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Closed Legendre geodesics in Sasaki manifolds

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Abstract. If $L \subset M$ is a Legendre submanifold in a Sasaki manifold, then the mean curvature flow does not preserve the Legendre condition. We define a kind of mean curvature flow for Legendre submanifolds which slightly differs from the standard one and then we prove that closed Legendre curves L in a Sasaki space form M converge to closed Legendre geodesics, if $k^2 + \sigma + 3 \leq 0$ and $\mathrm{rot}(L) = 0$, where σ denotes the sectional curvature of the contact plane ξ and k and $\mathrm{rot}(L)$ are the curvature respectively the rotation number of L. If $\mathrm{rot}(L) \neq 0$, we obtain convergence of a subsequence to Legendre curves with constant curvature. In case $\sigma + 3 \leq 0$ and if the Legendre angle α of the initial curve satisfies $\mathrm{osc}\,(\alpha) \leq \pi$, then we also prove convergence to a closed Legendre geodesic.

Contents

1. Introduction

Let (M, g) be a Riemannian manifold and $F_t : L \to M$ a smooth family of immersions such that

$$\frac{d}{dt}F_t = \overrightarrow{H}, \tag{1.1}$$

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where $\overrightarrow{H}$ is the mean curvature vector along $L_t := F_t(L)$. This equation is called mean curvature flow and it is the negative gradient flow of the volume functional of L_t. Hence the flow decreases the volume energy as fast as possible and stationary solutions are minimal submanifolds. There is a vast amount of literature on this equation which belongs to the most important equations in Geometric Analysis. For a detailed account of what is known, the reader is recommended to look at the survey article [14] where one can also find more references. If L is 1-dimensional, then (1.1) is called the curve shortening flow. Most of the known results have been obtained for hypersurfaces and in higher codimension only a few things have been done [2], [3], [4], [5], [8], [19], [20], [24], [25], [26]. One example in higher codimension is the Lagrangian mean curvature flow, in particular the Lagrangian condition is preserved if (M, g) is Kähler-Einstein [18]. Legendre and Lagrange submanifolds are closely related because any Legendre submanifold in a contact manifold M generates a Lagrangian submanifold in the symplectization of M, e.g., the Legendre submanifolds of S^{2n+1} (equipped with its standard contact structure) are precisely the intersections of S^{2n+1} with Lagrangian cones in $\mathbb{R}^{2n+2}$. In contrast to the situation for Lagrangian submanifolds, the mean curvature flow does not preserve the Legendre condition (see Section 3 for details). On the other hand one would like to minimize the volume energy in the class of Legendre immersions. The aim of this article is to establish such a flow for Legendre submanifolds. We will see that the flow preserves the Legendre condition, if the Sasaki manifold is pseudo-Einstein (see Definition 2.6). Then we apply this flow to deform closed Legendre curves into closed Legendre geodesics or more generally into Legendre curves of constant curvature, i.e., one of the main theorems states:

Theorem 1.1. *Let $L \subset (M, \xi, g, J)$ be a closed Legendre curve in a compact Sasaki manifold M with constant sectional curvature σ on the hyperplane distribution ξ. Suppose the curvature k of L satisfies*

$$k^2 + \sigma + 3 \leq 0. \tag{1.2}$$

Then the Legendrian curve shortening flow (3.11) admits a smooth solution for $t \in [0, \infty)$. If $\operatorname{rot}(L) = 0$, then the curves converge in the C^∞-topology to a closed Legendre geodesic and if $\operatorname{rot}(L) \neq 0$, then a subsequence of the flow converges in the C^∞-topology to a closed Legendre curve of constant nonvanishing curvature.

The rotation number of a Legendre curve vanishes if and only if the (mean) curvature form H (see Definition 2.5) is exact, i.e., if there exists a globally defined Legendre angle α with $d\alpha = H$. In particular the rotation number of a geodesic vanishes and the Legendre angle is constant. In case $\sigma + 3 \leq 0$ we will prove

Theorem 1.2. *Let $L \subset (M, \xi, g, J)$ be a closed Legendre curve in a compact Sasaki manifold M with constant sectional curvature $\sigma \leq -3$ on the hyperplane distribution ξ. Suppose the rotation number of L vanishes and the Legendre angle α satisfies*

$$\operatorname{osc}(\alpha) \leq \pi. \tag{1.3}$$

Then the Legendrian curve shortening flow (3.11) admits a smooth solution for $t \in [0, \infty)$ and the curves converge in the C^∞-topology to a closed Legendre geodesic.

Similar theorems for the curve shortening flow of curves on surfaces have been obtained earlier [10], [17] (see also [11] for more references).

In [9] the authors provide the classification of topologically trivial Legendrian knots in tight contact 3-manifolds. They prove that for two topologically trivial Legendrian knots, if their invariants tb, rot (Thurston-Bennequin invariant and rotation number) are equal, then these knots are Legendrian isotopic. This together with Theorem 1.2 implies the following: If a tight Sasaki manifold with $\sigma \leq -3$ admits a Legendre knot L with $\mathrm{rot}(L) = 0$ and $\mathrm{osc}\,(\alpha) \leq \pi$, then any Legendrian knot with the same rotation number and Thurston-Bennequin invariant is isotopic to a closed Legendrian geodesic.

In [15] a different and very natural volume decreasing flow of Legendrian immersions is introduced that can be compared with the Willmore flow. This flow is of fourth order whereas the flow defined here is a second order equation and stems from the projection of the L^2-gradient of the volume energy (the mean curvature vector) onto the tangent space of the space of Legendrian immersions.

In this article we will discuss general Legendrian isotopies as well. As a result we obtain the next theorem:

Theorem 1.3. *Let L_0 be a compact, oriented Legendrian immersion into a Sasaki pseudo-Einstein manifold (M, ξ, g, J) with*

$$\mathrm{Ric}(V, W) = Kg(V, W), \quad \forall\, V, W \in \xi.$$

Assume that the mean curvature form $H = d\alpha$ is exact, where α is the Legendre angle. Then we have

a) *If $K = -2$ and $\int_{L_0} \cos(\alpha) d\mu > 0$, then there exists a constant $c > 0$ depending only on $\int_{L_0} \cos(\alpha) d\mu$ such that*

$$\mathrm{Vol}\,(L_1) \geq c > 0$$

for any Legendrian immersion L_1 Legendrian isotopic to L_0.

b) *If $K < -2$ and α satisfies $\mathrm{osc}\,(\alpha) \leq \pi$, then the same result as in a) holds with a constant c depending only on $\mathrm{osc}\,(\alpha)$ and $\mathrm{Vol}\,(L_0)$ provided L_0, L_1 are isotopic by the Legendrian mean curvature flow.*

The organization of this article is as follows: Section 2 is seperated into 3 subsections. In the first subsection we explain our terminology and recall the fundamental material needed in contact geometry, the second subsection explains associated metrics, almost complex structures and Sasaki manifolds. Legendre submanifolds are discussed in Section 2.3. In Section 3 we investigate variations of Legendrian submanifolds, define the Legendrian mean curvature flow and prove Theorem 1.3. Our focus in Section 4 is the Legendrian curve shortening flow and the proof of Theorems 1.1, 1.2.

2. Basic material

2.1. Contact manifolds. A contact manifold (of restricted type)[1)] (M, λ) is an odd-dimensional manifold of dimension $2n+1$ together with a one-form λ such that

[1)] More generally a contact manifold M is a differentiable manifold of odd dimension $2n + 1$ with a completely nonintegrable distribution ξ of hyperplanes in the tangent space. Locally such hyperplane fields can be described as the kernel of a nonvanishing one-form λ. The nonintegrability then implies that $\lambda \wedge (d\lambda)^n$ locally defines a volume form. If this one-form λ exists globally then we speak of a contact manifold of restricted type. In this paper we will only consider contact manifolds of restricted type.

$\lambda \wedge (d\lambda)^n$ defines a volume form on M. One observes that a contact manifold is orientable and that the contact form λ defines a natural orientation.

Assume now that (M, λ) is a given contact manifold of dimension $2n+1$. λ defines a $2n$-dimensional vector bundle ξ over M, where at each point $p \in M$ the fiber ξ_p of ξ is given by

$$\xi_p = \ker\lambda_p.$$

Moreover, since $\lambda \wedge (d\lambda)^n$ is a volume form, we see that

$$\omega := d\lambda$$

is a closed nondegenerate 2-form on $\xi \oplus \xi$ and hence defines a symplectic product on ξ so that $(\xi, \omega_{|\xi\oplus\xi})$ becomes a symplectic vector bundle. Since the dimension of M is odd, the 2-form $\omega = d\lambda$ must be degenerate on TM. Therefore one obtains a line bundle l over M via the definition

$$l_p := \{V \in T_pM \mid \omega(V, W) = 0 \ \forall \ W \in \xi_p\}.$$

The Reeb vector field (sometimes called characteristic vector field) X_λ is given by the natural section X_λ in l defined by

$$\lambda(X_\lambda) = 1\ ,\ X_\lambda \lrcorner d\lambda = 0. \tag{2.1}$$

Thus a contact form λ on an odd-dimensional manifold M of dimension $2n+1$ defines a splitting of the tangent bundle TM into a line bundle l with canonical section X_λ and a symplectic vector bundle $(\xi, \omega_{|\xi\oplus\xi})$:

$$TM = (l, X_\lambda) \oplus (\xi, \omega_{|\xi\oplus\xi}).$$

We denote the projection of TM along l by π, i.e.,

$$\begin{aligned} \pi \quad &: \quad TM \to \xi, \\ \pi(V) \quad &:= \quad V - \lambda(V)X_\lambda. \end{aligned}$$

A submanifold L of a $(2n+1)$-dimensional contact manifold (M, λ) is called isotropic if it is tangent to ξ, i.e., if $\lambda_{|TL} = 0$. This implies that $d\lambda_{|TL} = \omega_{|TL} = 0$ also. An isotropic submanifold L of maximal dimension n is called Legendrian.

The following example shows that there exist closed Legendre curves:

Example 2.1. Consider $M = \mathbb{R}^3$ with its standard contact form

$$\lambda = dz - xdy.$$

Since $d\lambda = -dx \wedge dy$ we observe

$$X_\lambda = \frac{\partial}{\partial z}$$

and $\xi_x = \ker\lambda_{|x}$ is given by

$$\xi_x = \left[\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ x \end{pmatrix} \right].$$

Suppose $a, b \in \mathbb{Z}$ with $a \neq b$. For any $c, d \in \mathbb{R}$ we define the curve

$$\gamma_{\{a,b,c,d\}}(\phi) := \begin{pmatrix} c\cos(a\phi) \\ d\sin(b\phi) \\ \frac{bcd}{2}\left(\frac{\sin((a-b)\phi)}{a-b} + \frac{\sin((a+b)\phi)}{a+b}\right) \end{pmatrix}.$$

If a, b are chosen such that there do not exist two constants $k, l \in \mathbb{Z}$ with $2bk = (2l+1)a$, then $\gamma_{\{a,b,c,d\}}$ is a regular Legendre curve.

Proof. A curve γ is Legendre iff $\lambda(\gamma') = 0$. Here, this is the case if and only if $\gamma'_z - \gamma_x \gamma'_y = 0$ which is true. γ is regular if $\gamma' \neq 0, \forall \phi$. γ' can only vanish somewhere, if there exist constants $k, l \in \mathbb{Z}$ with $2bk = (2l+1)a$. $\square$

Figure 1 is $\gamma_{\{5,2,2,3\}}$. Figure 2 depicts the same curve projected onto the three coordinate planes.

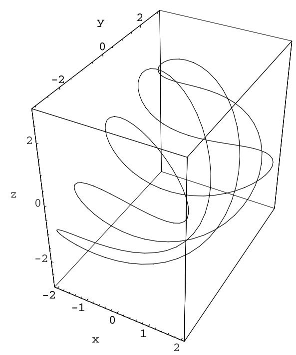

FIGURE 1. The curve $\gamma_{\{5,2,2,3\}}$

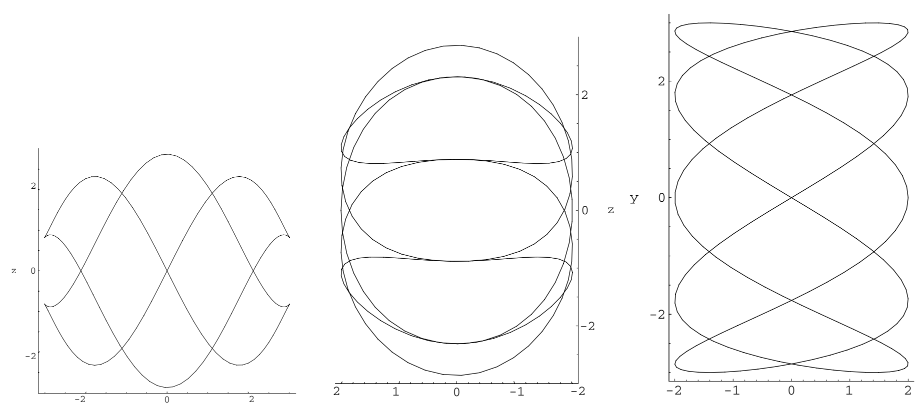

FIGURE 2. The projections of $\gamma_{\{5,2,2,3\}}$

2.2. Associated metrics, complex structures, etc. A Riemannian metric $g = g_{\alpha\beta}dy^\alpha \otimes dy^\beta \in \Gamma(T^*M \otimes T^*M)$ on a contact manifold (M, λ) is said to be associated, if

$$g^{\alpha\beta}\lambda_\beta = X_\lambda^\alpha, \tag{2.2}$$

i.e.,

$$g(X_\lambda, V) = \lambda(V), \ \forall\ V \in TM.$$

In the sequel we will always assume that a given contact manifold (M, λ) is equipped with an associated Riemannian metric and we will write

$$\lambda^\alpha = X_\lambda^\alpha.$$

If (M, λ, g) is a contact manifold with associated Riemannian metric, then

$$g(X_\lambda, X_\lambda) = 1, \tag{2.3}$$

and

$$g(X_\lambda, V) = 0, \ \forall\ V \in \xi. \tag{2.4}$$

If (M, λ) is a contact manifold and $\widetilde{J} \in \Gamma(\xi^* \otimes \xi)$ an almost complex structure on the symplectic subbundle ξ, then one can extend $\widetilde{J}$ to a section $J \in \Gamma(T^*M \otimes TM)$ by setting

$$J(V) := \widetilde{J}(\pi(V)),$$

where π is the projection from above. Since $\widetilde{J}^2(V) = -V, \ \forall\ V \in \xi$ we obtain

$$J^2 = -\pi, \ J_\alpha^\beta J_\gamma^\alpha = -\pi_\gamma^\beta. \tag{2.5}$$

From the definition of J it also follows

$$\ker J = l. \tag{2.6}$$

We introduce the bilinear form $\mathbf{L}$ by

$$\mathbf{L}(V, W) := \omega(V, JW) = d\lambda(V, JW). \tag{2.7}$$

J or $\widetilde{J}$ is said to be associated to ω, if $\mathbf{L}$ is symmetric and positive definite, so that by definition of X_λ the tensor

$$g := \mathbf{L} + \lambda \otimes \lambda$$

is an associated Riemannian metric on (M, λ). Thus, in this case

$$g_{\alpha\beta} = \lambda_\alpha\lambda_\beta + \omega_{\alpha\gamma}J_\beta^\gamma. \tag{2.8}$$

The torsion T of J is defined as

$$T(J) := N(J) + 2\omega \otimes X_\lambda,$$

where $N(J)$ denotes the Nijenhuis tensor of J, i.e.,

$$N(J)(X, Y) := J^2[X, Y] + [JX, JY] - J[X, JY] - J[JX, Y]$$

and J is called integrable, if $T(J) = 0$. A contact manifold (M, λ, J) with an integrable, associated complex structure J is called Sasaki. It turns out that the torsion of an associated almost complex structure on a contact manifold (M,λ) vanishes if and only if

$$\nabla_\alpha J^\gamma_\beta = \delta^\gamma_\alpha \lambda_\beta - g_{\alpha\beta}\lambda^\gamma. \tag{2.9}$$

Lemma 2.2. *Let (M,λ,J) be a Sasaki manifold and $g := \lambda\otimes\lambda + \omega(\cdot, J\cdot)$ the corresponding associated metric with $\omega := d\lambda$. Then the following relations hold:*

$$\nabla_\gamma \omega_{\alpha\beta} = g_{\gamma\beta}\lambda_\alpha - g_{\gamma\alpha}\lambda_\beta, \tag{2.10}$$

$$J^\gamma_\beta \lambda_\gamma = 0, \tag{2.11}$$

$$\lambda^\delta \nabla_\beta \lambda_\delta = 0, \tag{2.12}$$

$$\nabla_\alpha \lambda_\beta = \omega_{\alpha\beta}, \tag{2.13}$$

$$R^\epsilon_{\beta\alpha\gamma}\lambda_\epsilon = g_{\gamma\alpha}\lambda_\beta - g_{\gamma\beta}\lambda_\alpha, \tag{2.14}$$

$$R_\beta{}^\epsilon \lambda_\epsilon = 2n\lambda_\beta, \tag{2.15}$$

$$R_{\beta\alpha\epsilon\gamma}\lambda^\beta\lambda^\epsilon = g_{\alpha\gamma} - \lambda_\alpha\lambda_\gamma = \mathbf{L}_{\alpha\gamma}, \tag{2.16}$$

$$R^\epsilon_{\beta\alpha\gamma}\omega_{\epsilon\delta} + R^\epsilon_{\beta\alpha\delta}\omega_{\gamma\epsilon} = g_{\beta\delta}\omega_{\alpha\gamma} - g_{\beta\gamma}\omega_{\alpha\delta} - g_{\alpha\delta}\omega_{\beta\gamma} + g_{\alpha\gamma}\omega_{\beta\delta}, \tag{2.17}$$

$$J^\beta_\epsilon R^\epsilon_{\beta\alpha\gamma} = R_\alpha{}^\epsilon \omega_{\gamma\epsilon} + (2n-1)\omega_{\alpha\gamma}, \tag{2.18}$$

$$J^\beta_\epsilon R^\epsilon_{\gamma\alpha\beta} = 2\big(R_\alpha{}^\epsilon\omega_{\gamma\epsilon} + (2n-1)\omega_{\alpha\gamma}\big). \tag{2.19}$$

Proof. $\omega_{\alpha\beta} = J^\gamma_\alpha g_{\gamma\beta}$ and (2.9) imply (2.10). (2.11) follows from (2.1), (2.2), (2.5) and (2.8). Equation (2.12) follows from covariant differentiation of $\lambda^\delta\lambda_\delta = 1$. Then from (2.11) and (2.9) we obtain

$$\nabla_\alpha\lambda_\delta J^\delta_\gamma = -\lambda_\delta\nabla_\alpha J^\delta_\gamma = g_{\alpha\gamma} - \lambda_\alpha\lambda_\gamma.$$

We multiply this with J^γ_β and (2.5) implies

$$\omega_{\beta\alpha} = -\pi^\delta_\beta \nabla_\alpha\lambda_\delta = \lambda_\beta\lambda^\delta\nabla_\alpha\lambda_\delta - \nabla_\alpha\lambda_\beta.$$

Then (2.13) follows from (2.12) and $\omega_{\alpha\beta} = -\omega_{\beta\alpha}$. To prove (2.14) we observe that (2.10), $d\omega = 0$ and (2.13) imply

$$\begin{aligned} g_{\gamma\alpha}\lambda_\beta - g_{\gamma\beta}\lambda_\alpha &= \nabla_\gamma\omega_{\beta\alpha} \\ &= \nabla_\alpha\omega_{\beta\gamma} - \nabla_\beta\omega_{\alpha\gamma} \\ &= \nabla_\alpha\nabla_\beta\lambda_\gamma - \nabla_\beta\nabla_\alpha\lambda_\gamma \\ &= R^\epsilon_{\beta\alpha\gamma}\lambda_\epsilon. \end{aligned}$$

This is (2.14). Equations (2.15), (2.16) follow from $R^\epsilon_{\beta\alpha\gamma} = R_{\beta\alpha}{}^\epsilon{}_\gamma$ and by taking the trace of (2.14) resp. by multiplying this with λ^β. With the same method one can prove the last equation

$$\begin{aligned} R^\epsilon_{\beta\alpha\gamma}\omega_{\epsilon\delta} + R^\epsilon_{\beta\alpha\delta}\omega_{\gamma\epsilon} &= \nabla_\alpha\nabla_\beta\omega_{\gamma\delta} - \nabla_\beta\nabla_\alpha\omega_{\gamma\delta} \\ &= \nabla_\alpha\big(g_{\beta\delta}\lambda_\gamma - g_{\beta\gamma}\lambda_\delta\big) - \nabla_\beta\big(g_{\alpha\delta}\lambda_\gamma - g_{\alpha\gamma}\lambda_\delta\big) \\ &= g_{\beta\delta}\nabla_\alpha\lambda_\gamma - g_{\beta\gamma}\nabla_\alpha\lambda_\delta - g_{\alpha\delta}\nabla_\beta\lambda_\gamma + g_{\alpha\gamma}\nabla_\beta\lambda_\delta \end{aligned}$$

and (2.17) follows from (2.13). To prove (2.18) it suffices to take the trace of (2.17) w.r.t. β, δ. Finally, to prove (2.19) we use the Bianchi identity to obtain

$$J^\beta_\epsilon R^\epsilon_{\gamma\alpha\beta} = J^\beta_\epsilon\big(R^\epsilon_{\beta\alpha\gamma} + R^\epsilon_{\alpha\gamma\beta}\big) = 2J^\beta_\epsilon R^\epsilon_{\beta\alpha\gamma}$$

because $J^\beta_\epsilon = g^{\beta\delta}\omega_{\epsilon\delta} = -g^{\beta\delta}\omega_{\delta\epsilon}$. Then (2.19) is a consequence of (2.18). □

2.3. The geometry of Legendre immersions in Sasaki manifolds. Let L be a smooth manifold and

$$F : L \to (M, g)$$

a smooth Riemannian immersion into a smooth Riemannian manifold (M, g), i.e., the tensor $F^*g \in \Gamma(T^*L \otimes T^*L)$ is positive definite and defines a Riemannian metric on TL. We set

$$g_{ij} := g_{\alpha\beta}F^\alpha_i F^\beta_j,$$

where $F^\alpha_i := \frac{\partial F^\alpha}{\partial x^i}$ are the components of the differential $dF \in \Gamma(T^*L \otimes F^{-1}TM)$,

$$dF = F^\alpha_i dx^i \otimes \frac{\partial}{\partial y^\alpha}.$$

The second fundamental tensor $A \in \Gamma(T^*L \otimes T^*L \otimes F^{-1}TM)$ is then given by $A = \nabla dF$ and in local coordinates

$$A = A^\alpha_{ij} dx^i \otimes dx^j \otimes \frac{\partial}{\partial y^\alpha}$$

with

$$A^\alpha_{ij} = \nabla_i F^\alpha_j = \frac{\partial^2 F^\alpha}{\partial x^i \partial x^j} - \Gamma^k_{ij}\frac{\partial F^\alpha}{\partial x^k} + \Gamma^\alpha_{\beta\gamma}\frac{\partial F^\beta}{\partial x^i}\frac{\partial F^\gamma}{\partial x^j}. \tag{2.20}$$

Moreover, dF is normal, i.e.,

$$g_{\alpha\beta}F^\alpha_i A^\beta_{jk} = 0. \tag{2.21}$$

In addition, the Gauss equations and Codazzi-Mainardi equations are

$$R_{ijkl} = R_{\alpha\beta\gamma\delta}F^\alpha_i F^\beta_j F^\gamma_k F^\delta_l + g_{\alpha\beta}\big(A^\alpha_{ik}A^\beta_{jl} - A^\alpha_{il}A^\beta_{jk}\big), \tag{2.22}$$

$$\nabla_i A^\alpha_{jk} - \nabla_j A^\alpha_{ik} = -R^l_{ijk}F^\alpha_l + R^\alpha_{\beta\gamma\delta}F^\beta_i F^\gamma_j F^\delta_k. \tag{2.23}$$

In case where $F : L \to (M, \lambda, J)$ is a Riemannian immersion into a Sasaki manifold, we define the section

$$\nu = \nu^\alpha_i dx^i \otimes \frac{\partial}{\partial y^\alpha} \in \Gamma(T^*L \otimes F^{-1}TM)$$

and the second fundamental form

$$h = h_{ijk} dx^i \otimes dx^j \otimes dx^k \in \Gamma(T^*L \otimes T^*L \otimes T^*L)$$

by

$$\nu^\alpha_i := J^\alpha_\beta F^\beta_i \tag{2.24}$$

and

$$h_{ijk} := -\omega_{\alpha\beta}F^\alpha_i A^\beta_{jk}. \tag{2.25}$$

Now let

$$F^*\lambda := \lambda_i dx^i := \lambda_\alpha F_i^\alpha dx^i$$

and

$$F^*\omega := \omega_{ij} dx^i \otimes dx^j := \omega_{\alpha\beta} F_i^\alpha F_j^\beta dx^i \otimes dx^j$$

be the pull-backs of λ and $\omega = d\lambda$ on L. Then we have:

Lemma 2.3. *Let $F : L \to (M, \lambda, J)$ be a Riemannian immersion into a Sasaki manifold. Then the following relations hold:*

$$\nabla_j \nu_i^\alpha = \lambda_i F_j^\alpha - g_{ij}\lambda^\alpha + J_\beta^\alpha A_{ij}^\beta, \tag{2.26}$$

$$h_{kij} - h_{jik} = \nabla_i \omega_{kj} + g_{ij}\lambda_k - g_{ik}\lambda_j, \tag{2.27}$$

$$\nabla_l h_{ijk} = \lambda_\beta A_{jk}^\beta g_{li} - \omega_{\alpha\beta} A_{li}^\alpha A_{jk}^\beta - \omega_{\alpha\beta} F_i^\alpha \nabla_l A_{jk}^\beta, \tag{2.28}$$

$$\begin{aligned}\nabla_l h_{ijk} - \nabla_j h_{ilk} = {} & \lambda_\beta (A_{jk}^\beta g_{li} - A_{lk}^\beta g_{ji}) - \omega_{\alpha\beta}(A_{li}^\alpha A_{jk}^\beta - A_{ji}^\alpha A_{lk}^\beta) \\ & + \ \omega_{im} R_{ljk}^m - \omega_{\alpha\beta} R_{\gamma\delta\epsilon}^\beta F_i^\alpha F_l^\gamma F_j^\delta F_k^\epsilon,\end{aligned} \tag{2.29}$$

$$\lambda_\alpha A_{ij}^\alpha = \nabla_i \lambda_j - \omega_{ij}. \tag{2.30}$$

Proof. For (2.26) we compute

$$\begin{aligned}\nabla_i \nu_j^\alpha &= \nabla_i (J_\beta^\alpha F_j^\beta) \\ &= \nabla_\gamma J_\beta^\alpha F_i^\gamma F_j^\beta + J_\beta^\alpha \nabla_i F_j^\beta \\ &= (\delta_\beta^\alpha \lambda_\gamma - g_{\beta\gamma}\lambda^\alpha) F_i^\gamma F_j^\beta + J_\beta^\alpha A_{ij}^\beta \\ &= \lambda_i F_j^\alpha - g_{ij}\lambda^\alpha + J_\beta^\alpha A_{ij}^\beta.\end{aligned}$$

Also

$$\begin{aligned}\nabla_i \omega_{jk} &= \nabla_i (\omega_{\alpha\beta} F_j^\alpha F_k^\beta) \\ &= \nabla_\gamma \omega_{\alpha\beta} F_i^\gamma F_j^\alpha F_k^\beta + \omega_{\alpha\beta}(A_{ij}^\alpha F_k^\beta + F_j^\alpha A_{ik}^\beta) \\ &= (g_{\gamma\beta}\lambda_\alpha - g_{\gamma\alpha}\lambda_\beta) F_i^\gamma F_j^\alpha F_k^\beta + h_{kij} - h_{jik} \\ &= g_{ik}\lambda_j - g_{ij}\lambda_k + h_{kij} - h_{jik}\end{aligned}$$

which is (2.27). The covariant derivative of h_{ijk} is given by

$$\begin{aligned}\nabla_l h_{ijk} &= -\nabla_l (\omega_{\alpha\beta} F_i^\alpha A_{jk}^\beta) \\ &= -(g_{\gamma\beta}\lambda_\alpha - g_{\gamma\alpha}\lambda_\beta) F_l^\gamma F_i^\alpha A_{jk}^\beta - \omega_{\alpha\beta}(A_{li}^\alpha A_{jk}^\beta + F_i^\alpha \nabla_l A_{jk}^\beta)\end{aligned}$$

which due to (2.21) gives equation (2.28). Equation (2.29) then easily follows from the Codazzi equation (2.23) and (2.28). The last equation of the lemma follows from

$$\nabla_i \lambda_j = \nabla_\beta \lambda_\alpha F_i^\alpha F_j^\beta + \lambda_\alpha A_{ij}^\alpha$$

and (2.13). □

From now on we will assume that

$$F : L \to (M, \lambda, J)$$

is a Legendre immersion into a Sasaki manifold, i.e.,

$$F^*\lambda = \lambda_i dx^i = 0 \tag{2.31}$$

and $\dim(L) = n$, where $\dim(M) = 2n + 1$.

Corollary 2.4. *Let $F : L \to (M, \lambda, J)$ be a Legendre immersion into a Sasaki manifold. Then the following relations hold:*

$$\lambda_\alpha A^\alpha_{ij} = 0, \tag{2.32}$$

$$\nabla_i F^\alpha_j = A^\alpha_{ij} = -h^k{}_{ij}\nu^\alpha_k, \tag{2.33}$$

$$\nabla_i \nu^\alpha_j = -g_{ij}\lambda^\alpha + h^k{}_{ij}F^\alpha_k, \tag{2.34}$$

$$h_{kij} = h_{kji} = h_{jki}, \tag{2.35}$$

$$\nabla_l h_{ijk} - \nabla_j h_{ilk} = -\omega_{\alpha\beta}R^\beta_{\gamma\delta\epsilon}F^\alpha_i F^\gamma_l F^\delta_j F^\epsilon_k. \tag{2.36}$$

Proof. Since F is a Legendre immersion we must have $\lambda_i = \omega_{ij} = 0$. In particular

$$\omega_{ij} = \omega_{\alpha\beta}F^\alpha_i F^\beta_j = J^\gamma_\alpha g_{\gamma\beta}F^\alpha_i F^\beta_j = g_{\gamma\beta}\nu^\gamma_i F^\beta_j$$

and $\dim(L) = \frac{1}{2}(\dim(M) - 1)$ imply that the normal bundle NL of L can be decomposed as

$$NL = F^{-1}l \oplus JdF(TL),$$

where the fiber of the bundle $F^{-1}l$ (the line bundle along F) at a point $x \in L$ is given by $l_{F(x)}$. On the other hand the second fundamental tensor A^α_{ij} is normal and therefore there must exist p_{ij} and $s^k{}_{ij}$ such that

$$A^\alpha_{ij} = p_{ij}\lambda^\alpha + s^k{}_{ij}\nu^\alpha_k.$$

From (2.30) we get

$$p_{ij} = \lambda_\alpha A^\alpha_{ij} = 0$$

which is (2.32). Moreover

$$\begin{aligned} h_{lij} &= -\omega_{\alpha\beta}F^\alpha_l A^\beta_{ij} \\ &= -\omega_{\alpha\beta}F^\alpha_l s^k{}_{ij}\nu^\beta_k = -\omega_{\alpha\beta}J^\beta_\gamma F^\alpha_l F^\gamma_k s^k{}_{ij} \\ &= -g_{\alpha\gamma}F^\alpha_l F^\gamma_k s^k{}_{ij} = -g_{lk}s^k{}_{ij} = -s_{kij}, \end{aligned}$$

which by Lemma 2.3 proves (2.33) and (2.34). Then (2.35) and (2.36) are just equations (2.27) resp. (2.29) because the compatibility of J with ω implies

$$\omega_{\alpha\beta}\nu^\alpha_i\nu^\beta_j = \omega_{\alpha\beta}F^\alpha_i F^\beta_j = \omega_{ij} = 0.$$

□

Definition 2.5. Let $F : L \to (M, \lambda, J)$ be a Legendre immersion. The mean curvature form $H = H_i dx^i \in \Gamma(T^*L)$ is given by

$$H_i := g^{kl}h_{ikl}. \tag{2.37}$$

Since $NL = F^{-1}l \oplus JdF(TL)$ and $F^*\lambda = 0$ we can decompose a tangent vector $\frac{\partial}{\partial y^\sigma}$ along F so that

$$\frac{\partial}{\partial y^\sigma} = g^{ik} g_{\sigma\alpha} F_i^\alpha F_k + g^{ik} g_{\sigma\alpha} \nu_i^\alpha \nu_k + \lambda_\sigma X_\lambda,$$

with $F_k = F_k^\beta \frac{\partial}{\partial y^\beta}, \nu_k = \nu_k^\beta \frac{\partial}{\partial y^\beta}$. For later purposes we compute

$$\begin{aligned} g^{ik} R_{\gamma\delta\beta\epsilon} \nu_i^\beta F_k^\epsilon &= \frac{1}{2} g^{ik} R\left(\frac{\partial}{\partial y^\gamma}, \frac{\partial}{\partial y^\delta}, \frac{\partial}{\partial y^\beta}, \nu_i^\beta F_k - F_i^\beta \nu_k\right) \\ &= \frac{1}{2} g^{\sigma\beta} R\left(\frac{\partial}{\partial y^\gamma}, \frac{\partial}{\partial y^\delta}, \frac{\partial}{\partial y^\beta}, g^{ik} g_{\sigma\alpha}(\nu_i^\alpha F_k - F_i^\alpha \nu_k)\right) \\ &= -\frac{1}{2} g^{\sigma\beta} R\left(\frac{\partial}{\partial y^\gamma}, \frac{\partial}{\partial y^\delta}, \frac{\partial}{\partial y^\beta}, J\frac{\partial}{\partial y^\sigma}\right) \\ &= -\frac{1}{2} R^\beta_{\gamma\delta\sigma} J^\sigma_\beta \end{aligned}$$

and with (2.19)

$$g^{ik} R_{\gamma\delta\beta\epsilon} \nu_i^\beta F_k^\epsilon = -R_\delta{}^\epsilon \omega_{\gamma\epsilon} - (2n-1)\omega_{\delta\gamma}. \tag{2.38}$$

Definition 2.6. Let (M, λ, J) be a Sasaki manifold. Then (M, λ, J) is called pseudo-Einstein, if there exists a constant K such that

$$R_{\alpha\beta} V^\alpha W^\beta = K g_{\alpha\beta} V^\alpha W^\beta$$

for all $V, W \in \xi = \ker(\lambda)$, i.e., the associated metric g is Einstein on the symplectic subbundle ξ.

The following examples are taken from [6]:

Example 2.7. a) (Tanno [21], [22]). Let S^{2n+1} be equipped with the standard contact structure λ, almost complex structure J and metric g that are induced by $\mathbb{C}^{n+1}$. Suppose $c > 0$ is a constant and define

$$\widetilde{\lambda} := c\lambda,$$

$$\widetilde{g} := cg + c(c-1)\lambda \otimes \lambda.$$

Then $(S^{2n+1}, \widetilde{\lambda}, \widetilde{g}, J)$ is a Sasaki pseudo-Einstein manifold with

$$K = 1 + (2n-1)\left(\frac{4}{c} - 3\right).$$

b) (Okumura [16]). Let $\mathbb{R}^{2n+1}$ be equipped with the contact structure

$$\lambda = \frac{1}{2}(dz - y_i dx^i)$$

and the Riemannian metric

$$g = \frac{1}{4}\big(\lambda \otimes \lambda + \delta_{ij}(dx^i \otimes dx^j + dy^i \otimes dy^j)\big),$$

then $(\mathbb{R}^{2n+1}, \lambda, g)$ is Sasaki pseudo-Einstein with

$$K = 4 - 6n.$$

c) (Tanno [22]). Let $B^n \subset \mathbb{C}^n$ be a bounded, simply connected domain with a Kähler structure (J, g) of constant holomorphic sectional curvature $\theta < 0$. Let β be the real analytic 1-form such that $d\beta = \omega$ gives the Kähler form on B^n. We define a Sasaki structure $(\lambda, \widetilde{g})$ on $B^n \times \mathbb{R}$ by

$$\lambda := \pi^*\beta + dt$$

and

$$\widetilde{g} := \pi^* g + \lambda \otimes \lambda,$$

where

$$\pi : B^n \times \mathbb{R} \to B^n$$

is the projection and t denotes the coordinate in $\mathbb{R}$-direction. Then $(B^n, \lambda, \widetilde{g})$ is Sasaki pseudo-Einstein with

$$K = 1 + (2n-1)(\theta - 3).$$

Lemma 2.8. *Let $F : L \to (M, \lambda, J)$ be a Legendre immersion into a Sasaki pseudo-Einstein manifold. Then the mean curvature form H is closed.*

Proof. H is closed if and only if $\nabla_l H_j - \nabla_j H_l = 0$. We observe

$$\begin{aligned}
\nabla_l H_j - \nabla_j H_l &= g^{ik}(\nabla_l h_{ijk} - \nabla_j h_{ilk}) \\
&\overset{(2.36)}{=} -\omega_{\alpha\beta} g^{ik} R^{\beta}_{\gamma\delta\epsilon} F^{\alpha}_i F^{\gamma}_l F^{\delta}_j F^{\epsilon}_k \\
&= -g^{ik} R_{\gamma\delta\beta\epsilon} \nu^{\beta}_i F^{\epsilon}_k F^{\gamma}_l F^{\delta}_j \\
&\overset{(2.38)}{=} \big(R_{\delta}{}^{\epsilon}\omega_{\gamma\epsilon} + (2n-1)\omega_{\delta\gamma}\big) F^{\gamma}_l F^{\delta}_j \\
&= R_{\alpha\beta} F^{\alpha}_j \nu^{\beta}_l
\end{aligned}$$

and if (M, λ, J) is Sasaki pseudo-Einstein, then (because $\nu_l, F_j \in \xi$)

$$\nabla_l H_j - \nabla_j H_l = K g_{\alpha\beta} F^{\alpha}_j \nu^{\beta}_l = 0.$$

□

3. Variations of Legendre submanifolds

In this subsection we want to study necessary conditions for a variation to preserve the Legendre condition. Geometrical interesting variations are only given by normal variations because tangential deformations correspond to diffeomorphisms of the given submanifold. As we have already seen, there exists a natural splitting of the normal bundle for a Legendre submanifold. Hence a smooth normal vector field V can be identified with a pair (f, θ) consisting of a smooth function f on L and a smooth 1-form θ on L via the decomposition

$$V = fX_\lambda + JdF(\theta^\sharp),$$

where $\sharp$ denotes the identification of a 1-form with a tangent vector via the metric tensor g. Now assume that for $t \in \Omega := (-\epsilon, \epsilon),\ \epsilon > 0$ we are given a smooth family of Legendre immersions $F_t : L \to L_t \subset M$ such that

$$\frac{\partial F_t}{\partial t} = fX_\lambda + \theta^i \nu_i,$$

where (f, θ) is a smooth family of pairs consisting of functions f and 1-forms θ on L and $\nu_i = \nu\left(\frac{\partial}{\partial x^i}\right) = JF_i = J\frac{\partial F}{\partial x^i} = J^\beta_\alpha F^\alpha_i \frac{\partial}{\partial y^\beta}$. To compute time derivatives of tensor expressions on L it is useful to consider the manifold

$$\hat{L} := L \times \Omega$$

and the smooth map

$$F : \hat{L} \to M,$$
$$F(x,t) := F_t(x).$$

The canonical connections on tensor bundles over L can then be extended to connections on corresponding bundles over $\hat{L}$, e.g.,

$$\nabla_{\frac{\partial}{\partial t}} \frac{\partial}{\partial y^\alpha} = \dot{F}^\gamma \Gamma^\beta_{\gamma\alpha} \frac{\partial}{\partial y^\beta},$$
$$\nabla_{\frac{\partial}{\partial t}} dx^i = 0,$$

where here and in the following $\dot{F} = \dot{F}^\gamma \frac{\partial}{\partial y^\gamma} = \frac{\partial F}{\partial t}$. We have

$$\begin{aligned}
\nabla_{\frac{\partial}{\partial t}} dF_t &= \nabla_{\frac{\partial}{\partial t}} \left(F^\alpha_i dx^i \otimes \frac{\partial}{\partial y^\alpha} \right) \\
&= \left(\frac{\partial^2 F^\alpha}{\partial x^i \partial t} + \Gamma^\alpha_{\gamma\beta} \dot{F}^\gamma F^\beta_i \right) dx^i \otimes \frac{\partial}{\partial y^\alpha} \\
&= \nabla_i \dot{F}^\alpha dx^i \otimes \frac{\partial}{\partial y^\alpha},
\end{aligned}$$

i.e.,

$$\nabla_{\frac{\partial}{\partial t}} F^\alpha_i = \nabla_i \dot{F}^\alpha. \tag{3.1}$$

In addition, for a section $V \in \Gamma(T^*L \otimes F^{-1}TM)$

$$\nabla_{\frac{\partial}{\partial t}} \nabla_i V^\alpha_j = \nabla_i \nabla_{\frac{\partial}{\partial t}} V^\alpha_j + R^\alpha_{\beta\gamma\delta} \dot{F}^\beta F^\gamma_i V^\delta_j \tag{3.2}$$

because T^*L does not depend on t but $F^{-1}TM$ does. The condition for L_t being Legendre is $\lambda_i dx^i = F^*_t \lambda = 0$. We compute

$$\begin{aligned}
\nabla_{\frac{\partial}{\partial t}} \lambda_i &= \nabla_{\frac{\partial}{\partial t}} (\lambda_\alpha F^\alpha_i) \\
&= \nabla_\gamma \lambda_\alpha \dot{F}^\gamma F^\alpha_i + \lambda_\alpha \nabla_{\frac{\partial}{\partial t}} F^\alpha_i \\
&\overset{(2.13),(3.1)}{=} \omega_{\gamma\alpha} \dot{F}^\gamma F^\alpha_i + \lambda_\alpha \nabla_i \dot{F}^\alpha \\
&= \omega_{\gamma\alpha} (f\lambda^\gamma + \theta^k \nu^\gamma_k) F^\alpha_i + \lambda_\alpha \nabla_i (f\lambda^\alpha + \theta^k \nu^\alpha_k) \\
&\overset{(2.1)}{=} \theta^k \omega_{\gamma\alpha} \nu^\gamma_k F^\alpha_i + \nabla_i f \\
&\quad + \lambda_\alpha (\nabla_\gamma \lambda^\alpha F^\gamma_i + \nabla_i \theta^k \nu^\alpha_k + \theta^k \nabla_i \nu^\alpha_k) \\
&\overset{(2.11),(2.12),(2.13)}{=} \theta^k \omega_{\gamma\alpha} \nu^\gamma_k F^\alpha_i + \nabla_i f + \lambda_\alpha \theta^k \nabla_i \nu^\alpha_k \\
&\overset{(2.5),(2.34)}{=} \theta^k (\lambda_\beta \lambda_\alpha - g_{\beta\alpha}) F^\beta_k F^\alpha_i \\
&\quad + \nabla_i f + \lambda_\alpha \theta^k (h^l{}_{ik} F^\alpha_l - g_{ik} \lambda^\alpha) \\
&= \nabla_i f - 2\theta_i,
\end{aligned}$$

because L_t is Legendre. Therefore we have shown:

Lemma 3.1. *Let L be a smooth n-dimensional manifold and for $t \in [0, \epsilon)$, $\epsilon > 0$ let (f, θ) be a smooth family of pairs consisting of functions f and 1-forms θ on L. Moreover let $F_t : L \to M,\ t \in [0, \epsilon)$ be a smooth family of immersions into a Sasaki manifold (M, λ, g) such that $\frac{\partial F_t}{\partial t} = f X_\lambda + \theta^k \nu_k$ and assume that $L_0 := F_0(L)$ is Legendre. Then $L_t := F_t(L)$ is Legendre for all $t \in [0, \epsilon)$ if and only if $df = 2\theta$.*

In view of Lemma 3.1 we will from now on assume that $F_t : L \to M,\ t \in [0, \epsilon)$ is a smooth family of Legendre immersions into a Sasaki manifold (M, λ, g) such that

$$\dot{F} = 2f X_\lambda + \nabla^k f \nu_k \tag{3.3}$$

for a smooth family of functions $f : L \to \mathbb{R}$. Next we compute the evolution equations for various objects.

Lemma 3.2. *If a family of Legendre immersions into a Sasaki manifold evolves according to (3.3), then*

$$\nabla_{\frac{\partial}{\partial t}} g_{ij} = 2\nabla^k f h_{kij}, \tag{3.4}$$

$$\begin{aligned}\nabla_{\frac{\partial}{\partial t}} h_{ijk} = &-\nabla_j \nabla_k \nabla_i f + \nabla^l f (h_{lim} h^m{}_{jk} + h_{lkm} h^m{}_{ji}) \\ &- 2\nabla_j f g_{ik} - \nabla_k f g_{ij} - \nabla^l f \omega_{\alpha\beta} F_i^\alpha R^\beta_{\gamma\delta\epsilon} \nu_l^\gamma F_j^\delta F_k^\epsilon,\end{aligned} \tag{3.5}$$

$$\nabla_{\frac{\partial}{\partial t}} H_j = -\nabla_j \Delta f - 2\nabla_j f - \nabla^l f R_{\alpha\beta} F_l^\alpha F_j^\beta. \tag{3.6}$$

Proof.

$$\begin{aligned}
\nabla_{\frac{\partial}{\partial t}} g_{ij} &= \nabla_{\frac{\partial}{\partial t}} (g_{\alpha\beta} F_i^\alpha F_j^\beta) \\
&= \nabla_\gamma g_{\alpha\beta} \dot{F}^\gamma F_i^\alpha F_j^\beta + g_{\alpha\beta} (\nabla_{\frac{\partial}{\partial t}} F_i^\alpha F_j^\beta + F_i^\alpha \nabla_{\frac{\partial}{\partial t}} F_j^\beta) \\
&\overset{\nabla g = 0,\ (3.1)}{=} g_{\alpha\beta} (\nabla_i \dot{F}^\alpha F_j^\beta + F_i^\alpha \nabla_j \dot{F}^\beta) \\
&= g_{\alpha\beta} \big(\nabla_i (2f\lambda^\alpha + \nabla^k f \nu_k^\alpha) F_j^\beta \\
&\quad + F_i^\alpha \nabla_j (2f\lambda^\beta + \nabla^k f \nu_k^\beta)\big) \\
&= 2\nabla_i f \lambda_j + 2\nabla_j f \lambda_i \\
&\quad + 2f \nabla_\gamma \lambda_\beta F_i^\gamma F_j^\beta + 2f \nabla_\gamma \lambda_\alpha F_j^\gamma F_i^\alpha \\
&\quad + \nabla_i \nabla^k f g(\nu_k, F_j) + \nabla_j \nabla^k f g(F_i, \nu_k) \\
&\quad + \nabla^k f \big(g(\nabla_i \nu_k, F_j) + g(F_i, \nabla_j \nu_k)\big) \\
&\overset{F^*\lambda = F^*\omega = 0,\ (2.13)}{=} \nabla^k f \big(g(\nabla_i \nu_k, F_j) + g(F_i, \nabla_j \nu_k)\big) \\
&\overset{(2.34),\ F^*\lambda = 0}{=} 2\nabla^k f h_{kij}.
\end{aligned}$$

$$\begin{aligned}
\nabla_{\frac{\partial}{\partial t}} h_{ijk} &= -\nabla_{\frac{\partial}{\partial t}} (\omega_{\alpha\beta} F_i^\alpha \nabla_j F_k^\beta) \\
&\overset{(3.1)}{=} -\nabla_\gamma \omega_{\alpha\beta} \dot{F}^\gamma F_i^\alpha \nabla_j F_k^\beta \\
&\quad - \omega_{\alpha\beta} \big(\nabla_i \dot{F}^\alpha \nabla_j F_k^\beta + F_i^\alpha \nabla_{\frac{\partial}{\partial t}} \nabla_j F_k^\beta\big) \\
&\overset{(2.10),(2.33),(3.2)}{=} (g_{\gamma\alpha} \lambda_\beta - g_{\gamma\beta} \lambda_\alpha)(2f\lambda^\gamma + \nabla^l f \nu_l^\gamma) F_i^\alpha \nabla_j F_k^\beta \\
&\quad + \omega_{\alpha\beta} \nabla_i (2f\lambda^\alpha + \nabla^l f \nu_l^\alpha) h^m{}_{jk} \nu_m^\beta \\
&\quad - \omega_{\alpha\beta} F_i^\alpha \big(\nabla_j \nabla_{\frac{\partial}{\partial t}} F_k^\beta + R^\beta_{\gamma\delta\epsilon} \dot{F}^\gamma F_j^\delta F_k^\epsilon\big)
\end{aligned}$$

$$
\begin{aligned}
&\overset{F^*\lambda=0,\ (2.34),(3.1)}{=} && 2f\omega_{\alpha\beta}\nabla_\gamma\lambda^\alpha F_i^\gamma {h^m}_{jk}\nu_m^\beta \\
& && + \omega_{\alpha\beta}\nabla^l f({h^p}_{il}F_p^\alpha - g_{il}\lambda^\alpha){h^m}_{jk}\nu_m^\beta \\
& && - \omega_{\alpha\beta}F_i^\alpha\big(\nabla_j\nabla_k\dot{F}^\beta + R^\beta_{\gamma\delta\epsilon}\dot{F}^\gamma F_j^\delta F_k^\epsilon\big) \\
&\overset{(2.13),\ F^*\lambda=0}{=} && \nabla^l f h_{mil}{h^m}_{jk} - \omega_{\alpha\beta}F_i^\alpha\big(\nabla_j\nabla_k(2f\lambda^\beta + \nabla^l f\nu_l^\beta) \\
& && + R^\beta_{\gamma\delta\epsilon}\dot{F}^\gamma F_j^\delta F_k^\epsilon\big).
\end{aligned}
$$

Now

$$
\begin{aligned}
-\omega_{\alpha\beta}F_i^\alpha\nabla_j\nabla_k(2f\lambda^\beta) &= -\omega_{\alpha\beta}F_i^\alpha\big(2\nabla_j\nabla_k f\lambda^\beta + 2\nabla_k f J_\gamma^\beta F_j^\gamma \\
&\quad + 2\nabla_j f J_\gamma^\beta F_k^\gamma + 2f\nabla_j(J_\gamma^\beta F_k^\gamma)\big) \\
&= -2\nabla_k f g_{ij} - 2\nabla_j f g_{ik} \\
&\quad - 2f\omega_{\alpha\beta}\nabla_\delta J_\gamma^\beta F_i^\alpha F_j^\delta F_k^\gamma - 2f\omega_{\alpha\beta}J_\gamma^\beta F_i^\alpha\nabla_j F_k^\gamma \\
&\overset{(2.9)}{=} -2\nabla_k f g_{ij} - 2\nabla_j f g_{ik}
\end{aligned}
$$

and

$$
\begin{aligned}
-\omega_{\alpha\beta}F_i^\alpha\nabla_j\nabla_k(\nabla^l f\nu_l^\beta) &= -\omega_{\alpha\beta}F_i^\alpha\big(\nabla_j\nabla_k\nabla^l f\nu_l^\beta + \nabla_k\nabla^l f\nabla_j\nu_l^\beta \\
&\quad + \nabla_j\nabla^l f\nabla_k\nu_l^\beta + \nabla^l f\nabla_j\nabla_k\nu_l^\beta\big) \\
&= -\omega_{\alpha\beta}F_i^\alpha\nabla_j\nabla_k\nabla^l f\nu_l^\beta \\
&\quad - \omega_{\alpha\beta}F_i^\alpha\nabla_j({h^m}_{kl}F_m^\beta - g_{kl}\lambda^\beta)\nabla^l f \\
&= -\nabla_j\nabla_k\nabla_i f \\
&\quad + \omega_{\alpha\beta}F_i^\alpha{h^m}_{kl}{h^p}_{jm}\nu_p^\beta\nabla^l f + g_{kl}\omega_{\alpha\beta}F_i^\alpha J_\gamma^\beta F_j^\gamma\nabla^l f \\
&= -\nabla_j\nabla_k\nabla_i f + \nabla^l f{h^m}_{kl}h_{ijm} + \nabla_k f g_{ij}.
\end{aligned}
$$

Therefore in a first step

$$
\begin{aligned}
\nabla_{\frac{\partial}{\partial t}}h_{ijk} &= \nabla^l f h_{mil}{h^m}_{jk} - 2\nabla_k f g_{ij} - 2\nabla_j f g_{ik} \\
&\quad - \nabla_j\nabla_k\nabla_i f + \nabla^l f{h^m}_{kl}h_{ijm} + \nabla_k f g_{ij} \\
&\quad - \omega_{\alpha\beta}F_i^\alpha R^\beta_{\gamma\delta\epsilon}\dot{F}^\gamma F_j^\delta F_k^\epsilon \\
&= -\nabla_j\nabla_k\nabla_i f + \nabla^l f(h_{lim}{h^m}_{jk} + h_{lkm}{h^m}_{ji}) - 2\nabla_j f g_{ik} \\
&\quad - \nabla_k f g_{ij} - \omega_{\alpha\beta}F_i^\alpha R^\beta_{\gamma\delta\epsilon}\dot{F}^\gamma F_j^\delta F_k^\epsilon.
\end{aligned}
$$

It remains to compute

$$
\begin{aligned}
-\omega_{\alpha\beta}F_i^\alpha R^\beta_{\gamma\delta\epsilon}\dot{F}^\gamma F_j^\delta F_k^\epsilon &= -\omega_{\alpha\beta}F_i^\alpha R^\beta_{\gamma\delta\epsilon}(2f\lambda^\gamma + \nabla^l f\nu_l^\gamma)F_j^\delta F_k^\epsilon \\
&= -2f\omega_{\alpha\beta}F_i^\alpha {R^\gamma}_\delta{}^\beta{}_\epsilon\lambda_\gamma F_j^\delta F_k^\epsilon \\
&\quad - \nabla^l f\omega_{\alpha\beta}F_i^\alpha R^\beta_{\gamma\delta\epsilon}\nu_l^\gamma F_j^\delta F_k^\epsilon \\
&\overset{(2.14)}{=} -2f\omega_{\alpha\beta}F_i^\alpha(g_{\delta\epsilon}\lambda^\beta - \delta_\delta^\beta\lambda_\epsilon)F_j^\delta F_k^\epsilon \\
&\quad - \nabla^l f\omega_{\alpha\beta}F_i^\alpha R^\beta_{\gamma\delta\epsilon}\nu_l^\gamma F_j^\delta F_k^\epsilon \\
&= -\nabla^l f\omega_{\alpha\beta}F_i^\alpha R^\beta_{\gamma\delta\epsilon}\nu_l^\gamma F_j^\delta F_k^\epsilon
\end{aligned}
$$

so that finally

$$\begin{aligned}\nabla_{\frac{\partial}{\partial t}} h_{ijk} &= -\nabla_j\nabla_k\nabla_i f + \nabla^l f(h_{lim}h^m{}_{jk} + h_{lkm}h^m{}_{ji}) - 2\nabla_j f g_{ik}\\ &\quad - \nabla_k f g_{ij} - \nabla^l f \omega_{\alpha\beta}F_i^\alpha R^\beta_{\gamma\delta\epsilon}\nu_l^\gamma F_j^\delta F_k^\epsilon.\end{aligned}$$

For the mean curvature form we compute

$$\begin{aligned}\nabla_{\frac{\partial}{\partial t}} H_j &= \nabla_{\frac{\partial}{\partial t}}(g^{ki}h_{ijk})\\ &= -h_{ijk}g^{km}g^{ip}\nabla_{\frac{\partial}{\partial t}} g_{mp} + g^{ik}\nabla_{\frac{\partial}{\partial t}} h_{ijk}\\ &= -2\nabla^l f h_{lmp}h^p{}_j{}^m - \nabla_j\Delta f + 2\nabla^l f h_{lpm}h^m{}_j{}^p\\ &\quad - (2n+1)\nabla_j f - \nabla^l f g^{ik}\omega_{\alpha\beta}F_i^\alpha R^\beta_{\gamma\delta\epsilon}\nu_l^\gamma F_j^\delta F_k^\epsilon.\\ &= -\nabla_j\Delta f - (2n+1)\nabla_j f - \nabla^l f g^{ik}\omega_{\alpha\beta}F_i^\alpha R^\beta_{\gamma\delta\epsilon}\nu_l^\gamma F_j^\delta F_k^\epsilon\\ &= -\nabla_j\Delta f - (2n+1)\nabla_j f - \nabla^l f \nu_l^\gamma F_j^\delta g^{ik}R_{\gamma\delta\beta\epsilon}\nu_i^\beta F_k^\epsilon\\ &\overset{(2.38)}{=} -\nabla_j\Delta f - (2n+1)\nabla_j f + \nabla^l f \nu_l^\gamma F_j^\delta (R_\delta{}^\epsilon\omega_{\gamma\epsilon} + (2n-1)\omega_{\delta\gamma})\\ &= -\nabla_j\Delta f - 2\nabla_j f - \nabla^l f R_{\alpha\beta}F_l^\alpha F_j^\beta.\end{aligned}$$

□

Corollary 3.3. *If $F_t : L \to (M, \lambda, J)$, $t \in [0, \epsilon)$ is a smooth family of Legendre immersions into a Sasaki pseudo-Einstein manifold (with $J_\delta^\alpha R_{\alpha\beta} = K\omega_{\delta\beta}$) that evolves according to (3.3), then the mean curvature form H satisfies*

$$\nabla_{\frac{\partial}{\partial t}} H_j = -\nabla_j\big(\Delta f + (2+K)f\big). \tag{3.7}$$

In particular, the cohomology class of H is fixed. If H is exact at $t = 0$, then there exists a smooth family of functions α on L, smoothly depending on $t \in [0, \epsilon)$, such that

$$\nabla_{\frac{\partial}{\partial t}} \alpha = -\Delta f - (2+K)f \tag{3.8}$$

and

$$d\alpha = H. \tag{3.9}$$

The family α is unique up to adding a function depending only on t. The volume form $d\mu$ satisfies the evolution equation

$$\nabla_{\frac{\partial}{\partial t}} d\mu = H(\nabla f)d\mu. \tag{3.10}$$

Definition 3.4. If $F : L \to (M, \lambda, J)$ is a Legendre immersion such that the mean curvature form H is exact, then any function α with $H = d\alpha$ is called the Legendre angle of $F(L)$.
In general one can find a unique pair (β, θ) consisting of a smooth function $\beta : L \to \mathbb{R}$ and a harmonic 1-form on L such that

$$H = d\beta + \theta, \quad \int_L \beta = 0.$$

We call β the Legendre pseudo-angle of L.

Definition 3.5. Let $L \subset (M, \lambda, J)$ be a Legendre submanifold of a Sasaki pseudo-Einstein manifold. Then the Legendrian mean curvature flow is the solution of

$$\frac{d}{dt}F_t = -2\beta X_\lambda - \nabla^k \beta \nu_k, \tag{3.11}$$

where β is the Legendre pseudo-angle of $L_t := F_t(L)$.

Remark 3.6. To explain how β depends on F_t let H_t denote the mean curvature form of F_t. Assume $F_0 \in C^\infty(L, M)$ and fix a smooth 1-form θ_0 on L such that $[H_0] = [\theta_0]$. Define

$$C_0^\infty(L, \mathbb{R}) := C^\infty(L, \mathbb{R}) \cap \left\{ f : L \to \mathbb{R} \;\Big|\; \int_L f d\mu = 0 \right\}.$$

For $F \in C^\infty(L, M)$ we then obtain a map

$$\Xi : C^\infty(L, M) \to C_0^\infty(L, \mathbb{R})$$

by

$$\Xi(F) := \Delta^{-1}(d^\dagger \theta_0),$$

where $d^\dagger$ is the negative adjoint of d w.r.t. the metric g_{ij} induced by F and

$$\Delta^{-1} : C_0^\infty(L, \mathbb{R}) \to C_0^\infty(L, \mathbb{R})$$

is the Green's operator for the Laplacian $\Delta = g^{ij}\nabla_i \nabla_j$. Let $\alpha_t \in C_0^\infty(L, \mathbb{R})$ be the solution of

$$\Delta \alpha_t = d^\dagger \theta_0$$

and define $\beta_t \in C_0^\infty(L, \mathbb{R})$ through

$$d\beta_t := H_t - \theta_0 + d\alpha_t$$

for all F_t sufficiently close to F_0 so that $[H_t] = [H_0]$. Then $\Delta \beta_t = d^\dagger H_t$ and (3.11) can be written as

$$\frac{d}{dt}F = \overrightarrow{H} - 2\beta X_\lambda + (\theta_0^k - \nabla^k \alpha)\nu_k.$$

This differs from the mean curvature flow (1.1) only by $-2\beta X_\lambda + (\theta_0^k - \nabla^k \alpha)\nu_k$ which in view of $\alpha, \beta \in C_0^\infty(L, \mathbb{R})$ has a non-local nature. From standard PDE theory we deduce that (3.11) admits a smooth solution for a short time and that the Legendre condition is preserved during the evolution. In case $n = 1$, the flow is analogue to the volume preserving mean curvature flow of curves in $\mathbb{R}^2$ (compare with [13]) because then the ξ-component of the velocity has length equal to $\left| k - \frac{\int_L k d\mu}{\int_L d\mu} \right|$, where

$$k := g(H, d\mu)$$

denotes the curvature of the curve. In this case β is the function such that

$$d\beta = \left(k - \frac{\int_L k d\mu}{\int_L d\mu} \right) d\mu, \quad \int_L \beta = 0.$$

From Corollary 3.3 we get that

$$\frac{d}{dt} d\mu = -g(H, d\beta) d\mu, \tag{3.12}$$

provided L evolves according to (3.11) because $f = -\beta$. From this it follows that

$$\begin{aligned}\frac{d}{dt}\int_L d\mu &= -\int_L g(H, d\beta)\, d\mu = \int_L \beta\, d^\dagger H\, d\mu = \int_L \beta\Delta\beta\, d\mu \\ &= -\int_L |\nabla\beta|^2\, d\mu \,\leq\, 0,\end{aligned} \tag{3.13}$$

so that the Legendrian mean curvature flow is always volume decreasing.

Proof of Theorem 1.3. a) Let L_0, L_1 be Legendrian isotopic and assume that the isotopy is generated by a smooth family of functions f, i.e., by the flow

$$\frac{d}{dt}F = 2fX_\lambda + \nabla^k f\nu_k.$$

Since $K+2=0$, there exists a Legendrian angle α that satisfies

$$d\alpha = H, \quad \frac{d}{dt}\alpha = -\Delta f, \quad \int_{L_0}\cos(\alpha)d\mu > 0.$$

We compute

$$\frac{d}{dt}\int\cos(\alpha)d\mu = \int\left(\sin(\alpha)\Delta f + \cos(\alpha)\langle df, H\rangle\right)d\mu$$

and partial integration gives

$$\frac{d}{dt}\int\cos(\alpha)d\mu = \int\left(-\cos(\alpha)\langle d\alpha, df\rangle + \cos(\alpha)\langle df, H\rangle\right)d\mu = 0,$$

i.e., $\int\cos(\alpha)d\mu$ is a Legendrian isotopy invariant. Since

$$\mathrm{Vol}\,(L) \geq \int\cos(\alpha)d\mu$$

and $\int_{L_0}\cos(\alpha)d\mu > 0$ we are done.

b) Here, there exists a Legendre angle α with

$$d\alpha = H, \quad \frac{d}{dt}\alpha = \Delta\alpha + (2+K)\alpha,$$

$$\int_{L_0}\cos(\alpha)d\mu > 0, \quad \cos(\alpha) \geq 0 \quad \text{at } t = 0.$$

In particular α and β differ only by a function m that depends on t only. The maximum principle and $K+2<0$ imply

$$\cos(e^{-(2+K)t}\alpha) > 0, \forall t > 0$$

and as above we compute

$$\begin{aligned}&\frac{d}{dt}\int\cos(e^{-(2+K)t}\alpha)d\mu \\ &\quad= \int\Big(-\sin(e^{-(2+K)t}\alpha)\Delta(e^{-(2+K)t}\alpha) - \cos(e^{-(2+K)t}\alpha)|H|^2\Big)\,d\mu \\ &\quad= \int\Big(e^{-2(2+K)t} - 1\Big)\cos(e^{-(2+K)t}\alpha)|H|^2 d\mu \\ &\quad\geq 0\end{aligned}$$

so that

$$\mathrm{Vol}(L_t) = \int_{L_t} d\mu \geq \int_{L_t} \cos(e^{-(2+K)t}\alpha) d\mu \geq \int_{L_0} \cos(e^{-(2+K)t}\alpha) d\mu > 0.$$

□

4. Shortening Legendre curves

From now on we will only consider the case $n = 1$. Therefore L is always a closed Legendre curve. The (mean) curvature form can be decomposed as

$$H = d\beta + h d\mu,$$

where $d\mu$ is the line element and h is given by

$$h = \frac{\int_L H}{\int_L d\mu}.$$

Let us define the function

$$p := d\mu(\nabla\beta).$$

Then

$$p d\mu = d\beta \tag{4.1}$$

and

$$|\nabla\beta|^2 = p^2. \tag{4.2}$$

The curvature k of L as defined above becomes

$$k = p + h.$$

Next we compute

$$\begin{aligned}
\frac{d}{dt} d\beta &= \frac{d}{dt} H - \left(\frac{d}{dt} h\right) d\mu - h \frac{d}{dt} d\mu \\
&= d\big(\Delta\beta + (2+K)\beta\big) - \left(\frac{\int \frac{d}{dt} H - h \int \frac{d}{dt} d\mu}{\int d\mu} - hp(h+p)\right) d\mu \\
&= d\big(\Delta\beta + (2+K+h^2)\beta\big) + h\left(p^2 - \frac{\int p^2 d\mu}{\int d\mu}\right) d\mu.
\end{aligned}$$

Let q be the uniquely determined function with

$$\int_L q d\mu = 0, \quad dq = \left(p^2 - \frac{\int p^2 d\mu}{\int d\mu}\right) d\mu.$$

Then

$$\frac{d}{dt} d\beta = d\big(\Delta\beta + (2+K+h^2)\beta + hq\big). \tag{4.3}$$

Moreover

$$\begin{aligned}
\frac{d}{dt}\nabla\beta &= \nabla\Delta\beta + (2+K+h^2)\nabla\beta \\
&\quad + h\nabla q + 2H(\nabla\beta)\nabla\beta \\
&= \Delta\nabla\beta + (2+K+h^2)\nabla\beta \\
&\quad + h\nabla q + 2p(h+p)\nabla\beta.
\end{aligned} \tag{4.4}$$

Since $d\mu$ is parallel we compute

$$\Delta p = \Delta\left(d\mu(\nabla\beta)\right) = d\mu(\Delta\nabla\beta).$$

Hence

$$\begin{aligned}\frac{d}{dt}p &= \frac{d}{dt}(d\mu(\nabla\beta))\\ &= \Delta p + (2+K+h^2)p + h\left(p^2 - \frac{\int p^2 d\mu}{\int d\mu}\right)\\ &\quad + 2p^2(h+p) - p(h+p)p\\ &= \Delta p + (2+K+(h+p)^2)p - h\frac{\int p^2 d\mu}{\int d\mu}, \end{aligned}\tag{4.5}$$

where the term $-p(h+p)p$ in the second to last line occurs due to $\frac{d}{dt}d\mu = -p(h+p)d\mu$. Since

$$\frac{d}{dt}h = h\frac{\int p^2 d\mu}{\int d\mu}$$

we also observe

$$\begin{aligned}\frac{d}{dt}k &= \frac{d}{dt}(p+h)\\ &= \Delta k + (2+K+k^2)p.\end{aligned}\tag{4.6}$$

In particular

$$\frac{d}{dt}k^2 = \Delta k^2 - 2|\nabla k|^2 + 2kp(2+K+k^2).\tag{4.7}$$

The strong maximum principle now implies:

Corollary 4.1. *Assume $k^2+K+2 \le 0$ for $t=0$ and that $h=0$, i.e., the rotation number of L vanishes. Then there exist constants $c, \lambda > 0$ such that*

$$k^2 \le ce^{-\lambda t}, \quad \forall\, t.$$

Proof. In case $h = 0$ we have

$$\frac{d}{dt}k^2 \quad = \quad \Delta k^2 - 2|\nabla k|^2 + 2k^2(2+K+k^2).$$

and by the strong parabolic maximum principle we must either have $k \equiv 0, \forall t \ge 0$ or $k^2 + 2 + K < -\epsilon$ for a positive constant ϵ and for all $t \ge t_0$, where $t_0 > 0$ is fixed. But then

$$\frac{d}{dt}k^2 \quad \le \quad \Delta k^2 - 2\epsilon k^2$$

implies the result. □

If $h \neq 0$, then we can still derive a bound for k^2 because from (4.6) we obtain a nice evolution equation for the quantity

$$r := k^2 + K + 2,$$

namely

$$\frac{d}{dt}r = \Delta r - 2|\nabla k|^2 + 2pkr.\tag{4.8}$$

Again the maximum principle gives:

Corollary 4.2. *Assume $k^2 + K + 2 \leq 0$ for $t = 0$. Then this remains true during the evolution.*

Remark 4.3. Let e_1, e_2 be an orthonormal basis of ξ. Then the 3 sectional curvatures of M are

$$\sigma(X_\lambda, e_1) = 1 = \sigma(X_\lambda, e_2), \quad \sigma := \sigma(e_1, e_2) = K - 1.$$

In particular $K + 2 = 3 + \sigma$ and if we compute $K + 2$ for the three cases given in Example 2.7 we obtain:

a) $K + 2 = \sigma + 3 = \frac{4}{c} > 0$,
b) $K + 2 = \sigma + 3 = 0$,
c) $K + 2 = \sigma + 3 = \theta < 0$.

H does not change its cohomology class and therefore

$$h(t) = h(0)\frac{\int d\mu_0}{\int d\mu_t}, \tag{4.9}$$

in particular the sign of h remains the same. This implies that if

$$\mathrm{rot}(L) := \frac{1}{2\pi}\int H \neq 0,$$

then the stationary curves of the Legendrian curve shortening flow are no longer geodesics but Hamiltonian minimal curves, i.e., curves for which

$$H = h d\mu$$

is a harmonic 1-form and $k = h$ is constant.

Lemma 4.4. *Under the assumptions made in Theorem 1.1 there exists for each $m \geq 0$ a constant $c_m > 0$ such that*

$$||\nabla^m k||_t^2 \leq c_m, \quad \forall\, t \in [0, T), \tag{4.10}$$

where $[0, T)$ denotes the maximal time interval on which a smooth solution of (3.11) exists and

$$||\nabla^m k||_t^2 = \sup_{L_t} |\nabla^m k|^2.$$

Proof. We prove this by induction on m. The case $m = 0$ is Corollary 4.2. Suppose now that Lemma 4.4 holds for all $0 \leq l < m$, where $m > 0$. With the evolution equation for k we obtain for any $0 \leq l \leq m$ that there exists a constant $a_l > 0$ such that

$$\frac{d}{dt}|\nabla^l k|^2 \leq \Delta|\nabla^l k|^2 - 2|\nabla^{l+1}k|^2 + a_l\left(|\nabla^l k|^2 + 1\right).$$

Then define

$$\phi := |\nabla^m k|^2 + a_m|\nabla^{m-1}k|^2.$$

For ϕ we get

$$\begin{aligned}
\frac{d}{dt}\phi &\leq \Delta\phi + a_m\left(|\nabla^m k|^2 + 1\right) - 2a_m|\nabla^m k|^2 + a_m a_{m-1}\left(|\nabla^{m-1}k|^2 + 1\right)\\
&= \Delta\phi - a_m\phi + a_m(a_m + a_{m-1})|\nabla^{m-1}k|^2 + a_m(a_{m-1} + 1)\\
&\leq \Delta\phi - a_m\phi + a_m(a_m + a_{m-1})c_{m-1} + a_m(a_{m-1} + 1)
\end{aligned}$$

and the maximum principle implies that ϕ and then also $|\nabla^m k|^2$ must be uniformly bounded. $\square$

Lemma 4.5. *Under the assumptions made in Theorem 1.1 there exist constants $l_0, l_\infty > 0$ such that*

$$l_0 \geq l(L_t) := \int_{L_t} d\mu \geq l_\infty, \quad \forall\, t \in [0,T). \tag{4.11}$$

Proof. For l_0 we may choose $l(L_0)$ because by (3.13) we have $\frac{d}{dt} l(L_t) \leq 0$. Now we distinguish two cases:

(i) $\mathrm{rot}(L) \neq 0$: From (4.9) we deduce that h becomes unbounded if and only if $l(L_t)$ tends to zero. On the other hand $k = h + p$ is bounded by Corollary 4.2 and since $p = d\mu(\nabla\beta)$ we conclude that p must vanish in at least two points on L so that in these points $k = h$. Consequently, since h is constant in space, we have shown that h is uniformly bounded from above and $l(L_t)$ must admit a lower positive bound l_2.

(ii) $\mathrm{rot}(L) = 0$: Then we use Corollary 4.1 and (3.13) to estimate

$$\frac{d}{dt} l(L_t) = -\int_{L_t} k^2 d\mu_t \geq -ce^{-\lambda t} l(L_t)$$

so that

$$l(L_t) \geq l_0 e^{\frac{1}{\lambda}\left(e^{-\lambda c t} - 1\right)}.$$

$\square$

Corollary 4.6. *Under the assumptions made in Theorem 1.1 there exists for each $m \geq 0$ a constant $b_m > 0$ such that*

$$||\nabla^m F||_t^2 \leq b_m, \quad \forall\, t \in [0,T). \tag{4.12}$$

Proof. For $m \geq 2$ the estimates follow from Lemma 4.4. The case $m = 1$ follows from Lemma 4.5 and the compactness of M implies a bound for F as well. Here, the norm of F shall be defined as the distance from a fixed point $p \in M$. $\square$

Proof of Theorem 1.1. From Corollary 4.6 we know that F is uniformly bounded in C^∞. Thus $T = \infty$ and we can extract a convergent subsequence. The smooth limit curve must be a stationary solution of (3.11) and therefore a curve of constant curvature. In case $\mathrm{rot}(L) = 0$, the limit curve must be a geodesic. We show that the L^2-norm of β tends to zero if $\mathrm{rot}(L) = 0$. Let us first compute the evolution equation for β. From (4.3) we conlude that there exists a function $c = c(t)$ such that

$$\frac{d}{dt}\beta = \Delta\beta + (h^2 + K + 2)\beta + hq + c$$

and then $\int \beta\, d\mu = 0, \forall\, t$ implies

$$c = \frac{1}{\int d\mu} \int (pk\beta - hq)\, d\mu$$

and in particular

$$\begin{aligned}\frac{d}{dt}\int \beta^2 d\mu = &-2\int |\nabla\beta|^2\, d\mu + 2(h^2 + K + 2)\int \beta^2\, d\mu \\ &+ 2h\int q\beta\, d\mu - \int pk\beta^2\, d\mu.\end{aligned}$$

If $\mathrm{rot}(L) = 0$, then $h = 0, k = p, p^2 = |\nabla\beta|^2$ and the last equation simplifies to

$$\frac{d}{dt}\int \beta^2 d\mu = -2\int |\nabla\beta|^2\, d\mu + 2(K+2)\int \beta^2\, d\mu - \int |\nabla\beta|^2\beta^2\, d\mu.$$

Note that by assumption $K + 2 + k^2 \leq 0$, so that $K + 2 \leq 0$. Since by Lemma 4.5 all metrics are uniformly equivalent and by definition $\int \beta d\mu = 0$ we can apply the Poincare inequality to estimate

$$\frac{d}{dt}\int \beta^2 d\mu \leq -\epsilon \int \beta^2 d\mu$$

with a fixed constant $\epsilon > 0$. Hence there exists a positive constant c with

$$\int \beta^2 d\mu \leq ce^{-\epsilon t}, \quad \forall\, t \in [0, T)$$

and $T = \infty$ implies that $\int \beta^2 d\mu$ converges to 0 in L^2. Standard arguments (e.g., see [7]) then show that not only the subsequence but also the complete family of curves must converge to a geodesic in the C^∞-topology. □

Proof of Theorem 1.2. From (4.3) we deduce that there exists a smooth family of functions α such that $d\alpha = H$, $\cos(\alpha) \geq 0$ at $t = 0$ and

$$\frac{d}{dt}\alpha = \Delta\alpha + (3+\sigma)\alpha.$$

We see that β and α differ only by a function c depending only on time. Let us define the function

$$\varrho(t) := e^{-(3+\sigma)t}.$$

Then

$$\frac{d}{dt}(\varrho\alpha) = \Delta(\varrho\alpha)$$

and the maximum principle implies that $\cos(\varrho\alpha) > 0$ for all $t > 0$, in particular the oscillation of $\varrho\alpha$ is strictly bounded from above by π. In a next step we compute the evolution equation of the quantity

$$l := mk^2,$$

where $m = m(\alpha, t)$ is a function to be determined. We let

$$\dot{m} := \frac{\partial m}{\partial t}, \quad m' := \frac{\partial m}{\partial \alpha}$$

and obtain

$$\begin{aligned}
\frac{d}{dt}l &= \dot{m}k^2 + m'k^2\big(\Delta\alpha + (3+\sigma)\alpha\big) + 2mk\big(\Delta k + k(3+\sigma+k^2)\big)\\
&= \Delta l - \frac{1}{2mk^2}|\nabla l|^2 - \frac{m'}{m}\langle\nabla l, \nabla\alpha\rangle - lk^2\left(\frac{m''}{m} - \frac{3}{2}\left(\frac{m'}{m}\right)^2\right)\\
&\quad + \dot{m}k^2 + (3+\sigma)m'\alpha k^2 + 2l(3+\sigma+k^2)\\
&= \Delta l - \frac{1}{2mk^2}|\nabla l|^2 - \frac{m'}{m}\langle\nabla l, \nabla\alpha\rangle - lk^2\left(\frac{m''}{m} - \frac{3}{2}\left(\frac{m'}{m}\right)^2 - 2\right)\\
&\quad + l\left(\frac{\dot{m}}{m} + (3+\sigma)\frac{m'\alpha}{m} + 2(3+\sigma)\right)
\end{aligned}$$

at those points where $k \neq 0$. Now we choose

$$m := \frac{\varrho^2}{\cos^2(\varrho\alpha)}$$

and get

$$\frac{\dot{m}}{m} = -2(3+\sigma)\left(1 + \varrho\alpha\tan(\varrho\alpha)\right)$$

as well as

$$\frac{m''}{m} - \frac{3}{2}\left(\frac{m'}{m}\right)^2 - 2 = 2(\varrho^2 - 1)$$

so that if $\sigma + 3 \leq 0$ we deduce

$$\frac{d}{dt} l \leq \Delta l - \frac{m'}{m}\langle \nabla l, \nabla\alpha\rangle$$

at all points where $k \neq 0$. l is well-defined for $t > 0$ since $\cos(\varrho\alpha) > 0$ and l vanishes if k vanishes. Therefore the maximum principle implies that there exists a constant $c > 0$ with

$$k^2 \leq ce^{2(3+\sigma)t}\cos^2(e^{-(3+\sigma)t}\alpha) \leq ce^{2(3+\sigma)t} \leq c.$$

As in the proof of Theorem 1.1 this implies C^∞-bounds, that the curvature must tend to zero and that the Legendrian loops converge to a closed Legendrian geodesic. □

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