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The maximal and minimal ranks of $A - BXC$ with applications

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Abstract. We consider how to take X such that the linear matrix expression $A - BXC$ attains its maximal and minimal ranks, respectively. As applications, we investigate the rank invariance and the range invariance of $A - BXC$ with respect to the choice of X. In addition, we also give the general solution of the rank equation $\operatorname{rank}(A - BXC) + \operatorname{rank}(BXC) = \operatorname{rank}(A)$ and then determine the minimal rank of $A - BXC$ subject to this equation.

Contents

1. Introduction

Let

$$p(X) = A - BXC \tag{1.1}$$

be a linear matrix expression over an arbitrary field $\mathbb{F}$, where $A \in \mathbb{F}^{m\times n}$, $B \in \mathbb{F}^{m\times k}$ and $C \in \mathbb{F}^{l\times n}$ are given; $X \in \mathbb{F}^{k\times l}$ is a variant matrix. One of the fundamental problems for (1.1) is to determine the maximal and minimal ranks of (1.1) when X is running over $\mathbb{F}^{k\times l}$. Because the rank of a matrix is an integer between zero and the minimum of the row and column numbers of the matrix, the maximal and minimal values of the rank of (1.1) with respect to $X \in \mathbb{F}^{k\times l}$ must exist and these two values can be attained for some X in $\mathbb{F}^{k\times l}$. In fact, the maximal and minimal ranks of any linear or nonlinear matrix expression with respect to variant matrices in it always exist. $A - BXC$ is one of the simplest cases among various linear

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matrix expressions. The results on the extremal ranks of (1.1) can be used for finding extremal ranks of many other linear and nonlinear matrix expressions.

The investigation of extremal ranks of matrix expressions has many direct motivations in matrix analysis. For example, the matrix equation $BXC = A$ is consistent if and only if $\min_X \operatorname{rank}(A - BXC) = 0$; the matrix equation $B_1X_1C_1 + B_2X_2C_2 = A$ is consistent if and only if $\min_{X_1,X_2} \operatorname{rank}(A-B_1X_1C_1-B_2X_2C_2) = 0$; the two consistent matrix equations $B_1X_1C_1 = A_1$ and $B_2X_2C_2 = A_2$, where X_1 and X_2 have the same size, have a common solution if and only if $\min_{X_1,X_2} \operatorname{rank}(X_1 - X_2) = 0$; there is a matrix X such that the square block matrix $\begin{bmatrix} A & B \\ C & X \end{bmatrix}$ of order n is nonsingular if and only if $\max_X \operatorname{rank}\begin{bmatrix} A & B \\ C & X \end{bmatrix} = n$. In general, for any two matrix expressions $p(X_1, \dots, X_s)$ and $q(Y_1, \dots, Y_t)$ of the same size, there are $X_1, \dots, X_s$, $Y_1, \dots, Y_t$ such that $p(X_1, \dots, X_s) = q(Y_1, \dots, Y_t)$ if and only if

$$\min_{X_1,\dots,X_s,Y_1,\dots,Y_t} \operatorname{rank}[p(X_1, \dots, X_s) - q(Y_1, \dots, Y_t)] = 0;$$

$p(X_1, \dots, X_s)$ and $q(Y_1, \dots, Y_t)$ are identical if and only if

$$\max_{X_1,\dots,X_s,Y_1,\dots,Y_t} \operatorname{rank}[p(X_1, \dots, X_s) - q(Y_1, \dots, Y_t)] = 0.$$

Moreover, the rank invariance and the range invariance of any matrix expression with respect to its variant matrices can also be characterized by the extremal ranks of the matrix expression. These examples imply that the extremal ranks of matrix expressions have close links with many topics in matrix analysis and applications. Various statements on extremal ranks of matrix expressions are quite easy to understand for the people who know linear algebra. But the question now is how to give simple or closed forms for the extremal ranks of a matrix expression with respect to its variant matrices. This topic was noticed and studied in the late 1980s in the investigation of matrix completion problems; see, e.g., [5, 6, 10, 11, 25]. In these papers, minimal ranks of some partial matrices are derived in closed forms. But the methods used in these papers are not easy to understand for the people who just learn elementary linear algebra.

It is well-known (see, e.g., [12, 13]) that a powerful tool in the investigation of ranks of matrices is generalized inverses of matrices. An $n \times m$ matrix X is called a generalized inverse of an $m \times n$ matrix A if it satisfies $AXA = A$, and is denoted as usual by $X = A^-$. The collection of all generalized inverses of A is denoted by $\{A^-\}$. If A is decomposed as $A = P\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}Q$, where both P and Q are nonsingular, then the general expression of A^- can be written as $A^- = Q^{-1}\begin{bmatrix} I_r & V_2 \\ V_3 & V_4 \end{bmatrix}P^{-1}$, where V_2, V_3 and V_4 are arbitrary matrices. If a generalized inverse $A^\sim$ of A is known, then the general expression of A^- can be written as $A^- = A^\sim + (I_n - A^\sim A)U_1 + U_2(I_m - AA^\sim)$, where U_1 and U_2 are arbitrary matrices.

Generalized inverses of matrices can be used to establish various rank equalities for matrices. Some well-known rank equalities for block matrices due to Marsaglia and Styan [12] are presented in Lemma 1.1 of this paper. These rank equalities can further be used to establish or simplify various rank equalities for matrix expressions that involve generalized inverses.

It is well-known that the rank of a given nonzero matrix is a positive integer and it can be evaluated through row or column elementary operations for the matrix. This fact motivates us to establish some rank identities for (1.1) by block elementary operations for matrices, and then find the extremal ranks of (1.1) from these rank identities.

In a recent paper [23] on the commutativity of generalized inverses of matrices, Tian shows an identity for the rank of $A - BXC$ by block elementary operations as follows:

$$\begin{aligned}\text{rank}(A - BXC) = \text{rank}\begin{bmatrix} A \\ C \end{bmatrix} + \text{rank}[A,\ B] - \text{rank}(M) \\ + \text{rank}[E_{T_1}(X + TM^-S)F_{S_1}],\end{aligned} \tag{1.2}$$

where $[A,\ B]$ denotes a row block matrix consisting of A and B,

$$M = \begin{bmatrix} A & B \\ C & 0 \end{bmatrix}, \quad T = [0,\ I_k], \quad S = \begin{bmatrix} 0 \\ I_l \end{bmatrix},$$

$$T_1 = T - TM^-M, \quad S_1 = S - MM^-S, \quad E_{T_1} = I_l - T_1T_1^-, \quad F_{S_1} = I_k - S_1^-S_1.$$

Note that $\text{rank}[E_{T_1}(X + TM^-S)F_{S_1}]$ is a nonnegative term on the right-hand side of (1.2) and the matrix X in $X + TM^-S$ is free. Hence, one can take $X = -TM^-S$ such that $E_{T_1}(X + TM^-S)F_{S_1} = 0$. Thus, the minimal rank of $A - BXC$ with respect to X is $\text{rank}\begin{bmatrix} A \\ C \end{bmatrix} + \text{rank}[A,\ B] - \text{rank}(M)$, which looks quite simple and symmetric. Using this result and some other rank formulas, the first author of this paper gives a group of formulas for the minimal ranks of $AA^- - A^-A$, $A^kA^- - A^-A^k$ and $BB^-A - AC^-C$ with respect to A^-, B^- and C^-. But [23] just gives some introductory results on the extremal ranks of $A - BXC$. This leads us to give a compelte consideration for this problem. In Section 2, we shall give a new identity for the rank of $A - BXC$ and find the extremal ranks of $A - BXC$ with respect to X from this new rank identity. Through the extremal ranks of $A - BXC$, we also investigate the rank invariance and the range invariance of $A - BXC$ with respect to X, respectively. In Section 3, we shall solve the rank equation $\text{rank}(A - BXC) + \text{rank}(BXC) = \text{rank}(A)$ for X and then find the minimal rank of $A - BXC$ subject to this rank equation.

Throughout, $\mathbb{F}$ denotes an arbitrary field. For a matrix A over $\mathbb{F}$, the symbols E_A and F_A stand for the two oblique projectors $E_A = I - AA^-$ and $F_A = I - A^-A$ induced by A; A^T, $r(A)$, $\mathcal{R}(A)$ and $\mathcal{N}(A)$ denote the transpose, the rank, the range (column space) and the null space of A, respectively. A matrix $X \in \mathbb{F}^{n\times m}$ is called a reflexive generalized inverse of $A \in \mathbb{F}^{m\times n}$ if it satisfies both $AXA = A$ and $XAX = X$, and is denoted by $X = A_r^-$.

In order to find the extremal ranks of $A - BXC$ with respect to X and solve the rank equation $r(A - BXC) + r(BXC) = r(A)$, we need the following results on ranks of matrices and general solutions of some matrix equations.

Lemma 1.1 ([12])**.** *Let* $A \in \mathbb{F}^{m\times n}$, $B \in \mathbb{F}^{m\times k}$, $C \in \mathbb{F}^{l\times n}$ *and* $D \in \mathbb{F}^{l\times k}$. *Then:*

(a) $r[A,\ B] = r(A) + r(E_AB) = r(B) + r(E_BA)$.

(b) $r\begin{bmatrix} A \\ C \end{bmatrix} = r(A) + r(CF_A) = r(C) + r(AF_C)$.

(c) $r\begin{bmatrix} A & B \\ C & 0 \end{bmatrix} = r\begin{bmatrix} E_BA \\ C \end{bmatrix} + r(B)$

$= r[AF_C,\ B] + r(C) = r(B) + r(C) + r(E_BAF_C).$

(d) $r\begin{bmatrix} A & B \\ C & D \end{bmatrix} = r(A) + r\begin{bmatrix} 0 & E_AB \\ CF_A & S_A \end{bmatrix} = r\begin{bmatrix} A \\ C \end{bmatrix} + r[A,\ B] - r(A) + r(E_{C_1}S_AF_{B_1}),$ *where* $B_1 = E_AB$ *and* $C_1 = CF_A$, *the matrix* $S_A = D - CA^-B$ *is the Schur complement of* A *in* $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$.

Lemma 1.2 ([12]). *The rank of a triple matrix product* PAQ *satisfies the following identity:*

$$r(PAQ) = r(PA) + r(AQ) - r(A) + r(E_{AQ}AF_{PA}). \tag{1.3}$$

Lemma 1.3 ([16]). *Suppose* $BXC = A$ *is a linear matrix equation over* $\mathbb{F}$. *Then this equation is consistent if and only if* $\mathcal{R}(A) \subseteq \mathcal{R}(B)$ *and* $\mathcal{R}(A^T) \subseteq \mathcal{R}(C^T)$, *or equivalently*, $BB^-AC^-C = A$. *In this case, the general solution of* $BXC = A$ *can be expressed as*

$$X = B^-AC^- + U - B^-BUCC^-,$$

where U *is an arbitrary matrix. If, in particular, the matrix* A *has the form* $A = BJC$, *then the general solution of* $BXC = A$ *can also be expressed in the form*

$$X = JC(BJC)^-BJ + U - B^-BUCC^-.$$

The solution of $BXC = A$ *is unique if and only if* B *has full column rank and* C *has full row rank.*

The following result was shown in [17]; see also [19]:

Lemma 1.4. *The general solution of the homogeneous linear matrix equation*

$$AXB = CYD$$

can be decomposed as

$$X = X_1X_2 + X_3, \qquad Y = Y_1Y_2 + Y_3,$$

where $X_1,\ X_2,\ X_3$ *and* $Y_1,\ Y_2,\ Y_3$ *are, respectively, the general solutions of the following four homogeneous matrix equations:*

$$AX_1 = CY_1, \quad X_2B = Y_2D, \quad AX_3B = 0, \quad CY_3D = 0.$$

By making use of generalized inverses, the general solution of $AXB = CYD$ *can be written as*

$$X = F_{A_1}UE_{B_1} + U_1 - A^-AU_1BB^-, \quad Y = C^-AF_{A_1}UE_{B_1}BD^- + U_2 - C^-CU_2DD^-,$$

or equivalently,

$$X = A^-CF_{C_1}UE_{D_1}DB^- + U_1 - A^-AU_1BB^-, \quad Y = F_{C_1}UE_{D_1} + U_2 - C^-CU_2DD^-,$$

where $A_1 = E_CA$, $B_1 = BF_D$, $C_1 = E_AC$ *and* $D_1 = DF_B$; *the matrices* U, U_1 *and* U_2 *are arbitrary.*

Lemma 1.5. *Let $N \in \mathbb{F}^{n\times m}$, $B \in \mathbb{F}^{m\times k}$ and $C \in \mathbb{F}^{l\times n}$ be given. Then the general solution of the quadratic matrix equation*

$$(BXC)N(BXC) = BXC \tag{1.4}$$

can be expressed in the form

$$X = B^-BU_1(U_2CNBU_1)_r^-U_2CC^- + U - B^-BUCC^-, \tag{1.5}$$

where $U_1 \in \mathbb{F}^{k\times n}$, $U_2 \in \mathbb{F}^{m\times l}$ and $U \in \mathbb{F}^{k\times l}$ are arbitrary.

Proof. It is easy to verify that (1.5) satisfies (1.4). On the other hand, for any solution X_0 of (1.4), let $U_1 = X_0C$, $U_2 = BX_0$ and $U = X_0$ in (1.5). Then (1.5) becomes

$$\begin{aligned} X &= B^-BX_0C(BX_0CNBX_0C)_r^-BX_0CC^- + X_0 - B^-BX_0CC^- \\ &= B^-(BX_0C)(BX_0C)_r^-(BX_0C)C^- + X_0 - B^-BX_0CC^- \\ &= B^-BX_0CC^- + X_0 - B^-BX_0CC^- \\ &= X_0. \end{aligned}$$

This result implies that any solution of (1.4) can be represented through (1.5). Hence, (1.5) is the general solution of (1.4). □

2. The maximal and minimal ranks of $A - BXC$ with respect to X

Suppose that $p(X)$ is given by (1.1). If the corresponding linear matrix equation $BXC = A$ is consistent, then we say that $p(X)$ is a consistent linear matrix expression. In addition to (1.2), we are also able to derive another identity for the rank of $p(X)$ from Lemma 1.1 (a), (b) and (c).

Theorem 2.1. *Let $p(X)$ be given by (1.1). Then:*

(a) *$p(X)$ satisfies the following rank identity:*

$$\begin{aligned} r(A - BXC) = r[A,\ B] + r\begin{bmatrix} A \\ C \end{bmatrix} - r\begin{bmatrix} A & B \\ C & 0 \end{bmatrix} \\ + r(E_{A_2}AF_{A_1} - E_{A_2}BXCF_{A_1}), \end{aligned} \tag{2.1}$$

where $A_1 = E_BA$ and $A_2 = AF_C$.

(b) *The linear matrix expression*

$$\widehat{p}(X) = E_{A_2}AF_{A_1} - E_{A_2}BXCF_{A_1} \tag{2.2}$$

is consistent.

Proof. We first show the following rank equality:

$$r\begin{bmatrix} A & B \\ C & 0 \end{bmatrix} = r\begin{bmatrix} A \\ C \end{bmatrix} + r[A,\ B\,] - r(A) + r(E_{A_2}AF_{A_1}). \tag{2.3}$$

It is easy to see by block Gaussian elimination that

$$r\begin{bmatrix} A & AF_C \\ E_BA & 0 \end{bmatrix} = r\begin{bmatrix} A & 0 \\ 0 & E_BAF_C \end{bmatrix} = r(E_BAF_C) + r(A).$$

Also note by Lemma 1.1 (c) that

$$r\begin{bmatrix} A & AF_C \\ E_BA & 0 \end{bmatrix} = r(E_BA) + r(AF_C) + r(E_{A_2}AF_{A_1}).$$

Hence,

$$r(E_BAF_C) = r(E_BA) + r(AF_C) - r(A) + r(E_{A_2}AF_{A_1}).$$

Substituting this rank equality into Lemma 1.1 (c) and applying Lemma 1.1 (a) and (b) yields (2.3). Replace the matrix A in (2.3) with $p(X) = A - BXC$ and observe that

$$r\begin{bmatrix} A - BXC & B \\ C & 0 \end{bmatrix} = r\begin{bmatrix} A & B \\ C & 0 \end{bmatrix}, \quad r\begin{bmatrix} A - BXC \\ C \end{bmatrix} = r\begin{bmatrix} A \\ C \end{bmatrix},$$

$$r[A - BXC,\ B] = r[A,\ B], \quad E_B(A - BXC) = E_BA, \quad (A - BXC)F_C = AF_C.$$

Then (2.3) becomes

$$r\begin{bmatrix} A & B \\ C & 0 \end{bmatrix} = r\begin{bmatrix} A \\ C \end{bmatrix} + r[A,\ B] - r(A - BXC) + r(E_{A_2}AF_{A_1} - E_{A_2}BXCF_{A_1}),$$

as claimed in (2.1). On the other hand, one can obtain from $E_{A_2}A_2 = 0$ and $A_1F_{A_1} = 0$ that $E_{A_2}AC^-C = E_{A_2}A$ and $BB^-AF_{A_1} = AF_{A_1}$. Thus

$$\mathcal{R}(E_{A_2}AF_{A_1}) = \mathcal{R}(E_{A_2}BB^-AF_{A_1}) \subseteq \mathcal{R}(E_{A_2}B),$$
$$\mathcal{R}[(E_{A_2}AF_{A_1})^T] = \mathcal{R}[(E_{A_2}AC^-CF_{A_1})^T] \subseteq \mathcal{R}[(CF_{A_1})^T].$$

These two equalities imply that the matrix equation $E_{A_2}BXCF_{A_1} = E_{A_2}AF_{A_1}$ is consistent. Thus $\widehat{p}(X)$ is a consistent linear matrix expression. □

For convenience of statement, we call $\widehat{p}(X)$ in (2.2) the adjoint linear matrix expression of $p(X)$ in (1.1). If $p(X)$ is consistent, then $A_1 = 0$ and $A_2 = 0$ in (2.2). In this case, $\widehat{p}(X)$ and $p(X)$ are identical, since $A_1 = 0$ and $A_2 = 0$ in (2.2).

From (1.3), we can also derive another interesting rank identity:

Theorem 2.2. *Let $A \in \mathbb{F}^{m\times n}$, $P \in \mathbb{F}^{p\times m}$ and $Q \in \mathbb{F}^{n\times q}$ be given and let*

$$p(X) = A - F_PXE_Q, \tag{2.4}$$

where $X \in \mathbb{F}^{m\times n}$ is a variant matrix. Then:

(a) *$p(X)$ in (2.4) satisfies the following rank identity:*

$$r[p(X)] = r(PA) + r(AQ) - r(PAQ) + r(E_{AQ}AF_{PA} - E_{AQ}F_PXE_QF_{PA}). \tag{2.5}$$

(b) *The matrix expression $\widehat{p}(X) = E_{AQ}AF_{PA} - E_{AQ}F_PXE_QF_{PA}$ is consistent.*

Proof. Equality (2.5) follows from (1.3) by replacing A with $p(X) = A - F_PXE_Q$ and simplifying. The consistency of the matrix equation $E_{AQ}F_PXE_QF_{PA} = E_{AQ}AF_{PA}$ can be seen from the two simple facts $F_PAF_{PA} = AF_{PA}$ and $E_{AQ}AE_Q = E_{AQ}A$. □

Equality (2.1) implies that the rank of $A - BXC$ is the sum of a nonnegative constant and the rank of a consistent linear matrix expression. Thus, the extremal ranks of $A - BXC$ with respect to X can be determined through this consistent linear matrix expression.

Lemma 2.3. *Suppose that $p(X)$ in (1.1) is consistent. Then:*

(a) *The maximal rank of $p(X)$ with respect to X is*

$$\max_X r(A - BXC) = \min\{r(B),\ r(C)\}. \tag{2.6}$$

The general expression of X satisfying (2.6) can be written in the form

$$X = B^- AC^- - Y, \tag{2.7}$$

where Y is any matrix satisfying $r(BYC) = \min\{r(B),\ r(C)\}$.

(b) *The minimal rank of $p(X)$ with respect to X is*

$$\min_X r(A - BXC) = 0. \tag{2.8}$$

The general expression of X satisfying (2.8) is the general solution of the matrix equation $BXC = A$.

Proof. The consistency of the equation $BXC = A$ implies that $BB^- AC^- C = A$ by Lemma 1.3. Hence, we can rewrite $A - BXC$ as

$$A - BXC = BB^- AC^- C - BXC = B(B^- AC^- - X)C = BYC,$$

where $Y = B^- AC^- - X$. The results in this lemma follow from this expression. □

Applying Lemma 2.3 to the rank identity in (2.1), we obtain the following two results:

Theorem 2.4. *Let $p(X)$ be given by (1.1). Then the maximal rank of $p(X)$ with respect to X is*

$$\max_X r(A - BXC) = \min\left\{ r[A,\ B],\ r\begin{bmatrix} A \\ C \end{bmatrix} \right\}. \tag{2.9}$$

The general expression of X satisfying (2.9) is

$$X = (E_{A_2} B)^- E_{A_2} A F_{A_1} (C F_{A_1})^- - U, \tag{2.10}$$

where the matrix $U \in \mathbb{F}^{k\times l}$ is chosen such that

$$r(E_{A_2} BUCF_{A_1}) = \min\{r(E_{A_2} B),\ r(CF_{A_1})\}.$$

Proof. We see from (2.1) that

$$\max_X r[p(X)] = r[A,\ B] + r\begin{bmatrix} A \\ C \end{bmatrix} - r\begin{bmatrix} A & B \\ C & 0 \end{bmatrix} + \max_X r[\widehat{p}(X)].$$

Since $\widehat{p}(X)$ in (2.2) is consistent, its maximal rank by (2.6) is

$$\max_X r[\widehat{p}(X)] = \min\{\, r(E_{A_2} B),\quad r(CF_{A_1})\,\},$$

where

$$r(E_{A_2} B) = r[A_2,\ B] - r(A_2) = r[AF_C,\ B] - r(AF_C) = r\begin{bmatrix} A & B \\ C & 0 \end{bmatrix} - r\begin{bmatrix} A \\ C \end{bmatrix}$$

and

$$r(CF_{A_1}) = r\begin{bmatrix} A_1 \\ C \end{bmatrix} - r(A_1) = r\begin{bmatrix} E_B A \\ C \end{bmatrix} - r(E_B A) = r\begin{bmatrix} A & B \\ C & 0 \end{bmatrix} - r[A,\ B]$$

by Lemma 1.1 (a), (b) and (c). Thus (2.9) follows. Equality (2.10) is derived from (2.7). □

Theorem 2.5. *Let $p(X)$ be given by (1.1). Then the minimal rank of $p(X)$ with respect to X is*

$$\min_X r(A - BXC) = r[A,\ B] + r\begin{bmatrix} A \\ C \end{bmatrix} - r\begin{bmatrix} A & B \\ C & 0 \end{bmatrix}. \tag{2.11}$$

The matrix X satisfying (2.11) is the general solution of the following consistent matrix equation:

$$E_{A_2}BXCF_{A_1} = E_{A_2}AF_{A_1}, \tag{2.12}$$

where $A_1 = E_BA$ and $A_2 = AF_C$. Through generalized inverses, the general expression of X satisfying (2.11) can be written in the following two forms:

$$\begin{aligned} X &= (E_{A_2}B)^- E_{A_2}AF_{A_1}(CF_{A_1})^- + U \\ &\quad - (E_{A_2}B)^- E_{A_2}BUCF_{A_1}(CF_{A_1})^-, \end{aligned} \tag{2.13}$$

$$\begin{aligned} X &= B^- AF_{A_1}(E_{A_2}AF_{A_1})^- E_{A_2}AC^- + U \\ &\quad - (E_{A_2}B)^- E_{A_2}BUCF_{A_1}(CF_{A_1})^-, \end{aligned} \tag{2.14}$$

where $U \in \mathbb{F}^{k\times l}$ is arbitrary.

Proof. We see from (2.1) that

$$\min_X r[p(X)] = r[A,\ B] + r\begin{bmatrix} A \\ C \end{bmatrix} - r\begin{bmatrix} A & B \\ C & 0 \end{bmatrix} + \min_X r[\widehat{p}(X)].$$

Applying Lemma 2.3 (b) and Lemma 1.3 to the consistent linear matrix expression $\widehat{p}(X) = E_{A_2}AF_{A_1} - E_{A_2}BXCF_{A_1}$ yields the desired results in this theorem. □

From now on, we call any matrix X satisfying (2.11) a minimal rank solution of the matrix equation $BXC = A$. The two general expressions of minimal rank solutions of $BXC = A$ are given in (2.13) and (2.14).

From the rank identity (2.5), we are also able to find the extremal ranks of $p(X)$ in (2.4).

Theorem 2.6. *Let $p(X) = A - F_PXE_Q$ be given by (2.4). Then:*

(a) *The maximal rank of $A - F_PXE_Q$ with respect to X is*

$$\max_X r(A - F_PXE_Q) = \min\left\{m + r(PA) - r(P),\ n + r(AQ) - r(Q)\right\}. \tag{2.15}$$

The general expression of X satisfying (2.15) is

$$X = (E_{AQ}F_P)^- E_{AQ}AF_{PA}(E_QF_{PA})^- - U,$$

where the matrix $U \in \mathbb{F}^{k\times l}$ is chosen such that

$$r(E_{AQ}F_PUE_QF_{PA}) = \min\{r(E_{AQ}F_P),\ r(E_QF_{PA})\}.$$

(b) *The minimal rank of $A - F_PXE_Q$ with respect to X is*

$$\min_X r(A - F_PXE_Q) = r(PA) + r(AQ) - r(PAQ). \tag{2.16}$$

The general expression of X satisfying (2.16) is the general solution of the consistent matrix equation $E_{AQ}F_PXE_QF_{PA} = E_{AQ}AF_{PA}$. Through generalized inverses, the general expression of X satisfying (2.16) can be written in *the following two forms:*

$$X = (E_{AQ}F_P)^-(E_{AQ}AF_{PA})(E_QF_{PA})^- + U - G^-GUHH^-,$$
$$X = AF_{PA}(E_{AQ}AF_{PA})^-E_{AQ}A + U - G^-GUHH^-,$$

where $G = E_{AQ}F_P$ *and* $H = E_QF_{PA}$; *the matrix* U *is arbitrary.*

Proof. Observe from (2.5) that

$$\max_X r[p(X)] = r(PA) + r(AQ) - r(PAQ) + \max_X r(E_{AQ}AF_{PA} - E_{AQ}F_PXE_QF_{PA}),$$

$$\min_X r[p(X)] = r(PA) + r(AQ) - r(PAQ) + \min_X r(E_{AQ}AF_{PA} - E_{AQ}F_PXE_QF_{PA}).$$

Applying Lemma 2.3 to the consistent matrix expression $\widehat{p}(X) = E_{AQ}AF_{PA} - E_{AQ}F_PXE_QF_{PA}$ yields the results in this theorem. $\square$

Several consequences of Theorems 2.4 and 2.5 are given below.

Corollary 2.7. *Let* $p(X)$ *be given by* (1.1). *Then the matrix* X *satisfying* (2.11) *is unique, i.e., the minimal rank solution of the matrix equation* $BXC = A$ *is unique, if and only if* $r(B) = k$, $r(C) = l$ *and*

$$r\begin{bmatrix} A & B \\ C & 0 \end{bmatrix} = r\begin{bmatrix} A \\ C \end{bmatrix} + r(B) = r[A,\ B\,] + r(C). \tag{2.17}$$

In such case, the unique matrix X *satisfying* (2.11) *can be expressed as*

$$X = (E_{A_2}B)^-E_{A_2}AF_{A_1}(CF_{A_1})^- = B^-AF_{A_1}(E_{A_2}AF_{A_1})^-E_{A_2}AC^-, \tag{2.18}$$

where $A_1 = E_BA$ *and* $A_2 = AF_C$. *The uniqueness of* X *satisfying* (2.11) *implies the two matrix expressions on the right-hand side of* (2.18) *are invariant with respect to the choice of* A^-, B^- *and* C^-.

Proof. The matrix X satisfying (2.11) is unique if and only if the solution to (2.12) is unique, which, from Lemma 1.3, is equivalent to

$$r(E_{A_2}B) = k \quad \text{and} \quad r(CF_{A_1}) = l. \tag{2.19}$$

Note that $r(B) \leqslant k$ and $r(C) \leqslant l$. Hence, (2.19) is equivalent to the following four rank equalities:

$$r(B) = k, \qquad r(C) = l, \qquad r(E_{A_2}B) = r(B), \qquad r(CF_{A_1}) = r(C). \tag{2.20}$$

Applying Lemma 1.1 (a), (b) and (c) to the last two rank equalities in (2.20) yields (2.17). The unique matrix X satisfying (2.11) is derived from (2.13) and (2.14). $\square$

Corollary 2.8. *Let* $p(X)$ *be given by* (1.1). *Then the following four statements are equivalent*:

(a) $\min_X r(A - BXC) = r(A)$.

(b) $r\begin{bmatrix} A & B \\ C & 0 \end{bmatrix} = r\begin{bmatrix} A \\ C \end{bmatrix} + r[A,\ B] - r(A)$.

(c) $E_{A_2}AF_{A_1} = 0$, *where* $A_1 = E_BA$ *and* $A_2 = AF_C$.

(d) $E_{C_1}CA^-BF_{B_1} = 0$, *where* $B_1 = E_AB$ *and* $C_1 = CF_A$.

In such cases, the matrix X *minimizing* $r(A - BXC)$ *is the general solution of the homogeneous matrix equation* $E_{A_2}BXCF_{A_1} = 0$.

Proof. It follows immediately from the combination of (2.11), (2.3) and Lemma 1.1 (d). □

Corollary 2.9. *Let $p(X)$ be given by (1.1). Then the rank of $p(X)$ is invariant with respect to the choice of X if and only if*

$$r\begin{bmatrix} A & B \\ C & 0 \end{bmatrix} = r\begin{bmatrix} A \\ C \end{bmatrix} \quad \text{or} \quad r\begin{bmatrix} A & B \\ C & 0 \end{bmatrix} = r[A,\ B\,]. \tag{2.21}$$

Proof. From (2.9) and (2.11),

$$\begin{aligned} &\max_X r[p(X)] - \min_X r[p(X)] \\ &= \min\left\{ r\begin{bmatrix} A & B \\ C & 0 \end{bmatrix} - r\begin{bmatrix} A \\ C \end{bmatrix},\ r\begin{bmatrix} A & B \\ C & 0 \end{bmatrix} - r[A,\ B\,] \right\}. \end{aligned} \tag{2.22}$$

Letting the right-hand side of (2.22) be zero gives the result in this corollary. □

Using (2.9) and (2.11), we are also able to characterize the range invariance of $A - BXC$ with respect to the choice of X. The analogous problems were examined in [3] and [9] for the range invariance of the product AB^-C with respect to the choice of B^-.

Corollary 2.10. *Let $p(X)$ be given by (1.1). Then:*

(a) *The range of $p(X)$ is invariant with respect to the choice of X if and only if $C = 0$ or*

$$\mathcal{R}\begin{bmatrix} B \\ 0 \end{bmatrix} \subseteq \mathcal{R}\begin{bmatrix} A \\ C \end{bmatrix}. \tag{2.23}$$

(b) *The range of $p^T(X)$ is invariant with respect to the choice of X if and only if $B = 0$ or*

$$\mathcal{R}([C,\ 0\,]^T) \subseteq \mathcal{R}([A,\ B]^T). \tag{2.24}$$

Proof. Notice a simple fact that two matrices A_1 and A_2 have the same range, i.e., $\mathcal{R}(A_1) = \mathcal{R}(A_2)$, if and only if $r[A_1,\ A_2] = r(A_1) = r(A_2)$. Applying this result to $A - BXC$, we see that the range of $A - BXC$ is invariant with respect to the choice of X if and only if

$$r[A - BXC,\ A - BYC] = r(A - BXC) = r(A - BYC) \tag{2.25}$$

for any X and Y. Obviously, this rank equality holds for any X and Y if and only if

$$r(A - BXC) = r(A) \tag{2.26}$$

for any X and

$$r[A - BXC,\ A - BYC\,] = r\left([\,A,\ A\,] - B[X,\ Y\,]\begin{bmatrix} C & 0 \\ 0 & C \end{bmatrix} \right) = r(A) \tag{2.27}$$

for any X and Y. From Corollary 2.9, the equality (2.26) holds for any X if and only if (2.21) holds. Also from Corollary 2.9, the equality (2.27) holds for any $[X,\ Y]$ if and only if

$$r\begin{bmatrix} A & B \\ C & 0 \end{bmatrix} = r\begin{bmatrix} A \\ C \end{bmatrix} \quad \text{or} \quad r\begin{bmatrix} A & B \\ C & 0 \end{bmatrix} + r(C) = r[A,\ B\,]. \tag{2.28}$$

Combining (2.21) with (2.28), we see that (2.25) for any X and Y holds if and only if $C = 0$ or

$$r\begin{bmatrix} A & B \\ C & 0 \end{bmatrix} = r\begin{bmatrix} A \\ C \end{bmatrix},$$

which is, in turn, equivalent to (2.23). Similarly, one can show (b) of this corollary. □

Combining the above two corollaries gives the following result:

Corollary 2.11. *Let $p(X)$ be given by (1.1) with $B \neq 0$ and $C \neq 0$. Then the rank of $p(X)$ is invariant with respect to the choice of X if and only if the range of $p(X)$ is invariant with respect to the choice of X or the range of $p^T(X)$ is invariant with respect to the choice of X.*

In the remainder of this section, we present some equivalent statements for the results in Theorems 2.4, 2.5 and 2.6.

Let $B \in \mathbb{F}^{m\times k}$, $C \in \mathbb{F}^{l\times n}$, $P \in \mathbb{F}^{p\times m}$, $Q \in \mathbb{F}^{n\times q}$ and let Θ and Ω be the following two matrix sets:

$$\Theta \overset{\mathrm{d}}{=} \left\{ Z \in \mathbb{F}^{m\times n} \mid \mathcal{R}(Z) \subseteq \mathcal{R}(B) \ \text{ and } \ \mathcal{R}(Z^T) \subseteq \mathcal{R}(C^T) \right\}, \tag{2.29}$$

$$\Omega \overset{\mathrm{d}}{=} \left\{ Z \in \mathbb{F}^{m\times n} \mid \mathcal{R}(Z) \subseteq \mathcal{N}(P) \ \text{ and } \ \mathcal{R}(Z^T) \subseteq \mathcal{N}(Q^T) \right\}. \tag{2.30}$$

Then we have the following results:

Theorem 2.12. *Let $A \in \mathbb{F}^{m\times n}$ and Θ be defined in (2.29). Then:*

(a) *The maximal rank of $A - Z$ subject to $Z \in \Theta$ is*

$$\max_{Z\in\Theta} r(A - Z) = \min\left\{ r[A,\ B],\ r\begin{bmatrix} A \\ C \end{bmatrix} \right\}. \tag{2.31}$$

The general expression of Z satisfying (2.31) can be written in the form

$$Z = B(E_{A_2}B)^- E_{A_2}AF_{A_1}(CF_{A_1})^- C - BUC,$$

where the matrix U is chosen such that

$$r(E_{A_2}BUCF_{A_1}) = \min\left\{ r(E_{A_2}B),\ r(CF_{A_1}) \right\}.$$

(b) *The minimal rank of $A - Z$ subject to $Z \in \Theta$ is*

$$\min_{Z\in\Theta} r(A - Z) = r[A,\ B\,] + r\begin{bmatrix} A \\ C \end{bmatrix} - r\begin{bmatrix} A & B \\ C & 0 \end{bmatrix}. \tag{2.32}$$

The general expression of Z satisfying (2.32) can be written in the following two forms:

$$\begin{aligned} Z = {} & B(E_{A_2}B)^- E_{A_2}AF_{A_1}(CF_{A_1})^- C + V \\ & - B(E_{A_2}B)^- E_{A_2}VF_{A_1}(CF_{A_1})^- C, \end{aligned} \tag{2.33}$$

$$\begin{aligned} Z = {} & AF_{A_1}(E_{A_2}AF_{A_1})^- E_{A_2}A + V \\ & - B(E_{A_2}B)^- E_{A_2}VF_{A_1}(CF_{A_1})^- C, \end{aligned} \tag{2.34}$$

where $A_1 = E_BA$ and $A_2 = AF_C$; the matrix $V \in \Theta$ is arbitrary.

Proof. From (2.29) and Lemma 1.3, we easily see that any $Z \in \Theta$ can be expressed as $Z = BXC$. Hence, the matrix set Θ in (2.29) can equivalently be rewritten as

$$\Theta = \left\{ Z = BXC \mid X \in \mathbb{F}^{k\times l} \right\}.$$

Thus we see from (2.1) that the rank of $A - Z$ with $Z \in \Theta$ satisfies the following identity:

$$\begin{aligned} r(A - Z) &= r(A - BXC) \\ &= r[A,\ B] + r\begin{bmatrix} A \\ C \end{bmatrix} - r\begin{bmatrix} A & B \\ C & 0 \end{bmatrix} + r(E_{A_2}AF_{A_1} - E_{A_2}BXCF_{A_1}) \\ &= r[A,\ B] + r\begin{bmatrix} A \\ C \end{bmatrix} - r\begin{bmatrix} A & B \\ C & 0 \end{bmatrix} + r(E_{A_2}AF_{A_1} - E_{A_2}ZF_{A_1}). \end{aligned}$$

Applying Theorems 2.4 and 2.5 to this equality yields the results in this theorem. □

The matrix Z satisfying (2.32) is called a shorted matrix of A relative to Θ in the literature. Equations (2.33) and (2.34) are two general expressions of shorted matrices of A relative to Θ. We can also derive from (2.33) and (2.34) a necessary and sufficient condition for the uniqueness of the matrix $Z \in \Theta$ satisfying (2.32). This problem was studied by several authors (see, e.g., [1, 8, 14, 15]).

Corollary 2.13 ([14]). *Let $A \in \mathbb{F}^{m\times n}$ and let Θ be defined in (2.29). Then shorted matrix of A relative to Θ is unique if and only if*

$$r\begin{bmatrix} A & B \\ C & 0 \end{bmatrix} = r\begin{bmatrix} A \\ C \end{bmatrix} + r(B) = r[A,\ B] + r(C).$$

In this case, the unique shorted matrix can be written in the following three forms:

$$\begin{aligned} Z &= B(E_{A_2}B)^{-}E_{A_2}AF_{A_1}(CF_{A_1})^{-}C \\ &= AF_{A_1}(E_{A_2}AF_{A_1})^{-}E_{A_2}A \\ &= A - A(E_BAF_C)^{-}A, \end{aligned}$$

where $A_1 = E_BA$ and $A_2 = AF_C$. These matrix expressions are invariant with respect to the choice of the generalized inverses in them.

Theorem 2.14. *Let $A \in \mathbb{F}^{m\times n}$ and let Ω be defined in (2.30). Then:*

(a) *The maximal rank of $A - Z$ subject to $Z \in \Omega$ is*

$$\max_{Z\in\Omega} r(A - Z) = \min\left\{r(PA) + r(F_P),\ r(AQ) + r(E_Q)\right\}. \tag{2.35}$$

The general expression of Z satisfying (2.35) can be written as

$$Z = F_P(E_{AQ}F_P)^{-}E_{AQ}AF_{PA}(E_QF_{PA})^{-}E_Q - F_PUE_Q,$$

where the matrix U is chosen such that

$$r(E_{AQ}F_PUE_QF_{PA}) = \min\left\{r(E_{AQ}F_P),\ r(E_QF_{PA})\right\}.$$

(b) *The minimal rank of $A - Z$ subject to $Z \in \Omega$ is*

$$\min_{Z\in\Omega} r(A - Z) = r(PA) + r(AQ) - r(PAQ). \tag{2.36}$$

The general expression of the matrix Z satisfying (2.36) can be written in the following two forms:

$$\begin{aligned} Z &= F_P(E_{AQ}F_P)^- E_{AQ}AF_{PA}(E_QF_{PA})^- E_Q + V \\ &\quad - F_P(E_{AQ}F_P)^- E_{AQ}VF_{PA}(E_QF_{PA})^- E_Q, \\ Z &= AF_{PA}(E_{AQ}AF_{PA})^- E_{AQ}A + V \\ &\quad - F_P(E_{AQ}F_P)^- E_{AQ}VF_{PA}(E_QF_{PA})^- E_Q, \end{aligned}$$

where $V \in \Omega$ is arbitrary.

Proof. From Lemma 1.3, Ω in (2.30) can also be written as

$$\Omega = \left\{ Z = F_PXE_Q \mid X \in \mathbb{F}^{m\times n} \right\}.$$

Hence, we see from (2.5) that the rank of $A - Z$ satisfies the following identity:

$$\begin{aligned} r(A - Z) &= r(A - F_PXE_Q) \\ &= r(PA) + r(AQ) - r(PAQ) + r(E_{AQ}AF_{PA} - E_{AQ}F_PXE_QF_{PA}) \\ &= r(PA) + r(AQ) - r(PAQ) + r(E_{AQ}AF_{PA} - E_{AQ}ZF_{PA}). \end{aligned}$$

Applying Theorem 2.6 to this equality yields the results in this theorem. □

Theorem 2.15 ([8])**.** *Let $A \in \mathbb{F}^{m\times n}$ and Ω be defined in (2.30). Then shorted matrix of A relative to Ω is unique if and only if*

$$r(PAQ) = r(PA) = r(AQ).$$

In this case, the unique shorted matrix can be expressed in the following three forms:

$$\begin{aligned} Z &= F_P(E_{AQ}F_P)^- E_{AQ}AF_{PA}(E_QF_{PA})^- E_Q \\ &= AF_{PA}(E_{AQ}AF_{PA})^- E_{AQ}A \\ &= A - AQ(PAQ)^- PA, \end{aligned}$$

These expressions are invariant with respect to the choice of the generalized inverses in them.

3. Solutions to the equation rank($A - BXC$) + rank(BXC) = rank(A)

Let $p(X) = A - BXC$ be given by (1.1). In this section, we solve the following rank equation induced by $p(X)$:

$$r(A - BXC) + r(BXC) = r(A), \tag{3.1}$$

and then consider the minimal rank of $p(X)$ subject to (3.1) and some related topics.

To solve the rank equation (3.1), we need the following result due to Marsaglia and Styan [12]:

Lemma 3.1. *Two matrices $A,\ S \in \mathbb{F}^{m\times n}$ satisfy the following rank equality:*

$$r(A - S) + r(S) = r(A)$$

if and only if

$$\mathcal{R}(S) \subseteq \mathcal{R}(A), \quad \mathcal{R}(S^T) \subseteq \mathcal{R}(A^T) \quad \text{and} \quad \{A^-\} \subseteq \{S^-\}.$$

Applying Lemma 3.1 to (3.1) gives the following result:

Lemma 3.2. *The rank equation (3.1) and the system of matrix equations*

$$(BXC)A^-(BXC) = BXC, \qquad BXC = AYA \tag{3.2}$$

have the same solution for X, where $A^- \in \{A^-\}$ is arbitrary.

Proof. From Lemma 3.1, Equation (3.1) is equivalent to

$$\mathcal{R}(BXC) \subseteq \mathcal{R}(A), \quad \mathcal{R}[(BXC)^T] \subseteq \mathcal{R}(A^T) \ \text{ and } \ \{A^-\} \subseteq \{(BXC)^-\}. \tag{3.3}$$

From Lemma 1.3, the first two range inclusions in (3.3) hold if and only if the matrix equation $BXC = AYA$ is solvable for Y. By definition of generalized inverse, the third set inclusion in (3.3) holds if and only if $(BXC)A^-(BXC) = BXC$ for any $A^- \in \{A^-\}$. Thus we have (3.2). □

Theorem 3.3. *The general solution of the rank equation (3.1) can be expressed in the following two forms:*

$$X = B^-AF_{A_1}U_1(U_2E_{A_2}AF_{A_1}U_1)_r^-U_2E_{A_2}AC^- + U - B^-BUCC^-, \tag{3.4}$$

$$X = B^-BF_{B_1}U_3(U_4E_{C_1}CA^-BF_{B_1}U_3)_r^-U_4E_{C_1}CC^- + U - B^-BUCC^-, \tag{3.5}$$

where $A_1 = E_BA$, $A_2 = AF_C$, $B_1 = E_AB$ and $C_1 = CF_A$; the matrices $U \in \mathbb{F}^{k\times l}$, $U_1 \in \mathbb{F}^{n\times n}$, $U_2 \in \mathbb{F}^{m\times m}$, $U_3 \in \mathbb{F}^{k\times n}$ and $U_4 \in \mathbb{F}^{m\times l}$ are arbitrary.

Proof. From Lemma 1.4, the general solution X of $BXC = AYA$ in (3.2) can be expressed in the following two forms:

$$X = B^-AF_{A_1}V_1E_{A_2}AC^- + U - B^-BUCC^-, \tag{3.6}$$

$$X = F_{B_1}V_2E_{C_1} + U - B^-BUCC^-, \tag{3.7}$$

where $U \in \mathbb{F}^{k\times l}$, $V_1 \in \mathbb{F}^{n\times m}$ and $V_2 \in \mathbb{F}^{k\times l}$ are arbitrary. Substituting (3.6) into the first equation in (3.2) and observing that $BB^-AF_{A_1} = AF_{A_1}$ and $E_{A_2}AC^-C = E_{A_2}A$ gives

$$(AF_{A_1}V_1E_{A_2}A)A^-(AF_{A_1}V_1E_{A_2}A) = AF_{A_1}V_1E_{A_2}A.$$

From Lemma 1.5, the general solution of this equation is

$$V_1 = (AF_{A_1})^-(AF_{A_1})U_1(U_2E_{A_2}AF_{A_1}U_1)_r^-U_2(E_{A_2}A)(E_{A_2}A)^- + Z, \tag{3.8}$$

where $U_1 \in \mathbb{F}^{n\times n}$ and $U_2 \in \mathbb{F}^{m\times m}$ are arbitrary, Z is the general solution of the matrix equation $AF_{A_1}ZE_{A_2}A = 0$. Substituting (3.8) into (3.6) yields (3.4) for the general solution X to (3.1). Similarly, one can obtain (3.5) from (3.7). □

Some special cases of Theorem 3.3 are listed below.

Corollary 3.4. *The rank equation (3.1) and the matrix equation $BXC = 0$ have the same solution if and only if*

$$r\begin{bmatrix} A & B \\ C & 0 \end{bmatrix} = r\begin{bmatrix} A \\ C \end{bmatrix} + r[A,\ B] - r(A). \tag{3.9}$$

Proof. Any solution of the equation $BXC = 0$ also satisfies the equation (3.1). Conversely, substituting the general solution (3.4) of (3.1) into BXC gives

$$BXC = AF_{A_1}U_1(U_2E_{A_2}AF_{A_1}U_1)_r^-U_2E_{A_2}A.$$

Hence, both (3.1) and $BXC = 0$ have the same solution if and only if $E_{A_2}AF_{A_1} = 0$, which is equivalent to (3.9) by Corollary 2.8. □

Corollary 3.5. *The rank equation* (3.1) *only has null solution if and only if B has full column rank, C has full row rank and* (3.9) *holds.*

Proof. It follows from the general solution (3.4) of (3.1). □

Corollary 3.6. *Suppose that the matrix equation $BXC = A$ is consistent. Then the general solution of rank equation* (3.1) *is*

$$X = B^- AU_1(U_2A^-U_1)_r^- U_2AC^- + U - B^-BUCC^-, \tag{3.10}$$

where U, U_1 and U_2 are as in (3.4).

Proof. Note that the consistency of $BXC = A$ is equivalent to $A_1 = E_BA = 0$ and $A_2 = AF_C = 0$. Hence, (3.4) is reduced to (3.10). □

Corollary 3.7. *If the matrices A, B and C satisfy*

$$\mathcal{R}(B) \subseteq \mathcal{R}(A) \quad \textit{and} \quad \mathcal{R}(C^T) \subseteq \mathcal{R}(A^T), \tag{3.11}$$

then the general solution of (3.1) *is*

$$X = B^-BU_1(U_2CA^-BU_1)_r^- U_2CC^- + U - B^-BUCC^-, \tag{3.12}$$

where $U_1 \in \mathbb{F}^{k\times n}$, $U_2 \in \mathbb{F}^{m\times l}$ and $U \in \mathbb{F}^{k\times l}$ are arbitrary.

Proof. Clearly, (3.11) is equivalent to $B_1 = E_AB = 0$ and $C_1 = CF_A = 0$. Hence, (3.5) is reduced to (3.12). □

In the remainder of this section, we consider the minimal rank of $A - BXC$ when X satisfies the rank equation (3.1). For this purpose, let

$$\Delta \overset{\mathrm{d}}{=} \left\{ X \in \mathbb{F}^{k\times l} \mid r(A - BXC) + r(BXC) = r(A) \right\}. \tag{3.13}$$

Obviously, Δ is a nonempty set.

Theorem 3.8. *Let Δ be defined in* (3.13). *Then*

$$\max_{X\in\Delta} r(BXC) = r(A) - r[A,\ B] - r\begin{bmatrix} A \\ C \end{bmatrix} + r\begin{bmatrix} A & B \\ C & 0 \end{bmatrix}, \tag{3.14}$$

$$\min_{X\in\Delta} r(A - BXC) = r[A,\ B\,] + r\begin{bmatrix} A \\ C \end{bmatrix} + r\begin{bmatrix} A & B \\ C & 0 \end{bmatrix}. \tag{3.15}$$

The matrix X satisfying (3.14) *and* (3.15) *is given by* (3.4) *and* (3.5), *where the matrices $U_1, \ldots, U_4$ in them are choosen such that*

$$r(U_2E_{A_2}AF_{A_1}U_1) = r(E_{A_2}AF_{A_1}),$$
$$r(U_4E_{C_1}CA^-BF_{B_1}U_3) = r(E_{C_1}CA^-BF_{B_1}).$$

Proof. Substituting the two general solutions (3.4) and (3.5) of (3.1) into BXC yields

$$BXC = AF_{A_1}U_1(U_2E_{A_2}AF_{A_1}U_1)_r^- U_2E_{A_2}A,$$
$$BXC = BF_{B_1}U_3(U_4E_{C_1}CA^-BF_{B_1}U_3)_r^- U_4E_{C_1}C,$$

respectively. In such cases, the rank of BXC is

$$r(BXC) = r(U_2E_{A_2}AF_{A_1}U_1) \quad \text{or} \quad r(BXC) = r(U_4E_{C_1}CA^-BF_{B_1}U_3).$$

Thus,

$$\max_{X\in\Delta} r(BXC) = r(E_{A_2}AF_{A_1}) = r(E_{C_1}CA^-BF_{B_1}).$$

Combining this result with (2.3) and Lemma 1.1 (d) yields (3.14). Again from (3.13) we see that

$$\min_{X\in\Delta} r(A - BXC) = r(A) - \max_{X\in\Delta} r(BXC). \tag{3.16}$$

Hence, (3.15) follows from (3.16) and (3.14). $\square$

The results in Theorems 2.5 and 3.8 show that

$$\min_{X\in\Delta} r(A - BXC) = \min_{X} r(A - BXC).$$

Theorem 3.3 can be restated as follows:

Corollary 3.9. *Let Θ be defined in (2.29). Then the general solution of the rank equation*

$$r(A - Z) + r(Z) = r(A) \quad \textit{subject to} \quad Z \in \Theta$$

can be expressed in the two forms

$$Z = AF_{A_1}U_1(U_2E_{A_2}AF_{A_1}U_1)_r^- U_2E_{A_2}A,$$
$$Z = BF_{B_1}U_3(U_4E_{C_1}CA^- BF_{B_1}U_3)_r^- U_4E_{C_1}C,$$

where $U_1, \ldots, U_4$ are as in (3.4) and (3.5).

Proof. It follows from Theorem 3.3. $\square$

Let

$$\Theta_A \overset{\mathrm{d}}{=} \left\{ Z \in \mathbb{F}^{m\times n} \mid r(A - Z) + r(Z) = r(A) \ \text{ and } \ Z \in \Theta \right\}, \tag{3.17}$$

where Θ is defined in (2.29). Obviously, $\Theta_A \subseteq \Theta$.

Theorem 3.10. *Let Θ_A be defined in (3.17). Then*

$$\min_{Z\in\Theta_A} r(A - Z) = r[A,\ B\,] + r\begin{bmatrix} A \\ C \end{bmatrix} + r\begin{bmatrix} A & B \\ C & 0 \end{bmatrix}. \tag{3.18}$$

The general expression of Z satisfying (3.18) can be expressed in the following two forms:

$$Z = AF_{A_1}U_1(U_2E_{A_2}AF_{A_1}U_1)_r^- U_2E_{A_2}A \tag{3.19}$$

and

$$Z = BF_{B_1}U_3(U_4E_{C_1}CA^- BF_{B_1}U_3)_r^- U_4E_{C_1}C, \tag{3.20}$$

where $U_1, \ldots, U_4$ satisfy

$$r(U_2E_{A_2}AF_{A_1}U_1) = r(E_{A_2}AF_{A_1}), \quad r(U_4E_{C_1}CA^- BF_{B_1}U_3) = r(E_{C_1}CA^- BF_{B_1}).$$

Proof. It follows from Theorem 3.8. $\square$

Any matrix $Z \in \Theta_A$ satisfying (3.18) is also a shorted matrix of A relative to Θ_A, and is called a shorted matrix relative to Θ_A. Two general expressions of shorted matrices of A relative to Θ_A are given in (3.19) and (3.20). The conclusion on the uniqueness of shorted matrix of A relative to Θ_A is the same as that in Corollary 2.13.

As a special case of (3.1), one now is able to solve the following rank equation:

$$r\begin{bmatrix} A & B \\ C & D \end{bmatrix} = r\begin{bmatrix} A & B \\ C & D - X \end{bmatrix} + r(X)$$

for block matrices. This equation can also be rewritten as the following standard form:

$$r\begin{bmatrix} A & B \\ C & D \end{bmatrix} = r\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix} - \begin{bmatrix} 0 \\ I_l \end{bmatrix} X[0,\ I_k]\right) + r\left(\begin{bmatrix} 0 \\ I_l \end{bmatrix} X[0,\ I_k]\right).$$

4. Summary

In this paper, we show a new identity for the rank of the linear matrix $A - BXC$ and derive from the identity the maximal and minimal ranks of $A - BXC$ with respect to X. Many consequences and applications of these results are also presented. These results are so elementary that any people with some knowledge of linear algebra can understand. Besides the work in the previous sections, the first author of this paper also gives the closed forms for the extremal ranks of the matrix expressions $A - BX - YC$, $A - B_1X_1C_1 - B_2X_2C_2$, $A - (A_1 - B_1X_1C_1)D(A_2 - B_2X_2C_2)$, etc., where X, Y, X_1 and X_2 are variant matrices; see [18, 20, 21, 22]. Besides these matrix expressions, one can also consider extremal ranks of matrix expressions that involve generalized inverses. One of the simplest matrix expressions that involve a generalized inverse is the Schur complement $D - CA^-B$. This expression is well-known in matrix analysis and applications; see, e.g., [2, 4, 7]. If A^- is not unique, the rank of $D - CA^-B$ is variant with respect to the choice of A^-. It is easy to derive the extremal ranks of $D - CA^-B$ with respect to A^- from the extremal ranks of $A - B_1X_1C_1 - B_2X_2C_2$. The corresponding results are given in [22]. It should be pointed out that it is an interesting and fruitful research topic to determine extremal ranks of matrix expressions that involve generalized inverses. As an example, we present a simple formula for the minimal rank of a block matrix consisting of three generalized inverses

$$\min_{A^-,\, B^-,\, C^-} r\begin{bmatrix} A^- & C^- \\ B^- & 0 \end{bmatrix} = \max\{r(A),\quad r(B) + r(C)\}.$$

This formula is proposed as a problem in [24]. Its solutions will appear in Bull. International Linear Algebra Society, **31**.

Finally, we mention two open problems related to the results in this paper: it is well-known that the least squares solution of the matrix equation $BXC = A$ over the field of complex numbers is $X = B^\dagger AC^\dagger + (I_k - B^\dagger B)V_1 + V_2(I_l - CC^\dagger)$, where $(\cdot)^\dagger$ denotes the Moore-Penrose inverse of a matrix; V_1 and V_2 are arbitrary. In such case, $\min_X ||\, A - BXC\, ||_F = ||\, A - BB^\dagger AC^\dagger C\, ||_F$, where $||\cdot||_F$ denotes the Frobenius norm of a matrix. On the other hand, minimal rank solutions to $BXC = A$ are given in (2.13) and (2.14). Then:

(a) What is the relationship between least squares solutions and minimal rank solutions to the matrix equation $BXC = A$?
(b) What is $\min_X ||\, A - BXC\, ||_F$ subject to $r(A - BXC) = \min$?

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