New York Journal of Mathematics

New York J. Math. **8** (2002) 215–234.

Algebras of Singular Integral Operators on Rearrangement-Invariant Spaces and Nikolski Ideals

Alexei Yu. Karlovich

Abstract. We construct a presymbol for the Banach algebra $\mathrm{Alg}\,(\Omega, S)$ generated by the Cauchy singular integral operator S and the operators of multiplication by functions in a Banach subalgebra Ω of L^∞. This presymbol is a homomorphism $\mathrm{Alg}\,(\Omega, S) \to \Omega \oplus \Omega$ whose kernel coincides with the commutator ideal of $\mathrm{Alg}\,(\Omega, S)$. In terms of the presymbol, necessary conditions for Fredholmness of an operator in $\mathrm{Alg}\,(\Omega, S)$ are proved. All operators are considered on reflexive rearrangement-invariant spaces with nontrivial Boyd indices over the unit circle.

Contents

1. Introduction 216
2. Rearrangement-invariant spaces and their indices 217
   2.1. Rearrangement-invariant spaces 217
   2.2. Boyd indices 218
   2.3. Singular integral operators, Toeplitz and Hankel operators 219
3. Nikolski ideals associated with Douglas algebras 219
   3.1. Definition of the Nikolski ideals 219
   3.2. Properties of the Nikolski ideals 221
   3.3. Estimates for quotient norms 225
4. The presymbol of the algebra $\mathrm{Alg}\,(\Omega, S)$ 227
   4.1. The construction of a presymbol 227
   4.2. Necessary conditions for Fredholmness 229
   4.3. The commutator ideal of the algebra $\mathrm{Alg}\,(\Omega, S)$ 231
   4.4. Singular integral operators with quasicontinuous coefficients 232
References 233

Received July 25, 2002.

Mathematics Subject Classification. Primary 47B35, 47B38, 47A53; Secondary 46E30.

Key words and phrases. Douglas algebra, Nikolski ideal, singular integral operator, Fredholmness, rearrangement-invariant space.

The author is partially supported by F.C.T. (Portugal) grants POCTI 34222/MAT/2000 and PRAXIS XXI/BPD/22006/99.

ISSN 1076-9803/02

1. Introduction

Let $\mathbb{T}$ be the unit circle equipped with the normalized Lebesgue measure $dm = |d\tau|/(2\pi)$. For a function $\varphi \in L^1 = L^1(\mathbb{T}, dm)$, the Cauchy singular integral is defined by

$$(S\varphi)(t) := \frac{1}{\pi i} v.p. \int_{\mathbb{T}} \frac{\varphi(\tau)d\tau}{\tau - t}, \quad t \in \mathbb{T}.$$

Let $X = X(\mathbb{T}, dm)$ be a reflexive rearrangement-invariant space with nontrivial Boyd indices (for the definitions, see Section 2) and let Ω be an arbitrary Banach subalgebra of L^∞. We denote by $\mathcal{L}(X)$ the Banach algebra of all bounded linear operators on X and by $\mathcal{K}(X)$ the closed two-sided ideal of all compact operators on X. The smallest Banach subalgebra of $\mathcal{L}(X)$ containing the Cauchy singular integral operator S and the operators of multiplication M_φ by functions $\varphi \in \Omega$ is denoted by $\mathrm{Alg}\,(\Omega, S)$. The commutator ideal of $\mathrm{Alg}\,(\Omega, S)$, that is, the closed two-sided ideal generated by all commutators $AB - BA$ with $A, B \in \mathrm{Alg}\,(\Omega, S)$ is denoted by $\mathrm{Com\,Alg}\,(\Omega, S)$.

S. G. Mikhlin suggested [22, 23] an idea of symbol calculus for investigation of Fredholm properties of singular integral operators on Lebesgue spaces. Recall that an operator acting on a Banach space is said to be Fredholm if its image is closed and the dimensions of its kernel and cokernel are finite. In particular, S. G. Mikhlin proved [24] (see also [25]) that every operator $F \in \mathrm{Alg}\,(C, S) \subset \mathcal{L}(L^p)$, $1 < p < \infty$, where $C = C(\mathbb{T})$ stands for the C^*-algebra of all continuous functions on $\mathbb{T}$, admits a canonical representation of the form

$$F = M_\varphi P_+ + M_\psi P_- + K, \tag{1.1}$$

where

$$P_+ := (I + S)/2, \quad P_- := (I - S)/2 \tag{1.2}$$

are the *Riesz projections*, I is the identity operator, $\varphi, \psi \in C$ and $K \in \mathcal{K}(L^p)$. Moreover, in this case $\mathcal{K}(L^p) = \mathrm{Com\,Alg}\,(C, S)$ and F is Fredholm if and only if $\varphi(t) \neq 0, \psi(t) \neq 0$ for all $t \in \mathbb{T}$. The representation (1.1) allows us to construct a canonical homomorphism (*symbol*) $\mathrm{Alg}\,(C, S) \to C \oplus C$ with the kernel $\mathcal{K}(L^p)$, where $\mathcal{A} \oplus \mathcal{B}$ stands for the direct sum of Banach algebras $\mathcal{A}$ and $\mathcal{B}$ equipped with the operations $(a, b) + (c, d) = (a + c, b + d), (a, b) \cdot (c, d) = (ac, bd)$ and the norm $\|(a, b)\|_{\mathcal{A}\oplus\mathcal{B}} := \max\{\|a\|_{\mathcal{A}}, \|b\|_{\mathcal{B}}\}$.

The situation becomes more difficult if Ω is wider than C and X is more general than a Lebesgue space L^p, $1 < p < \infty$. In this paper some necessary conditions for Fredholmness of $F \in \mathrm{Alg}\,(\Omega, S) \subset \mathcal{L}(X)$ are obtained in terms of a *presymbol* of $\mathrm{Alg}\,(\Omega, S)$. The presymbol is a canonical homomorphism of $\mathrm{Alg}\,(\Omega, S)$ onto the quotient algebra $\mathrm{Alg}\,(\Omega, S)/\mathrm{Com\,Alg}\,(\Omega, S)$ modulo the commutator ideal $\mathrm{Com\,Alg}\,(\Omega, S)$. In general, the latter ideal is wider than $\mathcal{K}(X)$. Some specific algebras $\Omega \subset L^\infty$ were treated earlier in the case of (weighted) Lebesgue spaces in [2, 3, 4, 5, 7, 10, 19, 25, 29] (see also the references therein). For more general rearrangement-invariant spaces, only the algebra $\Omega = PC$ of piecewise-continuous functions was considered earlier in [13] (see also [15]).

In this paper we follow the approach of [9] and construct a presymbol of the algebra $\mathrm{Alg}\,(\Omega, S) \subset \mathcal{L}(X)$ for any Banach subalgebra Ω of L^∞ and any reflexive rearrangement-invariant space X with nontrivial Boyd indices. More precisely, we describe a Banach algebra homomorphism $\mathrm{Alg}\,(\Omega, S) \to \Omega \oplus \Omega$ with the kernel $\mathrm{Com\,Alg}\,(\Omega, S)$ and obtain the representation (1.1) for an arbitrary operator F in $\mathrm{Alg}\,(\Omega, S)$ with $\varphi, \psi \in \Omega$ and $K \in \mathrm{Com\,Alg}\,(\Omega, S)$. In this construction a collection of so-called Nikolski ideals $\mathfrak{J}^{\pm}(\mathbb{A})$ (see [9, Section 2] and also [26, 27, 28]) associated with a Douglas algebra $\mathbb{A}$ (see, e.g., [8, Ch. 9]) plays an important role. Another important ingredients in the construction are two-sided estimates for the norms of the Toeplitz operators $P_+M_\varphi P_+, P_-M_\varphi P_-$ and the Hankel operators $P_-M_\varphi P_+, P_+M_\varphi P_-$ with a symbol $\varphi \in L^\infty$. These estimates were recently obtained in [16] for reflexive rearrangement-invariant spaces with nontrivial Boyd indices.

The paper is organized as follows. In Section 2 we give necessary preliminaries on rearrangement-invariant spaces and their Boyd indices. We conclude this section with the estimates for the norms of Toeplitz and Hankel operators. In Section 3 we study properties of Nikolski ideals associated with Douglas algebras. This allows us to give estimates for quotient norms modulo these ideals for Hankel and singular integral operators of the form $M_\varphi P_+ + M_\psi P_-$. Our main results are concentrated in Section 4. First, we construct the presymbol for the algebra $\mathrm{Alg}\,(\Omega, S) \subset \mathcal{L}(X)$, where Ω is an arbitrary Banach subalgebra of L^∞. Secondly, we prove necessary conditions for Fredholmness of an arbitrary operator $F \in \mathrm{Alg}\,(\Omega, S)$ and describe the commutator ideal of the algebra $\mathrm{Alg}\,(\Omega, S)$. Finally, we discuss commutator ideals of algebras $\mathrm{Alg}\,(\Omega, S)$ for Ω between C and QC, where QC is the algebra of all quasicontinuous functions, and give a criterion for the Fredholmness of an operator $A \in \mathrm{Alg}\,(\Omega, S)$ in this case.

The presentation is selfcontained. We complement and extend [9] giving details in the cases which were omitted in [9] and vice versa. In places we consider topics in the same sequence in which they are considered in [9]. As a reader of both papers will see, in some cases we are able to adapt the proofs there directly to our context, however in other places we have to involve more delicate arguments, for instance, such as new analogues of classical estimates for the norms of Hankel and Toeplitz operators (see [16]). We refine also some minor inaccuracies of [9].

2. Rearrangement-invariant spaces and their indices

2.1. Rearrangement-invariant spaces. For a general discussion of rearrangement-invariant spaces, see [1, 18, 20]. In this section we collect necessary facts.

Denote by $\mathcal{M}$ the set of all measurable complex-valued functions on $\mathbb{T}$, and let $\mathcal{M}^+$ be the subset of functions in $\mathcal{M}$ whose values lie in $[0, \infty]$. The characteristic function of a measurable set $E \subset \mathbb{T}$ will be denoted by χ_E. A mapping $\rho : \mathcal{M}^+ \to [0, \infty]$ is called a *function norm* if for all functions $f, g, f_n \in \mathcal{M}^+$ ($n \in \mathbb{N}$), for all constants $a \geq 0$, and for all measurable subsets E of $\mathbb{T}$, the following properties hold:

(a) $\quad \rho(f) = 0 \Leftrightarrow f = 0$ a.e., $\quad \rho(af) = a\rho(f), \quad \rho(f+g) \leq \rho(f) + \rho(g)$,

(b) $\quad 0 \leq g \leq f$ a.e. $\Rightarrow \rho(g) \leq \rho(f)$ (the lattice property),

(c) $\quad 0 \leq f_n \uparrow f$ a.e. $\Rightarrow \rho(f_n) \uparrow \rho(f)$ (the Fatou property),

(d) $\quad \rho(\chi_E) < \infty, \quad \displaystyle\int_E f\,dm \leq C_E\rho(f)$

with $C_E \in (0,\infty)$ depending on E and ρ but independent of f. When functions differing only on a set of measure zero are identified, the set X of all functions $f \in \mathcal{M}$ for which $\rho(|f|) < \infty$, is a Banach space under the norm $\|f\|_X := \rho(|f|)$. Such a space X is called a Banach function space. If ρ is a function norm, its associate norm ρ' is defined on $\mathcal{M}^+$ by

$$\rho'(g) := \sup\left\{\int_{\mathbb{T}} fg\,dm \;:\; f \in \mathcal{M}^+,\ \rho(f) \le 1\right\}, \quad g \in \mathcal{M}^+.$$

The Banach function space X' determined by the function norm ρ' is called the *associate space* (*Köthe dual*) of X. The associate space X' is a subspace of the dual space X^*.

Let $\mathcal{M}_0$ and $\mathcal{M}_0^+$ be the classes of a.e. finite functions in $\mathcal{M}$ and $\mathcal{M}^+$, respectively. Two functions $f, g \in \mathcal{M}_0$ are said to be equimeasurable if

$$m\{\tau \in \mathbb{T} : |f(\tau)| > \lambda\} = m\{\tau \in \mathbb{T} : |g(\tau)| > \lambda\} \quad \text{for all} \quad \lambda \ge 0.$$

A function norm $\rho : \mathcal{M}^+ \to [0,\infty]$ is called rearrangement-invariant if for every pair of equimeasurable functions $f, g \in \mathcal{M}_0^+$ the equality $\rho(f) = \rho(g)$ holds. In that case, the Banach function space X generated by ρ is said to be a *rearrangement-invariant space*. A Banach function space X is rearrangement-invariant if and only if its associate space X' is rearrangement-invariant too [1, p. 60].

The Lebesgue space L^p, $1 \le p \le \infty$, is the simplest example of a rearrangement-invariant space. Orlicz and Lorentz spaces are other important classical examples of rearrangement-invariant spaces. For every rearrangement-invariant space X (see, e.g., [1, p. 78]), we have $L^\infty \subset X \subset L^1$.

2.2. Boyd indices. By the Luxemburg representation theorem [1, Ch. 2, Theorem 4.10], there is a unique rearrangement-invariant function norm $\overline{\rho}$ over $[0,1]$ with the Lebesgue measure dt such that $\rho(f) = \overline{\rho}(f^*)$ for all $f \in \mathcal{M}_0^+$, where f^* is the non-increasing rearrangement of f (see, e.g., [1, p. 39]). The rearrangement-invariant space over $([0,1], dt)$ generated by $\overline{\rho}$ is called the Luxemburg representation of X. For each $s \in \mathbb{R}_+ := (0,\infty)$, let E_s denote the dilation operator defined on $\mathcal{M}_0([0,1], dt)$ by

$$(E_s f)(t) := \begin{cases} f(st), & st \in [0,1] \\ 0, & st \notin [0,1] \end{cases}, \quad t \in [0,1].$$

For every $s \in \mathbb{R}_+$, the operator $E_{1/s}$ is bounded on the Luxemburg representation of X [1, p. 165], its norm is denoted by $h_X(s)$. The function $h_X : \mathbb{R}_+ \to \mathbb{R}_+$ is submultiplicative and non-decreasing. From [18, Ch. 2, Theorem 1.3] it follows that the limits

$$\alpha_X := \lim_{s\to 0} \frac{\log h_X(s)}{\log s}, \quad \beta_X := \lim_{s\to\infty} \frac{\log h_X(s)}{\log s}$$

exist and $\alpha_X \le \beta_X$. The numbers α_X and β_X are called the *lower and upper Boyd indices* of the rearrangement-invariant space X, respectively [6]. For the Lebesgue spaces $L^p, 1 \le p \le \infty$, the Boyd indices coincide and equal $1/p$. For an arbitrary rearrangement-invariant space, its Boyd indices belong to $[0,1]$. We will say that the Boyd indices are *nontrivial* if $\alpha_X, \beta_X \in (0,1)$. In the case of Orlicz spaces the latter condition is equivalent to the reflexivity of the space (see, e.g., [21]). One can find properties of the Boyd indices in [1, 6, 20, 21].

2.3. Singular integral operators, Toeplitz and Hankel operators. Let M_φ be the operator of multiplication by a function $\varphi \in L^\infty$. The Calderón-Mitjagin interpolation theorem (see, e.g., [20, Theorem 2.a.10]) implies that M_φ is bounded on arbitrary rearrangement-invariant space and

$$\|M_\varphi\|_{\mathcal{L}(X)} \le \|\varphi\|_\infty. \tag{2.1}$$

The Cauchy singular integral operator S is bounded on a rearrangement-invariant space X if and only if X has nontrivial Boyd indices (see, e.g., [18, Ch. 2, Section 8.6] and also [1, Ch. 3, Theorem 5.18]).

Lemma 2.1 (see [16, Lemma 4.2 and Proposition 4.3])**.** *If X is a reflexive rearrangement-invariant space with nontrivial Boyd indices, then the operators P_+ and P_- given by* (1.2) *are bounded projections on X and on X' and their norms are equal*

$$\gamma := \|P_+\|_{\mathcal{L}(X)} = \|P_-\|_{\mathcal{L}(X)} = \|P_+\|_{\mathcal{L}(X')} = \|P_-\|_{\mathcal{L}(X')}.$$

The exact value of γ for Lebesgue spaces $L^p, 1 < p < \infty$, was recently found by B. Hollenbeck and I. E. Verbitsky [11, Theorem 2.1]: $\gamma = \gamma_{L^p} = 1/\sin(\pi/p)$. A lower estimate of γ for an arbitrary reflexive rearrangement-invariant space with nontrivial Boyd indices was obtained in [14, Theorem 4.5]. The exact value of this constant is unknown even for reflexive Orlicz spaces.

In the following we will always assume that X *is a reflexive rearrangement-invariant space with nontrivial Boyd indices*.

For a set $\mathcal{F} \subset L^\infty$, put $\overline{\mathcal{F}} := \{\overline{f} : f \in \mathcal{F}\}$. Let H^∞ be the Hardy space of all bounded analytic functions in the open unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$. Consider the *Toeplitz operators*

$$T_\varphi^+ := P_+ M_\varphi P_+, \quad T_\varphi^- := P_- M_\varphi P_-$$

and the *Hankel operators*

$$H_\varphi^+ := P_- M_\varphi P_+, \quad H_\varphi^- := P_+ M_\varphi P_-.$$

Their norms admit the following estimates.

Theorem 2.2 (see [16, Corollaries 4.6 and 5.10])**.** *If $\varphi \in L^\infty$, then*

$$\|\varphi\|_\infty \le \|T_\varphi^+\|_{\mathcal{L}(X)} \le \gamma^2 \|\varphi\|_\infty, \tag{2.2}$$

$$\|\varphi\|_\infty \le \|T_\varphi^-\|_{\mathcal{L}(X)} \le \gamma^2 \|\varphi\|_\infty, \tag{2.3}$$

$$\inf_{\psi \in H^\infty} \|\varphi - \psi\|_\infty \le \|H_\varphi^+\|_{\mathcal{L}(X)} \le \gamma^2 \inf_{\psi \in H^\infty} \|\varphi - \psi\|_\infty, \tag{2.4}$$

$$\inf_{\psi \in \overline{H^\infty}} \|\varphi - \psi\|_\infty \le \|H_\varphi^-\|_{\mathcal{L}(X)} \le \gamma^2 \inf_{\psi \in \overline{H^\infty}} \|\varphi - \psi\|_\infty. \tag{2.5}$$

3. Nikolski ideals associated with Douglas algebras

3.1. Definition of the Nikolski ideals. Consider the set of all *inner functions*, that is, the set

$$\mathfrak{B} := \Big\{ b \in H^\infty \ : \ |b(t)| = 1 \quad \text{a.e. on } \mathbb{T} \Big\}.$$

A Banach subalgebra $\mathbb{A}$ of L^∞ generated by H^∞ and $\overline{\mathcal{B}}$ with $\mathcal{B} \subset \mathfrak{B}$ is called a *Douglas algebra* (see, e.g., [8, Ch. 9, Section 1]). For a Douglas algebra $\mathbb{A}$, put

$$\mathfrak{B}_{\mathbb{A}} := \left\{ b \in \mathfrak{B} \ : \ \overline{b} \in \mathbb{A} \right\}, \quad \mathbb{A}_+ := \mathbb{A}, \quad \mathbb{A}_- := \overline{\mathbb{A}}, \quad Q_{\mathbb{A}} := \mathbb{A}_+ \cap \mathbb{A}_-.$$

The following characteristic property of the Douglas algebras can be easily deduced from the definition.

Lemma 3.1. *A function $f \in L^\infty$ belongs to a Douglas algebra $\mathbb{A}$ if and only if for every $\varepsilon > 0$ there exist $h \in H^\infty$ and $b \in \mathfrak{B}_{\mathbb{A}}$ such that $\|f - h\overline{b}\|_\infty < \varepsilon$.*

Example 3.2 (see [8, Ch. 9, Sections 1–2])**.**

(a) If $\mathbb{A} = L^\infty$, then $\mathfrak{B}_{\mathbb{A}} = \mathfrak{B}$ and $Q_{\mathbb{A}} = L^\infty$;
(b) if $\mathbb{A} = H^\infty$, then $\mathfrak{B}_{\mathbb{A}} = \mathbb{T}$ and $Q_{\mathbb{A}} = \mathbb{C}$;
(c) if $\mathbb{A} = H^\infty + C$, then $\mathfrak{B}_{\mathbb{A}}$ is the set of all finite Blaschke products and $Q_{\mathbb{A}} = QC$ is the algebra of all quasicontinuous functions.

For a Douglas algebra $\mathbb{A}$, following [9, Section 2], put

$$\mathfrak{J}^-(\mathbb{A}) := \left\{ F \in \mathcal{L}(X) \ : \ \inf_{b \in \mathfrak{B}_{\mathbb{A}}} \|P_- M_b F\|_{\mathcal{L}(X)} = 0 \right\}, \tag{3.1}$$

$$\mathfrak{J}^+(\mathbb{A}) := \left\{ F \in \mathcal{L}(X) \ : \ \inf_{b \in \mathfrak{B}_{\mathbb{A}}} \|P_+ M_{\overline{b}} F\|_{\mathcal{L}(X)} = 0 \right\}, \tag{3.2}$$

$$\mathfrak{J}(\mathbb{A}) := \mathfrak{J}^-(\mathbb{A}) \cap \mathfrak{J}^+(\mathbb{A}). \tag{3.3}$$

If $\mathbb{A}_1, \mathbb{A}_2$ are Douglas algebras and $\mathbb{A}_1 \subset \mathbb{A}_2$, then $\mathfrak{B}_{\mathbb{A}_1} \subset \mathfrak{B}_{\mathbb{A}_2}$. Hence, form the definitions of the sets $\mathfrak{J}^\pm(\mathbb{A}_i)$ and $\mathfrak{J}(\mathbb{A}_i)$, where $i = 1, 2$, we get

$$\mathfrak{J}^-(\mathbb{A}_1) \subset \mathfrak{J}^-(\mathbb{A}_2), \quad \mathfrak{J}^+(\mathbb{A}_1) \subset \mathfrak{J}^+(\mathbb{A}_2), \quad \mathfrak{J}(\mathbb{A}_1) \subset \mathfrak{J}(\mathbb{A}_2).$$

Lemma 3.3. *The sets $\mathfrak{J}^-(\mathbb{A}), \mathfrak{J}^+(\mathbb{A})$, and $\mathfrak{J}(\mathbb{A})$ are closed right ideals in $\mathcal{L}(X)$.*

Proof. Let $F_1 \in \mathfrak{J}^+(\mathbb{A})$, $F_2 \in \mathcal{L}(X)$, and $b \in \mathfrak{B}_{\mathbb{A}}$. Then

$$\|(P_+ M_{\overline{b}})(F_1 F_2)\|_{\mathcal{L}(X)} \le \|P_+ M_{\overline{b}} F_1\|_{\mathcal{L}(X)} \|F_2\|_{\mathcal{L}(X)}.$$

Taking the infimum over all $b \in \mathfrak{B}_{\mathbb{A}}$, we get

$$\inf_{b \in \mathfrak{B}_{\mathbb{A}}} \|(P_+ M_{\overline{b}})(F_1 F_2)\|_{\mathcal{L}(X)} \le \|F_2\|_{\mathcal{L}(X)} \left(\inf_{b \in \mathfrak{B}_{\mathbb{A}}} \|P_+ M_{\overline{b}} F_1\|_{\mathcal{L}(X)} \right) = 0.$$

Hence, $F_1 F_2 \in \mathfrak{J}^+(\mathbb{A})$, that is, $\mathfrak{J}^+(\mathbb{A})$ is a right ideal.

Now we prove that $\mathfrak{J}^+(\mathbb{A})$ is closed. Let $F \in \mathcal{L}(X)$ and let $\{F_n\}_{n=1}^\infty \subset \mathfrak{J}^+(\mathbb{A})$ satisfy

$$\lim_{n \to \infty} \|F - F_n\|_{\mathcal{L}(X)} = 0.$$

Given $\varepsilon > 0$, we choose $N \in \mathbb{N}$ such that

$$\|F - F_n\|_{\mathcal{L}(X)} < \frac{\varepsilon}{2\|P_+\|_{\mathcal{L}(X)}} \quad \text{for every} \quad n > N. \tag{3.4}$$

Take $m > N$. Since $F_m \in \mathfrak{J}^+(\mathbb{A})$, by the definition of $\mathfrak{J}^+(\mathbb{A})$, there exists $b \in \mathfrak{B}_{\mathbb{A}}$ such that

$$\|P_+ M_{\overline{b}} F_m\|_{\mathcal{L}(X)} < \frac{\varepsilon}{2}. \tag{3.5}$$

Then, taking into account (3.4), (3.5), and (2.1), we get

$$\begin{aligned}\|P_+M_{\overline{b}}F\|_{\mathcal{L}(X)} &\le \|P_+M_{\overline{b}}(F-F_m)\|_{\mathcal{L}(X)} + \|P_+M_{\overline{b}}F_m\|_{\mathcal{L}(X)}\\ &\le \|P_+\|_{\mathcal{L}(X)}\|M_{\overline{b}}\|_{\mathcal{L}(X)}\|F-F_m\|_{\mathcal{L}(X)} + \|P_+M_{\overline{b}}F_m\|_{\mathcal{L}(X)}\\ &< \frac{\varepsilon}{2}+\frac{\varepsilon}{2} = \varepsilon.\end{aligned}$$

Since ε is arbitrary, this means that

$$\inf_{b\in\mathfrak{B}_{\mathbb{A}}} \|P_+M_{\overline{b}}F\|_{\mathcal{L}(X)} = 0.$$

Thus, $F \in \mathfrak{J}^+(\mathbb{A})$, which shows that $\mathfrak{J}^+(\mathbb{A})$ is closed.

In the case of $\mathfrak{J}^-(\mathbb{A})$ the proof is similar. Since $\mathfrak{J}^-(\mathbb{A})$ and $\mathfrak{J}^+(\mathbb{A})$ are closed right ideals, $\mathfrak{J}(\mathbb{A}) = \mathfrak{J}^-(\mathbb{A}) \cap \mathfrak{J}^+(\mathbb{A})$ is also a closed right ideal of $\mathcal{L}(X)$. □

We use here the terminology of [9, Section 2] and call the ideals $\mathfrak{J}^\pm(\mathbb{A})$ and $\mathfrak{J}(\mathbb{A})$ as the *Nikolski ideals associated with the Douglas algebra* $\mathbb{A}$. Analogous ideals were used by N. K. Nikolski [26, 27, 28] for studying of Toeplitz and Hankel operators on the Hardy space H^2.

3.2. Properties of the Nikolski ideals. In this subsection we study properties of Nikolski ideals.

Lemma 3.4. *Let* $\mathbb{A}$ *be a Douglas algebra and let* $F \in \mathcal{L}(X)$. *Then*

(a) $P_\mp F \in \mathfrak{J}^\pm(\mathbb{A})$;
(b) $F \in \mathfrak{J}^\pm(\mathbb{A})$ *if and only if* $P_\pm F \in \mathfrak{J}^\pm(\mathbb{A})$;
(c) $F \in \mathfrak{J}(\mathbb{A})$ *if and only if* $P_+F \in \mathfrak{J}(\mathbb{A})$ *and* $P_-F \in \mathfrak{J}(A)$;
(d) *if* $F \in \mathfrak{J}(\mathbb{A})$, *then* $SF \in \mathfrak{J}(\mathbb{A})$.

Proof. (a) If $b \in \mathfrak{B}_{\mathbb{A}}$, then $b \in H^\infty$ and $\overline{b} \in \overline{H^\infty}$. From (2.5) and (2.4) we deduce that, respectively,

$$P_+M_{\overline{b}}P_-F = 0, \quad P_-M_bP_+F = 0. \tag{3.6}$$

Then from (3.6) and the definition of $\mathfrak{J}^+(\mathbb{A})$ and $\mathfrak{J}^-(\mathbb{A})$ we get $P_-F \in \mathfrak{J}^+(\mathbb{A})$ and $P_+F \in \mathfrak{J}^-(\mathbb{A})$, respectively. Part (a) is proved.

(b) From (3.6) it follows that

$$P_+M_{\overline{b}}F = P_+M_{\overline{b}}P_+F, \quad P_-M_bF = P_-M_bP_-F. \tag{3.7}$$

From (3.7) and the definition of $\mathfrak{J}^\pm(\mathbb{A})$ we infer that $F \in \mathfrak{J}^\pm(\mathbb{A})$ if and only if $P_\pm F \in \mathfrak{J}^\pm(\mathbb{A})$. Part (b) is proved.

(c) *Necessity.* By Part (a), $P_-F \in \mathfrak{J}^+(\mathbb{A})$ and $P_+F \in \mathfrak{J}^-(\mathbb{A})$. Due to Part (b), if $F \in \mathfrak{J}(\mathbb{A}) = \mathfrak{J}^-(\mathbb{A}) \cap \mathfrak{J}^+(\mathbb{A})$, then $P_+F \in \mathfrak{J}^+(\mathbb{A})$ and $P_-F \in \mathfrak{J}^-(\mathbb{A})$. Thus, P_-F and P_+F belong to $\mathfrak{J}(\mathbb{A})$. Necessity of (c) is proved.

Sufficiency. If P_-F and P_+F belong to $\mathfrak{J}(\mathbb{A})$, then by Part (b), $F \in \mathfrak{J}^-(\mathbb{A})$ and $F \in \mathfrak{J}^+(\mathbb{A})$. Thus, $F \in \mathfrak{J}(\mathbb{A})$. Part (c) is proved.

(d) By Part (c), if $F \in \mathfrak{J}(\mathbb{A})$, then P_+F and P_-F belong to $\mathfrak{J}(\mathbb{A})$. Hence, $SF = P_+F - P_-F \in \mathfrak{J}(\mathbb{A})$. □

Lemma 3.5. *Let* $\mathbb{A}$ *be a Douglas algebra.*

(a) *If* $f \in H^\infty_\mp$ *and* $F \in \mathfrak{J}^\pm(\mathbb{A})$, *then* $M_fF \in \mathfrak{J}^\pm(\mathbb{A})$.
(b) *Suppose* $f \in \mathfrak{B}_{\mathbb{A}}$. *If* $F \in \mathfrak{J}^-(\mathbb{A})$ (*resp.* $F \in \mathfrak{J}^+(\mathbb{A})$), *then* $M_{\overline{f}}F \in \mathfrak{J}^-(\mathbb{A})$ (*resp.* $M_fF \in \mathfrak{J}^+(\mathbb{A})$).

(c) *If* $f \in \mathbb{A}_{\mp}$ *and* $F \in \mathfrak{J}^{\pm}(\mathbb{A})$, *then* $M_f F \in \mathfrak{J}^{\pm}(\mathbb{A})$.

Proof. (a) If $f(\tau) = 0$ a.e. on $\mathbb{T}$ and $F \in \mathfrak{J}^{\pm}(\mathbb{A})$, then $M_f F = 0 \in \mathfrak{J}^{\pm}(\mathbb{A})$.

Suppose $f \in H^\infty_- \setminus \{0\}$ and $F \in \mathfrak{J}^+(\mathbb{A})$. Then for any $\varepsilon > 0$ there exists $b \in \mathfrak{B}_{\mathbb{A}}$ such that

$$\|P_+ M_{\overline{b}} F\|_{\mathcal{L}(X)} < \frac{\varepsilon}{\|P_+\|_{\mathcal{L}(X)} \|f\|_\infty}. \tag{3.8}$$

Since $f \in H^\infty_- = \overline{H^\infty}$, from (2.5) we get $P_+ M_f P_- = 0$. Therefore,

$$P_+ M_{\overline{b}} M_f F = P_+ M_f (P_+ + P_-) M_{\overline{b}} F = P_+ M_f P_+ M_{\overline{b}} F. \tag{3.9}$$

From (3.8), (3.9), and (2.1) we get

$$\|P_+ M_{\overline{b}} M_f F\|_{\mathcal{L}(X)} \le \|P_+\|_{\mathcal{L}(X)} \|M_f\|_{\mathcal{L}(X)} \|P_+ M_{\overline{b}} F\|_{\mathcal{L}(X)} < \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, the latter inequality means that $M_f F \in \mathfrak{J}^+(\mathbb{A})$.

Analogously, one can prove that $f \in H^\infty_+$ and $F \in \mathfrak{J}^-(\mathbb{A})$ imply $M_f F \in \mathfrak{J}^-(\mathbb{A})$. Part (a) is proved.

(b) Suppose $f \in \mathfrak{B}_{\mathbb{A}}$ and $F \in \mathfrak{J}^+(\mathbb{A})$. Then for any $\varepsilon > 0$ there exists $b \in \mathfrak{B}_{\mathbb{A}}$ such that $\|P_+ M_{\overline{b}} F\|_{\mathcal{L}(X)} < \varepsilon$. Since $b \in \mathfrak{B}_{\mathbb{A}}$ and $f \in \mathfrak{B}_{\mathbb{A}}$, we have $bf \in \mathfrak{B}_{\mathbb{A}}$. Therefore, for $F_1 = M_f F$ and any $\varepsilon > 0$ there exists $b_1 = bf \in \mathfrak{B}_{\mathbb{A}}$ such that

$$\|P_+ M_{\overline{b_1}} F_1\|_{\mathcal{L}(X)} = \|P_- M_{\overline{bf}} M_f F\|_{\mathcal{L}(X)} = \|P_+ M_{\overline{b}} F\|_{\mathcal{L}(X)} < \varepsilon.$$

Since ε is arbitrary, the latter inequality means that $F_1 = M_f F \in \mathfrak{J}^+(\mathbb{A})$.

Analogously one can show that $f \in \mathfrak{B}_{\mathbb{A}}$ and $F \in \mathfrak{J}^-(\mathbb{A})$ imply $M_{\overline{f}} F \in \mathfrak{J}^-(\mathbb{A})$. Part (b) is proved.

(c) Suppose $f \in \mathbb{A}_- = \overline{\mathbb{A}}$ and $F \in \mathfrak{J}^+(\mathbb{A})$. By Lemma 3.1, for every $\varepsilon > 0$ there exists $b \in \mathfrak{B}_{\mathbb{A}}$ and $h \in H^\infty_-$ such that

$$\|\overline{f} - \overline{bh}\|_\infty = \|f - hb\|_\infty < \varepsilon. \tag{3.10}$$

In view of Part (a), since $h \in H^\infty_-$, we have $M_h F \in \mathfrak{J}^+(\mathbb{A})$. Further, by Part (b), $M_{bh} F = M_b(M_h F) \in \mathfrak{J}^+(\mathbb{A})$. From (3.10) and (2.1) it follows that

$$\|M_f F - M_{bh} F\|_{\mathcal{L}(X)} \le \|f - bh\|_\infty \|F\|_{\mathcal{L}(X)} < \varepsilon \|F\|_{\mathcal{L}(X)}. \tag{3.11}$$

Since $M_{bh} F \in \mathfrak{J}^+(\mathbb{A})$, Lemma 3.3 and (3.11) imply that $M_f F \in \mathfrak{J}^+(\mathbb{A})$.

Analogously, one can show that $f \in \mathbb{A}_+$ and $F \in \mathfrak{J}^-(\mathbb{A})$ imply $M_f F \in \mathfrak{J}^-(\mathbb{A})$. Part (c) and the lemma are proved. □

Lemma 3.6. *Let* $\mathbb{A}$ *be a Douglas algebra.*

(a) *If* $\varphi \in \mathbb{A}_\pm$, *then* $H^\pm_\varphi \in \mathfrak{J}(\mathbb{A})$.
(b) *If* $\varphi \in Q_{\mathbb{A}}$, *then* $H^\pm_\varphi \in \mathfrak{J}(\mathbb{A})$.

Proof. (a) Let $\varphi \in \mathbb{A}_- = \overline{\mathbb{A}}$. By Lemma 3.1, for any $\varepsilon > 0$ there exist $h \in H^\infty_-$ and $b \in \mathfrak{B}_{\mathbb{A}}$ such that $\|\overline{\varphi} - \overline{hb}\|_\infty = \|\varphi - hb\|_\infty < \varepsilon$. In view of (3.7),

$$P_+ M_{\overline{b}} H^-_\varphi = P_+ M_{\overline{b}} P_+ M_\varphi P_- = P_+ M_{\overline{b}} M_\varphi P_- = H^-_{\overline{b}\varphi}. \tag{3.12}$$

From (3.12) and (2.5) it follows that

$$\|P_+ M_{\overline{b}} H^-_\varphi\|_{\mathcal{L}(X)} = \|H^-_{\overline{b}\varphi}\|_{\mathcal{L}(X)} \le \gamma^2 \inf_{\psi \in H^\infty_-} \|\overline{b}\varphi - \psi\|_\infty. \tag{3.13}$$

Since $|b(\tau)| = 1$ a.e. on $\mathbb{T}$, we have

$$\inf_{\psi\in H^\infty_-} \|\bar{b}\varphi - \psi\|_\infty = \inf_{\psi\in H^\infty_-} \|\varphi - b\psi\|_\infty \le \|\varphi - hb\|_\infty < \varepsilon. \tag{3.14}$$

Combining (3.13) and (3.14), we infer that for any $\varepsilon > 0$ there exists $b \in \mathfrak{B}_{\mathbb{A}}$ such that $\|P_+ M_{\bar{b}} H^-_\varphi\|_{\mathcal{L}(X)} < \gamma^2\varepsilon$. This means that

$$\inf_{b\in\mathfrak{B}_{\mathbb{A}}} \|P_+ M_{\bar{b}} H^-_\varphi\|_{\mathcal{L}(X)} = 0,$$

that is, $H^-_\varphi \in \mathfrak{J}^+(\mathbb{A})$. On the other hand, applying Lemma 3.4(a) to $F = M_\varphi P_-$, we obtain $H^-_\varphi = P_+(M_\varphi P_-) \in \mathfrak{J}^-(\mathbb{A})$. Thus, $H^-_\varphi \in \mathfrak{J}^-(\mathbb{A}) \cap \mathfrak{J}^+(\mathbb{A}) = \mathfrak{J}(\mathbb{A})$.

The proof for $\varphi \in \mathbb{A}_+$ is similar. Part (a) is proved.

Statement (b) is a direct consequence of (a) because $Q_{\mathbb{A}} = \mathbb{A}_- \cap \mathbb{A}_+$. □

Corollary 3.7. *For every $\varphi \in L^\infty$, we have $M_\varphi P_\mp \in \mathfrak{J}^\pm(L^\infty)$.*

Proof. From the definitions of $T^\pm_\varphi$ and $H^\pm_\varphi$ it follows that

$$M_\varphi P_\mp = T^\mp_\varphi + H^\mp_\varphi. \tag{3.15}$$

In view of Example 3.2(a), $L^\infty = Q_{L^\infty}$. Then, by Lemma 3.6(a),

$$H^\pm_\varphi \in \mathfrak{J}(L^\infty) = \mathfrak{J}^-(L^\infty) \cap \mathfrak{J}^+(L^\infty). \tag{3.16}$$

Applying Lemma 3.4(a) to $F_1 = M_\varphi P_-$ and to $F_2 = M_\varphi P_+$, we get

$$T^-_\varphi = P_- F_1 = P_- M_\varphi P_- \in \mathfrak{J}^+(L^\infty), \quad T^+_\varphi = P_+ F_2 = P_+ M_\varphi P_+ \in \mathfrak{J}^-(L^\infty), \tag{3.17}$$

respectively. Combining (3.15)–(3.17), we obtain $M_\varphi P_\mp \in \mathfrak{J}^\pm(L^\infty)$. □

Theorem 3.8. *Let Ω be a Banach subalgebra of L^∞ and let $\mathbb{A}$ be a Douglas algebra. If $\Omega \subset Q_{\mathbb{A}}$, then $\mathrm{Alg}\,(\Omega, S) \cap \mathfrak{J}(\mathbb{A})$ is a closed two-sided ideal of $\mathrm{Alg}\,(\Omega, S)$.*

Proof. From Lemma 3.3 it follows that $\mathrm{Alg}\,(Q_{\mathbb{A}}, S) \cap \mathfrak{J}(\mathbb{A})$ is a closed right ideal of $\mathrm{Alg}\,(Q_{\mathbb{A}}, S)$. On the other hand, if $F \in \mathfrak{J}(\mathbb{A})$, then, by Lemma 3.4(b), $SF \in \mathfrak{J}(\mathbb{A})$. If $f \in Q_{\mathbb{A}} = \mathbb{A}_- \cap \mathbb{A}_+$ and $F \in \mathfrak{J}(\mathbb{A}) = \mathfrak{J}^-(\mathbb{A}) \cap \mathfrak{J}^+(\mathbb{A})$, then, due to Lemma 3.5(c), $M_f F \in \mathfrak{J}^-(\mathbb{A}) \cap \mathfrak{J}^+(\mathbb{A})$. This means that for every $F \in \mathfrak{J}(\mathbb{A})$ and every generator B of $\mathrm{Alg}\,(Q_{\mathbb{A}}, S)$ we have $BF \in \mathfrak{J}(\mathbb{A})$. Therefore, for every $F \in \mathrm{Alg}\,(Q_{\mathbb{A}}, S) \cap \mathfrak{J}(\mathbb{A})$ and every $C \in \mathrm{Alg}\,(Q_{\mathbb{A}}, S)$ we have $CF \in \mathrm{Alg}\,(Q_{\mathbb{A}}, S) \cap \mathfrak{J}(\mathbb{A})$, that is, $\mathrm{Alg}\,(Q_{\mathbb{A}}, S) \cap \mathfrak{J}(\mathbb{A})$ is also a left ideal of $\mathrm{Alg}\,(Q_{\mathbb{A}}, S)$. Thus, $\mathrm{Alg}\,(Q_{\mathbb{A}}, S) \cap \mathfrak{J}(\mathbb{A})$ is a closed two-sided ideal of $\mathrm{Alg}\,(Q_{\mathbb{A}}, S)$.

By Lemma 3.3, $\mathrm{Alg}\,(\Omega, S) \cap \mathfrak{J}(\mathbb{A})$ is a closed right ideal of $\mathrm{Alg}\,(\Omega, S)$.

On the other hand, let $F_1 \in \mathrm{Alg}\,(\Omega, S)$ and $F_2 \in \mathrm{Alg}\,(\Omega, S) \cap \mathfrak{J}(\mathbb{A})$. Then, obviously, $F_1 F_2 \in \mathrm{Alg}\,(\Omega, S)$. Since $\Omega \subset Q_{\mathbb{A}}$ and $\mathrm{Alg}\,(Q_{\mathbb{A}}, S) \cap \mathfrak{J}(\mathbb{A})$ is a closed two-sided ideal of $\mathrm{Alg}\,(Q_{\mathbb{A}}, S)$, we have $F_1 F_2 \in \mathrm{Alg}\,(Q_{\mathbb{A}}, S) \cap \mathfrak{J}(\mathbb{A})$. Therefore,

$$F_1 F_2 \in \mathrm{Alg}\,(\Omega, S) \cap \Big(\mathrm{Alg}\,(Q_{\mathbb{A}}, S) \cap \mathfrak{J}(\mathbb{A})\Big) = \mathrm{Alg}\,(\Omega, S) \cap \mathfrak{J}(\mathbb{A}),$$

that is, $\mathrm{Alg}\,(\Omega, S) \cap \mathfrak{J}(\mathbb{A})$ is a left ideal of $\mathrm{Alg}\,(\Omega, S)$. Thus, $\mathrm{Alg}\,(\Omega, S) \cap \mathfrak{J}(\mathbb{A})$ is a closed two-sided ideal of $\mathrm{Alg}\,(\Omega, S)$. □

From Example 3.2(b) and the definition of $\mathfrak{J}(H^\infty)$ one can straightforwardly deduce that $\mathfrak{J}(H^\infty) = \{0\}$. A more interesting example is the ideal $\mathfrak{J}(H^\infty + C)$.

Lemma 3.9. *We have*

$$\mathfrak{J}(H^\infty + C) = \mathcal{K}(X). \tag{3.18}$$

Proof. First, we show that

$$\mathfrak{J}(H^\infty + C) \subset \mathcal{K}(X). \tag{3.19}$$

Let $F \in \mathfrak{J}(H^\infty + C)$. By Lemma 3.4(c), $P_+F \in \mathfrak{J}(H^\infty + C) \subset \mathfrak{J}^+(H^\infty + C)$. Therefore, by the definition of $\mathfrak{J}^+(H^\infty + C)$ and Example 3.2(c), for an arbitrary $\varepsilon > 0$ there exists a finite Blaschke product b such that

$$\|P_+ M_{\overline{b}} P_+ F\|_{\mathcal{L}(X)} < \varepsilon. \tag{3.20}$$

On the other hand,

$$P_+F - M_b H^+_{\overline{b}} F = M_b(M_{\overline{b}} - P_- M_{\overline{b}})P_+F = M_b P_+ M_{\overline{b}} P_+ F. \tag{3.21}$$

From (3.20), (3.21), and (2.1) it follows that

$$\|P_+F - M_b H^+_{\overline{b}} F\|_{\mathcal{L}(X)} \le \|M_b\|_{\mathcal{L}(X)} \|P_+ M_{\overline{b}} P_+ F\|_{\mathcal{L}(X)} < \varepsilon. \tag{3.22}$$

Since the finite Blaschke product b is continuous on $\mathbb{T}$, by [13, Lemma 6.4], the operator $M_{\overline{b}}S - SM_{\overline{b}}$ is compact on X. Hence, the operator

$$M_b H^+_{\overline{b}} F = M_b P_-(M_{\overline{b}} P_+ - P_+ M_{\overline{b}})F$$

is compact on X. From this and (3.22), taking into account that ε is arbitrary, we obtain $P_+F \in \mathcal{K}(X)$. Analogously one can show that $P_-F \in \mathcal{K}(X)$. Thus, $F = P_-F + P_+F \in \mathcal{K}(X)$, and we have proved (3.19).

Let $\mathcal{F}(X)$ be the ideal of all operators of finite rank on X. Let us show that

$$\mathcal{F}(X) \subset \mathfrak{J}(H^\infty + C). \tag{3.23}$$

Every operator $K \in \mathcal{F}(X)$ has the form

$$(Kf)(t) = \sum_{j=1}^{m} a_j(t) \int_{\mathbb{T}} b_j(\tau) f(\tau)\, d\tau, \quad t \in \mathbb{T}, \tag{3.24}$$

where $a_j \in X$ and $b_j \in X'$ for $j \in \{1, \dots, m\}$. Since X is reflexive, the set $\mathcal{P}$ of all trigonometric polynomials is dense in X (see, e.g., [16, Corollary 3.2]). Hence, every operator of the form (3.24) can be approximated in the operator norm by the operators of the form (3.24) with $a_j \in \mathcal{P}$. This means that it is sufficient to prove that the operator of the form

$$(K_i f)(t) = \chi_i(t) \int_{\mathbb{T}} f(\tau) g(\tau)\, d\tau, \quad g \in X', \quad t \in \mathbb{T},$$

belongs to $\mathfrak{J}(H^\infty + C)$ for every $i \in \mathbb{Z}$.

Obviously, $\chi_j \in \{f \in C : |f| = 1\} \subset \mathfrak{B}_{H^\infty + C}$ for each $j \in \mathbb{Z}$. For every $i \in \mathbb{Z}$, we take $j_1, j_2 \in \mathbb{Z}$ such that $j_1 < i < j_2$. Then $P_-(\chi_{j_1}\chi_i) = 0$ and $P_+(\overline{\chi_{j_2}}\chi_i) = 0$. Therefore, $P_- M_{\chi_{j_1}} K_i = 0$ and $P_+ M_{\overline{\chi_{j_2}}} K_i = 0$. This means that

$$K_i \in \mathfrak{J}^-(H^\infty + C) \cap \mathfrak{J}^+(H^\infty + C) = \mathfrak{J}(H^\infty + C) \quad \text{for every} \quad i \in \mathbb{Z}.$$

Thus, we have proved (3.23).

Since X is reflexive and its Boyd indices are nontrivial, Corollary 6.11 of [1, Ch. 3] says that every function in X can be approximated in the norm of X by the partial sums of its Fourier series. That is, there exists a sequence of finite-rank operators on X converging strongly to the identity operator. Consequently, every operator in $\mathcal{K}(X)$ can be approximated in the operator norm by operators in $\mathcal{F}(X)$. On the other hand, by Lemma 3.3, $\mathfrak{J}(H^\infty + C)$ is a closed ideal. Thus, (3.23) implies

$$\mathcal{K}(X) \subset \mathfrak{J}(H^\infty + C). \tag{3.25}$$

Combining (3.19) and (3.25), we arrive at (3.18). □

3.3. Estimates for quotient norms. Let $\mathcal{N}$ be a closed subspace of $\mathcal{L}(X)$. We denote by $|F|_{\mathcal{N}}$ the quotient norm of $F \in \mathcal{L}(X)$ modulo $\mathcal{N}$, that is, the norm of the image of F in the quotient algebra $\mathcal{L}(X)/\mathcal{N}$. In other words,

$$|F|_{\mathcal{N}} := \inf_{N \in \mathcal{N}} \|F - N\|_{\mathcal{L}(X)}.$$

Theorem 3.10. *Let $\mathbb{A}$ be a Douglas algebra. If $\varphi \in L^\infty$, then:*

$$\frac{1}{\gamma} \inf_{\psi \in \mathbb{A}_+} \|\varphi - \psi\|_\infty \le |H_\varphi^+|_{\mathfrak{J}^-(\mathbb{A})} \le |H_\varphi^+|_{\mathfrak{J}(\mathbb{A})} \le \gamma^2 \inf_{\psi \in \mathbb{A}_+} \|\varphi - \psi\|_\infty, \tag{3.26}$$

$$\frac{1}{\gamma} \inf_{\psi \in \mathbb{A}_-} \|\varphi - \psi\|_\infty \le |H_\varphi^-|_{\mathfrak{J}^+(\mathbb{A})} \le |H_\varphi^-|_{\mathfrak{J}(\mathbb{A})} \le \gamma^2 \inf_{\psi \in \mathbb{A}_-} \|\varphi - \psi\|_\infty. \tag{3.27}$$

Proof. Since $\mathfrak{J}(\mathbb{A}) = \mathfrak{J}^-(\mathbb{A}) \cap \mathfrak{J}^+(\mathbb{A})$, we immediately get

$$|H_\varphi^+|_{\mathfrak{J}^-(\mathbb{A})} \le |H_\varphi^+|_{\mathfrak{J}(\mathbb{A})}, \quad |H_\varphi^-|_{\mathfrak{J}^+(\mathbb{A})} \le |H_\varphi^-|_{\mathfrak{J}(\mathbb{A})}. \tag{3.28}$$

By Lemma 3.6(a), if $\psi \in \mathbb{A}_-$, then $H_\psi^- \in \mathfrak{J}(\mathbb{A})$. Hence, taking into account (2.1),

$$\begin{aligned} |H_\varphi^-|_{\mathfrak{J}(\mathbb{A})} &= \inf_{F \in \mathfrak{J}(\mathbb{A})} \|H_\varphi^- - F\|_{\mathcal{L}(X)} \le \inf_{\psi \in \mathbb{A}_-} \|H_\varphi^- - H_\psi^-\|_{\mathcal{L}(X)} \\ &= \inf_{\psi \in \mathbb{A}_-} \|H_{\varphi-\psi}^-\|_{\mathcal{L}(X)} \le \inf_{\psi \in \mathbb{A}_-} \Big(\|P_+\|_{\mathcal{L}(X)} \|M_{\varphi-\psi}\|_{\mathcal{L}(X)} \|P_-\|_{\mathcal{L}(X)} \Big) \\ &\le \gamma^2 \inf_{\psi \in \mathbb{A}_-} \|\varphi - \psi\|_\infty. \end{aligned} \tag{3.29}$$

Let us prove that

$$\inf_{\psi \in \mathbb{A}_-} \|\varphi - \psi\|_\infty \le \gamma |H_\varphi^-|_{\mathfrak{J}^+(\mathbb{A})}. \tag{3.30}$$

For any $F \in \mathfrak{J}^+(\mathbb{A})$ and $b \in \mathfrak{B}_{\mathbb{A}}$ from (3.12) we deduce that

$$P_+ M_{\overline{b}}(H_\varphi^- - F) = H_{\overline{b}\varphi}^- - P_+ M_{\overline{b}} F. \tag{3.31}$$

Then, taking into account (2.1), from (3.31) we get

$$\begin{aligned} \gamma \|H_\varphi^- - F\|_{\mathcal{L}(X)} &\ge \|P_+\|_{\mathcal{L}(X)} \|M_{\overline{b}}\|_{\mathcal{L}(X)} \|H_\varphi^- - F\|_{\mathcal{L}(X)} \\ &\ge \|H_{\overline{b}f}^- - P_+ M_{\overline{b}} F\|_{\mathcal{L}(X)} \\ &\ge \|H_{\overline{b}f}^-\|_{\mathcal{L}(X)} - \|P_+ M_{\overline{b}} F\|_{\mathcal{L}(X)}. \end{aligned} \tag{3.32}$$

Since $\overline{H^\infty} \subset \overline{\mathbb{A}} = \mathbb{A}_-$ and $\overline{b} \in \mathbb{A}$ for any Douglas algebra $\mathbb{A}$, from (2.5) we get

$$\|H_{\overline{b}\varphi}^-\|_{\mathcal{L}(X)} \ge \inf_{\psi \in \overline{H^\infty}} \|\overline{b}\varphi - \psi\|_\infty = \inf_{\psi \in \overline{H^\infty}} \|\varphi - b\psi\|_\infty \ge \inf_{\psi \in \mathbb{A}_-} \|\varphi - \psi\|_\infty. \tag{3.33}$$

From (3.32) and (3.33) we obtain for any $F \in \mathfrak{J}^+(\mathbb{A})$ and $b \in \mathfrak{B}_{\mathbb{A}}$,

$$\inf_{\psi \in \mathbb{A}_-} \|\varphi - \psi\|_\infty \le \gamma \|H_\varphi^- - F\|_{\mathcal{L}(X)} + \|P_+ M_{\overline{b}} F\|_{\mathcal{L}(X)}. \tag{3.34}$$

Then (3.34) and (3.2) imply

$$\begin{aligned}\inf_{\psi\in\mathbb{A}_-}\|\varphi-\psi\|_\infty &\le \inf_{F\in\mathfrak{J}^+(\mathbb{A})}\left(\inf_{b\in\mathfrak{B}_\mathbb{A}}\left(\gamma\|H_\varphi^- - F\|_{\mathcal{L}(X)}+\|P_+M_{\bar b}F\|_{\mathcal{L}(X)}\right)\right)\\ &= \inf_{F\in\mathfrak{J}^+(\mathbb{A})}\left(\gamma\|H_\varphi^- - F\|_{\mathcal{L}(X)}+\inf_{b\in\mathfrak{B}_\mathbb{A}}\|P_+M_{\bar b}F\|_{\mathcal{L}(X)}\right)\\ &= \gamma\inf_{F\in\mathfrak{J}^+(\mathbb{A})}\|H_\varphi^- - F\|_{\mathcal{L}(X)}=\gamma|H_\varphi^-|_{\mathfrak{J}^+(\mathbb{A})}.\end{aligned}$$

So, we have proved (3.30).

Combining (3.28)–(3.30), we arrive at (3.27). Inequalities (3.26) are proved similarly to (3.27). □

Theorem 3.11. *If $\varphi,\psi\in L^\infty$, then*

$$|M_\varphi P_+ + M_\psi P_-|_{\mathfrak{J}(L^\infty)} \ge \frac{1}{\gamma}\max\{\|\varphi\|_\infty,\|\psi\|_\infty\}. \tag{3.35}$$

Proof. Let us prove that

$$|M_\varphi P_+ + M_\psi P_-|_{\mathfrak{J}(L^\infty)} \ge \frac{\|\psi\|_\infty}{\gamma}. \tag{3.36}$$

Let $F\in\mathfrak{J}^-(L^\infty)$ and $b\in\mathfrak{B}_{L^\infty}$. Then, taking into account (2.1), we get

$$\begin{aligned}\gamma\|M_\psi P_- - F\|_{\mathcal{L}(X)} &\ge \|P_-\|_{\mathcal{L}(X)}\|M_b\|_{\mathcal{L}(X)}\|M_\psi P_- - F\|_{\mathcal{L}(X)}\\ &\ge \|P_-M_b(M_\psi P_- - F)\|_{\mathcal{L}(X)}\\ &\ge \|P_-M_bM_\psi P_-\|_{\mathcal{L}(X)}-\|P_-M_bF\|_{\mathcal{L}(X)}\\ &= \|T_{b\psi}^-\|_{\mathcal{L}(X)}-\|P_-M_bF\|_{\mathcal{L}(X)}.\end{aligned} \tag{3.37}$$

By (2.3), taking into account that $|b(\tau)|=1$ a.e. on $\mathbb{T}$,

$$\|T_{b\psi}^-\|_{\mathcal{L}(X)} \ge \|b\psi\|_\infty = \|\psi\|_\infty. \tag{3.38}$$

From (3.37) and (3.38), for every $F\in\mathfrak{J}^-(L^\infty)$ and every $b\in\mathfrak{B}_{L^\infty}$, we get

$$\frac{\|\psi\|_\infty}{\gamma} \le \|M_\psi P_- - F\|_{\mathcal{L}(X)} + \frac{1}{\gamma}\|P_-M_bF\|_{\mathcal{L}(X)}. \tag{3.39}$$

From (3.39) and (3.1) we deduce that

$$\begin{aligned}\frac{\|\psi\|_\infty}{\gamma} &\le \inf_{F\in\mathfrak{J}^-(L^\infty)}\left(\inf_{b\in\mathfrak{B}_{L^\infty}}\left(\|M_\psi P_- - F\|_{\mathcal{L}(X)}+\frac{1}{\gamma}\|P_-M_bF\|_{\mathcal{L}(X)}\right)\right)\\ &= \inf_{F\in\mathfrak{J}^-(L^\infty)}\left(\|M_\psi P_- - F\|_{\mathcal{L}(X)}+\frac{1}{\gamma}\inf_{b\in\mathfrak{B}_{L^\infty}}\|P_-M_bF\|_{\mathcal{L}(X)}\right)\\ &= \inf_{F\in\mathfrak{J}^-(L^\infty)}\|M_\psi P_- - F\|_{\mathcal{L}(X)} = |M_\psi P_-|_{\mathfrak{J}^-(L^\infty)}.\end{aligned} \tag{3.40}$$

By Corollary 3.7, $M_\varphi P_+\in\mathfrak{J}^-(L^\infty)$. Therefore,

$$|M_\varphi P_+ + M_\psi P_-|_{\mathfrak{J}^-(L^\infty)} = |M_\psi P_-|_{\mathfrak{J}^-(L^\infty)}. \tag{3.41}$$

Since $\mathfrak{J}(L^\infty)\subset\mathfrak{J}^-(L^\infty)$, we have

$$|M_\varphi P_+ + M_\psi P_-|_{\mathfrak{J}(L^\infty)} \ge |M_\varphi P_+ + M_\psi P_-|_{\mathfrak{J}^-(L^\infty)}. \tag{3.42}$$

Combining (3.40)–(3.42), we arrive at (3.36).

Analogously one can prove that

$$|M_\varphi P_+ + M_\psi P_-|_{\mathfrak{J}(L^\infty)} \ge \frac{\|\varphi\|_\infty}{\gamma}. \tag{3.43}$$

From (3.36) and (3.43) we obtain (3.35). □

4. The presymbol of the algebra Alg (Ω, S)

4.1. The construction of a presymbol. For a Banach subalgebra Ω of L^∞, we denote by $\mathcal{H}(\Omega)$ the closed two-sided ideal of Alg (Ω, S) generated by all Hankel operators H_φ^+ and H_ψ^- with $\varphi, \psi \in \Omega$.

Lemma 4.1. *If Ω is a Banach subalgebra of L^∞, then $\mathcal{H}(\Omega) = \mathrm{Com\,Alg}\,(\Omega, S)$.*

This lemma follows from the straightforwardly checked identities

$$M_\varphi M_\psi = M_\psi M_\varphi, \quad 2(H_\varphi^+ - H_\varphi^-) = M_\varphi S - SM_\varphi, \quad (M_\varphi S - SM_\varphi)P_\pm = \pm 2H_\varphi^\pm.$$

Lemma 4.2. *If Ω is a Banach subalgebra of L^∞, then*

$$\mathrm{Com\,Alg}\,(\Omega, S) \subset \mathrm{Alg}\,(\Omega, S) \cap \mathfrak{J}(L^\infty).$$

Proof. In view of Example 3.2(a), we have $\Omega \subset L^\infty = Q_{L^\infty}$. Due to Lemma 3.6(b), if $\varphi, \psi \in \Omega$, then $H_\varphi^+ \in \mathfrak{J}(L^\infty)$ and $H_\psi^- \in \mathfrak{J}(L^\infty)$. On the other hand, obviously, H_φ^+, H_ψ^- belong to Alg (Ω, S). Thus, $\mathcal{H}(\Omega) \subset \mathrm{Alg}\,(\Omega, S) \cap \mathfrak{J}(L^\infty)$. From the latter imbedding and Lemma 4.1 it follows that $\mathrm{Com\,Alg}\,(\Omega, S) \subset \mathrm{Alg}\,(\Omega, S) \cap \mathfrak{J}(L^\infty)$. □

Lemma 4.3. *Let Ω be a Banach subalgebra of L^∞. For any $\varphi, \psi \in \Omega$ we have*

$$|M_\varphi P_+ + M_\psi P_-|_{\mathrm{Com\,Alg}\,(\Omega,S)} \ge \frac{1}{\gamma} \max\{\|\varphi\|_\infty, \|\psi\|_\infty\}. \tag{4.1}$$

Proof. From Lemma 4.2 it follows that

$$\mathrm{Com\,Alg}\,(\Omega, S) \subset \mathrm{Alg}\,(\Omega, S) \cap \mathfrak{J}(L^\infty) \subset \mathfrak{J}(L^\infty).$$

Therefore,

$$|M_\varphi P_+ + M_\psi P_-|_{\mathrm{Com\,Alg}\,(\Omega,S)} \ge |M_\varphi P_+ + M_\psi P_-|_{\mathfrak{J}(L^\infty)}. \tag{4.2}$$

On the other hand, by Theorem 3.11,

$$|M_\varphi P_+ + M_\psi P_-|_{\mathfrak{J}(L^\infty)} \ge \frac{1}{\gamma} \max\{\|\varphi\|_\infty, \|\psi\|_\infty\}. \tag{4.3}$$

From (4.2) and (4.3) we get (4.1). □

Let $\mathrm{Alg}_0(\Omega, S)$ denote the linear subspace of Alg (Ω, S) consisting of all operators of the form

$$F = M_\varphi P_+ + M_\psi P_- + K, \tag{4.4}$$

where $\varphi, \psi \in \Omega$ and $K \in \mathrm{Com\,Alg}\,(\Omega, S)$.

Lemma 4.4. *If Ω is a Banach subalgebra of L^∞, then $\mathrm{Alg}_0(\Omega, S)$ is a Banach subalgebra of Alg (Ω, S).*

Proof. Let $a, b, c, d \in \Omega$ and $K_1, K_2 \in \operatorname{Com}\operatorname{Alg}(\Omega, S)$. Then

$$F_1 := M_a P_+ + M_b P_- + K_1, \quad F_2 := M_c P_+ + M_d P_- + K_2$$

belong to $\operatorname{Alg}_0(\Omega, S)$ and

$$F_1 F_2 = M_{ac} P_+ + M_{bd} P_- + K, \tag{4.5}$$

where

$$\begin{aligned} K = {} & (M_a P_+ + M_b P_-) K_2 + K_1 (M_c P_+ + M_d P_-) \\ & + K_1 K_2 + M_a (H_d^- - H_c^+) + M_b (H_c^+ - H_d^-). \end{aligned}$$

From the properties of the two-sided ideal $\operatorname{Com}\operatorname{Alg}(\Omega, S)$ and Lemma 4.1 it follows that $K \in \operatorname{Com}\operatorname{Alg}(\Omega, S)$. Therefore, $F_1 F_2 \in \operatorname{Alg}_0(\Omega, S)$, that is, $\operatorname{Alg}_0(\Omega, S)$ is a subalgebra of $\operatorname{Alg}(\Omega, S)$.

Now we show that $\operatorname{Alg}_0(\Omega, S)$ is closed in $\operatorname{Alg}(\Omega, S)$. Suppose that a sequence $\{F_n\}_{n=1}^\infty$ converges to $F \in \operatorname{Alg}(\Omega, S)$, where

$$F_n = M_{\varphi_n} P_+ + M_{\psi_n} P_- + K_n, \quad K_n \in \operatorname{Com}\operatorname{Alg}(\Omega, S), \quad \varphi_n, \psi_n \in \Omega. \tag{4.6}$$

By Lemma 4.3, for any $m, n \in \mathbb{N}$, we have

$$\begin{aligned} \|F_m - F_n\|_{\mathcal{L}(X)} & \ge |M_{\varphi_m - \varphi_n} P_+ + M_{\psi_m - \psi_n} P_-|_{\operatorname{Com}\operatorname{Alg}(\Omega, S)} \\ & \ge \frac{1}{\gamma} \max\{\|\varphi_m - \varphi_n\|_\infty, \|\psi_m - \psi_n\|_\infty\}. \end{aligned} \tag{4.7}$$

Since $\{F_n\}_{n=1}^\infty$ is a Cauchy sequence in $\mathcal{L}(X)$, from (4.7) it follows that $\{\varphi_n\}_{n=1}^\infty$ and $\{\psi_n\}_{n=1}^\infty$ are Cauchy sequences in L^∞. But Ω is closed in L^∞, therefore, the limits

$$\varphi := \lim_{n \to \infty} \varphi_n, \quad \psi := \lim_{n \to \infty} \psi_n$$

belong to Ω and

$$M_\varphi P_+ = \lim_{n \to \infty} M_{\varphi_n} P_+, \quad M_\psi P_- = \lim_{n \to \infty} M_{\psi_n} P_-. \tag{4.8}$$

Put $K := F - (M_\varphi P_+ + M_\psi P_-)$. From (4.6) and (4.8) we get

$$K = \lim_{n \to \infty} \Big[F_n - (M_{\varphi_n} P_+ + M_{\psi_n} P_-) \Big] = \lim_{n \to \infty} K_n. \tag{4.9}$$

Since $\operatorname{Com}\operatorname{Alg}(\Omega, S)$ is a closed two-sided ideal in $\operatorname{Alg}(\Omega, S)$, we infer from (4.9) that $K \in \operatorname{Com}\operatorname{Alg}(\Omega, S)$. Thus, $F \in \operatorname{Alg}_0(\Omega, S)$, which proves that $\operatorname{Alg}_0(\Omega, S)$ is closed in $\operatorname{Alg}(\Omega, S)$. □

Now we are in a position to prove the main result of this paper.

Theorem 4.5. *Let Ω be a Banach subalgebra of L^∞. An operator $F \in \operatorname{Alg}(\Omega, S)$ admits a unique representation (4.4), where $\varphi, \psi \in \Omega$ and $K \in \operatorname{Com}\operatorname{Alg}(\Omega, S)$. The mapping $F \mapsto (\varphi, \psi)$ defines a Banach algebra homomorphism (presymbol)*

$$\mu_\Omega : \operatorname{Alg}(\Omega, S) \to \Omega \oplus \Omega.$$

The kernel $\ker \mu_\Omega$ of this homomorphism coincides with $\operatorname{Com}\operatorname{Alg}(\Omega, S)$ and the norm of this homomorphism satisfies the inequality $\|\mu_\Omega\| \le \gamma$.

Proof. Since $0 \in \operatorname{Com Alg}(\Omega, S)$ and $\pm 1 \in \Omega$, we have

$$M_\varphi = M_\varphi P_+ + M_\varphi P_- \in \operatorname{Alg}_0(\Omega, S), \quad S = P_+ - P_- \in \operatorname{Alg}_0(\Omega, S).$$

So, the generators of $\operatorname{Alg}(\Omega, S)$ lie in $\operatorname{Alg}_0(\Omega, S)$. Then, in view of Lemma 4.4, $\operatorname{Alg}_0(\Omega, S) = \operatorname{Alg}(\Omega, S)$. Consequently, every operator F belonging to $\operatorname{Alg}(\Omega, S)$ admits a representation of the form (4.4).

This representation is unique. Indeed, assume the contrary. Then there exist $\varphi_1, \varphi_2, \psi_1, \psi_2 \in \Omega$ and $K_1, K_2 \in \operatorname{Com Alg}(\Omega, S)$ such that the triple $\{\varphi_1, \psi_1, K_1\}$ does not coincide with the triple $\{\varphi_2, \psi_2, K_2\}$ and

$$F = M_{\varphi_1} P_+ + M_{\psi_1} P_- + K_1 = M_{\varphi_2} P_+ + M_{\psi_2} P_- + K_2.$$

Clearly, the situation $\varphi_1 = \varphi_2, \psi_1 = \psi_2, K_1 \neq K_2$ is impossible. Therefore, $\varphi_1 \neq \varphi_2$ or $\psi_1 \neq \psi_2$. From Lemma 4.3 it follows that

$$\begin{aligned} 0 &= \|(M_{\varphi_1} - M_{\varphi_2}) P_+ + (M_{\psi_1} - M_{\psi_2}) P_- + K_1 - K_2\|_{\mathcal{L}(X)} \\ &\geq |M_{\varphi_1 - \varphi_2} P_+ + M_{\psi_1 - \psi_2} P_-|_{\operatorname{Com Alg}(\Omega, S)} \\ &\geq \frac{1}{\gamma} \max\{\|\varphi_1 - \varphi_2\|_\infty, \|\psi_1 - \psi_2\|_\infty\}. \end{aligned}$$

Hence, $\varphi_1 = \varphi_2$ and $\psi_1 = \psi_2$, so we arrive at a contradiction.

The fact that μ_Ω is a homomorphism with kernel $\operatorname{Com Alg}(\Omega, S)$ follows from the first statement of the theorem and the relation (4.5).

Let $F \in \operatorname{Alg}(\Omega, S)$ and $F \neq 0$. Then, by just proved, F admits a unique representation $F = M_\varphi P_+ + M_\psi P_- + K$ with $(\varphi, \psi) \in \Omega \oplus \Omega$ and $K \in \operatorname{Com Alg}(\Omega, S)$. In view of Lemma 4.3,

$$\|F\|_{\mathcal{L}(X)} \geq |M_\varphi P_+ + M_\psi P_-|_{\operatorname{Com Alg}(\Omega, S)} \geq \frac{1}{\gamma} \max\{|\varphi\|_\infty, \|\psi\|_\infty\}.$$

Then

$$\frac{\|\mu_\Omega(F)\|_{\Omega \oplus \Omega}}{\|F\|_{\mathcal{L}(X)}} \leq \frac{\max\{\|\varphi\|_\infty, \|\psi\|_\infty\}}{\frac{1}{\gamma} \max\{\|\varphi\|_\infty, \|\psi\|_\infty\}} = \gamma.$$

Taking the supremum over all $F \neq 0$ in the latter inequality, we get $\|\mu_\Omega\| \leq \gamma$. □

Remark 4.6. Theorem 4.1 in [9] contains wrong estimate for $\|\mu_\Omega\|$.

4.2. Necessary conditions for Fredholmness. In this subsection we obtain necessary conditions for Fredholmness of an operator $F \in \operatorname{Alg}(\Omega, S)$ in terms of its presymbol.

It is well-known that an operator A is Fredholm if and only if there exists a *regularizer* R of A, that is, $AR - I$ and $RA - I$ are compact operators (see, e.g., [10, Vol. 1, Section 4.7]). Recall that a Banach subalgebra Ω of L^∞ is said to be *inverse closed in* L^∞ if for every $f \in \Omega$ satisfying $\operatorname{ess\,inf}_{\tau \in \mathbb{T}} |f(\tau)| > 0$, we have $1/f \in \Omega$. The group of invertible elements of an algebra $\mathcal{A}$ is denoted by $\mathcal{GA}$.

Theorem 4.7. (a) *If* $F \in \operatorname{Alg}(L^\infty, S)$ *is Fredholm, then*

$$\mu_{L^\infty}(F) \in \mathcal{G}(L^\infty \oplus L^\infty).$$

(b) *Suppose* Ω *is an inverse closed subalgebra of* L^∞. *If* $F \in \operatorname{Alg}(\Omega, S)$ *is Fredholm, then*

$$\mu_\Omega(F) \in \mathcal{G}(\Omega \oplus \Omega).$$

Proof. (a) Assume the contrary, that is $F \in \operatorname{Alg}(L^\infty, S)$ is Fredholm, but its presymbol $\mu_{L^\infty}(F) = (\varphi, \psi)$ is not invertible in $L^\infty \oplus L^\infty$. Then there exists a regularizer R of F and

$$\operatorname*{ess\,inf}_{\tau\in\mathbb{T}} |\varphi(\tau)| = 0, \quad \operatorname*{ess\,inf}_{\tau\in\mathbb{T}} |\psi(\tau)| = 0.$$

Assume for definiteness that $\psi \notin \mathcal{G}L^\infty$, then there exists a set of positive measure on $\mathbb{T}$ with the characteristic function χ satisfying the inequality

$$\|\chi\psi\|_\infty < \frac{1}{\gamma^3\|R\|_{\mathcal{L}(X)}}. \tag{4.10}$$

By the definition of regularizer, we have

$$FR = I + K_1 \tag{4.11}$$

with $K_1 \in \mathcal{K}(X)$. In view of Theorem 4.5,

$$F = M_\varphi P_+ + M_\psi P_- + K_2, \quad \varphi, \psi \in L^\infty, \quad K_2 \in \operatorname{Com}\operatorname{Alg}(L^\infty, S). \tag{4.12}$$

Multiplying (from the left) both sides of (4.11) by $P_- M_\chi$ and taking into account (4.12), we get

$$P_- M_\chi (M_\varphi P_+ + M_\psi P_- + K_2)R = P_- M_\chi + P_- M_\chi K_1.$$

This equality can be rewritten in the form

$$T^-_{\chi\psi} R = M_\chi P_- + K, \tag{4.13}$$

where

$$K = P_- M_\chi K_1 - P_- M_\chi K_2 R - H^+_{\chi\varphi} R + (P_- M_\chi - M_\chi P_-).$$

In view of Lemma 3.9, $\mathcal{K}(X) = \mathfrak{J}(H^\infty + C)$. Since $H^\infty + C \subset L^\infty$, we have $\mathfrak{J}(H^\infty + C) \subset \mathfrak{J}(L^\infty)$. Therefore, $\mathcal{K}(X) \subset \mathfrak{J}(L^\infty)$. Since $K_1 \in \mathcal{K}(X)$, we deduce that

$$P_- M_\chi K_1 \in \mathcal{K}(X) \subset \mathfrak{J}(L^\infty). \tag{4.14}$$

Further, K_2 belongs to the closed two-sided ideal $\operatorname{Com}\operatorname{Alg}(L^\infty, S)$. Consequently, $-P_- M_\chi K_2 R$ belongs to $\operatorname{Com}\operatorname{Alg}(L^\infty, S)$ too. Taking into account that $\chi, \varphi \in L^\infty$, from Lemma 4.1 we obtain $H^+_{\chi\varphi} \in \mathcal{H}(L^\infty) = \operatorname{Com}\operatorname{Alg}(L^\infty, S)$, whence, $-H^+_{\chi\varphi} R \in \operatorname{Com}\operatorname{Alg}(L^\infty, S)$. By the definition of the ideal $\operatorname{Com}\operatorname{Alg}(L^\infty, S)$, we get $P_- M_\chi - M_\chi P_- \in \operatorname{Com}\operatorname{Alg}(L^\infty, S)$. Thus,

$$-P_- M_\chi K_2 R - H^+_{\chi\varphi} R + (P_- M_\chi - M_\chi P_-) \in \operatorname{Com}\operatorname{Alg}(L^\infty, S). \tag{4.15}$$

Due to Lemma 4.2, $\operatorname{Com}\operatorname{Alg}(L^\infty, S) \subset \mathfrak{J}(L^\infty)$. Then from (4.14) and (4.15) we obtain $K \in \mathfrak{J}(L^\infty)$. Applying Theorem 3.11, we get

$$\|M_\chi P_- + K\|_{\mathcal{L}(X)} \ge |M_\chi P_-|_{\mathfrak{J}(L^\infty)} \ge \frac{1}{\gamma}\|\chi\|_\infty = \frac{1}{\gamma}. \tag{4.16}$$

On the other hand, from (4.13), (4.10), and (2.1) we obtain

$$\begin{aligned} \|M_\chi P_- + K\|_{\mathcal{L}(X)} &= \|T^-_{\chi\psi} R\|_{\mathcal{L}(X)} = \|P_- M_{\chi\psi} P_- R\|_{\mathcal{L}(X)} \\ &\le \gamma^2 \|\chi\varphi\|_\infty \|R\|_{\mathcal{L}(X)} < \frac{\gamma^2\|R\|_{\mathcal{L}(X)}}{\gamma^3\|R\|_{\mathcal{L}(X)}} = \frac{1}{\gamma}. \end{aligned} \tag{4.17}$$

Combining (4.16) and (4.17), we arrive at the wrong inequality $1/\gamma < 1/\gamma$. Therefore, $\psi \in \mathcal{G}L^\infty$. Analogously one can prove that $\varphi \in \mathcal{G}L^\infty$. Part (a) is proved.

(b) If Ω is inverse closed in L^∞, then $\Omega \oplus \Omega$ is inverse closed in $L^\infty \oplus L^\infty$. Clearly, $F \in \operatorname{Alg}(\Omega, S) \subset \operatorname{Alg}(L^\infty, S)$ and, by Theorem 4.5, $\mu_\Omega(F) = \mu_{L^\infty}(F)$. Due to Part (a), if F is Fredholm, then $\mu_{L^\infty}(F) \in \mathcal{G}(L^\infty \oplus L^\infty)$. Since $\Omega \oplus \Omega$ is inverse closed in $L^\infty \oplus L^\infty$, the inclusion $\mu_\Omega(F) = \mu_{L^\infty}(F) \in \mathcal{G}(L^\infty \oplus L^\infty)$ implies the inclusion $\mu_\Omega(F) \in \mathcal{G}(\Omega \oplus \Omega)$. Part (b) and the theorem is proved. □

4.3. The commutator ideal of the algebra Alg (Ω, S). In this subsection we describe the commutator ideal of the algebra $\operatorname{Alg}(\Omega, S)$.

Theorem 4.8. *Let Ω be a Banach subalgebra of L^∞ and let $\mathbb{A}$ be a Douglas algebra. Then $\Omega \subset Q_\mathbb{A}$ if and only if*

$$\operatorname{Alg}(\Omega, S) \cap \mathfrak{J}(\mathbb{A}) = \operatorname{Com}\operatorname{Alg}(\Omega, S). \tag{4.18}$$

Proof. *Necessity.* By Theorem 3.8, the set $\operatorname{Alg}(\Omega, S) \cap \mathfrak{J}(\mathbb{A})$ is a closed two-sided ideal of $\operatorname{Alg}(\Omega, S)$. Moreover, by Lemma 3.6(b), this ideal contains all operators H_φ^+ and H_ψ^- with $\varphi, \psi \in \Omega$. But Lemma 4.1 states that $\operatorname{Com}\operatorname{Alg}(\Omega, S)$ is the smallest closed two-sided ideal containing all such operators. This means that

$$\operatorname{Com}\operatorname{Alg}(\Omega, S) \subset \operatorname{Alg}(\Omega, S) \cap \mathfrak{J}(\mathbb{A}). \tag{4.19}$$

To prove the reverse inclusion, assume that $F \in \operatorname{Alg}(\Omega, S) \cap \mathfrak{J}(\mathbb{A})$. By Theorem 4.5, there exist $\varphi, \psi \in \Omega$ and $K \in \operatorname{Com}\operatorname{Alg}(\Omega, S)$ such that $F = M_\varphi P_+ + M_\psi P_- + K$. Therefore, in view of Lemma 4.2, $K \in \operatorname{Alg}(\Omega, S) \cap \mathfrak{J}(L^\infty)$. On the other hand, since $\mathbb{A} \subset L^\infty$, we have $\mathfrak{J}(\mathbb{A}) \subset \mathfrak{J}(L^\infty)$. Then $F \in \operatorname{Alg}(\Omega, S) \cap \mathfrak{J}(L^\infty)$. Thus,

$$M_\varphi P_+ + M_\psi P_- = F - K \in \operatorname{Alg}(\Omega, S) \cap \mathfrak{J}(L^\infty). \tag{4.20}$$

From Example 3.2(a) and Theorem 3.8 it follows that $\operatorname{Alg}(\Omega, S) \cap \mathfrak{J}(L^\infty)$ is a closed two-sided ideal of $\operatorname{Alg}(\Omega, S)$. Since $\operatorname{Alg}(\Omega, S) \cap \mathfrak{J}(L^\infty) \subset \mathfrak{J}(L^\infty)$, we have

$$|M_\varphi P_+ + M_\psi P_-|_{\operatorname{Alg}(\Omega,S)\cap\mathfrak{J}(L^\infty)} \ge |M_\varphi P_+ + M_\psi P_-|_{\mathfrak{J}(L^\infty)}.$$

From here and Theorem 3.11 it follows that

$$|M_\varphi P_+ + M_\psi P_-|_{\operatorname{Alg}(\Omega,S)\cap\mathfrak{J}(L^\infty)} \ge \frac{1}{\gamma} \max\{\|\varphi\|_\infty, \|\psi\|_\infty\}. \tag{4.21}$$

From (4.20) and (4.21) we get $\varphi = \psi = 0$ a.e. on $\mathbb{T}$. Thus, $F = K \in \operatorname{Com}\operatorname{Alg}(\Omega, S)$. This means that

$$\operatorname{Alg}(\Omega, S) \cap \mathfrak{J}(\mathbb{A}) \subset \operatorname{Com}\operatorname{Alg}(\Omega, S). \tag{4.22}$$

Combining (4.19) and (4.22), we arrive at (4.18). Necessity is proved.

Sufficiency. Let $\varphi \in \Omega$. From (4.18) and Lemma 4.1 it follows that

$$H_\varphi^\pm \in \mathcal{H}(\Omega) = \operatorname{Com}\operatorname{Alg}(\Omega, S) = \operatorname{Alg}(\Omega, S) \cap \mathfrak{J}(\mathbb{A}) \subset \mathfrak{J}(\mathbb{A}). \tag{4.23}$$

By Theorem 3.10,

$$\inf_{\psi \in \mathbb{A}_+} \|\varphi - \psi\|_\infty \le \gamma |H_\varphi^+|_{\mathfrak{J}(\mathbb{A})}, \quad \inf_{\psi \in \mathbb{A}_-} \|\varphi - \psi\|_\infty \le \gamma |H_\varphi^-|_{\mathfrak{J}(\mathbb{A})}. \tag{4.24}$$

From (4.23) it follows that $|H_\varphi^+|_{\mathfrak{J}(\mathbb{A})} = |H_\varphi^-|_{\mathfrak{J}(\mathbb{A})} = 0$. Therefore, (4.24) implies $\varphi \in \mathbb{A}_+ \cap \mathbb{A}_- = Q_\mathbb{A}$. Thus, $\Omega \subset Q_\mathbb{A}$. □

Corollary 4.9. *Let Ω be a Banach subalgebra of L^∞ and let $\mathbb{A}$ be a Douglas algebra. If $\Omega \subset Q_\mathbb{A}$, then*

$$\operatorname{Com}\operatorname{Alg}(\Omega, S) = \operatorname{Alg}(\Omega, S) \cap \operatorname{Com}\operatorname{Alg}(Q_\mathbb{A}, S). \tag{4.25}$$

Proof. Since $\Omega \subset Q_{\mathbb{A}}$, we have

$$\operatorname{Alg}(\Omega, S) = \operatorname{Alg}(\Omega, S) \cap \operatorname{Alg}(Q_{\mathbb{A}}, S). \tag{4.26}$$

By Theorem 4.8 and (4.26),

$$\begin{aligned} \operatorname{Com}\operatorname{Alg}(\Omega, S) &= \operatorname{Alg}(\Omega, S) \cap \mathfrak{J}(\mathbb{A}) = \Big(\operatorname{Alg}(\Omega, S) \cap \operatorname{Alg}(Q_{\mathbb{A}}, S)\Big) \cap \mathfrak{J}(\mathbb{A}) \\ &= \operatorname{Alg}(\Omega, S) \cap \Big(\operatorname{Alg}(Q_{\mathbb{A}}, S) \cap \mathfrak{J}(\mathbb{A})\Big). \end{aligned} \tag{4.27}$$

Applying Theorem 4.8 to the trivial situation $Q_{\mathbb{A}} \subset Q_{\mathbb{A}}$, we get

$$\operatorname{Alg}(Q_{\mathbb{A}}, S) \cap \mathfrak{J}(\mathbb{A}) = \operatorname{Com}\operatorname{Alg}(Q_{\mathbb{A}}, S). \tag{4.28}$$

Combining (4.27) and (4.28), we arrive at (4.25). □

4.4. Singular integral operators with quasicontinuous coefficients. In the following lemma we characterize the commutator ideal of Banach algebras $\operatorname{Alg}(\Omega, S)$ for Ω being a subalgebra between C and QC.

Lemma 4.10.

(a) *For a Banach subalgebra Ω of L^∞ which lies between C and QC,*

$$\operatorname{Com}\operatorname{Alg}(\Omega, S) = \mathcal{K}(X). \tag{4.29}$$

(b) *For every Banach subalgebra Ω of L^∞ such that $C \subset \Omega$ and $QC \setminus \Omega \neq \emptyset$, the ideal of compact operators $\mathcal{K}(X)$ is properly contained in the commutator ideal* $\operatorname{Com}\operatorname{Alg}(\Omega, S)$.

Proof. (a) Repeating the proof of [12, Lemma 9.1] (see also [3, Lemma 8.23]), one can show that

$$\mathcal{K}(X) \subset \operatorname{Alg}(C, S). \tag{4.30}$$

On the other hand, consider the Douglas algebra $\mathbb{A} = H^\infty + C$. Then $Q_{\mathbb{A}} = QC$ (see Example 3.2(c)). By Lemma 3.9, $\mathfrak{J}(H^\infty + C) = \mathcal{K}(X)$. Combining these facts with Theorem 4.8, we get

$$\operatorname{Com}\operatorname{Alg}(\Omega, S) = \operatorname{Alg}(\Omega, S) \cap \mathfrak{J}(H^\infty + C) = \operatorname{Alg}(\Omega, S) \cap \mathcal{K}(X). \tag{4.31}$$

Since $C \subset \Omega \subset QC$, we have

$$\operatorname{Alg}(C, S) \subset \operatorname{Alg}(\Omega, S) \subset \operatorname{Alg}(QC, S). \tag{4.32}$$

From (4.30)–(4.32) we get

$$\mathcal{K}(X) \subset \operatorname{Alg}(C, S) \cap \mathcal{K}(X) \subset \operatorname{Alg}(\Omega, S) \cap \mathcal{K}(X) = \operatorname{Com}\operatorname{Alg}(\Omega, S) \subset \mathcal{K}(X).$$

Thus, we arrive at (4.29).

(b) Since $C \subset \Omega$, we have $\mathcal{K}(X) = \operatorname{Com}\operatorname{Alg}(C, S) \subset \operatorname{Com}\operatorname{Alg}(\Omega, S)$. On the other hand, by [17, Theorem 4.1], for a function $\varphi \in L^\infty$, the commutator $M_\varphi S - SM_\varphi$ is compact on X if and only if $\varphi \in QC$. Thus, for $\varphi \in \Omega \setminus QC$, we have $M_\varphi S - SM_\varphi \in \operatorname{Com}\operatorname{Alg}(\Omega, S) \setminus \mathcal{K}(X)$. □

Lemma 4.10(b) shows that if the set $QC \setminus \Omega$ is nonempty, then Theorem 4.7 gives only necessary conditions for Fredholmness of an operator $F \in \operatorname{Alg}(\Omega, S)$. However, in view of Lemma 4.10(a), they become sufficient if Ω lies between C and QC and it is inverse closed in L^∞. More precisely, the following criterion is true.

Theorem 4.11. *Suppose a Banach subalgebra Ω of L^∞ is inverse closed in L^∞ and lies between C and QC. An operator $F \in \mathrm{Alg}\,(\Omega, S)$ admits a unique representation of the form (4.4) with $\varphi, \psi \in \Omega$ and $K \in \mathcal{K}(X)$, and it is Fredholm if and only if*

$$\operatorname*{ess\,inf}_{\tau\in\mathbb{T}} |\varphi(\tau)| > 0, \quad \operatorname*{ess\,inf}_{\tau\in\mathbb{T}} |\psi(\tau)| > 0. \tag{4.33}$$

Proof. From Theorem 4.5 and Lemma 4.10(a) it follows that $F \in \mathrm{Alg}\,(\Omega, S)$ admits a unique representation of the form (4.4) with $\varphi, \psi \in \Omega$ and $K \in \mathcal{K}(X) = \mathrm{Com}\,\mathrm{Alg}\,(\Omega, S)$. Theorem 4.7 implies that conditions (4.33) are necessary for Fredholmness of F.

Let us prove that these conditions are also sufficient. Since Ω is inverse closed in L^∞, conditions (4.33) imply $1/\varphi, 1/\psi \in \Omega$. Then, taking into account that $\mathcal{K}(X) = \mathrm{Com}\,\mathrm{Alg}\,(\Omega, S)$, it is not difficult to show that $R := M_{1/\varphi}P_+ + M_{1/\psi}P_-$ is a regularizer for F. Hence, F is Fredholm. □

References

[1] C. Bennett and R. Sharpley, *Interpolation of Operators*, Pure and Applied Mathematics, **129**, Academic Press, Inc., Boston, MA, 1988, MR 89e:46001, Zbl 0647.46057.

[2] A. Böttcher and S. M. Grudsky, *Toeplitz operators with discontinuous symbols: phenomena beyond piecewise continuity*, in "Singular Integral Operators and Related Topics (Tel Aviv, 1995)", Operator Theory: Advances and Applications, **90**, Birkhäuser Verlag, Basel, 1996, 55-118, MR 97k:47025, Zbl 0887.47026.

[3] A. Böttcher and Yu. I. Karlovich, *Carleson Curves, Muckenhoupt Weights, and Toeplitz Operators*, Progress in Mathematics, **154**, Birkhäuser Verlag, Basel, 1997, MR 98m:47031, Zbl 0889.47001.

[4] A. Böttcher, Yu. I. Karlovich and I. M. Spitkovsky, *Convolution Operators and Factorization of Almost Periodic Matrix Functions*, Operator Theory: Advances and Applications, **131**, Birkhäuser Verlag, Basel, 2002, CMP 1 898 405.

[5] A. Böttcher and B. Silbermann, *Analysis of Toeplitz Operators*, Springer-Verlag, Berlin, 1990, MR 92e:47001, Zbl 0732.47029.

[6] D. W. Boyd, *Indices of function spaces and their relationship to interpolation*, Canad. J. Math. **21** (1969), 1245–1254, MR 54 #909, Zbl 0184.34802.

[7] V. M. Deundyak and K. A. Georgiev, *An outer derivation construction on the algebra of singular integral operators with general coefficients in weighted spaces and its applications*, in "Operator Theory and Related Topics, Vol. II (Odessa, 1997)", Operator Theory: Advances and Applications, **118**, Birkhäuser Verlag, Basel, 2000, 91–103, MR 2001f:47118, Zbl 0988.47019.

[8] J. B. Garnett, *Bounded Analytic Functions*, Pure and Applied Mathematics, **96**, Academic Press, Inc., New York-London, 1981, MR 83g:30037, MR 0469.30024.

[9] K. A. Georgiev and V. M. Deundyak, *Nikol'skii ideals and their application to the study of algebras of singular integral operators*, Algebra i Analiz **11** (1999), 88–108 (in Russian), English translation: St. Petersburg Math. J. **11** (2000), 269–284, MR 2000d:47106, Zbl 0945.47052.

[10] I. Gohberg and N. Krupnik, *One-Dimensional Linear Singular Integral Equations*, Vols. I–II, Operator Theory: Advances and Applications, **53**, **54**, Birkhäuser Verlag, Basel, 1992, MR 93c:47061, Zbl 0743.45004, MR 93k:47059, Zbl 0781.47038. Russian original: Shtiintsa, Kishinev, 1973, MR 53 #8971, Zbl 0271.47017.

[11] B. Hollenbeck and I. E. Verbitsky, *Best constants for the Riesz projection*, J. Funct. Anal. **175** (2000), 370–392, MR 2001i:42010, Zbl 0963.42006.

[12] A. Karlovich, *Algebras of singular integral operators with piecewise continuous coefficients on reflexive Orlicz spaces*, Math. Nachr. **179** (1996), 187–222, MR 97b:47020, Zbl 0873.47020.

[13] A. Karlovich, *Singular integral operators with piecewise continuous coefficients in reflexive rearrangement-invariant spaces*, Integral Equations and Operator Theory **32** (1998), 436–481, MR 99k:47118, Zbl 0923.47016.

[14] A. Karlovich, *On the essential norm of the Cauchy singular integral operator in weighted rearrangement-invariant spaces*, Integral Equations and Operator Theory **38** (2000), 28–50, MR 2001g:47095.

[15] A. Karlovich, *Algebras of singular integral operators with PC coefficients in rearrangement-invariant spaces with Muckenhoupt weights*, Journal of Operator Theory **47** (2002), 303–323, CMP 1 911 849.

[16] A. Karlovich, *Norms of Toeplitz and Hankel operators on Hardy type subspaces of rearrangement-invariant spaces*, to appear in Integral Equations and Operator Theory. Also in Departamento de Matemática, Instituto Superior Técnico, Preprint 18, 2002.

[17] A. Karlovich and Yu. Karlovich, *Compactness of commutators arising in the Fredholm theory of singular integral operators with shifts*, to appear in "Factorization, Singular Integral Operators and Related Problems; Proceedings of the Conference in Honour of Georgui Litvinchuk", Kluwer Academic Publishers. Also in Departamento de Matemática, Instituto Superior Técnico, Preprint 10, 2002.

[18] S. G. Krein, Ju. I. Petunin and E. M. Semenov, *Interpolation of Linear Operators*, Nauka, Moscow, 1978 (Russian), MR 81f:46086, Zbl 0499.46044, English translation: AMS Translations of Mathematical Monographs **54**, Providence, R.I., 1982, MR 84j:46103, Zbl 0493.46058.

[19] N. Ya. Krupnik, *Banach Algebras with Symbol and Singular Integral Operators*, Operator Theory: Advances and Applications, **26**, Birkhäuser Verlag, Basel, 1987, MR 90g:47092, Zbl 0641.47031, Russian original: Shtiintsa, Kishinev, 1984, MR 86j:47071, Zbl 0641.47030.

[20] J. Lindenstrauss and L. Tzafriri. *Classical Banach Spaces.* II: *Function Spaces*, Springer Verlag, New York, Berlin, 1979, MR 81c:46001, Zbl 0403.46022.

[21] L. Maligranda, *Indices and Interpolation*, Dissertationes Math. (Rozprawy Mat.), **234** (1985), 1–49, MR 87k:46059, Zbl 0566.46038.

[22] S. G. Mikhlin, *Composition of singular integrals*, Dokl. Akad. Nauk SSSR, **2**(**11**) (1936), 3–6 (in Russian).

[23] S. G. Mikhlin, *Singular integral equations with two independent variables*, Mat. Sb., **1**(**43**) (1936), 535–550 (in Russian).

[24] S. G. Mikhlin, *Singular integral equations*, Uspehi Matem. Nauk (N.S.) **3**(**25**) (1948), 29–112 (in Russian), MR 10,305a. English translation: Amer. Math. Soc. Translation 1950, (1950) no. 24, MR 12,107d.

[25] S. G. Mikhlin and S. Prössdorf, *Singular Integral Operators*, Springer-Verlag, Berlin, 1986, MR 88e:47097, Zbl 0612.47024.

[26] N. K. Nikolski, *Hankel and Toeplitz operators. The spectral theory*, Preprint LOMI no. R-1-82, LOMI, Leningrad, 1982 (in Russian).

[27] N. K. Nikolski, *Hankel and Toeplitz operators. The algebraic approach*, Preprint LOMI no. R-2-82, LOMI, Leningrad, 1982 (in Russian).

[28] N. K. Nikolski, *Treatise on the Shift Operator. Spectral Function Theory*, Springer-Verlag, Berlin, 1986, MR 87i:47042, Zbl 0587.47036.

[29] S. Roch and B. Silbermann, *Algebras of convolution operators and their image in the Calkin algebra*, Report MATH, 90-05. Akademie der Wissenschaften der DDR, Karl-Weierstrass-Institut für Mathematik, Berlin, 1990, MR 92d:47067, Zbl 0717.47019.

Departamento de Matemática, Instituto Superior Técnico, Av. Rovisco Pais, 1049–001 Lisboa, Portugal

akarlov@math.ist.utl.pt http://www.math.ist.utl.pt/~akarlov

This paper is available via http://nyjm.albany.edu:8000/j/2002/8-14.html.