Scalar curvature, covering spaces, and Seiberg-Witten theory

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ABSTRACT. The *Yamabe invariant* $\mathcal{Y}(M)$ of a smooth compact manifold is roughly the supremum of the scalar curvatures of unit-volume constant-scalar-curvature Riemannian metrics g on M. (To be precise, one only considers those constant-scalar-curvature metrics which are *Yamabe minimizers*, but this technicality does not, e.g., affect the *sign* of the answer.) In this article, it is shown that many 4-manifolds M with $\mathcal{Y}(M) < 0$ have have finite covering spaces $\widetilde{M}$ with $\mathcal{Y}(\widetilde{M}) > 0$.

Two decades ago, Lionel Bérard Bergery [2] pointed out that there are high-dimensional smooth compact manifolds M which do not admit metrics of positive scalar curvature, but which nevertheless have finite coverings that *do* admit such metrics. For example, let Σ be an exotic 9-sphere which does not bound a spin manifold, and consider the connected sum $M = (S^2 \times \mathbb{RP}^7)\#\Sigma$. On one hand, M is a spin manifold with nonzero Hitchin invariant $\hat{\mathfrak{a}}(M) \in \mathbb{Z}_2$, so [7] there are harmonic spinors on M for every choice of metric; the Lichnerowicz Weitzenböck formula for the Dirac operator therefore tells us that no metric on M can have positive scalar curvature. On the other hand, the universal cover $\widetilde{M} = (S^2 \times S^7)\#2\Sigma$ of M is diffeomorphic to $S^2 \times S^7$, on which the obvious product metric certainly has positive scalar curvature.

As will be shown here, the same phenomenon also occurs in dimension four. Indeed, far more is true. In the process of passing from a 4-manifold to a finite cover, it is even possible to change the sign of the *Yamabe invariant*.

The Yamabe invariant is a diffeomorphism invariant that historically arose from an attempt to construct Einstein metrics (metrics of constant Ricci curvature) on smooth compact manifolds. A standard computation [3] shows that the Einstein metrics on any given smooth compact manifold M of dimension $n > 2$ are exactly the critical points of the *normalized total scalar curvature*

$$\mathfrak{S}(g) = V_g^{(2-n)/n} \int_M s_g d\mu_g,$$

considered as a functional on the space of all Riemannian metrics g on M; here s, $d\mu$, and V respectively denote the scalar curvature, volume measure, and total volume of the relevant metric. However, one cannot possibly hope to find a critical point of $\mathfrak{S}$ by either maximizing or minimizing it, as it is bounded neither above nor below. Nevertheless, as was first pointed out by Hidehiko Yamabe [23], the restriction of $\mathfrak{S}$ to any *conformal class*

$$\gamma = [g] = \{ug \mid u : M \xrightarrow{C^\infty} \mathbb{R}^+\}$$

of metrics *is* always bounded below. The trail blazed by Yamabe eventually led [20, 1, 15] to a proof of the fact that, for each conformal class γ, this infimum is actually achieved, by a constant-scalar-curvature metric known as a *Yamabe minimizer*. Yamabe's ultimate goal was to construct Einstein metrics by maximizing the restriction of $\mathfrak{S}$ to the set of these Yamabe minimizers. This last idea turns out to be unworkable in practice, but it nonetheless gives rise to a beautiful, real-valued diffeomorphism invariant [8, 16, 10]

$$\mathcal{Y}(M) = \sup_\gamma \inf_{g \in \gamma} \mathfrak{S}(g),$$

called the *Yamabe invariant* (or *sigma constant*) of M. It is not hard to show that $\mathcal{Y}(M) > 0$ if and only if M admits a metric of positive scalar curvature; thus the problem of computing the Yamabe invariant may be thought of as a quantitative refinement of the question of whether a given manifold admits positive-scalar-curvature metrics. On the other hand, if M does *not* admit metrics of positive scalar curvature, the number $\mathcal{Y}(M) \leq 0$ is just the supremum of the scalar curvatures of all unit-volume constant-scalar-curvature metrics on M. Dimension 4 turns out to be remarkably special so far as this invariant is concerned. Indeed, Seiberg-Witten theory allows one to show [10] that there are many simply connected 4-manifolds with $\mathcal{Y}(M) < 0$. By contrast, however, Petean [13] has shown that every simply connected compact manifold M^n of dimension $n \geq 5$ has $\mathcal{Y}(M) \geq 0$.

The main construction used in this paper primarily depends on properties of the oriented 4-manifold

$$N = (S^2 \times S^2)/\mathbb{Z}_2,$$

where the $\mathbb{Z}_2$ action is generated by the double antipodal map

$$(\vec{x}, \vec{y}) \mapsto (-\vec{x}, -\vec{y}).$$

Let X be any non-spin compact complex surface of general type which can be expressed as a *complete intersection* of complex hypersurfaces in some complex projective space; for example, one could take X to be the Fermat hypersurface

$$\left\{ [v : w : x : y] \in \mathbb{CP}_3 \mid v^m + w^m + x^m + y^m = 0 \right\}$$

for any odd $m \geq 5$.

Theorem 1. *Let X and N be as above, and let $M = X \# N$. Then M has negative Yamabe invariant, but its universal cover $\widetilde{M}$ has positive Yamabe invariant.*

Proof. Since X is a complex algebraic surface with $b_+(X) > 1$, the Seiberg-Witten invariant of X is well-defined and nonzero for the canonical spin^c structure determined by the complex structure. On the other hand, N satisfies $b_2(N) = b_1(N) = 0$, and a gluing result of Kotschick-Morgan-Taubes [9] thus implies that the Seiberg-Witten invariant is nonzero for the associated spin^c structure on $M = X \# N$ with

$c_1 = c_1(X)$. This tells us [22] that M does not admit any metrics of positive scalar curvature, and that [10], moreover,

$$\mathcal{Y}(X \# N) \leq -4\pi\sqrt{2c_1^2(X)}.$$

In particular, $\mathcal{Y}(X \# N) < 0$.

On the other hand, the universal cover of M is $\widetilde{M} = X \# X \# (S^2 \times S^2)$. But Gompf [5], inspired by the earlier work of Mandelbaum and Moishezon [12], has used a handle-slide argument to show that $X \# (S^2 \times S^2)$ *dissolves*, in the sense that

$$X \# (S^2 \times S^2) \overset{\text{diff}}{\approx} k_1 \mathbb{CP}_2 \# \ell_1 \overline{\mathbb{CP}}_2,$$

where $k_1 > 2$ and $\ell_1 \neq 0$. Since $(S^2 \times S^2) \# \mathbb{CP}_2 \approx 2\mathbb{CP}_2 \# \overline{\mathbb{CP}}_2$, it follows that

$$X \# 2\mathbb{CP}_2 \# \overline{\mathbb{CP}}_2 \approx (k_1 + 1)\mathbb{CP}_2 \# \ell_1 \overline{\mathbb{CP}}_2,$$

and hence that

$$\widetilde{M} = 2X \# (S^2 \times S^2) \approx X \# k_1 \mathbb{CP}_2 \# \ell_1 \overline{\mathbb{CP}}_2 \approx k\mathbb{CP}_2 \# \ell\overline{\mathbb{CP}}_2,$$

where $k = 2b_+(X) + 1$ and $\ell = 2b_-(X) + 1$. Since a connected sum of positive-scalar-curvature manifolds admits metrics of positive scalar curvature [6, 17], we thus conclude that $\mathcal{Y}(\widetilde{M}) > 0$. □

It is unclear whether an analogous change in the sign of the Yamabe invariant ever occurs in higher dimensions. At any rate, this phenomenon certainly *does not* occur in Bérard Bergery's examples. For example, we obviously have $\mathcal{Y}(S^2 \times \mathbb{RP}^7) > 0$; and we also know that $\mathcal{Y}(\Sigma^9) = 0$ by Petean's theorem [13]. The Petean-Yun surgery theorem [14] therefore implies that their connected sum has $\mathcal{Y} \geq 0$, too; and since $(S^2 \times \mathbb{RP}^7) \# \Sigma^9$ does not admit metrics of positive scalar curvature, this shows that $\mathcal{Y}([S^2 \times \mathbb{RP}^7] \# \Sigma) = 0$. (Indeed, so far as we seem to know at present, *every* compact n-manifold with $|\pi_1| < \infty$ and $n \geq 5$ could turn out to have nonnegative Yamabe invariant; for an interesting partial result in this direction, see [4].)

It is also perhaps worth mentioning that one can actually compute the exact value of the Yamabe invariant for any of the manifolds $M = X \# N$ considered in Theorem 1. Indeed, as already noted, the Seiberg-Witten argument tells us that $\mathcal{Y}(M) \leq -4\pi\sqrt{2c_1^2(X)}$. On the other hand, $\mathcal{Y}(N) > 0$ and $\mathcal{Y}(X) < 0$, a general inequality due to Osamu Kobayashi [8] tells us that

$$\mathcal{Y}(X \# N) \geq \mathcal{Y}(X).$$

However, because X is a minimal complex surface of general type, its Yamabe invariant is given [10] by $\mathcal{Y}(X) = -4\pi\sqrt{2c_1^2(X)}$. The above inequalities therefore allow us to ascertain the exact value

$$\mathcal{Y}(X \# N) = -4\pi\sqrt{2c_1^2(X)}$$

of the Yamabe invariant for any of the manifolds in question.

By contrast, however, exact calculations of the Yamabe invariant are notoriously difficult in the positive case, owing to the fact that in the positive regime a constant-scalar-curvature metric need not be a Yamabe minimizer. However, we do know [11] that $\mathcal{Y}(\mathbb{CP}_2) = \mathcal{Y}(\overline{\mathbb{CP}}_2) = 12\pi\sqrt{2}$. Thus Kobayashi's inequality [8] predicts that any connected sum of $\mathbb{CP}_2$'s and $\overline{\mathbb{CP}}_2$'s satisfies

$$\mathcal{Y}(k\mathbb{CP}_2 \# \ell\overline{\mathbb{CP}}_2) \in [\mathcal{Y}(\mathbb{CP}_2), \mathcal{Y}(S^4)] = [12\pi\sqrt{2}, 8\pi\sqrt{6}],$$

and we can thus at least conclude that the Yamabe invariant of the corresponding universal cover $\widetilde{M}$ is always somewhere in this narrow range.

Finally, let us observe that the examples in Theorem 1 can be greatly generalized, provided one does not insist on passing to the *universal* cover.

Theorem 2. *Let Y be a symplectic 4-manifold with $b_+ > 1$, $|\pi_1| < \infty$, and non-spin universal cover. Let $N = (S^2 \times S^2)/\mathbb{Z}_2$, as before. Then $M = Y\#2N$ does not admit metrics of positive scalar curvature, but nonetheless has finite coverings $\widetilde{M} \to M$ which* do *carry such metrics. Moreover, if the symplectic minimal model of Y has $c_1^2 \neq 0$, the Yamabe invariant reverses sign as one passes from M to $\widetilde{M}$.*

Proof. By a celebrated theorem of Taubes [18], the canonical spin^c structure of the symplectic manifold Y has nonzero Seiberg-Witten invariant, and since $N\#N$ has $b_1 = b_2 = 0$, the same gluing argument [9] as before implies that the Seiberg-Witten invariant is nonzero for a spin^c structure on $M = Y\#2N$ with $c_1 = c_1(Y)$. It thus follows that M does not admit metrics of positive scalar curvature. But even more is true. By another remarkable result of Taubes [19], we can express Y as an iterated symplectic blow-up of a symplectic manifold Y_0, called the *symplectic minimal model of Y*, which contains no symplectic (-1)-spheres, and satisfies $c_1^2(Y_0) \geq 0$. On the other hand, the same argument used in the proof of [10, Theorem 2] then shows that, for every metric g on M, there is a Seiberg-Witten basic class for which $(c_1^+)^2 \geq c_1^2(Y_0)$. The estimate $\mathcal{Y}(M) \leq -4\pi\sqrt{2c_1^2(Y_0)}$ then follows immediately. This shows that M actually has negative Yamabe invariant whenever $c_1^2(Y_0) \neq 0$.

Next, let X denote the universal cover of Y, and observe that $Y\#2N$ has an n-fold cover of the form $X\#2nN$, where $n = |\pi_1(Y)|$. Thus, unfolding copies of N one by one, we obtain a sequence of covering spaces $\widetilde{M}_\ell \to M$ with

$$\widetilde{M}_\ell = 2^\ell X \ \# \ \left(2^\ell - 1\right)(S^2 \times S^2) \ \# \ \left[2^{\ell+1}(n-1) + 2\right] N$$

for each $\ell \geq 1$. On the other hand, since X is simply connected and non-spin by assumption, a justly famous result of Wall [21] asserts that there is an integer k_0 such that

$$X\#k_0(S^2 \times S^2) \approx [k_0 + b_+(X)]\mathbb{CP}_2\#[k_0 + b_-(X)]\overline{\mathbb{CP}}_2.$$

Since $(S^2 \times S^2)\#\mathbb{CP}_2 \approx 2\mathbb{CP}_2\#\overline{\mathbb{CP}}_2$, it then follows that

$$X\#(k_0 + 1)\mathbb{CP}_2\#k_0\overline{\mathbb{CP}}_2 \approx [k_0 + 1 + b_+(X)]\mathbb{CP}_2\#[k_0 + b_-(X)]\overline{\mathbb{CP}}_2.$$

Induction therefore gives us

$$mX\#k(S^2 \times S^2) \approx [k + mb_+(X)]\mathbb{CP}_2\#[k + mb_-(X)]\overline{\mathbb{CP}}_2$$

for any $k \geq k_0$ and $m \geq 1$. Thus, setting $\widetilde{M} = \widetilde{M}_\ell$ for any $\ell > \log_2 k_0$, we have constructed a finite covering $\widetilde{M} \to M$ with

$$\widetilde{M} \approx p\mathbb{CP}_2\#q\overline{\mathbb{CP}}_2\#rN,$$

and such an $\widetilde{M}$ admits positive-scalar-curvature metrics because it is a connected sum of manifolds with positive scalar curvature. $\qquad\square$

It is perhaps worth noting that when the given Y is not simply-connected, essentially the same argument would also work for $Y\#N$.

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