the functions $\mathbf{x}_{i,j}$ form a basis for $H^2 \ominus qH^2$ (cf. [FF, Lemma X.1.1]). Write $k = k_1 + k_2$, where $k_1 \in H^2 \ominus qH^2$ and $k_2 \in qH^2$. Note that k_1 is entirely determined by the values of $k_1^{(j)}(\alpha_i)$ $(1 \le i \le n,\ 0 \le j < n_i)$, and also that

$$k^{(j)}(\alpha_i) = k_1^{(j)}(\alpha_i) \quad \text{for } 1 \le i \le n \text{ and } 0 \le j < n_i.$$

Therefore the problem of finding a function b in $k + qH^\infty$ with $||b||_\infty \le 1$ is equivalent to the problem of finding a function $b \in H^\infty$ satisfying

(a) $b^{(j)}(\alpha_i) = k_1^{(j)}(\alpha_i)$ for $1 \le i \le n$ and $0 \le j < n_i$;

(b) $||b||_\infty \le 1$.

This is exactly the classical Hermite-Fejér Interpolation Problem (HFIP) (If $n = 1$, this is the Carathéodory–Schur Interpolation Problem and if $n_i = 1$ for all i, this is the Nevanlinna-Pick Interpolation Problem; cf. [FF]). Then by [FF, Theorem X.5.6 and Corollary X.5.7], there exists a solution to HFIP if and only if the Hermite-Fejér matrix M_{k_1} associated with k_1 is a contraction, and furthermore the solution is unique if and only if $||M_{k_1}|| = 1$. (M_{k_1} is the $d \times d$ lower triangular matrix whose entries involve the values of $k_1^{(j)}(\alpha_i)$, where $d = \sum_{i=1}^n n_i$.) Suppose that G contains two functions. Then the Hermite-Fejér matrix M_{k_1} has norm less than 1. We can then choose a positive number $\lambda > 1$ for which $||M_{\lambda k_1}|| < 1$. This implies that $||\lambda k_1 + qh||_\infty \le 1$ for some $h \in H^\infty$. Let $b := k_1 + \frac{1}{\lambda} qh$; then $b \in k + qH^\infty$ and $||b||_\infty \le \frac{1}{\lambda} < 1$. This proves Lemma 3. □

In Section 1 we noticed that if $\varphi \equiv \varphi_+ + \overline{\varphi_-} \in L^\infty$ is such that T_φ is a hyponormal operator then $\ker H_{\overline{\varphi_+}} = \ker H_{\overline{\varphi}} \subseteq \ker\,[T_\varphi^*, T_\varphi]$. Thus we can see that if $\varphi = f + \bar{g}$, where $f \in H^\infty$ and $g \in G'_f$ and if $\operatorname{rank} H_{\overline{f}} < \infty$ then $\operatorname{rank}\,[T_\varphi^*, T_\varphi] \le \operatorname{rank} H_{\overline{f}}^* = \operatorname{rank} H_{\overline{f}}$.

We now have:

Theorem 4. *If $f \in H^\infty$ is such that $\operatorname{rank} H_{\overline{f}} < \infty$ then $\nabla G'_f \subseteq \operatorname{ext} G'_f$.*

Proof. Suppose that $\operatorname{rank} H_{\overline{f}} = N$. By the above considerations, if $g \in G'_f$ and $\varphi := f + \bar{g}$ then $\operatorname{rank}\,[T_\varphi^*, T_\varphi] \le N$. We observe that if $g \in \nabla G'_f$ then every solution k of the equation $H_{\overline{f}} k = \bar{z}\widetilde{g}$ has exactly norm 1; for, if k is a solution of the equation $H_{\overline{f}} k = \bar{z}\widetilde{g}$ with $||k||_\infty < 1$ then $\frac{k}{||k||_\infty} \in \mathcal{E}(\psi)$ for $\psi := f + \overline{g/||k||_\infty}$, and hence $\frac{1}{||k||_\infty} \cdot g = \frac{g}{||k||_\infty} \in G'_f$, a contradiction. We now claim that if $g \in \nabla G'_f$ then $\mathcal{E}(f + \bar{g})$ consists of exactly one finite Blaschke product. To see this observe that by Beurling's Theorem, $\ker H_{\overline{f}} = q\,H^2$ for some inner function q. (Recall that the second identity in (1) implies that $z(\ker H_\varphi) \subseteq \ker H_\varphi$ for all $\varphi \in L^\infty$.) Since $\operatorname{rank} H_{\overline{f}} < \infty$, q must be a finite Blaschke product. Furthermore if k is in $\mathcal{E}(f + \bar{g})$, that is, k is a solution of the equation $H_{\overline{f}} k = \bar{z}\widetilde{g}$ and $||k||_\infty \le 1$, then $\mathcal{E}(f + \bar{g}) = G(q, k) = \{b \in k + q\,H^\infty : \ ||b||_\infty \le 1\}$. By the above considerations and Lemma 3, $\mathcal{E}(f + \bar{g})$ then contains exactly one element. Since $[T_\varphi^*, T_\varphi]$ is of finite rank it follows from an argument of T. Nakazi and K. Takahashi [NT, Theorem 10] that $\mathcal{E}(f + \bar{g})$ contains a finite Blaschke product, and consequently, $\mathcal{E}(f + \bar{g})$ consists of one finite Blaschke product.

To prove $\nabla G'_f \subseteq \operatorname{ext} G'_f$, we now assume, without loss of generality, that g_1, g_2, $\frac{1}{2}(g_1 + g_2) \in \nabla G'_f$; it will suffice to show that $g_1 = g_2$. By what we have just discussed, there exist finite Blaschke products b_1 and b_2 corresponding to g_1 and g_2, respectively. Since $H_{\overline{f}} b_i = \bar{z}\widetilde{g_i}$ for $i = 1, 2$, it follows that $\frac{1}{2}(b_1 + b_2)$ is a solution of the equation $H_{\overline{f}} k = \frac{1}{2}\bar{z}(\widetilde{g_1} + \widetilde{g_2})$. Further since $||\frac{1}{2}(b_1 + b_2)||_\infty \le 1$, we have that $\frac{1}{2}(b_1 + b_2) \in \mathcal{E}(f + \overline{\frac{1}{2}(g_1 + g_2)})$. But since $\frac{1}{2}(g_1 + g_2) \in \nabla G'_f$, it follows that $\frac{1}{2}(b_1 + b_2)$ is a finite Blaschke product. However since Blaschke products are extreme points of the unit ball of H^∞ (cf. [Ga, p. 179]), we can conclude that $b_1 = b_2$, which implies $g_1 = g_2$. (In fact, by an argument of K. deLeeuw and W. Rudin [dLR], if $f \in H^\infty$, $||f||_\infty = 1$, then f is an extreme point of the unit ball of H^∞ if and only if $\int \log(1 - |f(e^{i\theta})|)d\theta = -\infty$.) This completes the proof of Theorem 4. □

References

[Ab] M. B. Abrahamse, *Subnormal Toeplitz operators and functions of bounded type*, Duke Math. J. **43** (1976), 597–604, MR 55 #1126, Zbl 0332.47017.

[CCL] M. Chō, R. E. Curto and W. Y. Lee, *Triangular Toeplitz contractions and Cowen sets for analytic polynomials*, preprint 2000.

[Co1] C. C. Cowen, *Hyponormal and subnormal Toeplitz operators*, Surveys of Some Recent Results in Operator Theory, I (J.B. Conway and B.B. Morrel, eds.), Pitman Research Notes in Mathematics, Vol 171, Longman, 1988; pp. 155–167, MR 90j:47022, Zbl 0677.47017.

[Co2] C. C. Cowen, *Hyponormality of Toeplitz operators*, Proc. Amer. Math. Soc. **103** (1988), 809–812, MR 89f:47038, Zbl 0668.47021.

[CoL] C. C. Cowen and J. J. Long, *Some subnormal Toeplitz operators*, J. Reine Angew. Math. **351** (1984), 216–220, MR 86h:47034.

[CuL] R. E. Curto and W. Y. Lee, *Joint Hyponormality of Toeplitz Pairs*, Memoirs Amer. Math. Soc. no. 712, Amer. Math. Soc., Providence, 2001, CMP 1 810 770.

[dLR] K. de Leeuw and W. Rudin, *Extreme points and extremum problems in* H_1, Pacific J. Math. **8** (1958), 467–485, MR 20 #5426, Zbl 0084.27503.

[FF] C. Foiaş and A. Frazho, *The Commutant Lifting Approach to Interpolation Problems*, Operator Theory: Adv. Appl., no. 44, Birkhäuser-Verlag, Boston, 1990, MR 92k:47033, Zbl 0718.47010.

[Ga] J. B. Garnett, *Bounded Analytic Functions*, Academic Press, New York, 1981, MR 83g:30037, Zbl 0469.30024.

[NT] T. Nakazi and K. Takahashi, *Hyponormal Toeplitz operators and extremal problems of Hardy spaces*, Trans. Amer. Math. Soc. **338** (1993), 753–769, MR 93j:47040, Zbl 0798.47018.

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