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Weak-L^1 estimates and ergodic theorems

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Abstract. We prove that for any dynamical system (X, Σ, m, T), the maximal operator defined by

$$N^* f(x) = \sup_n \frac{1}{n} \# \left\{ 1 \leq i : \frac{f(T^i x)}{i} \geq \frac{1}{n} \right\}$$

is almost everywhere finite for f in the Orlicz class $L \log\log L(X)$, extending a result of Assani [2]. As an application, a weighted return times theorem is also proved.

Contents

1. Introduction

Let T be a measure preserving transformation of a probability space (X, Σ, m). We call (X, Σ, m, T) a dynamical system. The following return times theorem was proved in [4]:

Theorem 1 (Bourgain). *Let $1 \leq p \leq \infty$ and let $1/p + 1/q = 1$. For each dynamical system (X, Σ, m, T) and $f \in L^p(X)$, there is a set $X_0 \subset X$ of full measure, such that for any other dynamical system $(Y, \mathcal{F}, \mu, S)$, $g \in L^q(Y)$ and $x \in X_0$, the limit,*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} f(T^k x) g(S^k y),$$

exists for μ a.e. y.

One of the most interesting unanswered questions that emerges from this result is whether or not the fact that f and g lie in dual spaces is in general necessary in order to have a positive result. Neither of the existing proofs of Theorem 1 gives any indication on this, since each of them relies on Hölder's inequality.

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On the other hand, if (gS^k) is replaced with a sequence (ξ_k) of independent identically distributed random variables such that $\mathbb{E}(|\xi_1|) < \infty$, then the following criterion of B. Jamison, S. Orey and W. Pruitt [5] proves to be an excellent tool to break the duality.

Theorem 2 (Jamison, Orey and Pruitt)**.** *Let (a_k) be a sequence of positive real numbers and let $N^* = \sup_n \frac{1}{n}\#\{k : a_k/\sum_{i=1}^k a_i \geq 1/n\}$, then the following are equivalent:*

1. $N^* < \infty$.
2. *For any i.i.d. sequence of random variables (ξ_k) such that $\mathbb{E}(|\xi_1|) < \infty$, defining a new sequence (Ξ_n) of random variables by*

$$\Xi_n(\omega) = \sum_{k=1}^n a_k \xi_k(\omega) / \sum_{k=1}^n a_k,$$

the sequence (Ξ_n) converges pointwise almost surely.

Motivated by this criterion, Assani [1] introduced the following maximal function: given $f \in L^1(X)$, consider

$$N^* f(x) = \sup_n \frac{1}{n} \# \left\{ 1 \leq i : \frac{f(T^i x)}{i} \geq \frac{1}{n} \right\}.$$

He proved in [2] for $f \in L \log L(X)$, $N^* f \in L^1$ and in particular $N^* f(x) < \infty$ for a.e. x. Based on this and Theorem 2, the following "duality-breaking" version of Theorem 1 follows almost immediately:

Corollary 3 (Assani)**.** *Let (X, Σ, m, T) be a measure-preserving transformation and let the function f satisfy $\int |f| \log^+ |f| \, dm < \infty$, that is $f \in L \log L(X)$. Then there is a set $X_0 \subset X$ of full measure, such that for any sequence (ξ_k) of i.i.d. random variables on the probability space $(\Omega, \mathcal{F}, \mu)$ with $\xi_1 \in L^1(\Omega)$ and any $x \in X_0$*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^n f(T^k x) \xi_k(\omega)$$

exists for μ a.e. ω.

Moreover in [1] it is proved that if Theorem 1 is true for $p = q = 1$, then $N^* f(x)$ must be finite almost everywhere for all $f \in L^1(X)$. This connection sheds more light on the importance of the operator N^* and motivates its further study.

In the next section we will prove the finiteness of N^* for functions in the larger class $L \log \log L$. Note that while Assani shows that $N^* f \in L^1$ for $f \in L \log L$, our result establishes that $N^* f \in L^{1,\infty}$ for $f \in L \log \log L$ (i.e., that $\sup_t tm\{x \colon N^* f(x) > t\} < \infty$) so that while our hypothesis is weaker, so is our conclusion. Note however that since our conclusion implies that $N^* f(x) < \infty$ for almost every x, it is sufficient to imply a corollary like Corollary 3 in the case where $f \in L \log \log L$.

In a preprint that appeared at around the time this paper was submitted, Assani, Buczolich, and Mauldin [3] show that there exists an $f \in L^1(X)$ such that $N^* f(x) = \infty$ almost everywhere.

2. Main results

Throughout this section we will denote the natural logarithm of x by $\log x$ and the weak-L^1 norm of f by

$$\|f\|_{1,\infty} = \sup_{\lambda>0} \lambda m\{x : |f(x)| > \lambda\}.$$

We will also need to refer to the *entropy* of a sequence of positive real numbers. Specifically, for a sequence (a_n) of nonnegative real numbers (not all 0), define the entropy by

$$H((a_n)) = \sum_n -\frac{a_n}{\sum_j a_j} \log\left(\frac{a_n}{\sum_j a_j}\right),$$

under the convention $0 \log 0 = 0$.

We define f^* to be the ergodic maximal function, $f^*(x) = \sup_n \left|\frac{1}{n}\sum_{k=1}^n f(T^k x)\right|$. The maximal ergodic theorem asserts that $\|f^*\|_{1,\infty} \le \|f\|_1$ for all $f \in L^1(X)$. The following inequality from [7] turns out to be extremely useful to our investigation:

Lemma 4. *Suppose that for $i = 1, 2, \dots,$ $g_i(x)$ is an $L_{1,\infty}$ function on a measure space such that $\sum \|g_i\|_{1,\infty} < \infty$. Then*

$$\left\|\sum_{i=1}^{\infty} g_i\right\|_{1,\infty} \le 2(K+2)\sum_{i=1}^{\infty}\|g_i\|_{1,\infty},$$

where K is the entropy of the sequence $(\|g_n\|_{1,\infty})$.

We can now prove our main result.

Theorem 5. *For each dynamical system (X, Σ, m, T) and each $f \in L\log\log L(X)$ (that is f satisfying $\int |f| \log^+ \log^+ |f|\, dm < \infty$), $N^* f(x) < \infty$ for a.e. x.*

Proof. It is enough to consider f positive. Making use of the fact that $f(x) \le \sum_{i=1}^{\infty} 2^i \chi_{A_i}(x)$, where $A_i = \{x : 2^{i-1} < f(x) \le 2^i\}$ for $i \ge 2$ and $A_1 = \{x : f(x) \le 2\}$, it easily follows that for each n,

$$\frac{1}{n}\#\left\{k \ge 1 : \frac{f(T^k x)}{k} \ge 1/n\right\} \le \frac{1}{n}\sum_{i=1}^{\infty}\sum_{k=1}^{n2^i}\chi_{A_i}(T^k x) \le \sum_{i=1}^{\infty} 2^i (\chi_{A_i})^*(x).$$

We will show that the last term in the above inequality is finite a.e. by proving that its $L_{1,\infty}$ norm is finite.

Let $a_i = 2^i\|\chi_{A_i}\|_{1,\infty}$. By the maximal ergodic theorem, we see that $a_i \le 2^i m(A_i)$. The fact that $f \in L\log\log L$ implies that $\sum_i a_i \log i < \infty$ (and hence clearly M, which we define to be $\sum_i a_i$, is finite). From the lemma above, we see that it is sufficient to show that the entropy of the sequence (a_i) is finite: $\sum_i -a_i/M \log(a_i/M) < \infty$. One quickly sees that this is equivalent to establishing $\sum_i -a_i \log a_i < \infty$.

Consider now $S_1 = \{i : a_i \le 1/i^2\}$ and $S_2 = \{i : a_i > 1/i^2\}$. Now

$$\sum_{i\in S_1} -a_i \log a_i \le 1 + \sum_{j=2}^{\infty} \log(j^2)/j^2 < \infty$$

since $\psi(t) = -t \log t$ is increasing on $[0, 1/e]$. On the other hand

$$\sum_{i \in S_2} -a_i \log a_i < 2 \sum_{i \in S_2} a_i \log i < \infty.$$

□

Corollary 6. *For each dynamical system (X, Σ, m, T) and nonnegative function $f \in L \log\log L(X)$, there is a set $X_0 \subset X$ of full measure, such that for any sequence (ξ_k) of i.i.d. random variables on the probability space $(\Omega, \mathcal{F}, \mu)$ with $\xi_1 \in L^1(\Omega)$ and any $x \in X_0$*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} f(T^k x)\xi_k(\omega)$$

exists for μ a.e. ω.

There does not seem to be better way of exploiting Lemma 4 in order to extend even more the class of functions for which $N^* f$ is almost everywhere finite. Moreover, as we show in the following proposition, the inequality in Lemma 4 is sharp up to a constant. We note that a more general version of this proposition appears in work of Kalton [6].

Proposition 7. *Given positive numbers $a_1, \ldots, a_n$, there exist functions $g_1, \ldots, g_n$ with $\|g_i\|_{1,\infty} = a_i$ such that $\|g_1 + \ldots g_n\|_{1,\infty} \geq \frac{1}{6}(2 + K) \sum \|g_i\|_{1,\infty}$, where K is the entropy of the sequence (a_i).*

Proof. For each i, let ξ_i be a random variable taking the value $1/n$ with probability $(1 - a_i)^{n-1} a_i$. Moreover, the ξ_i's will be chosen to be independent. One can then check that $\mathbb{P}(\xi_i > \lambda) \leq a_i/\lambda$ while $\mathbb{P}(\xi_i \geq 1 - \epsilon) = a_i$ for ϵ small enough, so that $\|\xi_i\|_{1,\infty} = a_i$.

We see that

$$\begin{aligned} \mathbb{E}(\xi_i) &= \sum_{n=1}^{\infty} a_i \frac{(1 - a_i)^{n-1}}{n} \\ &= -\frac{a_i}{1 - a_i} \log a_i \geq -a_i \log a_i. \end{aligned}$$

Similarly, we see that

$$\mathbb{E}(\xi_i^2) = a_i \sum_{n=1}^{\infty} \frac{1}{n^2} (1 - a_i)^{n-1} \leq a_i \sum_{n=1}^{\infty} \frac{1}{n^2} \leq 2a_i.$$

In particular, setting $\Xi = \xi_1 + \cdots + \xi_n$, we see that $\mathbb{E}(\Xi) \geq K$ but $\mathrm{Var}(\Xi) \leq 2$.

Using Tchebychev's inequality, we see that

$$\mathbb{P}(\Xi \geq K - 2) \geq \mathbb{P}(|\Xi - \mathbb{E}(\Xi)| \leq 2) \geq 1 - \frac{\mathrm{Var}(\Xi)}{2^2} \geq \frac{1}{2}.$$

If $K > 4$, we have $\mathbb{P}(\Xi \geq K/2) \geq \frac{1}{2}$ so that the weak-L^1 norm exceeds $K/4$, which in turn exceeds $(K + 2)/6$. If $K \leq 4$, take f to be any function of weak L^1 norm 1 and let $f_n = a_n f$, so that $\|f_n\|_{1,\infty} = a_n$. Then $\sum f_i = f$, so that $\|\sum f_i\|_{1,\infty} = 1 \geq \frac{1}{6}(K + 2) \sum \|f_i\|_{1,\infty}$. This completes the proof of the proposition. □

Remark 8. Note that although $f \in L\log\log L$ is sufficient to guarantee that $N^*f < \infty$ almost everywhere, there are functions f outside $L\log\log L(X)$, for which $N^*f(x) < \infty$ for a.e. x. In particular, it is easy to construct functions outside $L\log\log L$ for which the entropy computed in Theorem 5 is finite, guaranteeing the finiteness of N^*f.

Further, if we are willing to restrict the system, we see that no condition on the distribution of f ensures the divergence of $N^*f(x)$. Specifically, Lemma 1 of [1] guarantees that whenever T^kf are independent (identically distributed) random variables with an arbitrary L^1 distribution, then $N^*f(x) < \infty$ for a.e. x.

Another consequence of Theorem 5 is the following weighted version of Corollary 3.

Theorem 9. *For each dynamical system (X, Σ, m, T) and $f \in L^1(X)$, there is a set $X_0 \subset X$ of full measure, such that for any sequence (ξ_k) of i.i.d. random variables on the probability space $(\Omega, \mathcal{F}, \mu)$ with $\xi_1 \in L^1(\Omega)$ and any $x \in X_0$*

$$\lim_{n\to\infty} \frac{1}{n\log\log n}\sum_{k=1}^{n} f(T^kx)\xi_k(\omega) = 0$$

for μ a.e. ω.

The proof will be based on the following relative of Theorem 5. Define

$$L^*f(x) = \sup_n \frac{1}{n}\#\left\{1 \le i : \frac{f(T^ix)}{i\log\log i} \ge \frac{1}{n}\right\}.$$

Lemma 10. *Let (X, Σ, m, T) be a measure-preserving system. For each $f \in L_1(X)$, $L^*f(x) < \infty$ for a.e. x.*

Proof. As usual, we can assume f is positive. Fix an $n \in \mathbb{N}$. Using the fact that $f(x) \le \sum_{i=1}^{\infty} 2^i \chi_{A_i}(x)$ we get that

$$\frac{1}{n}\#\left\{1 \le k : \frac{f(T^kx)}{k\log\log k} \ge 1/n\right\} \le \frac{1}{n}\sum_{i=1}^{\infty}\sum_{k=1}^{p_i}\chi_{A_i}(T^kx) \le \frac{1}{n}\sum_{i=1}^{\infty} p_i(\chi_{A_i})^*(x)$$

where p_i is the largest integer such that $p_i(\log\log p_i) \le n2^i$. Letting $\phi\colon (1,\infty) \to \mathbb{R}$ be the increasing function $\phi(x) = x\log\log x$, we see that $p_i \le \phi^{-1}(n2^i)$. We claim that there exists a $C > 0$ such that $p_i \le C\frac{2^in}{\log(i+1)}$ for all $i, n \in \mathbb{N}$. To see this, we check the existence of a C such that $\phi^{-1}(2^x) \le C\frac{2^x}{\log(x+1)}$ or equivalently $2^x \le \phi(C\frac{2^x}{\log(x+1)})$ for all $x \ge 0$. Hence

$$\sup_n \frac{1}{n}\#\left\{1 \le i : \frac{f(T^ix)}{i\log\log i} \ge \frac{1}{n}\right\} \le C\sum_{i=1}^{\infty}\left(\frac{2^i}{\log(i+1)}\right)(\chi_{A_i})^\star(x).$$

Based on Lemma 4 and on the maximal ergodic theorem, it suffices to prove that $\sum_{i=1}^{\infty} -\left(\frac{2^i\|\chi_{A_i}\|_1}{\log(i+1)}\right)\log\left(\frac{2^i\|\chi_{A_i}\|_1}{\log(i+1)}\right) < \infty$. By splitting the sum in two parts depending on whether or not $\frac{2^i\|\chi_{A_i}\|_1}{\log(i+1)} < \frac{1}{i^2}$ and reasoning as in the proof of Theorem 5, it easily follows that the sum from above is finite. ☐

Proof of Theorem 9. It suffices to assume that both f and ξ_1 are positive. According to the previous lemma, let X_0 the subset of full measure of X containing all the points x for which $L^*f(x) < \infty$. For a fixed $x \in X_0$ denote $w_k := f(T^k x)$ and also $W_k := k \log\log k$. The argument of Jamison, Orey and Pruitt from [5] can be extended with really no essential changes to this case, to conclude that since

$$\sup_n \frac{1}{n} \# \left\{ 1 \le i : \frac{w_i}{W_i} \ge \frac{1}{n} \right\} < \infty,$$

$$\lim_{n\to\infty} \frac{1}{W_n} \sum_{k=1}^{n} w_k \xi_k(\omega) = 0$$

for μ a.e. ω. □

Remark 11. It is not known whether in Theorem 9 the weight $n \log\log n$ can be replaced with a smaller one, like $n \log\log\log n$. Any improvement on this weight will necessarily have behind it an extension of the result of Theorem 5 to a larger Orlicz class. On the other hand, a combination of Theorem 2 and the result from [3], shows that this weight can not be chosen to be n.

Remark 12. It would be interesting to find the largest Orlicz class that would guarantee that $N^*f(x) < \infty$ almost everywhere. The above establishes that such an Orlicz class would contain $L \log\log L$.

A careful examination of the proof of [3] demonstrates that in any Orlicz class with an essentially smaller weight than the class $L \log\log\log L$, there exists a function f such that $N^*f(x) = \infty$ almost everywhere.

In particular, these two results demonstrate that the largest Orlicz class that would guarantee that $N^*f(x) < \infty$ almost everywhere lies between $L \log\log L$ and $L \log\log\log L$.

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