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New Prime-Producing Quadratic Polynomials Associated with Class Number One or Two

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Abstract. This article provides necessary and sufficient conditions for a real quadratic field to have class number one or two in terms of a new set of prime-producing quadratic polynomials

Contents

1. Introduction

This section is devoted to the elucidation of several facts on ideal theory which we will require in the balance of the paper.

Let $D > 1$ be a square-free positive integer and set:

$$\sigma = \begin{cases} 2 & \text{if } D \equiv 1 \pmod 4, \\ 1 & \text{otherwise.} \end{cases}$$

Define $\omega_\Delta = (\sigma - 1 + \sqrt{D})/\sigma$, and $\Delta = (\omega_\Delta - \omega'_\Delta)^2 = 4D/\sigma^2$. The value Δ is called a *fundamental discriminant* or *field discriminant* with associated *radicand* D, and ω_Δ is called the *principal fundamental surd associated with* Δ.

There is a family of discriminants upon which we will concentrate in this paper, defined as follows (see [9] for complete details on their properties and background).

Definition 1.1. If $\Delta = \ell^2 + r$ is a discriminant with $r \mid 4\ell$, then Δ is said to be of extended Richaud-Degert type (ERD-type).

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Now we develop the notation required for the balance of our discussion. If $[\alpha, \beta] = \alpha\mathbb{Z} + \beta\mathbb{Z}$, then $\mathcal{O}_\Delta = [1, \omega_\Delta]$ is called the *maximal order* or *ring of integers* of $K = \mathbb{Q}(\sqrt{D})$. It may be shown that any $\mathbb{Z}$-module $I \neq (0)$ of $\mathcal{O}_\Delta$ has a representation of the form $[a, b + c\omega_\Delta]$, where $a, c \in \mathbb{N}$ with $0 \leq b < a$. We will only be concerned with *primitive* ones, namely those for which $c = 1$. In other words, I is a primitive $\mathbb{Z}$-submodule of $\mathcal{O}_\Delta$ if whenever $I = (z)J$ for some $z \in \mathbb{Z}$ and some $\mathbb{Z}$-submodule J of $\mathcal{O}_\Delta$, then $|z| = 1$. Thus, a canonical representation of a primitive $\mathbb{Z}$-submodule of $\mathcal{O}_\Delta$ is obtained by setting $\sigma a = Q$ and $b = (P-1)/2$ if $\sigma = 2$, while $b = P$ if $\sigma = 1$ for $P, Q \in \mathbb{Z}$, namely

$$I = [Q/\sigma, (P + \sqrt{D})/\sigma]. \tag{1.1}$$

A nonzero $\mathbb{Z}$-module I as given in (1.1) is called a primitive $\mathcal{O}_\Delta$-ideal if and only if $P^2 \equiv D \pmod{Q}$ (see [11, Theorem 3.5.1, p. 173]). *Henceforth, when we refer to an $\mathcal{O}_\Delta$-ideal it will be understood that we mean a* primitive *$\mathcal{O}_\Delta$-ideal.* Also, the value Q/σ is called the *norm of* I, denoted by $N(I)$. A *reduced ideal* I is one which contains an element $\beta = (P + \sqrt{D})/\sigma$ such that $I = [N(I), \beta]$, where $\beta > N(I)$ and $-N(I) < \beta' < 0$. In fact, the following holds.

Theorem 1.1. *If $\Delta > 0$ is a discriminant and I is an $\mathcal{O}_\Delta$-ideal with $N(I) < \sqrt{\Delta}/2$, then I is reduced. Conversely, if I is reduced, then $N(I) < \sqrt{\Delta}$.*

In particular, for ERD-types, we need to know what the norms of principal reduced ideals happen to be as in the following, each of which can be verified using [9, Theorem 3.2.1, pp. 78–80].

Theorem 1.2. *Suppose that $\Delta = 4(t^2 \pm 2) = 4D$ with $t > 3$. Then I is a principal, primitive $\mathcal{O}_\Delta$-ideal with $N(I) < \sqrt{D}$ if and only if $N(I) = 1$ or $N(I) = 2$.*

Theorem 1.3. *If $D = t^2 + 4q \equiv 1 \pmod{4}$ is a fundamental discriminant such that $t > 3$ and $|q|$ divides t, then I is a primitive principal $\mathcal{O}_D$-ideal with $N(I) < \sqrt{D}/2$ if and only if $N(I) = 1$ or $N(I) = |q|$.*

Theorem 1.4. *If $D = 4t^2 + q \equiv 1 \pmod{4}$ is a fundamental discriminant with $q|t$, then I is a primitive principal $\mathcal{O}_\Delta$-ideal if and only if $N(I)$ is one of 1, $|q|$, $t + (q-1)/4$, and if $q > 0$, also $t - (q-1)/4$.*

We will also need the following theorem on class groups of quadratic fields.

Theorem 1.5. *If Δ is the discriminant of a real quadratic field and $\mathcal{C}_\Delta$ is the class group of $\mathbb{Q}(\sqrt{\Delta})$, then $\mathcal{C}_\Delta$ is generated by the non-inert primitive prime ideals with norm less than $\sqrt{\Delta}/2$.*

Proof. See [9, Theorem 1.3.1, p. 15]. □

We will also be referring throughout to the *exponent* of $\mathcal{C}_\Delta$, denoted by e_Δ, which is the smallest natural number such that $I^{e_\Delta} = 1$ for all $I \in \mathcal{C}_\Delta$. Ideals I, J in the same class of $\mathcal{C}_\Delta$ are denoted by $I \sim J$. The following will also be relevant to our discussion in what follows.

Definition 1.2. If $I = [a, (b + \sqrt{\Delta})/2]$ is an $\mathcal{O}_\Delta$-ideal, then the ideal $I' = [a, (b - \sqrt{\Delta})/2]$ is called the conjugate of I. When $I = I'$, I is called an ambiguous ideal, and if $I \sim I'$, then I is said to be in an ambiguous class of ideals.

Remark 1.1. It is possible that an ambiguous class of ideals may **not** contain an ambiguous ideal. This phenomenon and its consequences are explored in detail in [9, Chapter 6, pp. 187–198]. See also the 1989 paper by Louboutin [2].

2. Prime-producers and class number one

This section is devoted to class number one criteria for real quadratic fields of ERD-type in terms of a new family of prime-producing quadratic polynomials. In [12], we provided class number one criteria for arbitrary discriminants in terms of certain prime-producing quadratic polynomials. The results of this section extend those results. However, for ease of exposition, we state our results for only fundamental discriminants. The reader may use the techniques of [12] and the theory developed therein to generalize these results to arbitrary discriminants.

Theorem 2.1. *Let $\Delta = 4D$ be a fundamental discriminant with radicand $D = t^2 \pm 2$, for some natural number $t > 3$. Then $h_\Delta = 1$ if and only if*

$$f_t(x) = -2x^2 + 2tx \pm 1$$

is prime for all natural numbers $x < t$. Also, $f_t(x)$ has discriminant Δ.

Proof. To begin, we notice that the prime ideal $(t+\sqrt{D})$ over the rational prime 2 is principal. If $h_\Delta > 1$, then by Theorem 1.5 there exists a non-principle primitive ideal J of odd norm a where $1 < a < \sqrt{D}$. Set $I = (t + \sqrt{D})$ and write $I = [2a, b + \sqrt{D}]$ with $-a \leq b < a$. Hence, with $-t < b < t$ and a natural number x with $x < t$, we may let $b = t - 2x$, since $b \equiv t \pmod 2$ given that $2a \mid (b^2 - D)$. Since $I \not\sim 1$, then there exists a $c > 1$ such that $N(b + \sqrt{D}) = -2ac$, for otherwise we would have $I = (b + \sqrt{D}) \sim 1$. Therefore,

$$2ac = D - b^2 = -4x^2 + 4tx \pm 2 = 2f_t(x),$$

so $f_t(x) = ac$ is composite. We have shown that when $f_t(x)$ is prime for all natural numbers $x < t$, $h_\Delta = 1$.

Conversely, if $f_t(x) = -2x^2 + 2tx \pm 1$ is composite for some natural number $x < t$, then $c = c_1c_2 = f_t(x)$ with $1 < c_1 \leq c_2$. Let $\alpha = 2x - t + \sqrt{D}$, which is primitive with norm $N(\alpha) = -2c$. Thus, $I = [c_1, \alpha]$ is a primitive $\mathcal{O}_\Delta$-ideal with $N(I) = c_1$. If $c_1 > \sqrt{D}$, then $2c > c > D$, so $-4x^2 + 4tx \pm 2 > t^2 \pm 2$ from which it follows that $0 > (2x - t)^2$, a contradiction. Hence, by Theorem 1.1, I is reduced. If $I \sim 1$, then it follows from Theorem 1.2 that $2c_1 = 2$, contradicting that $c_1 > 1$. Hence, $I \not\sim 1$, so $h_\Delta > 1$. □

In [13], we provided criteria for arbitrary discriminants to have cyclic subgroups in the class groups of real quadratic orders. The following extends the ideas used therein.

Theorem 2.2. *If $D = t^2 \pm 2$, $t > 3$, is a fundamental radicand and if there exists an $x \in \mathbb{N}$ such that $-2x^2 + 2tx \pm 1 = c^n$ for some integers $c > 1$ and $n > 1$ with $\gcd(D, c) = 1$, then $\mathcal{C}_\Delta$ has a cyclic subgroup of order n.*

Proof. Set $\alpha = 2x - t + \sqrt{D}$. Then $N(\alpha) = -2c^n$ and $I = [c, \alpha]$ is an $\mathcal{O}_\Delta$-ideal where $\Delta = 4D$. Since $\gcd(D, c) = 1$, then $I^n = [c^n, \alpha]$. Also, since $[2, \alpha] = (t + \sqrt{D}) \sim 1$, then $I^n \sim 1 \sim [2c^n, \alpha] = [2, \alpha][c^n, \alpha]$. By the same reasoning as in the above proof, $c^{n/2} < \sqrt{D}$. If there exists a $j \mid n$ with $j \neq n$ such that $I^j \sim 1$, then since

$c^j \leq c^{n/2} < \sqrt{D}$, I^j is reduced. Since $t > 3$, it follows from Theorem 1.2 that $c^j = 1$, a contradiction. Therefore, the smallest value of j such that $I^j \sim 1$ is $j = n$, so $\mathcal{C}_\Delta$ has a cyclic subgroup of order n. □

Remark 2.1. The above is related to results obtained by this author in 1987 (see [5], as well as [9, Theorem 4.2.4, p. 132], and [15, Theorems 2.2–2.3, p. 475]) as follows. For $D = t^2 \pm 2$ a fundamental discriminant, let δ be defined by

$$\delta = \begin{cases} 1 & \text{if } D \equiv 3 \pmod 4, \\ 0 & \text{if } D \equiv 2 \pmod 4. \end{cases}$$

Notice that if we perform the translation $x \to x + (t - \delta)/2$ on $f_t(x) = -2x^2 + 2tx \pm 1$, we get $f_\Delta(x) = -2x^2 + 2\delta x + (D - \delta)/2$. The aforementioned result obtained in 1987 is that $f_\Delta(x)$ is 1 or prime for all $x \in \mathbb{N}$ with $x < (\sqrt{D} + \delta)/2$ if and only if $h_\Delta = 1$. This translates into the result in Theorem 2.1.

In the 1989 publication [16] (see also [9, Conjecture 4.2.1, p. 140]) we posed the following.

Conjecture 2.1 (Mollin-Williams–1989)**.** *Let $D = pq \equiv 5 \pmod 8$ where $p \equiv q \equiv 3 \pmod 4$ with $q < p$ are primes. Then the following are equivalent.*

(a) *$|f_q(x)| = |qx^2 + qx + (q - p)/4|$ is 1 or prime for all nonnegative integers $x < \sqrt{D}/4 - 1/2$.*

(b) *$h_D = 1$ and D is of the form $D = q^2s^2 \pm 4q$ or $D = 4q^2s^2 - q$ for some $s \in \mathbb{N}$.*

Remark 2.2. In [3], Louboutin proved that (a) implies $h_D = 1$ and D is of the form $D = q^2s^2 \pm 4q$ if we extend the range of values of x to $0 \leq x \leq \sqrt{D}/2 - 1/2$, and states: "This result is our first step towards Mollin-Williams' conjecture..." Then in [3, Theorem 10] and [4, Theorem 10′] it is proved that if the range of x is $0 \leq x \leq \sqrt{D}/3 - 1$ in (a), then (b) holds. However, in [21], Srinivasan (unconditionally) proved that if (a) holds, then $h_D = 1$ and the period length of the simple continued fraction expansion of $\sqrt{D}$ is at most 10. This implies, she notes as a corollary, that Conjecture 2.1 holds with one possible exceptional value of D, whose existence would be a counterexample to the GRH, since for this value of D, (b) would hold but not (a). More recently, in [22], Srinivasan proved (modulo the GRH assumption) that if q is allowed to be any divisor of D and $F_q(x) = |(p - qx^2)/4|$ is 1 or prime for all odd positive integers $x < \sqrt{D/5}$, then $D \leq 4245$. (Note that $f_q(x)$ and $F_q(x)$ are equivalent since $f_q(x) = ((2x + 1)^2q - p)/4$.) This establishes (modulo GRH) a conjecture made by this author in [8, Conjecture 3.1, p. 359] (see also [9, Conjecture 4.2.2, p. 143]). It also shows that if (a) of Conjecture 2.1 holds, then $h_D = 1$ and D is of ERD type (modulo GRH). She also proved, unconditionally, that if q is any divisor of D and $F_q(x)$ is 1 or prime for all odd positive integers $x < \sqrt{D/5}$, then either $h_D \leq 2$ or D is of ERD type. Under the assumption of the GRH, she proved that if q is any divisor of D and $F_q(x)$ is 1 or prime for all odd positive integers $x < \sqrt{D/5}$, then $h_D \leq 2$. Another result of interest that she verified (without the GRH assumption) is that if $F_q(x)$ is 1 or prime for all odd positive integers $x < \sqrt{D}/2$ and there exists at least one split prime less than $\sqrt{D}/2$, then $h_D \leq 2$ or $h_D = 4$.

A complete list (with one possible GRH-ruled-out exception) of ERD-types having class number one is given in [9, Theorem 5.4.3, p. 176], which first appeared

in the 1990 publication [17], although the announcement of it was made in a note added in proof at the end of the 1987 paper [6].

We address the class number two problem in the next section.

3. Prime-producers and class number two

We begin with a result that provides necessary and sufficient conditions for $h_\Delta \leq 2$ and extends the result in [7, Proposition 3.1, p. 89] where continued fraction techniques were used.

Theorem 3.1. *Let $\Delta = 4D = 4(t^2 \pm 2)$ be a fundamental discriminant, set*

$$\mathcal{S} = \{\text{odd primes } p < \sqrt{D} : (D/p) \neq -1\}$$

where the symbol on the right is the Legendre symbol, and let

$$f_t(x) = -2x^2 + 2tx \pm 1.$$

Then the following are equivalent.

(a) *Either $h_\Delta = 1$ and $\mathcal{S} = \varnothing$, or $h_\Delta = 2$.*

(b) *For each $p \in \mathcal{S}$ there exists an $x \in \mathbb{N}$ with $x \leq t/2$ such that $f_t(x) = pr_p$ where r_p is a prime which is the norm of $\mathcal{O}_\Delta$-ideal $\mathcal{R}_p$ in an ambiguous class and $\mathcal{R}_p \sim \mathcal{R}_{p'}$ for all $p, p' \in \mathcal{S}$.*

Also, $f_t(x)$ has discriminant Δ.

Proof. Given that the result is vacuously true when $t \leq 3$ (since in those cases $h_\Delta = 1$ and $\mathcal{S} = \varnothing$), we will assume throughout that $t > 3$.

First suppose that (b) holds. If $h_\Delta = 1$, then by Theorem 2.1, $\mathcal{S} = \varnothing$. Thus, we may assume that $h_\Delta > 1$. By Theorem 1.5, $\mathcal{C}_\Delta$ is generated by the non-inert primitive prime ideals $\mathcal{P}_p$ with $N(\mathcal{P}_p) = p < \sqrt{D}$. Since $[2, t+\sqrt{D}] = (t+\sqrt{D}) \sim 1$, then we may assume that $p \in \mathcal{S}$. Given such a p, the hypothesis tells us that there exists a natural number $x \leq t/2$ such that $f_t(x) = pr_p$ for some prime r_p which is the norm of an ambiguous $\mathcal{O}_\Delta$-ideal. If we set $b = t - 2x$ and consider $[2r_p p, b + \sqrt{D}] = (b + \sqrt{D}) \sim 1$, then $\mathcal{P}_p\mathcal{R}_p \sim 1$ where $\mathcal{R}_p = [2r_p, b + \sqrt{D}]$ and $\mathcal{P}_p = [p, b + \sqrt{D}]$. Therefore, by (b), $\mathcal{R}_p \sim \mathcal{R}'_p \sim \mathcal{P}_p\mathcal{R}_p\mathcal{R}'_p \sim \mathcal{P}_p$. Since (b) also tells us that $\mathcal{R}_p \sim \mathcal{R}_{p'}$ for all $p, p' \in \mathcal{S}$, then $\mathcal{P}_p \sim \mathcal{P}_{p'}$ for all $p, p' \in \mathcal{S}$. Moreover, since $\mathcal{R}_p^2 \sim 1$ for each $p \in \mathcal{S}$, then $\mathcal{P}_p^2 \sim 1$ for all $p \in \mathcal{S}$. Hence, $h_\Delta = 2$ (since the only ideals in the principal class of $\mathcal{C}_\Delta$ with norms less than $\sqrt{D}$ are the trivial ideal and the ideal over 2 by Theorem 1.2, which we may invoke since $t > 3$). We have shown that (b) implies (a).

Conversely, we may assume that (a) holds and that $\mathcal{S} \neq \varnothing$ since the result is vacuously true otherwise. Thus, $h_\Delta = 2$. Let $p \in \mathcal{S}$ and set $[2p, b + \sqrt{D}] \sim [p, b + \sqrt{D}] = \mathcal{P}$ with $0 \leq b < p$, which is a generator of $\mathcal{C}_\Delta$ by Theorem 1.5. Since $p < \sqrt{D}$, then $b < 2t$, so we may set $b = t - 2x$ where $1 \leq x \leq t$. Since $\mathcal{P} \not\sim 1$, then as in previous arguments, there exists a $c > 1$ such that $2pc = D - b^2 = -4x^2 + 4xt \pm 2$, so $pc = -2x^2 + 2xt \pm 1 = f_t(x)$.

We now show that c is prime. If $c = c_1c_2$ with $1 < c_1 \leq c_2$, then we may set $\alpha = 2x - t + \sqrt{D}$ and it follows that $I = [c_1, \alpha]$ is a primitive $\mathcal{O}_\Delta$-ideal with $N(I) = c_1$. If $c_1 > \sqrt{D}$, then $c_1c_2 = c \geq c_1^2 > D$. However, $D \geq 2f_t(x)$, so $c > 2pc$, a contradiction. Thus, $c_1 < \sqrt{D}$ which means that I is reduced by Theorem 1.1.

If $I \sim 1$, then by Theorem 1.2, $c_1 = 1$ or $c_1 = 2$, both of which are contradictions since $c_1 > 1$ by choice and c is odd. Hence, $I \not\sim 1$. Since $[pc, \alpha] \sim 1$, then

$$[p, \alpha] \sim [c, \alpha], \tag{3.2}$$

given that $h_\Delta = 2$. Since $[c, \alpha] = [c_1, \alpha][c_2, \alpha]$, then if $[c_2, \alpha] \not\sim 1$, we must have that $[c, \alpha] \sim 1$ since $h_\Delta = 2$. However, this contradicts that $\mathcal{P} \not\sim 1$ in view of (3.2). Hence, $[c_2, \alpha] \sim 1$. Assume that $c_2 > \sqrt{D}$. If $pc_1 > \sqrt{D}$, then $D < pc_1c_2 = pc = f_t(x) \leq D/2$, a contradiction, so $pc_1 < \sqrt{D}$. Given that $\mathcal{P} \not\sim 1$, $[c_1, \alpha] \not\sim 1$, and $h_\Delta = 2$, then $[pc_1, \alpha] \sim 1$. Thus, by Theorems 1.1–1.2, $pc_1 \in \{1, 2\}$, a contradiction. Hence, $c_2 < \sqrt{D}$ and $[c_2, \alpha] \sim 1$, so as above, $c_2 = 1$ or $c_2 = 2$, both of which are contradictions. We have therefore shown that $c = r_p$ is prime. By (3.2), $\mathcal{R}_p = [r_p, \alpha] \sim \mathcal{P}$. Since $t > 3$, we may invoke Theorem 1.2 which tells us that no element of $\mathcal{S}$ can be the norm of a principal $\mathcal{O}_\Delta$-ideal, this then is sufficient to show that (a) implies (b). □

Remark 3.1. In 1991, we established class number 2 criteria for ERD-types in [7], using the polynomial $f_\Delta(x)$ defined in Remark 2.1, which involved simple continued fraction expansions of quadratic irrationals. However, the criterion did not include the types $D = t^2 \pm 2$, which were problematic for the continued fraction approach. Recently, in [14], we were able to provide such continued fraction criteria for the latter types as well as some new such criteria for other ERD-types. However, some of the ERD-types were also excluded in [14], namely those of the form $D = t^2 \pm 2 = 2p$ where p is prime. The reason is that these types are those for which there exist ambiguous classes of ideals in $\mathcal{O}_\Delta$ without ambiguous ideals in them. The approach given in Theorem 3.1 does not suffer from this defect and so is more general. Thus, as with the class number one criteria given in the previous section, we have new class number two criteria in terms in the polynomials $f_t(x)$.

In [14], we posed the conjecture that if $\Delta = 4(t^2 \pm 2) = 4D$ is a fundamental discriminant, then $h_\Delta = 2$ if and only if D is one of $34, 66, 102, 119, 123, 146, 194, 258, 287, 402, 482, 527, 623, 678, 782, 843, 902, 1022, 1298$. Moreover, given the aforementioned techniques, we know that the list is complete with one GRH-ruled-out exception. Given the comments in Remark 1.1, it is worthy of note that the only values of D in the aforementioned list where the class group is generated by an ambiguous class without ambiguous ideals are $D \in \{34, 146, 194, 482\}$.

We now look at criteria for class number two for the remaining ERD-types in terms of quadratic polynomials which behave in a fashion similar to Theorem 3.1. We do not provide proofs for the following since the arguments are analogous to those presented above by using Theorems 1.3–1.4.

Theorem 3.2. *If $\Delta = t^2 + 4q \equiv 5 \pmod 8$ is a fundamental discriminant with $q \mid t$, and $\mathcal{S} = \{\text{primes } p \neq |q| < \sqrt{\Delta}/2 : (\Delta/q) \neq -1\}$, then the following are equivalent.*

(a) *Either $h_\Delta = 1$ and $\mathcal{S} = \varnothing$ or $h_\Delta = 2$.*
(b) *For all $p \in \mathcal{S}$ there exists a natural number $x < t/2$ such that*

$$|f_t(x)| = |-x^2 + tx + q| = pr_p,$$

where r_p is a prime that is the norm of an ideal $\mathcal{R}_p$ in an ambiguous class of $\mathcal{C}_\Delta$ and $\mathcal{R}_p \sim \mathcal{R}_{p'}$ for all $p, p' \in \mathcal{S}$.

Theorem 3.3. *If* $\Delta = 4t^2 + q \equiv 1 \pmod 4$ *is a fundamental discriminant with* $|q| \mid t$ *and*

$$\mathcal{S} = \{\text{primes } p < \sqrt{\Delta}/2 : p \neq |q|, t \pm (q-1)/4, \text{ and } (\Delta/p) \neq -1\},$$

then the following are equivalent.

(a) *Either* $h_\Delta = 1$ *and* $\mathcal{S} = \varnothing$ *or* $h_\Delta = 2$.

(b) *For all* $p \in \mathcal{S}$ *there exists a natural number* $x \leq t$ *such that*

$$f_{t,q}(x) = -x^2 + +(2t+1)x + (q - 4t - 1)/4 = pr_p,$$

where r_p *is a prime that is the norm of an ideal* $\mathcal{R}_p$ *in an ambiguous class of* $\mathcal{C}_\Delta$ *and* $\mathcal{R}_p \sim \mathcal{R}_{p'}$ *for all* $p, p' \in \mathcal{S}$.

Also, $f_{t,q}(x)$ *has discriminant* Δ.

In the 1991 publication [18] (see also [9, Table A9, p. 286]) we established the complete list of ERD type discriminants Δ with $h_\Delta = 2$ (with one GRH-ruled out exception). The largest of these is $\Delta = 14405$ and the second largest is $\Delta = 9005$. Also, in [19]–[20], we established under the assumption of the GRH that if Δ is *any* fundamental discriminant, $h_\Delta \leq 2$, and the period length of the simple continued fraction expansion of ω_Δ is at most 25, then $\Delta \leq 248093$ (see also [9, Table A3, pp. 274–277]).

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