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Nevanlinna–Pick interpolation for Schur–Agler class functions on domains with matrix polynomial defining function in $\mathbb{C}^n$

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Abstract. We consider a bitangential interpolation problem for operator-valued functions defined on a general class of domains in $\mathbb{C}^n$ (including as particular cases, Cartan domains of types I, II and III) which satisfy a type of von Neumann inequality associated with the domain. The compact formulation of the interpolation conditions via a functional calculus with operator argument includes prescription of various combinations of functional values and of higher-order partial derivatives along left or right directions at a prescribed subset of the domain as particular examples. Using realization results for such functions in terms of unitary colligation and the defining polynomial for the domain, necessary and sufficient conditions for the problem to have a solution were established recently in Ambrozie and Eschmeier (preprint, 2002), and Ball and Bolotnikov, 2004. In this paper we present a parametrization of the set of all solutions in terms of a Redheffer linear fractional transformation acting on a free-parameter function from the class subject to no interpolation conditions. In the finite-dimensional case when functions are matrix-valued, the matrix of the linear fractional transformation is given explicitly in terms of the interpolation data.

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1. Introduction

In this paper we pursue our work on interpolation theory for Schur–Agler functions that are a far-reaching operator-valued multivariable analogue of classical Schur functions (that is, analytic and mapping the unit disk $\mathbb{D}$ into the closed unit disk $\overline{\mathbb{D}}$. The operator-valued *Schur class* $\mathcal{S}(\mathcal{E}, \mathcal{E}_*)$ consists, by definition, of analytic functions F on $\mathbb{D}$ with values $F(z)$ equal to contraction operators between two Hilbert spaces $\mathcal{E}$ and $\mathcal{E}_*$. In what follows, the symbol $\mathcal{L}(\mathcal{E}, \mathcal{E}_*)$ stands for the algebra of bounded linear operators mapping $\mathcal{E}$ into $\mathcal{E}_*$. The class $\mathcal{S}(\mathcal{E}, \mathcal{E}_*)$ admits several remarkable characterizations. We mention in particular that any such function $F(z)$ can be realized in the form

$$F(z) = D + zC(I - zA)^{-1}B$$

where the connecting operator (or *colligation*)

$$U = \begin{bmatrix} A & B \\ C & D \end{bmatrix} : \begin{bmatrix} \mathcal{H} \\ \mathcal{E} \end{bmatrix} \to \begin{bmatrix} \mathcal{H} \\ \mathcal{E}_* \end{bmatrix}$$

is unitary, and where $\mathcal{H}$ is some auxiliary Hilbert space (the *internal space* for the colligation). From the point of view of system theory, $F(z) = D + zC(I - zA)^{-1}B$ is the *transfer function* of the linear system

$$\Sigma = \Sigma(U) \colon \begin{cases} x(n+1) = Ax(n) + Bu(n) \\ \quad y(n) = Cx(n) + Du(n) \end{cases}$$

in the sense that any solution (u, x, y) of Σ defined on the nonnegative integers $\mathbb{Z}^+$ with $x(0) = 0$ satisfies

$$\widehat{y}(z) = F(z) \cdot \widehat{u}(z).$$

Here in general we denote by

$$\widehat{x}(z) = \sum_{n=0}^{\infty} x(n) z^n$$

the *Z-transform* of the sequence $\{x(n)\}_{n=0}^{\infty}$. It is also well-known that the Schur class functions satisfy a von Neumann inequality: *$F \in \mathcal{S}(\mathcal{E}, \mathcal{E}_*)$ and $T \in \mathcal{L}(\mathcal{H}')$ a contraction operator $\Longrightarrow \|F(rT)\| \le 1$ for all $r < 1$.* Here $F(rT)$ can be defined, e.g., by

$$F(rT) = \sum_{n=0}^{\infty} r^n F_n \otimes T^n \in \mathcal{L}(\mathcal{E} \otimes \mathcal{H}', \mathcal{E}_* \otimes \mathcal{H}') \quad \text{if} \quad F(z) = \sum_{n=0}^{\infty} F_n z^n.$$

Multivariable generalizations of these and many other related results have been obtained recently in the following way: let $\mathbf{Q}$ be a $p \times q$ matrix-valued polynomial

$$\mathbf{Q}(z) = \begin{bmatrix} \mathbf{q}_{11}(z) & \dots & \mathbf{q}_{1q}(z) \\ \vdots & & \vdots \\ \mathbf{q}_{p1}(z) & \dots & \mathbf{q}_{pq}(z) \end{bmatrix} : \ \mathbb{C}^n \to \mathbb{C}^{p \times q} \tag{1.1}$$

and let $\mathcal{D}_{\mathbf{Q}} \in \mathbb{C}^n$ be the domain defined by

$$\mathcal{D}_{\mathbf{Q}} = \{ z \in \mathbb{C}^n \colon \|\mathbf{Q}(z)\|_{\mathbb{C}^{p \times q}} < 1 \}. \tag{1.2}$$

(Here $\|\cdot\|_{\mathbb{C}^{p \times q}}$ refers to the induced operator norm arising by considering a $p \times q$ matrix M as an operator from $\mathbb{C}^q$ into $\mathbb{C}^p$.) Now we recall the *Schur–Agler* *class* $\mathcal{SA}_{\mathbf{Q}}(\mathcal{E},\,\mathcal{E}_*)$ that consists, by definition, of $\mathcal{L}(\mathcal{E},\,\mathcal{E}_*)$-valued functions $S(z) = S(z_1,\dots,z_n)$ analytic on $\mathcal{D}_{\mathbf{Q}}$ and such that

$$\|S(T_1,\dots,T_n)\| \le 1$$

for any collection of n commuting operators $(T_1,\dots,T_n)$ on a Hilbert space $\mathcal{K}$, subject to

$$\|\mathbf{Q}(T_1,\dots,T_n)\| < 1.$$

By [9, Lemma 1], the Taylor joint spectrum of the commuting n-tuple $(T_1,\dots,T_n)$ is contained in $\mathcal{D}_{\mathbf{Q}}$ whenever $\|\mathbf{Q}(T_1,\dots,T_n)\| < 1$, and hence $S(T_1,\dots,T_n)$ is well-defined by the Taylor functional calculus (see [28]) for any $\mathcal{L}(\mathcal{E},\mathcal{E}_*)$-valued function S which is analytic on $\mathcal{D}_{\mathbf{Q}}$. Upon using $\mathcal{K} = \mathbb{C}$ and $T_j = z_j$ for $j = 1,\dots,n$ where $(z_1,\dots,z_n)$ is a point in $\mathcal{D}_{\mathbf{Q}}$ we conclude that any $\mathcal{L}(\mathcal{E},\,\mathcal{E}_*)$ function is contractive valued, and thus, the class $\mathcal{SA}_{\mathbf{Q}}(\mathcal{E},\,\mathcal{E}_*)$ is the subclass of the *Schur class* $\mathcal{S}_{\mathcal{D}_{\mathbf{Q}}}(\mathcal{E},\mathcal{E}_*)$ of contractive valued functions analytic on $\mathcal{D}_{\mathbf{Q}}$. By the von Neumann result, in the case when $\mathbf{Q}(z) = z$, these classes coincide; in general, $\mathcal{SA}_{\mathbf{Q}}(\mathcal{E},\,\mathcal{E}_*)$ is a proper subclass of $\mathcal{S}_{\mathcal{D}_{\mathbf{Q}}}(\mathcal{E},\mathcal{E}_*)$.

Special choices of

$$\mathbf{Q}(z) = \begin{bmatrix} z_1 & & \\ & \ddots & \\ & & z_n \end{bmatrix} \quad \text{and} \quad \mathbf{Q}(z) = \begin{bmatrix} z_1 & z_2 & \dots & z_n \end{bmatrix}$$

lead to the unit polydisk $\mathcal{D}_{\mathbf{Q}} = \mathbb{D}^n$ and the unit ball $\mathcal{D}_{\mathbf{Q}} = \mathbb{B}^n$ of $\mathbb{C}^n$, respectively. The classes $\mathcal{SA}_{\mathbf{Q}}(\mathcal{E},\,\mathcal{E}_*)$ for these two generic cases have been known for a while. The polydisk setting was first presented by J. Agler in [2] and then extended to the operator valued case in [19, 22]; see also [3, 15, 20]. The Schur–Agler functions on the unit ball appeared in [30] and later in [1, 47, 40] in connection with complete Nevanlinna–Pick kernels and in [13, 46] in connection with the study of dilation theory for commutative row contractions; we refer to [23] for a thorough review of the operator-valued case. The general setting introduced above unifies these two generic settings and besides, covers some other interesting cases including Cartan domains of the first three types, their cartesian products and their intersections. General domains $\mathcal{D}_{\mathbf{Q}}$ and classes $\mathcal{SA}_{\mathbf{Q}}(\mathcal{E},\,\mathcal{E}_*)$ (for $\mathcal{E} = \mathcal{E}_* = \mathbb{C}$) have been introduced in [9]. The operator-valued version of this class has appeared in [8], [17].

The following theorem gives several equivalent characterizations of the class $\mathcal{SA}_{\mathbf{Q}}(\mathcal{E},\,\mathcal{E}_*)$; the proof can be found in [8, 17]; the proof for the scalar-valued case (where $\mathcal{E} = \mathcal{E}_* = \mathbb{C}$) can be found in [9] in a somewhat different form.

Theorem 1.1. *Let S be a $\mathcal{L}(\mathcal{E},\,\mathcal{E}_*)$-valued function defined on $\mathcal{D}_{\mathbf{Q}}$. The following statements are equivalent:*

(1) *S belongs to $\mathcal{SA}_{\mathbf{Q}}(\mathcal{E},\,\mathcal{E}_*)$.*
(2) *There exist an auxiliary Hilbert space $\mathcal{H}$ and a function*

$$H(z) = \begin{bmatrix} H_1(z) & \dots & H_p(z) \end{bmatrix} \tag{1.3}$$

analytic on $\mathcal{D}_{\mathbf{Q}}$ with values in $\mathcal{L}(\mathbb{C}^p \otimes \mathcal{H},\,\mathcal{E}_*)$ so that

$$I_{\mathcal{E}_*} - S(z)S(w)^* = H(z)\left(I_{\mathbb{C}^p\otimes\mathcal{H}} - \mathbf{Q}(z)\mathbf{Q}(w)^*\right)H(w)^*. \tag{1.4}$$

(3) *There exist an auxiliary Hilbert space $\mathcal{H}$ and a function*

$$G(z) = \begin{bmatrix} G_1(z) \\ \vdots \\ G_q(z) \end{bmatrix} \tag{1.5}$$

analytic on $\mathcal{D}_{\mathbf{Q}}$ with values in $\mathcal{L}(\mathbb{C}^q \otimes \mathcal{H}, \mathcal{E})$ so that

$$I_{\mathcal{E}} - S(z)^* S(w) = G(z)^* \left(I_{\mathbb{C}^q \otimes \mathcal{H}} - \mathbf{Q}(z)^* \mathbf{Q}(w) \right) G(w). \tag{1.6}$$

(4) *There exist an auxiliary Hilbert space $\mathcal{H}$ and analytic functions $H(z)$ and $G(z)$ as in* (1.3) *and* (1.5), *so that relations* (1.4) *and* (1.6) *hold along with*

$$S(z) - S(w) = H(z) \left(\mathbf{Q}(z) - \mathbf{Q}(w) \right) G(w) \quad (z,\, w \in \mathcal{D}_{\mathbf{Q}}). \tag{1.7}$$

(5) *There is a unitary operator*

$$\mathbf{U} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} : \begin{bmatrix} \mathbb{C}^p \otimes \mathcal{H} \\ \mathcal{E} \end{bmatrix} \to \begin{bmatrix} \mathbb{C}^q \otimes \mathcal{H} \\ \mathcal{E}_* \end{bmatrix} \tag{1.8}$$

such that

$$S(z) = D + C \left(I_{\mathbb{C}^p \otimes \mathcal{H}} - \mathbf{Q}(z) A \right)^{-1} \mathbf{Q}(z) B \quad \textit{for all } z \in \Omega. \tag{1.9}$$

Moreover, if S is of the form (1.9), *then it holds that*

$$\begin{aligned} I_{\mathcal{E}_*} - S(z) S(w)^* &= C \left(I - \mathbf{Q}(z) A \right)^{-1} \left(I - \mathbf{Q}(z) \mathbf{Q}(w)^* \right) \left(I - A^* \mathbf{Q}(w)^* \right)^{-1} C^*, \\ S(z) - S(w) &= C \left(I - \mathbf{Q}(z) A \right)^{-1} \left(\mathbf{Q}(z) - \mathbf{Q}(w) \right) \left(I - A \mathbf{Q}(w) \right)^{-1} B, \\ I_{\mathcal{E}} - S(z)^* S(w) &= B^* \left(I - \mathbf{Q}(z)^* A^* \right)^{-1} \left(I - \mathbf{Q}(z)^* \mathbf{Q}(w) \right) \left(I - A \mathbf{Q}(w) \right)^{-1} B. \end{aligned}$$

Hence the representations (1.4), (1.6) *and* (1.7) *are valid with*

$$H(z) = C \left(I - \mathbf{Q}(z) A \right)^{-1} \quad \textit{and} \quad G(z) = \left(I - A \mathbf{Q}(z) \right)^{-1} B. \tag{1.10}$$

The representation (1.9) is called *a unitary realization* of $S \in \mathcal{SA}_{\mathbf{Q}}(\mathcal{E}, \mathcal{E}_*)$ and can be viewed as a realization of S as the *transfer function* of a certain type of multidimensional system; see Section 4.

Remark 1.2. In formulas (1.9) and (1.10) we abused notations and used $\mathbf{Q}(z)$ instead of $\mathbf{Q}(z) \otimes I_{\mathcal{H}}$.

Let $\mathcal{H}_{\mathcal{D}_{\mathbf{Q}}}(\mathcal{E}, \mathcal{K})$ be the set of all $\mathcal{L}(\mathcal{E}, \mathcal{K})$-valued functions F which are analytic on $\mathcal{D}_{\mathbf{Q}}$. Given $F \in \mathcal{H}_{\mathcal{D}_{\mathbf{Q}}}(\mathcal{E}, \mathcal{K})$ and $T = (T_1, \dots, T_n)$ an n-tuple of commuting bounded operators on $\mathcal{K}$ for which the Taylor joint spectrum $\sigma_{\text{Taylor}}(T)$ is contained in $\mathcal{D}_{\mathbf{Q}}$, one can use the Taylor functional calculus (details below in Section 2) to define a left evaluation map $F \mapsto F^{\wedge L}(T) \in \mathcal{L}(\mathcal{E}, \mathcal{K})$. Similarly, if $F \in \mathcal{H}_{\mathcal{D}_{\mathbf{Q}}}(\mathcal{K}, \mathcal{E}_*)$ and $T' = (T'_1, \dots, T'_n)$ is an n-tuple of commuting bounded operators on $\mathcal{K}$ with $\sigma_{\text{Taylor}}(T) \subset \mathcal{D}_{\mathbf{Q}}$, one can use the Taylor functional calculus to define a right evaluation map $F \mapsto F^{\wedge R}(T') \in \mathcal{L}(\mathcal{K}, \mathcal{E}_*)$.

Let $\mathcal{K}$ and $\mathcal{K}'$ be two Hilbert spaces and let

$$T = (T_1, \dots, T_n) \quad \text{and} \quad T' = (T'_1, \dots, T'_n) \tag{1.11}$$

be commutative n-tuples of operators $T_j \in \mathcal{L}(\mathcal{K})$ and $T'_j \in \mathcal{L}(\mathcal{K}')$ such that

$$\sigma_{\text{Taylor}}(T) \subset \mathcal{D}_{\mathbf{Q}} \quad \text{and} \quad \sigma_{\text{Taylor}}(T') \subset \mathcal{D}_{\mathbf{Q}}. \tag{1.12}$$

We shall consider bitangential interpolation problems with the data sets consisting of two Hilbert spaces $\mathcal{K}$ and $\mathcal{K}'$, two commutative n-tuples of the form (1.11) and satisfying (1.12), and bounded operators

$$X_L\colon \mathcal{E}_* \to \mathcal{K}, \quad Y_L\colon \mathcal{E} \to \mathcal{K}, \quad X_R\colon \mathcal{K}' \to \mathcal{E}_*, \quad Y_R\colon \mathcal{K}' \to \mathcal{E}.$$

Given this data set

$$\mathcal{D} = \{T,\ T',\ X_L,\ Y_L,\ X_R,\ Y_R\}\,, \tag{1.13}$$

the formal statement of the associated bitangential interpolation problem is:

Problem 1.3. *Find necessary and sufficient conditions for existence of a function $S \in \mathcal{SA}_{\mathbf{Q}}(\mathcal{E},\, \mathcal{E}_*)$ such that*

$$\left(X_L S\right)^{\wedge L}(T) = Y_L \quad \textit{and} \quad \left(S Y_R\right)^{\wedge R}(T') = X_R. \tag{1.14}$$

To formulate the solution criterion we need some additional notation. Define operators

$$E_1 = \begin{bmatrix} I_{\mathcal{K}} \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 0 \\ I_{\mathcal{K}} \\ \vdots \\ 0 \end{bmatrix}, \ \ldots,\ E_p = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ I_{\mathcal{K}} \end{bmatrix}, \tag{1.15}$$

$$E'_1 = \begin{bmatrix} I_{\mathcal{K}'} \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad E'_2 = \begin{bmatrix} 0 \\ I_{\mathcal{K}'} \\ \vdots \\ 0 \end{bmatrix}, \ \ldots,\ E'_q = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ I_{\mathcal{K}'} \end{bmatrix}, \tag{1.16}$$

$$\mathbf{Q}_{j\cdot}(T') = \begin{bmatrix} \mathbf{q}_{j1}(T') \\ \vdots \\ \mathbf{q}_{jq}(T') \end{bmatrix}, \qquad \mathbf{Q}_{\cdot k}(T) = \begin{bmatrix} \mathbf{q}_{1k}(T)^* \\ \vdots \\ \mathbf{q}_{pk}(T)^* \end{bmatrix} \tag{1.17}$$

and the operators

$$M_j = M_j(T') = \begin{bmatrix} E_j & 0 \\ 0 & \mathbf{Q}_{j\cdot}(T') \end{bmatrix} \quad \text{for } \ j = 1, \ldots, p, \tag{1.18}$$

$$N_k = N_k(T) = \begin{bmatrix} \mathbf{Q}_{\cdot k}(T) & 0 \\ 0 & E'_k \end{bmatrix} \quad \text{for } \ k = 1, \ldots, q. \tag{1.19}$$

Theorem 1.4. *There is a function $S \in \mathcal{SA}_{\mathbf{Q}}(\mathcal{E},\, \mathcal{E}_*)$ satisfying interpolation conditions (1.14) if and only if there exists a positive semidefinite operator*

$$P \in \mathcal{L}\left((\mathbb{C}^p \otimes \mathcal{K}) \oplus (\mathbb{C}^q \otimes \mathcal{K}')\right)$$

subject to the Stein identity

$$\sum_{j=1}^{p} M_j^* P M_j - \sum_{k=1}^{q} N_k^* P N_k = X^* X - Y^* Y \tag{1.20}$$

where M_j and N_k are the operators defined via formulas (1.15)–(1.19) and where

$$X = \begin{bmatrix} X_L^* & X_R \end{bmatrix} \quad \textit{and} \quad Y = \begin{bmatrix} Y_L^* & Y_R \end{bmatrix}. \tag{1.21}$$

The special case where only a set of left tangential interpolation conditions $(X_L S)^{\wedge L}(T) = Y_L$ or only right tangential interpolation conditions $(SY_k)^{\wedge R}(T') = X_R$ is considered corresponds to the special case of Problem 1.3 where one takes $\mathcal{K}' = \{0\}$ (respectively, $\mathcal{K} = \{0\}$). As a corollary to Theorem 1.4 we therefore have the following.

Corollary 1.5. (1) *Suppose that $\mathcal{D}_L = \{T, X_L, Y_L\}$ is the data set for a left tangential interpolation problem (i.e., $\mathbb{D}_L$ is as in (1.13) with $\mathcal{K}' = \{0\}$). Then there exists an $S \in \mathcal{SA}_{\mathbf{Q}}(\mathcal{E}, \mathcal{E}_*)$ satisfying the interpolation condition*

$$(X_L S)^{\wedge L}(T) = Y_L$$

if and only if there exists a positive semidefinite solution

$$P = [P_{ij}]_{i,j=1}^{p} \in \mathcal{L}(\mathcal{K} \otimes \mathbb{C}^p)$$

to the Stein equation

$$\sum_{j=1}^{p} P_{jj} - \sum_{k=1}^{q} \sum_{i,j=1}^{p} \mathbf{q}_{ik}(T) P_{ij} \mathbf{q}_{jk}(T)^* = X_L X_L^* - Y_l Y_L^*.$$

(2) *Suppose that $\mathcal{D}_R = \{T; , Y_R, X_R\}$ is the data set for a right tangential interpolation problem (i.e., $\mathbb{D}_R$ is as in (1.13) with $\mathcal{K} = \{0\}$). Then there exists an $S \in \mathcal{SA}_{\mathbf{Q}}(\mathcal{E}, \mathcal{E}_*)$ satisfying the interpolation condition*

$$(SY_R)^{\wedge R}(T') = X_R$$

if and only if there exists a positive semidefinite solution

$$P' = [P'_{ij}]_{i,j=1}^{q} \in \mathcal{L}(\mathcal{K}' \otimes \mathbb{C}^q)$$

of the Stein equation

$$\sum_{j=1}^{q} P'_{jj} - \sum_{k=1}^{p} \sum_{i,j=1}^{q} \mathbf{q}_{ki}(T')^* P'_{ij} \mathbf{q}_{jk}(T') = Y_R^* Y_R - X_R^* X_R.$$

In the special case in Corollary 1.5 where $\mathcal{K} = \oplus_{\omega \in \Omega} \mathcal{E}_*$ for some subset Ω of $\mathcal{D}_{\mathbf{Q}}$ and one takes

$$T = \operatorname{diag}_{\omega \in \Omega}[\omega I_{\mathcal{E}_*}], \qquad X_L = \operatorname{col}_{\omega \in \Omega}[I_{\mathcal{E}_*}], \qquad Y_L = \operatorname{col}_{\omega \in \Omega}[F(\omega)]$$

for some given function $F \colon \Omega \to \mathcal{L}(\mathcal{E}, \mathcal{E}_*)$, the left interpolation condition with operator argument $(X_L F)^{\wedge L}(T) = Y_L$ gives rise to full-operator-value interpolation along the subset Ω of $\mathcal{D}_{\mathbf{Q}}$. The interpolation problem then is: *given $F \colon \Omega \to \mathcal{L}(\mathcal{E}, \mathcal{E}_*)$, find $S \in \mathcal{SA}_{\mathbf{Q}}(\mathcal{E}, \mathcal{E}_*)$ so that*

$$S(\omega) = F(\omega) \text{ for all } \omega \in \Omega \subset \mathcal{D}_{\mathbf{Q}}.$$

This case of part (1) of Corollary 1.5 can already be found in [17]. We note that a more general version of this problem, where the matrix polynomial $\mathbf{Q}(z)$ is replaced by a continuum $z \to \mathbf{Q}_\lambda(z)$ of matrix-valued analytic functions indexed by λ in a separable compact Hausdorff space Λ, has been worked out in [7].

Let P be any operator satisfying the conditions in Theorem 1.4. Let us represent its block entries explicitly as

$$P = \begin{bmatrix} P_L & P_{LR} \\ P_{LR}^* & P_R \end{bmatrix} \tag{1.22}$$

where

$$P_L = \begin{bmatrix} \Psi_{11} & \dots & \Psi_{1p} \\ \vdots & & \vdots \\ \Psi_{p1} & \dots & \Psi_{pp} \end{bmatrix}, \quad P_R = \begin{bmatrix} \Phi_{11} & \dots & \Phi_{1q} \\ \vdots & & \vdots \\ \Phi_{q1} & \dots & \Phi_{qq} \end{bmatrix}, \tag{1.23}$$

$$P_{LR} = \begin{bmatrix} \Lambda_{11} & \dots & \Lambda_{1q} \\ \vdots & & \vdots \\ \Lambda_{p1} & \dots & \Lambda_{pq} \end{bmatrix}, \tag{1.24}$$

with

$$\begin{cases} \Psi_{j\ell} \in \mathcal{L}(\mathcal{K}) & \text{for } j, \ell = 1, \dots, p, \\ \Phi_{j\ell} \in \mathcal{L}(\mathcal{K}') & \text{for } j, \ell = 1, \dots, q, \\ \Lambda_{j\ell} \in \mathcal{L}(\mathcal{K}', \mathcal{K}) & \text{for } j = 1, \dots, p; \ \ell = 1, \dots, q. \end{cases} \tag{1.25}$$

It turns out that for every positive semidefinite P satisfying (1.20), there is a solution S of the bitangential interpolation Problem 1.3 such that, for some choice of associated functions $H(z)$ and $G(z)$ of the form (1.3) and (1.5) in representations (1.4), (1.6), (1.7), it holds that

$$(X_L H_j)^{\wedge L}(T) \left[(X_L H_\ell)^{\wedge L}(T)\right]^* = \Psi_{j\ell} \text{ for } j, \ell = 1, \dots, p, \tag{1.26}$$

$$(X_L H_j)^{\wedge L}(T) \, (G_\ell Y_R)^{\wedge R}(T') = \Lambda_{j\ell} \text{ for } j = 1, \dots, p; \ \ell = 1, \dots, q, \tag{1.27}$$

$$\left[(G_j Y_R)^{\wedge R}(T')\right]^* (G_\ell Y_R)^{\wedge R}(T') = \Phi_{j\ell} \text{ for } j, \ell = 1, \dots q. \tag{1.28}$$

Furthermore, it turns out that conversely, for every solution S of Problem 1.3 with representations (1.4), (1.6), (1.7) (existence of these representations is guaranteed by Theorem 1.1), the operator P defined via (1.22)–(1.24) and (1.26)–(1.28) satisfies conditions of Theorem 1.4. These observations suggested the following modification of Problem 1.3 with the data set

$$\mathcal{D} = \{T, \ T', \ X_L, \ Y_L, \ X_R, \ Y_R, \ \Psi_{j\ell}, \ \Phi_{j\ell}, \ \Lambda_{j\ell}\}. \tag{1.29}$$

Problem 1.6. *Given the data $\mathcal{D}$ as in (1.29), find all functions $S \in \mathcal{SA}_{\mathbf{Q}}(\mathcal{E}, \mathcal{E}_*)$ satisfying interpolation conditions (1.14) and such that for some choice of associated functions H_j and G_ℓ in the representations (1.4), (1.6), (1.7), the equalities (1.26)–(1.28) hold.*

In contrast to Problem 1.3, the solvability criterion for Problem 1.6 can be given explicitly in terms of the interpolation data.

Theorem 1.7. *Problem 1.6 has a solution if and only if the operator P given by (1.22)–(1.24) is positive semidefinite and satisfies the Stein identity (1.20).*

Moreover, there exists defect subspaces $\widetilde{\Delta}$ and $\widetilde{\Delta}_*$ and an operator-valued function

$$z \mapsto \Sigma(z) = \begin{bmatrix} \Sigma_{11}(z) & \Sigma_{12}(z) \\ \Sigma_{21}(z) & \Sigma_{22}(z) \end{bmatrix} : \begin{bmatrix} \mathcal{E} \\ \widetilde{\Delta}_* \end{bmatrix} \to \begin{bmatrix} \mathcal{E}_* \\ \widetilde{\Delta} \end{bmatrix} \text{ for } z \in \mathcal{D}_{\mathbf{Q}}$$

of the form

$$\Sigma(z) = \begin{bmatrix} U_{22} & U_{23} \\ U_{32} & 0 \end{bmatrix} + \begin{bmatrix} U_{21} \\ U_{31} \end{bmatrix} (I_{\widetilde{\Delta}_*} - \mathbf{Q}(z) U_{11})^{-1} \mathbf{Q}(z) \begin{bmatrix} U_{12} & U_{13} \end{bmatrix}$$

with

$$\mathbf{U}_0 = \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ U_{21} & U_{22} & U_{23} \\ U_{31} & U_{32} & 0 \end{bmatrix} : \begin{bmatrix} \widehat{\mathcal{H}} \\ \mathcal{E} \\ \widetilde{\Delta}_* \end{bmatrix} \to \begin{bmatrix} \widehat{\mathcal{H}} \\ \mathcal{E}_* \\ \widetilde{\Delta} \end{bmatrix}$$

unitary and completely determined by the interpolation data set $\mathcal{D}$ (see (1.13)) so that S is a solution of Problem 1.6 if and only if S has the form

$$S(z) = \Sigma_{11}(z) + \Sigma_{12}(z)\left(I_{\widetilde{\Delta}_*} - \mathcal{T}(z)\Sigma_{22}(z)\right)^{-1}\mathcal{T}(z)\Sigma_{21}(z)$$

for a free-parameter function $\mathcal{T}(z) \in \mathcal{SA}_{\mathbf{Q}}(\widetilde{\Delta}, \widetilde{\Delta}_*)$.

As a corollary of Theorem 1.7 we get a description (albeit less satisfactory) of the set of all solutions of a Problem 1.3.

Corollary 1.8. *Suppose that $\mathcal{D}$ is an interpolation data set for Problem 1.3 as in (1.13). Given any positive semidefinite solution P of the Stein identity (1.20), define operators $\Psi^P_{j\ell} \in \mathcal{L}(\mathcal{K})$ for $j, \ell = 1, \dots, p$, $\Phi^P_{ij} \in \mathcal{L}(\mathcal{K}')$ for $j, \ell = 1, \dots, p$, and $\Lambda^p_{j\ell} \in \mathcal{L}(\mathcal{K}, \mathcal{K}')$ for $j = 1, \dots, p$ and $\ell = 1, \dots, q$, by (1.22), (1.23) and (1.24). Let*

$$\Sigma^P(z) = \begin{bmatrix} \Sigma^P_{11}(z) & \Sigma^P_{12}(z) \\ \Sigma^P_{21}(z) & \Sigma^P_{22}(z) \end{bmatrix} : \begin{bmatrix} \mathcal{E} \\ \widetilde{\Delta}^P_* \end{bmatrix} \to \begin{bmatrix} \mathcal{E}_* \\ \widetilde{\Delta}^P \end{bmatrix}$$

be the linear-fractional coefficient matrix function generated from the expanded interpolation data set

$$\mathcal{D}^P = \left\{T, T', X_L, Y_L, X_R, Y_R, \Psi^P_{j\ell}, \Phi^P_{j\ell}, \Lambda^P_{j\ell}\right\}$$

associated with a Problem 1.6 as in Theorem 1.7. Then S is a solution of Problem 1.3 if and only if S has the form

$$S(z) = \Sigma^P_{11}(z) + \Sigma^P_{12}(z)\left(I_{\widetilde{\Delta}^P_*} - \mathcal{T}^P(z)\Sigma^P_{22}(z)\right)^{-1}\mathcal{T}^P(z)\Sigma^P_{21}(z)$$

for some choice P of positive semidefinite solution of (1.20) and some choice of free-parameter function $\mathcal{T}^P(z) \in \mathcal{SA}_{\mathbf{Q}}(\widetilde{\Delta}^P, \widetilde{\Delta}^P_*)$.

The paper is organized as follows. Section 2 reviews material from [49, 51] (see also [28]) on Vasilescu's adaptation based on the Martinelli kernel (see [51]) of the Taylor functional calculus (see [49, 50]) to formulate and develop the basic properties of the left and right point evaluation operators $F \mapsto F^{\wedge L}(T)$ and $F \mapsto F^{\wedge R}(T')$ needed in the very formulation of the bitangential interpolation problem. Section 3 derives the necessity direction of the solvability criterion in Theorem 1.7. Section 4 discusses the connections with multidimensional system theory and delineates how solutions of Problem 1.6 are in correspondence with the characteristic functions of unitary colligations arising as unitary extensions of a certain partially defined isometry uniquely specified by the interpolation data. Section 5 sets up the universal unitary colligation completely determined by the interpolation data which leads to the linear fractional parametrization for the set of all solutions of Problem 1.6 asserted in the second part of Theorem 1.7; the ideas here adapt the earlier work of [11, 12] done for the classical one-variable setting. Section 7 makes explicit how the bitangential interpolation problems covered in [17] can be seen as examples of Problems 1.3 and 1.6 here and considers the special case of a bitangential Nevanlinna–Pick interpolation problem involving only finitely many interpolation nodes. In the latter case the coefficients of the Redheffer linear fractional map parametrizing the set of all solutions can be described more explicitly. For some particular choices of $\mathbf{Q}$ similar formulas were obtained in [26], [5] and [14], [27]. The results and techniques used in Sections 4–7 are an adaptation of the approach carried out for closely related multivariable interpolation problems for particular cases of the domains $\mathcal{D}_{\mathbf{Q}}$ in [14, 16, 15].

2. Right and left evaluation with operator argument

We begin with a review of the elements of the Taylor functional calculus for a tuple of commuting Hilbert space operators, as worked out in explicit form by use of an adaptation of the Bochner–Martinelli kernel by Vasilescu. A good survey of the general topic of joint spectra and functional calculus for operator tuples is [28]; more specific information can be found in [51, 52, 43].

Denote by $\Lambda[e] = \oplus_{k=0}^{n}\Lambda^k[e]$ the exterior algebra over $\mathbb{C}$ on n generators $e_1, \dots, e_n$. The linear space $\Lambda[e]$ becomes a Hilbert space if we declare the collection

$$\{e_{i_1} \wedge \cdots \wedge e_{i_k} \colon 1 \le i_1 < \cdots < i_k \le n\}$$

to be an orthonormal basis for $\Lambda^k[e]$ for each $k = 1, \dots, n$. Note that we identify $\Lambda^0[e]$ with $\mathbb{C}$. For $i = 1, \dots, n$ let E_i be the operator defined on $\Lambda[e]$ by $E_i \colon \xi \mapsto e_i \wedge \xi$ for $\xi \in \Lambda[e]$. Then one can check that

$$E_iE_j + E_jE_i = 0, \quad E_i^2 = 0, \quad E_i^*E_j + E_jE_i^* = \delta_{i,j}I_{\Lambda[e]}$$

where $\delta_{i,j}$ is the Kronecker delta. Therefore

$$E_iE_i^*E_i = E_i(I - E_iE_i^*) = E_i - E_i^2E_i^* = E_i,$$

so each E_i is a partial isometry. Now let $T = (T_1, \dots, T_n)$ be a commuting n-tuple of operators on a Hilbert space $\mathcal{K}$. On $\mathcal{K} \otimes \Lambda[e]$ define the operator

$$D_T = T_1 \otimes E_1 + \cdots + T_n \otimes E_n.$$

One can check that $D_T^2 = 0$, i.e., that $\operatorname{Ran} D_T \subset \operatorname{Ker} D_T$. We say that T *is invertible in the sense of Taylor* if we have the equality $\operatorname{Ran} D_T = \operatorname{Ker} D_T$, or equivalently (see [51, Lemma 2.1]), if $R_T := D_T + D_T^*$ *is invertible* (as an operator on $\mathcal{K} \otimes \Lambda[e]$). We define the *Taylor spectrum* of T to be the set of all $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n$ for which the n-tuple

$$\lambda - T := (\lambda_1 I_{\mathcal{K}} - T_1, \dots, \lambda_n I_{\mathcal{K}} - T_n)$$

is not invertible in the sense of Taylor.

Now suppose that $z \mapsto \beta(z) = (\beta_1(z), \dots, \beta_n(z))$ is an n-tuple of $\mathcal{L}(\mathcal{K})$-valued holomorphic functions on an open set $\Omega \subset \mathbb{C}^n$. (We shall eventually restrict to the case

$$\beta(z) = z - T := (z_1 I_{\mathcal{K}} - T_1, \dots, z_n I_{\mathcal{K}} - T_n) \tag{2.1}$$

for an n-tuple of operators $T = (T_1, \dots, T_n)$ in $\mathcal{L}(\mathcal{K})$, so the reader should keep this example in mind.) Define an operator $D_\beta \colon C^\infty(\Omega, \Lambda(\mathcal{K})) \to C^\infty(\Omega, \Lambda(\mathcal{K}))$ by

$$(D_\beta f)(z) := D_{\beta(z)} f(z) \text{ for } z \in \Omega.$$

Hence R_β is then defined by

$$(R_\beta f)(z) = (D_{\beta(z)} + D_{\beta(z)}^*) f(z).$$

In addition we need the so-called *Dolbeault complex* (see [39, page 268]), i.e., the exterior algebra $\Lambda[\bar{\partial}_z]$ generated by the indeterminants $d\bar{z} = (d\bar{z}_1, \dots, d\bar{z}_n)$. The operator $\bar{\partial}$ on $C^\infty(\Omega) \otimes \Lambda[\bar{\partial}_z]$ is then given by

$$\bar{\partial}\colon \xi \mapsto \bar{\partial} f \wedge d\bar{z}_{i_1} \wedge \cdots \wedge d\bar{z}_{i_k} \quad \text{if} \quad \xi = f d\bar{z}_{i_1} \wedge \cdots \wedge d\bar{z}_{i_k}$$

where

$$\bar{\partial} f := \frac{\partial f}{\partial \bar{z}_1} d\bar{z}_1 + \cdots + \frac{\partial f}{\partial \bar{z}_n} d\bar{z}_n.$$

For an n-tuple $\beta = (\beta_1, \dots, \beta_n)$ of functions in $\mathcal{H}_\Omega(\mathcal{K}, \mathcal{K})$, we declare the *Taylor spectrum of* β *in* Ω to be

$$\sigma^\Omega_{\text{Taylor}}(\beta, \mathcal{K}) := \left\{\lambda \in \Omega\colon\ \operatorname{Ker}(D_{\beta(\lambda)}) \neq \operatorname{Ran}(D_{\beta(\lambda)})\right\}.$$

Then the *Martinelli kernel associated with* β is defined by

$$M(\beta)(z) := R^{-1}_{\beta(z)} \big(\bar{\partial}_z R^{-1}_{\beta(z)}\big)^{n-1} E|_{\mathcal{K}\otimes\Lambda^0[e]}\colon \mathcal{K} \otimes \Lambda^0[e] \to \mathcal{K} \otimes \Lambda^0[e] \otimes \Lambda^{n-1}[d\bar{z}].$$

for all $z \notin \sigma^\Omega_{\text{Taylor}}(\beta, \mathcal{K})$. If we identify $\mathcal{K} \otimes \Lambda^0[e]$ with $\mathcal{K}$, then we view $M(\beta)(z)$ simply as an element of $\mathcal{L}(\mathcal{K}) \otimes \Lambda^{n-1}[d\bar{z}]$.

We now specialize to the case when $\beta(z)$ is of the form (2.1) for an n-tuple of operators $T = (T_1, \dots, T_n) \in \mathcal{L}(\mathcal{K})^n$. Assume that $\sigma_{\text{Taylor}}(T) \subset \Omega$. One can use the Martinelli kernel associated with $z - T$ to define a functional calculus for T for functions $f \in \mathcal{H}(\Omega, \mathbb{C})$ as follows (see [51]). Choose an open subset Ω' with smooth boundary $\partial\Omega'$ so that

$$\sigma_{\text{Taylor}}(T) \subset \Omega' \subset \overline{\Omega'} \subset \Omega. \tag{2.2}$$

Note that by definition $\sigma^{\mathcal{D}_{\mathbf{Q}}}_{\text{Taylor}}(z - T) = \sigma_{\text{Taylor}}(T) \subset \Omega'$ and hence $M(z - T)$ is defined on $\partial\Omega'$. Then, for f a scalar-valued holomorphic function on Ω, we can define $f(T)$ via

$$f(T) = \frac{1}{(2\pi i)^n} \int_{\partial\Omega'} M(z - T) \cdot f(z) \wedge dz.$$

For further details, we refer to [51, 28]. The definition of the Taylor spectrum originates in [49] and an equivalent formulation of the functional calculus using more homological algebra machinery can be found in [50].

For $F \in \mathcal{H}_\Omega(\mathcal{E}, \mathcal{E}_*)$, we can define a functional calculus

$$F \mapsto F(T) \in \mathcal{L}(\mathcal{E} \otimes \mathcal{K}, \mathcal{E}_* \otimes \mathcal{K})$$

by

$$F(T) = \frac{1}{(2\pi i)^n} \int_{\partial\Omega'} F(z) \otimes M(z - T) \wedge dz.$$

This is the functional calculus needed to define the Schur–Agler class above.

To formulate the general bitangential interpolation problem, we need to introduce left and right operator evaluation defined as follows. Suppose first that we are given a function $F \in \mathcal{H}_\Omega(\mathcal{E}, \mathcal{K})$ (i.e., F is holomorphic on Ω with values in $\mathcal{L}(\mathcal{E}, \mathcal{K})$) together with a commuting n-tuple $T = (T_1, \dots, T_n) \in \mathcal{L}(\mathcal{K})^n$ with $\sigma_{\text{Taylor}}(T) \subset \Omega$. We then define *the left evaluation of* F *with operator argument* T by

$$F^{\wedge L}(T) = \frac{1}{(2\pi i)^n} \int_{\partial\Omega'} M(z - T) \cdot F(z) \wedge dz \tag{2.3}$$

with Ω' chosen as in (2.2). Similarly, if $\widetilde{F} \in \mathcal{H}_\Omega(\mathcal{K}', \mathcal{E}_*)$ and $T' = (T'_1, \dots, T'_n) \in \mathcal{L}(\mathcal{K}')^n$ with $\sigma_{\text{Taylor}}(T') \subset \Omega$, define *the right evaluation of* $\widetilde{F}$ *with operator argument* T' by

$$\widetilde{F}^{\wedge R}(T') = \frac{1}{(2\pi i)^n} \int_{\partial\Omega'} \widetilde{F}(z) \cdot M(z - T') \wedge dz. \tag{2.4}$$

We need the following general result of Fubini type concerning this left and right functional calculus with operator argument.

Proposition 2.1. *Let $T = (T_1, \dots, T_n) \in \mathcal{L}(\mathcal{K})^n$, $T' = (T'_1, \dots, T'_m) \in \mathcal{L}(\mathcal{K})^m$ and let (T, T') be a commuting $(n+m)$-tuple of operators on the Hilbert space $\mathcal{K}$. Suppose that the functions*

$$F : \Omega \mapsto \mathcal{L}(\mathcal{E}', \mathcal{K}) \quad \text{and} \quad \widetilde{F} : \widetilde{\Omega} \mapsto \mathcal{L}(\mathcal{E}, \mathcal{E}')$$

are analytic on open sets Ω and $\widetilde{\Omega}$ containing $\sigma_{Taylor}(T)$ and $\sigma_{Taylor}(T')$ respectively. Define an analytic function of $n + m$ variables

$$(z, w) = (z_1, \dots, z_n, w_1, \dots, w_m)$$

by

$$H(z, w) = F(z)\widetilde{F}(w) : \Omega \times \widetilde{\Omega} \mapsto \mathcal{L}(\mathcal{E}, \mathcal{K}).$$

Then

$$H^{\wedge L}(T, T') = (F^{\wedge L}(T) \cdot \widetilde{F})^{\wedge L}(T'). \tag{2.5}$$

Similarly, if F takes values in $\mathcal{L}(\mathcal{E}, \mathcal{E}')$ and $\widetilde{F}$ takes values in $\mathcal{L}(\mathcal{K}, \mathcal{E})$ and if we set $H(z, w) = F(z)\widetilde{F}(w)$, then

$$H^{\wedge R}(T, T') = (F \cdot \widetilde{F}^{\wedge R}(T'))^{\wedge R}(T). \tag{2.6}$$

Proof. This result is a mild generalization of Theorem 3.8 in [52]. Alternatively, one can view it as a generalization of Proposition 12 in [43] (specialized to the Hilbert space case where one can take the generalized inverse V appearing there to be simply $(R_{z-T})^{-1}$ — see the concluding remark (2) in [43]). In these references, the result is given for the case where $\mathcal{E} = \mathcal{E}' = \mathcal{K}$ and the values of F and G are scalar operators. The same proof goes through for our setting, with proper attention to the order of writing of values of F, $\widetilde{F}$ and $M(z - T, w - T')$. $\square$

Remark 2.2. If Ω is a logarithmically convex Reinhardt domain (see [39, Section 2.3]), then $0 \in \Omega$ and any function $F \in \mathcal{H}_\Omega(\mathcal{E}, \mathcal{E}_*)$ is given by its power series expansion about the origin

$$F(z) = \sum_{\mathbf{j} \in \mathbb{N}^n} F_{\mathbf{j}} z^{\mathbf{j}} \quad F_{\mathbf{j}} \in \mathcal{L}(\mathcal{E}, \mathcal{E}_*)$$

uniformly converging on compact subsets of Ω. Then the left and right evaluation maps (2.5) and (2.6) are given explicitly by

$$F^{\wedge L}(T) = \sum_{\mathbf{j} \in \mathbb{N}^n} T^{\mathbf{j}} F_{\mathbf{j}} \quad \text{and} \quad \widetilde{F}^{\wedge R}(T) = \sum_{\mathbf{j} \in \mathbb{N}^n} \widetilde{F}_{\mathbf{j}} T^{\mathbf{j}},$$

for every choice of $F \in \mathcal{H}_\Omega(\mathcal{E}, \mathcal{K})$ and $\widetilde{F} \in \mathcal{H}_\Omega(\mathcal{K}, \mathcal{E}_*)$ and for any commuting n-tuple $T = (T_1, \dots, T_n) \in \mathcal{L}(\mathcal{K})^n$.

Similarly, if $\Omega = \Omega_1 \times \cdots \times \Omega_n$ is a polydomain (the Cartesian product of n 1-variable domains $\Omega_1, \dots, \Omega_n$), then we may write

$$F^{\wedge L}(T) = \frac{1}{(2\pi i)^n} \int_{\Omega_n} \cdots \int_{\Omega_1} (z_1 I - T_1)^{-1} \cdots (z_n I - T_n)^{-1} F(z)\, dz_1 \cdots dz_n,$$
$$\widetilde{F}^{\wedge R}(T) = \frac{1}{(2\pi i)^n} \int_{\Omega_n} \cdots \int_{\Omega_1} \widetilde{F}(z)(z_1 I - T_1)^{-1} \cdots (z_n I - T_n)^{-1}\, dz_1 \cdots dz_n.$$

We need to note the following elementary properties of evaluations (2.3) and (2.4).

Lemma 2.3. *Let T and T' be commuting n-tuples of the form (1.11) with Taylor spectrum contained in Ω. Then:*

(1) *For every constant function $W(z) \equiv W \in \mathcal{L}(\mathcal{K}', \mathcal{K})$,*

$$(W)^{\wedge L}(T) = (W)^{\wedge R}(T') = W. \tag{2.7}$$

(2) *For every $F \in \mathcal{H}_\Omega(\mathcal{E}, \mathcal{K})$, $\widetilde{F} \in \mathcal{H}_\Omega(\mathcal{K}', \mathcal{E}_*)$, $W \in \mathcal{L}(\mathcal{E}', \mathcal{E})$ and $\widetilde{W} \in \mathcal{L}(\mathcal{E}_*, \mathcal{E}'_*)$,*

$$(F \cdot W)^{\wedge L}(T) = F^{\wedge L}(T) \cdot W \quad \textit{and} \quad \left(\widetilde{W} \cdot \widetilde{F}\right)^{\wedge R}(T') = \widetilde{W} \cdot \widetilde{F}^{\wedge R}(T'). \tag{2.8}$$

(3) *For every $F \in \mathcal{H}_\Omega(\mathcal{E}, \mathcal{K})$, $\widetilde{F} \in \mathcal{H}_\Omega(\mathcal{K}', \mathcal{E}_*)$ and $j \in \{1, \dots, d\}$,*

$$(z_j F(z))^{\wedge L}(T) = T_j \cdot F^{\wedge L}(T) \quad \textit{and} \quad \left(z_j \widetilde{F}(z)\right)^{\wedge R}(T') = \widetilde{F}^{\wedge R}(T') \cdot T'_j. \tag{2.9}$$

(4) *For every choice of $F \in \mathcal{H}_\Omega(\mathcal{E}, \mathcal{K})$ and of $\widetilde{F} \in \mathcal{H}_\Omega(\mathcal{E}', \mathcal{E})$,*

$$\left(F \cdot \widetilde{F}\right)^{\wedge L}(T) = (F^{\wedge L}(T) \cdot \widetilde{F})^{\wedge L}(T). \tag{2.10}$$

(5) *For every choice of $F \in \mathcal{H}_\Omega(\mathcal{E}'_*, \mathcal{E}_*)$ and of $\widetilde{F} \in \mathcal{H}_\Omega(\mathcal{K}', \mathcal{E}'_*)$,*

$$\left(F \cdot \widetilde{F}\right)^{\wedge R}(T') = (F \cdot \widetilde{F}^{\wedge R}(T'))^{\wedge R}(T'). \tag{2.11}$$

Proof. Statement (1) is a consequence of the fact that the Martinelli–Vasilescu functional calculus reproduces constants. Statement (2) is an immediate consequence of equalities

$$\int_{\Omega'} H(z) \cdot W\, dz = \int_{\Omega'} H(z)\, dz \cdot W, \qquad \int_{\Omega'} \widetilde{W} \cdot \widetilde{H}(z)\, dz = \widetilde{W} \cdot \int_{\Omega'} \widetilde{H}(z)\, dz$$

for a $\mathcal{L}(\mathcal{E}, \mathcal{K})$-valued $(2n-1)$-form $H(z)$ and a $\mathcal{L}(\mathcal{K}', \mathcal{E}_*)$-valued $(2n-1)$-form $\widetilde{H}(z)$.

Alternatively, the first equality in (2.8) follows from (2.10) for the special case of $\widetilde{F}(z) \equiv W$ when combined with (2.7):

$$(F \cdot W)^{\wedge L}(T) = \left(F^{\wedge L}(T) \cdot W\right)^{\wedge L}(T) = F^{\wedge L}(T) \cdot W$$

and the second equality in (2.8) follows in much the same way from (2.11) for the special case of $F(z) \equiv \widetilde{W}$.

The first relation in (2.9) follows from (2.10) for the special case when $\mathcal{E}' = \mathcal{E}$ and $\widetilde{F}(z) = z_j I_{\mathcal{E}}$. Indeed, in this case (2.10) gives

$$\begin{aligned} (z_j F(z))^{\wedge L}(T) &= (F \cdot \widetilde{F})^{\wedge L} \\ &= \left(F^{\wedge L}(T) \cdot \widetilde{F}\right))^{\wedge L}(T) = \left(F_1 \cdot F^{\wedge L}(T)\right)^{\wedge L}(T) \end{aligned} \tag{2.12}$$

where $F_1(z) = z_j I_{\mathcal{K}}$. Applying the first relation in (2.8) (with $F = F_1$ and $W = F^{\wedge L}(T)$) to the right-hand side in (2.12) we get

$$(z_j F(z))^{\wedge L}(T) = F_1^{\wedge L}(T) \cdot F^{\wedge L}(T)$$

and since $F_1^{\wedge L}(T) = T_j$ by (2.3), the first equality in (2.9) follows. To get the second equality, one can apply (2.11) to the special case $\mathcal{E}'_* = \mathcal{E}_*$ and $F(z) = z_j I_{\mathcal{E}_*}$.

Finally, statement (4) follows from (2.5) and statement (5) from (2.6) upon setting $T = T'$ in (2.5) and (2.6). □

3. The solvability criterion

In this section we prove the necessity part of Theorem 1.7.

Proof of the necessity part in Theorem 1.7. Suppose that $S \in \mathcal{SA}_{\mathbf{Q}}(\mathcal{E}, \mathcal{E}_*)$ satisfies conditions (1.14) and (1.26)–(1.28) for some choice of associated functions H and G of the form (1.3) and (1.5) in the representation (1.4), (1.6), (1.7). Let P be defined as in (1.22)–(1.24). Interpolation conditions (1.26)–(1.28) mean that P can be represented as

$$P = \begin{bmatrix} \mathbb{T}_L^* \\ \mathbb{T}_R^* \end{bmatrix} \begin{bmatrix} \mathbb{T}_L & \mathbb{T}_R \end{bmatrix} \tag{3.1}$$

where the operators $\mathbb{T}_L : \mathbb{C}^p \otimes \mathcal{K} \to \mathcal{H}$ and $\mathbb{T}_R : \mathbb{C}^q \otimes \mathcal{K}' \to \mathcal{H}$ are given by

$$\mathbb{T}_L = \left[\left[(X_L H_1)^{\wedge L}(T) \right]^* \dots \left[(X_L H_p)^{\wedge L}(T) \right]^* \right] \tag{3.2}$$

and

$$\mathbb{T}_R = \left[(G_1 Y_R)^{\wedge R}(T') \dots (G_q Y_R)^{\wedge R}(T') \right]. \tag{3.3}$$

Comparing (3.1) with (1.22) we conclude that

$$P_L = \mathbb{T}_L^* \mathbb{T}_L, \quad P_R = \mathbb{T}_R^* \mathbb{T}_R, \quad P_{LR} = \mathbb{T}_L^* \mathbb{T}_R. \tag{3.4}$$

It follows from (3.1) that $P \geq 0$ and thus, it remains to show that P satisfies the Stein identity (1.20). To this end, note that by the first property in (2.9),

$$(\mathbf{p}F)^{\wedge L}(T) = \mathbf{p}(T) \cdot F^{\wedge L}(T)$$

for every polynomial $\mathbf{p}$ in n variables and every $F \in \mathcal{H}_{\mathbf{Q}}(\mathcal{K}, \mathcal{E})$. In particular, taking into account the block structure (1.3) of H and (1.1) of $\mathbf{Q}$, we get

$$\begin{aligned} (X_L H \mathbf{Q})^{\wedge L}(T) &= \left[\sum_{i=1}^p \mathbf{q}_{i1} X_L H_i \dots \sum_{i=1}^p \mathbf{q}_{iq} X_L H_i \right]^{\wedge L}(T) \\ &= \left[\sum_{i=1}^p \mathbf{q}_{i1}(T) (X_L H_i)^{\wedge L}(T) \dots \sum_{i=1}^p \mathbf{q}_{iq}(T) (X_L H_i)^{\wedge L}(T) \right] \end{aligned}$$

which can be written in terms of (1.17) and (3.2) as

$$(X_L H \mathbf{Q})^{\wedge L}(T) = \left[\mathbf{Q}_{\cdot 1}(T)^* \mathbb{T}_L^* \dots \mathbf{Q}_{\cdot p}(T)^* \mathbb{T}_L^* \right]. \tag{3.5}$$

Note also that according to decomposition (1.3),

$$(X_L H)^{\wedge L}(T) = \left[(X_L H_1)^{\wedge L}(T) \dots (X_L H_p)^{\wedge L}(T) \right],$$

which can be written in terms of (1.15) and (3.2) as

$$(X_L H)^{\wedge L}(T) = \left[E_1^* \mathbb{T}_L^* \dots E_p^* \mathbb{T}_L^*\right]. \tag{3.6}$$

Similarly, taking advantage of the second property in (2.9) we conclude that

$$(\mathbf{p}F)^{\wedge R}(T') = F^{\wedge R}(T') \cdot \mathbf{p}(T')$$

for every polynomial $\mathbf{p}$ in n variables and thus, on account of the block structure (1.5) of G and (1.1) of $\mathbf{Q}$,

$$(\mathbf{Q}GY_R)^{\wedge R}(T') = \begin{bmatrix} \sum_{i=1}^{q} (G_i Y_R)^{\wedge L}(T')\mathbf{q}_{1i}(T') \\ \vdots \\ \sum_{i=1}^{q} (G_i Y_R)^{\wedge L}(T')\mathbf{q}_{pi}(T') \end{bmatrix}$$

which can be written in terms of (1.17) and (3.3) as

$$(\mathbf{Q}GY_R)^{\wedge R}(T') = \begin{bmatrix} \mathbb{T}_R \mathbf{Q}_{1\cdot}(T') \\ \vdots \\ \mathbb{T}_R \mathbf{Q}_{p\cdot}(T') \end{bmatrix}. \tag{3.7}$$

Finally,

$$(GY_R)^{\wedge R}(T') = \begin{bmatrix} (G_1 Y_R)^{\wedge R}(T') \\ \vdots \\ (G_q Y_R)^{\wedge R}(T') \end{bmatrix} = \begin{bmatrix} \mathbb{T}_R E_1' \\ \vdots \\ \mathbb{T}_R E_q' \end{bmatrix}, \tag{3.8}$$

where $E_1', \dots, E_q'$ are given in (1.16).

Substituting the partitionings (1.18), (1.19), (1.21) and (1.22) into (1.20) we conclude that (1.20) is equivalent to the following three equalities:

$$\sum_{j=1}^{p} E_j^* P_L E_j - \sum_{k=1}^{q} \mathbf{Q}_{\cdot k}(T)^* P_L \mathbf{Q}_{\cdot k}(T) = X_L X_L^* - Y_L Y_L^*, \tag{3.9}$$

$$\sum_{j=1}^{p} E_j^* P_{LR} \mathbf{Q}_{j\cdot}(T') - \sum_{k=1}^{q} \mathbf{Q}_{\cdot k}(T)^* P_{LR} E_k' = X_L X_R - Y_L Y_R, \tag{3.10}$$

$$\sum_{j=1}^{p} \mathbf{Q}_{j\cdot}(T')^* P_R \mathbf{Q}_{j\cdot}(T') - \sum_{k=1}^{q} (E_k')^* P_R E_k' = X_R^* X_R - Y_R^* Y_R. \tag{3.11}$$

To check (3.9) we fix w and apply the left evaluation (2.3) to the equality

$$X_L X_L^* - X_L S(z) S(w)^* X_L^* = X_L H(z)\left(I - \mathbf{Q}(z)\mathbf{Q}(w)^*\right) H(w)^* X_L^*$$

which is an immediate corollary of (1.4). Making use of properties (2.7), (2.8) and of relation (3.5) and taking into account the first interpolation condition (1.14), we get

$$\begin{aligned} X_L X_L^* - Y_L S(w)^* X_L^* = (X_L H)^{\wedge L}(T) \cdot H(w)^* X_L^* \\ - (X_L H \mathbf{Q})^{\wedge L}(T) \cdot \mathbf{Q}(w)^* H(w)^* X_L^*. \end{aligned}$$

The last equality holds for all $w \in \mathcal{D}_{\mathbf{Q}}$. Taking adjoints and replacing w by z, we get

$$X_L X_L^* - X_L S(z) Y_L^* = X_L H(z) \left((X_L H)^{\wedge L}(T) \right)^* - X_L H(z) \mathbf{Q}(z) \left((X_L H \mathbf{Q})^{\wedge L}(T) \right)^*.$$

Applying again the left evaluation to the latter equality we get

$$X_L X_L^* - Y_L Y_L^* = (X_L H)^{\wedge L}(T) \left((X_L H)^{\wedge L}(T) \right)^* - (X_L H \mathbf{Q})^{\wedge L}(T) \left((X_L H \mathbf{Q})^{\wedge L}(T) \right)^*.$$

Substituting (3.5) and (3.6) into the right-hand side expression we come to

$$X_L X_L^* - Y_L Y_L^* = \sum_{j=1}^{p} E_j^* \mathbb{T}_L^* \mathbb{T}_L E_j - \sum_{k=1}^{p} \mathbf{Q}_{\cdot k}(T)^* \mathbb{T}_L^* \mathbb{T}_L \mathbf{Q}_{\cdot k}(T)$$

which is equivalent to (3.9), since $\mathbb{T}_L^* \mathbb{T}_L = P_L$.

To prove (3.10) we start with equality

$$X_L S(z) Y_R - X_L S(w) Y_R = X_L H(z) \left(\mathbf{Q}(z) - \mathbf{Q}(w) \right) G(w) Y_R$$

which is a consequence of (1.7). We fix $w \in \mathcal{D}_{\mathbf{Q}}$ in this equality and apply the left evaluation: by the first interpolation condition in (1.14) we have

$$Y_L Y_R - X_L S(w) Y_R = (X_L H \mathbf{Q})^{\wedge L}(T) G(w) Y_R - (X_L H)^{\wedge L}(T) \mathbf{Q}(w) G(w) Y_R.$$

The last identity holds true for all $w \in \mathcal{D}_{\mathbf{Q}}$ and we apply the right evaluation (2.4) to it. In view of the second interpolation condition in (1.14) and of properties (2.7), (2.8), we obtain

$$Y_L Y_R - X_L X_R = (X_L H \mathbf{Q})^{\wedge L}(T) \, (G Y_R)^{\wedge R}(T') - (X_L H)^{\wedge L}(T) \, (\mathbf{Q} G Y_R)^{\wedge R}(T').$$

Substituting equalities (3.5), (3.6), (3.7) and (3.8) into the right-hand side expression in the last equality we come to

$$Y_L Y_R - X_L X_R = \sum_{j=1}^{p} E_j^* \mathbb{T}_L^* \mathbb{T}_R \mathbf{Q}_{j\cdot}(T') - \sum_{k=1}^{q} \mathbf{Q}_{\cdot k}(T)^* \mathbb{T}_L^* \mathbb{T}_R E_k'$$

which is equivalent to (3.10), since $\mathbb{T}_L^* \mathbb{T}_R = P_{LR}$, by (3.4). The proof of (3.11) is quite similar: we start with equality

$$Y_R^* Y_R - Y_R^* S(z)^* S(w) Y_R = Y_R^* G(z)^* \left(I - \mathbf{Q}(z)^* \mathbf{Q}(w) \right) G(w) Y_R$$

(which follows from (1.6)) and apply the right evaluation assuming that z is fixed. Then we take adjoints in the resulting equality (in which z is again a variable) and apply again the right evaluation map. The obtained equality together with relations (3.7) and (3.8) leads to (3.11). We omit the complete details. □

4. Solutions to the interpolation problem and unitary extensions

We define a $\mathbf{Q}$-colligation as a quadruple

$$\mathcal{C} = \{\mathcal{H},\ \mathcal{E},\ \mathcal{E}_*,\ \mathbf{U}\} \tag{4.1}$$

consisting of three Hilbert spaces $\mathcal{H}$ (*the state space*), $\mathcal{E}$ (*the input space*) and $\mathcal{E}_*$ (*the output space*), together with a connecting operator

$$\mathbf{U} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} : \quad \begin{bmatrix} \mathbb{C}^p \otimes \mathcal{H} \\ \mathcal{E} \end{bmatrix} \to \begin{bmatrix} \mathbb{C}^q \otimes \mathcal{H} \\ \mathcal{E}_* \end{bmatrix}. \tag{4.2}$$

Associated with any such $\mathbf{Q}$-colligation is the discrete-time, multidimensional, linear system

$$\Sigma_{\mathcal{C}}\colon \left\{ \begin{aligned} \begin{bmatrix} x_1(i) \\ \vdots \\ x_p(i) \end{bmatrix} &= \mathbf{Q}(\sigma^*)A \begin{bmatrix} x_1(i) \\ \vdots \\ x_p(i) \end{bmatrix} + \mathbf{Q}(\sigma^*)Bu(i) \\ y(i) &= C \begin{bmatrix} x_1(i) \\ \vdots \\ x_p(i) \end{bmatrix} + Du(i). \end{aligned} \right. \tag{4.3}$$

Here $i = (i_1, \dots, i_n) \in \mathbb{Z}^n$, $\sigma^* = (\sigma_1^*, \dots, \sigma_n^*)$ is the n-tuple of operators defined on any element $v \in \ell(\mathbb{Z}^n, \mathcal{V})$ (the linear space of all $\mathcal{V}$-valued function $i \mapsto v(i)$ on $\mathbb{Z}^n$ where $\mathcal{V}$ is any vector space) defined as

$$\sigma_j^*\colon v(i_1, \dots, i_n) \mapsto v(i_1, \dots, i_{j-1}, i_j - 1, i_{j+1}, \dots, i_n)$$

and $\mathbf{Q}(\sigma^*)\colon \mathbb{C}^q \otimes \ell(\mathbb{Z}^n, \mathcal{V}) \to \mathbb{C}^p \otimes \mathcal{V}$ is the operator obtained by substituting $(\sigma_1^*, \dots, \sigma_n^*)$ for $(z_1, \dots, z_n)$, the arguments of $\mathbf{Q}$. It is convenient to introduce the formal Z-transform:

$$\{v(i)\}_{i \in \mathbb{Z}^n} \mapsto v^\wedge(z) := \sum_{i \in \mathbb{Z}^n} v(i) z^i$$

where z is the n-tuple of independent variables $z = (z_1, \dots, z_n)$ and we are using the standard multivariable notation

$$z^i = z_1^{i_1} \cdots z_n^{i_n} \quad \text{if } \ i = (i_1, \dots, i_n) \in \mathbb{Z}^n.$$

Assume that

$$(u, x, y) = \{u(i),\ x(i) = (x_1(i), \dots, x_p(i)),\ y(i)\}_{i \in \mathbb{Z}^n}$$

is a system trajectory for $\Sigma_{\mathcal{C}}$, i.e., (u, x, y) satisfies the system equations (4.3) over all $i \in \mathbb{Z}^n$. Application of the formal Z-transform to all the equations in (4.3) leads to the identities among formal power series

$$\begin{aligned} x^\wedge(z) &= \mathbf{Q}(z)Ax^\wedge(z) + \mathbf{Q}(z)Bu^\wedge(z) \\ y^\wedge(z) &= Cx^\wedge(z) + Du^\wedge(z). \end{aligned} \tag{4.4}$$

Using the first equation to solve for $x^\wedge(z)$ leads to

$$x^\wedge(z) = (I - \mathbf{Q}(z)A)^{-1}\mathbf{Q}(z)Bu^\wedge(z)$$

and then substitution of this identity in the second of equations (4.4) leads to

$$y^\wedge(z) = T_{\Sigma_{\mathcal{C}}}(z) \cdot u^\wedge(z)$$

where we have set $T_{\Sigma_{\mathcal{C}}}(z)$ equal to the *transfer function* of the system $\Sigma_{\mathcal{C}}$ (also known as the *characteristic function* of the $\mathbf{Q}$-colligation $\mathcal{C}$)

$$S_{\mathcal{C}}(z) = D + C\left(I_{\mathbb{C}^p\otimes\mathcal{H}} - \mathbf{Q}(z)A\right)^{-1}\mathbf{Q}(z)B. \tag{4.5}$$

As examples we mention the case where $\mathbf{Q}(z)$ is equal to the first-degree diagonal polynomial of the form

$$Z_{\text{diag}}(z) := \begin{bmatrix} z_1 & & \\ & \ddots & \\ & & z_n \end{bmatrix}$$

in which case the system of equations (4.3) can be written in the form

$$\begin{cases} \begin{bmatrix} x_1(\sigma_1(i)) \\ \vdots \\ x_n(\sigma_n(i)) \end{bmatrix} = A \begin{bmatrix} x_1(i) \\ \vdots \\ x_n(i) \end{bmatrix} + Bu(i) \\ \qquad\qquad y(i) = C \begin{bmatrix} x_1(i) \\ \vdots \\ x_n(i) \end{bmatrix} + Du(i) \end{cases}$$

with transfer function of the form

$$S_{\mathcal{C}}(z) = D + C(I - Z_{\text{diag}}(z)A)^{-1}Z_{\text{diag}}(z)B.$$

Here $\sigma_1, \dots, \sigma_n$ are the forward shift operators

$$\sigma_j \colon v(i) \mapsto v(i_1, \dots, i_{j-1}, i_j + 1, i_{j+1}, \dots, i_n).$$

Multidimensional systems of this form are known as Roesser (sometimes also as Givone–Roesser) systems in the literature (see [48, 35]). As another example, consider the case where $\mathbf{Q}(z)$ has the row matrix form

$$Z_{\text{row}}(z) := \begin{bmatrix} z_1 & \cdots & z_n \end{bmatrix}$$

in which case the system equations have the form

$$\begin{aligned} x(i) &= A_1x(\sigma_1^*(i)) + \cdots + A_nx(\sigma_n^*(i)) + B_1u(\sigma_1^*(i)) + \cdots + B_nu(\sigma_n^*(i)) \\ y(i) &= Cx(i) + Du(i) \end{aligned}$$

and the transfer function has the form

$$S_{\mathcal{C}}(z) = D + C(I - Z_{\text{row}}(z)A)^{-1}Z_{\text{row}}(z)B.$$

Systems of this form are known as *Fornasini–Marchesini* systems in the literature (see [31, 35]).

The colligation $\mathcal{C}$ is said to be *unitary* if the connecting operator $\mathbf{U}$ is unitary, in which case the associated system $\Sigma_{\mathcal{C}}$ is said to be *conservative*. Thus, one of the assertions of Theorem 1.1 is that a $\mathcal{L}(\mathcal{E}, \mathcal{E}_*)$-valued function S which is analytic on Ω belongs to the class $\mathcal{SA}_{\mathbf{Q}}(\mathcal{E}, \mathcal{E}_*)$ if and only if it is the characteristic function of some unitary $\mathbf{Q}$-colligation $\mathcal{C}$ (i.e., the transfer function of a conservative $\mathbf{Q}$-system $\Sigma_{\mathcal{C}}$) of the form (4.1).

System-theoretic properties of conservative systems for the Roesser case are discussed in [22] and in more definitive detail in [21], where connections between conservative Roesser systems and Fornasini–Marchesini systems are also explored.

A noncommutative version of a conservative Fornasini–Marchesini system is studied at length in [24]. Here we do not develop the system-theoretic properties of conservative systems of the form (4.3) except only to observe the following fact which will be used in the sequel. A colligation

$$\widetilde{\mathcal{C}} = \{\widetilde{\mathcal{H}},\ \mathcal{E},\ \mathcal{E}_*,\ \widetilde{\mathbf{U}}\} \tag{4.6}$$

is said to be *unitarily equivalent* to the colligation $\mathcal{C}$ if there is a unitary operator $\alpha:\ \mathcal{H} \to \widetilde{\mathcal{H}}$ such that

$$\begin{bmatrix} \alpha \otimes I_q & 0 \\ 0 & I_{\mathcal{E}_*} \end{bmatrix} \mathbf{U} = \widetilde{\mathbf{U}} \begin{bmatrix} \alpha \otimes I_p & 0 \\ 0 & I_{\mathcal{E}} \end{bmatrix}.$$

It is easily checked by the very definition (5.5) of the characteristic function that:

Remark 4.1. *Unitarily equivalent colligations have the same characteristic function.*

Remark 4.2. Note also that for a fixed point $z \in \Omega$, the action of $S_{\mathcal{C}}(z)$ on a vector $e \in \mathcal{E}$, namely

$$S_{\mathcal{C}}(z) = D + C\left(I_{\mathbb{C}^p \otimes \mathcal{H}} - \mathbf{Q}(z)A\right)^{-1} \mathbf{Q}(z)B:\ e \to e_*$$

is the result of the feedback connection

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} h \\ e \end{bmatrix} = \begin{bmatrix} h' \\ e_* \end{bmatrix}, \qquad h = \mathbf{Q}(z)h'.$$

The characteristic function of a $\mathbf{Q}$-colligation can be expressed directly in terms of the connecting operator $\mathbf{U}$.

Lemma 4.3. *Let $\mathbf{U}$ be a unitary operator of the form (4.2), let S, H and G be the operator valued functions defined via formulas (1.9) and (1.10). Then*

$$\mathbf{U}\left(I_{(\mathbb{C}^p \otimes \mathcal{H}) \oplus \mathcal{E}} - \mathbf{P}_{\mathbb{C}^p \otimes \mathcal{H}} \mathbf{Q}(z) \mathbf{P}_{\mathbb{C}^q \otimes \mathcal{H}} \mathbf{U}\right)^{-1} = \begin{bmatrix} A\left(I_{\mathcal{H}^p} - \mathbf{Q}(z)A\right)^{-1} & G(z) \\ H(z) & S(z) \end{bmatrix}, \tag{4.7}$$

where $\mathbf{P}_{\mathbb{C}^p \otimes \mathcal{H}}$ and $\mathbf{P}_{\mathbb{C}^q \otimes \mathcal{H}}$ stand for the orthogonal projections of $(\mathbb{C}^p \otimes \mathcal{H}) \oplus \mathcal{E}$ and $(\mathbb{C}^q \otimes \mathcal{H}) \oplus \mathcal{E}_*$ onto $\mathbb{C}^p \otimes \mathcal{H}$ and $\mathbb{C}^q \otimes \mathcal{H}$, respectively.

Proof. Upon making use of (1.8) we get

$$\begin{aligned}
&\left(I_{\mathcal{H}^p \oplus \mathcal{E}} - \mathbf{P}_{\mathcal{H}^p} \mathbf{Q}(z) \mathbf{P}_{\mathcal{H}^q} \mathbf{U}\right)^{-1} \\
&\qquad = \begin{bmatrix} I_{\mathbb{C}^p \otimes \mathcal{H}} - \mathbf{Q}(z)A & -\mathbf{Q}(z)B \\ 0 & I_{\mathcal{E}} \end{bmatrix}^{-1} \\
&\qquad = \begin{bmatrix} \left(I_{\mathbb{C}^p \otimes \mathcal{H}} - \mathbf{Q}(z)A\right)^{-1} & \left(I_{\mathbb{C}^p \otimes \mathcal{H}} - \mathbf{Q}(z)A\right)^{-1} \mathbf{Q}(z)B \\ 0 & I_{\mathcal{E}} \end{bmatrix}
\end{aligned}$$

and since

$$\begin{aligned}
&\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} \left(I_{\mathbb{C}^p \otimes \mathcal{H}} - \mathbf{Q}(z)A\right)^{-1} & \left(I_{\mathbb{C}^p \otimes \mathcal{H}} - \mathbf{Q}(z)A\right)^{-1} \mathbf{Q}(z)B \\ 0 & I_{\mathcal{E}} \end{bmatrix} \\
&\quad = \begin{bmatrix} A\left(I_{\mathbb{C}^p \otimes \mathcal{H}} - \mathbf{Q}(z)A\right)^{-1} & B + \left(I_{\mathbb{C}^p \otimes \mathcal{H}} - \mathbf{Q}(z)A\right)^{-1} \mathbf{Q}(z)B \\ C\left(I_{\mathbb{C}^p \otimes \mathcal{H}} - \mathbf{Q}(z)A\right)^{-1} & D + C\left(I_{\mathbb{C}^p \otimes \mathcal{H}} - \mathbf{Q}(z)A\right)^{-1} \mathbf{Q}(z)B \end{bmatrix} \\
&\quad = \begin{bmatrix} A\left(I_{\mathbb{C}^p \otimes \mathcal{H}} - \mathbf{Q}(z)A\right)^{-1} & \left(I_{\mathbb{C}^q \otimes \mathcal{H}} - A\mathbf{Q}(z)\right)^{-1} B \\ C\left(I_{\mathbb{C}^p \otimes \mathcal{H}} - \mathbf{Q}(z)A\right)^{-1} & D + C\left(I_{\mathbb{C}^p \otimes \mathcal{H}} - \mathbf{Q}(z)A\right)^{-1} \mathbf{Q}(z)B \end{bmatrix},
\end{aligned}$$

relation (4.7) follows by (1.9) and (1.10). □

Equality (4.7) allows us to define the functions S, H and G associated to a **Q**-colligation $\mathcal{C}$ directly in terms of the connecting operator $\mathbf{U}$ and projections onto input, output and state spaces. Indeed, it is readily seen from (4.7) that

$$\left\{\begin{aligned} S(z) &= \mathbf{P}_{\mathcal{E}_*}\mathbf{U}\left(I_{(\mathbb{C}^p\otimes\mathcal{H})\oplus\mathcal{E}} - \mathbf{P}_{\mathbb{C}^p\otimes\mathcal{H}}\mathbf{Q}(z)\mathbf{P}_{\mathbb{C}^q\otimes\mathcal{H}}\mathbf{U}\right)^{-1}|_{\mathcal{E}},\\ H(z) &= \mathbf{P}_{\mathcal{E}_*}\mathbf{U}\left(I_{(\mathbb{C}^p\otimes\mathcal{H})\oplus\mathcal{E}} - \mathbf{P}_{\mathbb{C}^p\otimes\mathcal{H}}\mathbf{Q}(z)\mathbf{P}_{\mathbb{C}^q\otimes\mathcal{H}}\mathbf{U}\right)^{-1}|_{\mathcal{H}^p},\\ G(z) &= \mathbf{P}_{\mathcal{H}^q}\mathbf{U}\left(I_{(\mathbb{C}^p\otimes\mathcal{H})\oplus\mathcal{E}} - \mathbf{P}_{\mathbb{C}^p\otimes\mathcal{H}}\mathbf{Q}(z)\mathbf{P}_{\mathcal{H}^q}\mathbf{U}\right)^{-1}|_{\mathcal{E}}, \end{aligned}\right. \tag{4.8}$$

where $\mathbf{P}_{\mathcal{E}_*}$ and $\mathbf{P}_{\mathbb{C}^q\otimes\mathcal{H}}$ are the orthogonal projections of the space $(\mathbb{C}^p\otimes\mathcal{H})\oplus\mathcal{E}_*$ onto $\mathcal{E}_*$ and $\mathbb{C}^q\otimes\mathcal{H}$, respectively.

From now on we assume that we are given an interpolation data set $\mathcal{D}$ as in (1.29) and that the necessary conditions for Problem 1.6 to have a solution are in force: the operator P defined in (1.22)–(1.24) is positive semidefinite on the space

$$\mathcal{H}_0 = (\mathbb{C}^p\otimes\mathcal{K})\oplus(\mathbb{C}^q\otimes\mathcal{K}'). \tag{4.9}$$

and satisfies the Stein identity (1.20) which we write now as

$$\sum_{j=1}^{p} M_j^* P M_j + Y^*Y = \sum_{k=1}^{q} N_k^* P N_k + X^*X. \tag{4.10}$$

Introduce the equivalence $\sim$ on $\mathcal{H}_0$ by

$$h_1 \sim h_2 \ \text{ if and only if } \ \langle P(h_1-h_2),\ y\rangle_{\mathcal{H}_0} = 0 \ \text{ for all } \ y\in\mathcal{H}_0,$$

denote $[h]$ the equivalence class of h with respect to the above equivalence and endow the linear space of equivalence classes with the inner product

$$\langle [h],\ [y]\rangle = \langle Ph,\ y\rangle_{\mathcal{H}_0}. \tag{4.11}$$

We get a prehilbert space whose completion is $\widehat{\mathcal{H}}$. It is readily seen from definitions (1.18), (1.19) of operators M_j and N_k that M_jx and N_kx belong to $\mathcal{H}_0$ for any $x\in\mathcal{K}\oplus\mathcal{K}'$. Furthermore, identity (4.10) can be written as

$$\sum_{j=1}^{p}\langle [M_jf],[M_jg]\rangle_{\widehat{\mathcal{H}}} + \langle Yf, Yg\rangle_{\mathcal{E}} = \sum_{k=1}^{q}\langle [N_kf],[N_kg]\rangle_{\widehat{\mathcal{H}}} + \langle Xf, Xg\rangle_{\mathcal{E}_*},$$

holding for every choice of $f,\ g\in\mathcal{K}\oplus\mathcal{K}'$. Therefore the linear map defined by the rule

$$\mathbf{V}:\ \begin{bmatrix} [M_1f] \\ \vdots \\ [M_pf] \\ Yf \end{bmatrix} \to \begin{bmatrix} [N_1f] \\ \vdots \\ [N_qf] \\ Xf \end{bmatrix} \tag{4.12}$$

extends by linearity to define an isometry from

$$\mathcal{D}_{\mathbf{V}} = \operatorname{Clos}\left\{\begin{bmatrix} [M_1f] \\ \vdots \\ [M_pf] \\ Yf \end{bmatrix},\ f\in\mathcal{K}\oplus\mathcal{K}'\right\} \subset \begin{bmatrix} \mathbb{C}^p\otimes\widehat{\mathcal{H}} \\ \mathcal{E} \end{bmatrix} \tag{4.13}$$

onto

$$\mathcal{R}_{\mathbf{V}} = \operatorname{Clos}\left\{\begin{bmatrix} [N_1 f] \\ \vdots \\ [N_q f] \\ Xf \end{bmatrix}, \ f \in \mathcal{K} \oplus \mathcal{K}'\right\} \subset \begin{bmatrix} \mathbb{C}^q \otimes \widehat{\mathcal{H}} \\ \mathcal{E}_* \end{bmatrix}. \tag{4.14}$$

The next two lemmas establish a correspondence between solutions S to Problem 1.6 and unitary extensions of the partially defined isometry $\mathbf{V}$ given in (4.12).

Lemma 4.4. *Any solution S to Problem 1.6 is a characteristic function of a unitary colligation*

$$\widetilde{\mathbf{U}} = \begin{bmatrix} \widetilde{A} & \widetilde{B} \\ \widetilde{C} & \widetilde{D} \end{bmatrix} : \begin{bmatrix} \mathbb{C}^p \otimes (\widehat{\mathcal{H}} \oplus \widetilde{\mathcal{H}}) \\ \mathcal{E} \end{bmatrix} \to \begin{bmatrix} \mathbb{C}^q \otimes (\widehat{\mathcal{H}} \oplus \widetilde{\mathcal{H}}) \\ \mathcal{E}_* \end{bmatrix}, \tag{4.15}$$

which is an extension of the isometry $\mathbf{V}$ given in (4.12).

Proof. Let S be a solution to Problem 1.6. In particular, S belongs to the class $\mathcal{SA}_{\mathbf{Q}}(\mathcal{E}, \mathcal{E}_*)$ and by Theorem 1.1, it is the characteristic function of some unitary colligation $\mathcal{C}$ of the form (4.1). In other words, S admits a unitary realization (1.9) with the state space $\mathcal{H}$ and representations (1.4), (1.6), (1.7) holds for functions H and G defined via (1.10) and decomposed as in (1.3) and (1.5). These functions are analytic and respectively $\mathcal{L}(\mathbb{C}^p \otimes \mathcal{H}, \mathcal{E}_*)$- and $\mathcal{L}(\mathcal{E}, \mathbb{C}^q \otimes \mathcal{H})$-valued on Ω and lead to the following two representations

$$S(z) = D + H(z)\mathbf{Q}(z)B = D + C\mathbf{Q}(z)G(z), \tag{4.16}$$

of S, each of which is equivalent to (1.9).

The interpolation conditions (1.14) and (1.26)–(1.28) which are assumed to be satisfied by S, force certain restrictions on the connecting operator $\mathbf{U} = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$. Substituting (4.16) into (1.14) we get equalities

$$(X_L D + X_L H\mathbf{Q}B)^{\wedge L}(T) = Y_L$$
$$(DY_R + C\mathbf{Q}GY_R)^{\wedge R}(T') = X_R$$

which are equivalent, due to properties (2.7) and (2.8), to

$$X_L D + (X_L H\mathbf{Q})^{\wedge L}(T)B = Y_L \tag{4.17}$$

$$DY_R + C\,(\mathbf{Q}GY_R)^{\wedge R}(T') = X_R, \tag{4.18}$$

respectively. It also follows from (1.10) that

$$C + H(z)\mathbf{Q}(z)A = H(z), \qquad B + A\mathbf{Q}(z)G(z) = G(z)$$

and therefore, that

$$X_L C + (X_L H\mathbf{Q})^{\wedge L}(T)A = (X_L H)^{\wedge L}(T) \tag{4.19}$$

$$BY_R + A\,(\mathbf{Q}GY_R)^{\wedge R}(T') = (GY_R)^{\wedge R}(T'). \tag{4.20}$$

The equalities (4.17) and (4.19) can be written in matrix form as

$$\begin{bmatrix}(X_L H\mathbf{Q})^{\wedge L}(T) & X_L\end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix}(X_L H)^{\wedge L}(T) & Y_L\end{bmatrix}, \tag{4.21}$$

whereas the equalities (4.18) and (4.20) are equivalent to

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} (\mathbf{Q}GY_R)^{\wedge R}(T') \\ Y_R \end{bmatrix} = \begin{bmatrix} (GY_R)^{\wedge R}(T') \\ X_R \end{bmatrix}. \tag{4.22}$$

Since the operator $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ is unitary, we conclude from (4.21) that

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} (X_L H)^{\wedge L}(T)^* \\ Y_L^* \end{bmatrix} = \begin{bmatrix} (X_L H\mathbf{Q})^{\wedge L}(T)^* \\ X_L^* \end{bmatrix}. \tag{4.23}$$

Combining (4.22) and (4.23) we conclude that for every choice of $f \in \mathcal{K} \oplus \mathcal{K}'$,

$$\begin{aligned} &\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} (X_L H)^{\wedge L}(T)^* & (\mathbf{Q}GY_R)^{\wedge R}(T') \\ Y_L^* & Y_R \end{bmatrix} f \\ &\qquad = \begin{bmatrix} (X_L H\mathbf{Q})^{\wedge L}(T)^* & (GY_R)^{\wedge R}(T') \\ X_L^* & X_R \end{bmatrix} f. \end{aligned} \tag{4.24}$$

Let $\mathbb{T}_L$ and $\mathbb{T}_R$ be given by (3.2) and (3.3), and let

$$\mathbb{T} := \begin{bmatrix} \mathbb{T}_L & \mathbb{T}_R \end{bmatrix} : \mathcal{H}_0 \to \mathcal{H}. \tag{4.25}$$

(Recall that the space $\mathcal{H}_0$ is introduced in (4.9).) Now we use the interpolation conditions (1.26)–(1.28), which provide the factorization (3.1) of the operator P. Thus,

$$P = \mathbb{T}^*\mathbb{T}$$

and

$$\langle [h],\ [y] \rangle_{\widehat{\mathcal{H}}} = \langle Ph,\ y \rangle_{\mathcal{H}_0} = \langle \mathbb{T}h,\ \mathbb{T}y \rangle_{\mathcal{H}_0}$$

for every $h,\, y \in \mathcal{H}_0$. Therefore, the linear transformation U defined by the rule

$$U\ :\ \mathbb{T}h \to [h] \qquad (h \in \mathcal{H}_0) \tag{4.26}$$

can be extended to the unitary map (which still is denoted by U) from $\overline{\operatorname{Ran}\mathbb{T}}$ onto $\widehat{\mathcal{H}}$. Noticing that $\overline{\operatorname{Ran}\mathbb{T}}$ is a subspace of $\mathcal{H}$ and setting

$$\mathcal{N} := \mathcal{H} \ominus \overline{\operatorname{Ran}\mathbb{T}} \quad \text{and} \quad \widetilde{\mathcal{H}} := \widehat{\mathcal{H}} \oplus \mathcal{N},$$

we define the unitary map $\widetilde{U}\ :\ \mathcal{H} \to \widetilde{\mathcal{H}}$ by the rule

$$\widetilde{U}\, g = \begin{cases} Ug & \text{for } g \in \overline{\operatorname{Ran}\mathbb{T}}, \\ g & \text{for } g \in \mathcal{N}. \end{cases} \tag{4.27}$$

Introducing the operators

$$\widetilde{A} = (\widetilde{U} \otimes I_q)A(\widetilde{U} \otimes I_p)^*, \quad \widetilde{B} = (\widetilde{U} \otimes I_q)B, \quad \widetilde{C} = C(\widetilde{U} \otimes I_p)^*, \quad \widetilde{D} = D$$

we construct the colligation $\widetilde{\mathcal{C}}$ via (4.6) and (4.15). By definition, $\widetilde{\mathcal{C}}$ is unitarily equivalent to the initial colligation $\mathcal{C}$ defined in (4.1). By Remark 4.1, $\widetilde{\mathcal{C}}$ has the same characteristic function as $\mathcal{C}$, that is, $S(z)$. It remains to check that the connecting operator of $\widetilde{\mathcal{C}}$ is an extension of $\mathbf{V}$, that is

$$\begin{bmatrix} \widetilde{A} & \widetilde{B} \\ \widetilde{C} & \widetilde{D} \end{bmatrix} \begin{bmatrix} [M_1 f] \\ \vdots \\ [M_p f] \\ Yf \end{bmatrix} = \begin{bmatrix} [N_1 f] \\ \vdots \\ [N_q f] \\ Xf \end{bmatrix}, \text{ for every } f \in \mathcal{K} \oplus \mathcal{K}'. \tag{4.28}$$

To this end, note that by (4.26), (4.27) and block partitionings (1.18) and (4.25) of M_j and $\mathbb{T}$, it holds that

$$\widetilde{U}^*\left([M_j f]\right) = \mathbb{T}(M_j f) = \begin{bmatrix} \mathbb{T}_L E_j & \mathbb{T}_R \mathbf{Q}_{j\cdot}(T') \end{bmatrix} f$$

for every $f \in \mathcal{K} \oplus \mathcal{K}'$ and for $j = 1, \dots, p$. Therefore,

$$(\widetilde{U} \otimes I_p)^* \begin{bmatrix} [M_1 f] \\ \vdots \\ [M_p f] \end{bmatrix} = \begin{bmatrix} \mathbb{T}_L E_1 & \mathbb{T}_R \mathbf{Q}_{1\cdot}(T') \\ \vdots & \vdots \\ \mathbb{T}_L E_p & \mathbb{T}_R \mathbf{Q}_{p\cdot}(T') \end{bmatrix} f \tag{4.29}$$

which, on account of (3.6) and (3.7) can be written as

$$(\widetilde{U} \otimes I_p)^* \begin{bmatrix} [M_1 f] \\ \vdots \\ [M_p f] \end{bmatrix} = \begin{bmatrix} (X_L H)^{\wedge L}(T)^* & (\mathbf{Q} G Y_R)^{\wedge R}(T') \end{bmatrix} f. \tag{4.30}$$

Similarly, by (4.26), (4.27) and block partitionings (1.19) and (4.25) of N_k and $\mathbb{T}$, it holds that

$$[N_k f] = \widetilde{U}\mathbb{T}(N_k f) = \widetilde{U} \begin{bmatrix} \mathbb{T}_L \mathbf{Q}_{\cdot k}(T) & \mathbb{T}_R E'_k \end{bmatrix} f$$

for $k = 1, \dots, q$. Therefore,

$$(\widetilde{U} \otimes I_q)^* \begin{bmatrix} [N_1 f] \\ \vdots \\ [N_q f] \end{bmatrix} = \begin{bmatrix} \mathbb{T}_L \mathbf{Q}_{\cdot 1}(T) & \mathbb{T}_R E'_1 \\ \vdots & \vdots \\ \mathbb{T}_L \mathbf{Q}_{\cdot q}(T) & \mathbb{T}_R E'_q \end{bmatrix} f \tag{4.31}$$

which, on account of (3.5) and (3.8) can be written as

$$(\widetilde{U} \otimes I_q)^* \begin{bmatrix} [N_1 f] \\ \vdots \\ [N_q f] \end{bmatrix} = \begin{bmatrix} (X_L H \mathbf{Q})^{\wedge L}(T)^* & (G Y_R)^{\wedge R}(T') \end{bmatrix} f. \tag{4.32}$$

Thus, by (4.24) and in view of (1.21), (4.30) and (4.32),

$$\begin{aligned}
&\begin{bmatrix} \widetilde{A} & \widetilde{B} \\ \widetilde{C} & \widetilde{D} \end{bmatrix} \begin{bmatrix} [M_1 f] \\ \vdots \\ [M_p f] \\ Y f \end{bmatrix} \qquad (4.33) \\
&= \begin{bmatrix} \widetilde{U} \otimes I_q & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} (\widetilde{U} \otimes I_p)^* & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} [M_1 f] \\ \vdots \\ [M_p f] \\ Y f \end{bmatrix} \\
&= \begin{bmatrix} \widetilde{U} \otimes I_q & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} (X_L H)^{\wedge L}(T)^* & (\mathbf{Q} G Y_R)^{\wedge R}(T') \\ Y_L^* & Y_R \end{bmatrix} f \\
&= \begin{bmatrix} \widetilde{U} \otimes I_q & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} (X_L H \mathbf{Q})^{\wedge L}(T)^* & (G Y_R)^{\wedge R}(T') \\ X_L^* & X_R \end{bmatrix} = \begin{bmatrix} [N_1 f] \\ \vdots \\ [N_q f] \\ X f \end{bmatrix},
\end{aligned}$$

which proves (4.28) and completes the proof of the lemma. □

Lemma 4.5. *Let $\widetilde{\mathbf{U}}$ of the form (4.15) be a unitary extension of the isometry $\mathbf{V}$ given in (4.12). Then the characteristic function S of the colligation $\widetilde{\mathcal{C}} = \{\widehat{\mathcal{H}} \oplus \widetilde{\mathcal{H}},\ \mathcal{E},\ \mathcal{E}_*,\ \widetilde{\mathbf{U}}\}$,*

$$S(z) = \widetilde{D} + \widetilde{C}\left(I_{\mathbb{C}^p\otimes(\widehat{\mathcal{H}}\oplus\widetilde{\mathcal{H}})} - \mathbf{Q}(z)\widetilde{A}\right)^{-1}\mathbf{Q}(z)\widetilde{B},$$

is a solution to Problem 1.6.

Proof. We use the arguments from the proof of the previous lemma in the reverse order. We fix a factorization

$$P = \mathbb{T}^*\mathbb{T} \tag{4.34}$$

of the positive operator P with an operator

$$\mathbb{T} = \begin{bmatrix}\mathbb{T}_L & \mathbb{T}_R\end{bmatrix} : \mathcal{H}_0 \to \mathcal{G}$$

where $\mathcal{G}$ is an auxiliary Hilbert space,

$$\mathbb{T}_L = \begin{bmatrix}\mathbb{T}_{L,1} & \dots & \mathbb{T}_{L,p}\end{bmatrix}, \qquad \mathbb{T}_{L,1}, \dots, \mathbb{T}_{L,p} \in \mathcal{L}(\mathcal{K}, \mathcal{G})$$

and

$$\mathbb{T}_R = \begin{bmatrix}\mathbb{T}_{R,1} & \dots & \mathbb{T}_{R,q}\end{bmatrix}, \qquad \mathbb{T}_{R,1}, \dots, \mathbb{T}_{R,q} \in \mathcal{L}(\mathcal{K}', \mathcal{G}).$$

We use operators $\mathbb{T}_{L,j}$ and $\mathbb{T}_{R,k}$ to define operators $\mathbb{F}_L : \mathcal{K} \to \mathbb{C}^p \times \mathcal{G}$ and $\mathbb{F}_R : \mathcal{K}' \to \mathbb{C}^q \times \mathcal{G}$ as follows:

$$\mathbb{F}_L := \begin{bmatrix}\mathbb{T}_{L,1}\\ \vdots \\ \mathbb{T}_{L,p}\end{bmatrix} = \begin{bmatrix}\mathbb{T}_L E_1\\ \vdots \\ \mathbb{T}_L E_p\end{bmatrix} \quad \text{and} \quad \mathbb{F}_R := \begin{bmatrix}\mathbb{T}_{R,1}\\ \vdots \\ \mathbb{T}_{R,q}\end{bmatrix} = \begin{bmatrix}\mathbb{T}_R E_1'\\ \vdots \\ \mathbb{T}_R E_q'\end{bmatrix}. \tag{4.35}$$

We shall make use of the following two formulas:

$$\begin{bmatrix}\mathbb{T}_L\mathbf{Q}_{\cdot 1}(T)\\ \vdots \\ \mathbb{T}_L\mathbf{Q}_{\cdot q}(T)\end{bmatrix} = \left[(\mathbb{F}_L^* \cdot \mathbf{Q})^{\wedge L}(T)\right]^*, \tag{4.36}$$

$$\begin{bmatrix}\mathbb{T}_R\mathbf{Q}_{1\cdot}(T')\\ \vdots \\ \mathbb{T}_R\mathbf{Q}_{p\cdot}(T')\end{bmatrix} = (\mathbf{Q} \cdot \mathbb{F}_R)^{\wedge R}(T'), \tag{4.37}$$

which are similar to formulas (3.5) and (3.7) and are verified in much the same way.

Let $\widetilde{U}$ be the unitary map defined via formulas (4.26), (4.27). Then relations (4.29) and (4.31) hold by construction; in view of (4.35)–(4.37) these relations can be written as

$$(\widetilde{U} \otimes I_p)^* \begin{bmatrix}[M_1 f]\\ \vdots \\ [M_p f]\end{bmatrix} = \begin{bmatrix}\mathbb{F}_L & (\mathbf{Q} \cdot \mathbb{F}_R)^{\wedge R}(T')\end{bmatrix} f, \tag{4.38}$$

$$(\widetilde{U} \otimes I_q)^* \begin{bmatrix}[N_1 f]\\ \vdots \\ [N_q f]\end{bmatrix} = \begin{bmatrix}\left[(\mathbb{F}_L^* \cdot \mathbf{Q})^{\wedge L}(T)\right]^* & \mathbb{F}_R\end{bmatrix} f. \tag{4.39}$$

Since $\widetilde{\mathbf{U}}$ extends $\mathbf{V}$:

$$\begin{bmatrix} \widetilde{A} & \widetilde{B} \\ \widetilde{C} & \widetilde{D} \end{bmatrix} \begin{bmatrix} [M_1 f] \\ \vdots \\ [M_p f] \\ Yf \end{bmatrix} = \begin{bmatrix} [N_1 f] \\ \vdots \\ [N_q f] \\ Xf \end{bmatrix} \quad \text{for every } \; f \in \mathcal{K} \oplus \mathcal{K}',$$

it follows from (4.38) and (4.39) that the operator

$$\mathbf{U} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} (\widetilde{U} \otimes I_q)^* & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \widetilde{A} & \widetilde{B} \\ \widetilde{C} & \widetilde{D} \end{bmatrix} \begin{bmatrix} \widetilde{U} \otimes I_p & 0 \\ 0 & I \end{bmatrix}$$

satisfies

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} \mathbb{F}_L & (\mathbf{Q}\mathbb{F}_R)^{\wedge R}(T') \\ Y_L^* & Y_R \end{bmatrix} = \begin{bmatrix} \left[(\mathbb{F}_L^* \mathbf{Q})^{\wedge L}(T)\right]^* & \mathbb{F}_R \\ X_L^* & X_R \end{bmatrix}. \tag{4.40}$$

By Remark 4.1, the colligations $\mathcal{C}$ and $\widetilde{\mathcal{C}}$ defined in (4.1) and (4.6) have the same characteristic functions and thus, S can be taken in the form (1.9). Let $H(z)$ and $G(z)$ be defined as in (1.10) and decomposed as in (1.3) and (1.5). We shall use the representations (4.16) of $S(z)$ which are equivalent to (1.9).

Since $\mathbf{U}$ is unitary, it follows from (4.40) that

$$A^* \left(\mathbb{F}_L^* \mathbf{Q}\right)^{\wedge L}(T)^* + C^* X_L^* = \mathbb{F}_L, \tag{4.41}$$

$$B^* \left(\mathbb{F}_L^* \mathbf{Q}\right)^{\wedge L}(T)^* + D^* X_L^* = Y_L^*, \tag{4.42}$$

$$A \left(\mathbf{Q}\mathbb{F}_R\right)^{\wedge R}(T') + BY_R = \mathbb{F}_R, \tag{4.43}$$

$$C \left(\mathbf{Q}\mathbb{F}_R\right)^{\wedge R}(T') + DY_R = X_R. \tag{4.44}$$

Taking adjoints in (4.41) we get

$$X_L C = \mathbb{F}_L^* - \left(\mathbb{F}_L^* \mathbf{Q}\right)^{\wedge L}(T) A$$

which can be written, by properties (2.7) and (2.8) of the left evaluation map, as

$$X_L C = \left(\mathbb{F}_L^* (I - \mathbf{Q}A)\right)^{\wedge L}(T).$$

Multiplying both sides in the last equality by $(I - \mathbf{Q}(z)A)^{-1}$ on the right and applying the left evaluation map to the resulting identity

$$X_L H(z) = \left(\mathbb{F}_L^* (I - \mathbf{Q}A)\right)^{\wedge L}(T) \cdot (I - \mathbf{Q}(z)A)^{-1},$$

we get

$$\begin{aligned} (X_L H)^{\wedge L}(T) &= \left(\left(\mathbb{F}_L^* (I - \mathbf{Q}A)\right)^{\wedge L}(T)(I - \mathbf{Q}A)^{-1} \right)^{\wedge L}(T) \\ &= \left(\mathbb{F}_L^* (I - \mathbf{Q}A)(I - \mathbf{Q}A)^{-1}\right)^{\wedge L}(T) \\ &= \left(\mathbb{F}_L^*\right)^{\wedge L}(T) = \mathbb{F}_L^*. \end{aligned} \tag{4.45}$$

Note that the second equality in the last chain has been obtained upon applying (2.10) to functions $F(z) = \mathbb{F}_L^* (I - \mathbf{Q}(z)A)$ and $\widetilde{F}(z) = (I - \mathbf{Q}(z)A)^{-1}$, whereas the third equality follows by the property (2.7). Now we take adjoints in (4.42) to get

$$Y_L = \left(\mathbb{F}_L^* \mathbf{Q}\right)^{\wedge L}(T) B + X_L D = \left(\mathbb{F}_L^* \mathbf{Q} B\right)^{\wedge L}(T) + X_L D. \tag{4.46}$$

By (4.45),

$$\left(\mathbb{F}_L^*\mathbf{Q}B\right)^{\wedge L}(T) = \Big(\left(X_LH\right)^{\wedge L}(T)\cdot \mathbf{Q}B\Big)^{\wedge L}(T)$$

and applying (2.10) to functions $F(z) = X_LH(z)$ and $\widetilde{F}(z) = \mathbf{Q}(z)B$ leads us to

$$\left(\mathbb{F}_L^*\mathbf{Q}B\right)^{\wedge L}(T) = \left((X_LH\mathbf{Q}B\right)^{\wedge L}(T).$$

Substituting the latter equality into the left-hand side expression in (4.46) and making use of the first representation of S in (4.16), we get

$$\begin{aligned} Y_L &= \left(X_LH\mathbf{Q}B\right)^{\wedge L}(T) + X_LD \\ &= \left(X_LH\mathbf{Q}B + X_LD\right)^{\wedge L}(T) = \left(X_LS\right)^{\wedge L}(T), \end{aligned}$$

which proves the first interpolation condition in (1.14).

To get the second interpolation condition in (1.14) write (4.43) in the form

$$BY_R = \left(I - A\mathbf{Q})\mathbb{F}_R\right)^{\wedge R}(T'),$$

multiply the latter equality by $(I - A\mathbf{Q}(z))^{-1}$ on the left and apply the right evaluation map to the resulting identity

$$G(z)Y_R = \left(I - A\mathbf{Q}(z)\right)^{-1}\left(I - A\mathbf{Q})\mathbb{F}_R\right)^{\wedge R}(T').$$

We have

$$\begin{aligned} \left(GY_R\right)^{\wedge R}(T') &= \Big(\left(I - A\mathbf{Q}\right)^{-1}\left(\left(I - A\mathbf{Q}\right)\mathbb{F}_R\right)^{\wedge R}(T')\Big)^{\wedge R}(T') \\ &= \left((I - A\mathbf{Q})^{-1}(I - A\mathbf{Q})\mathbb{F}_R\right)^{\wedge R}(T') \\ &= \left(\mathbb{F}_R\right)^{\wedge R}(T') = \mathbb{F}_R. \end{aligned} \tag{4.47}$$

Note that the third equality in the last chain has been obtained upon applying (2.11) to functions $F(z) = (I - A\mathbf{Q}(z))^{-1}$ and $\widetilde{F}(z) = (I - A\mathbf{Q}(z))\mathbb{F}_R$. Substituting (4.47) into (4.44) and applying (2.11) to functions $F(z) = C\mathbf{Q}(z)$ and $\widetilde{F}(z) = G(z)Y_R$, we get

$$\begin{aligned} X_R &= \Big(C\mathbf{Q}\left(GY_R\right)^{\wedge R}(T')\Big)^{\wedge R}(T') + DY_R \\ &= \left(C\mathbf{Q}GY_R\right)^{\wedge R}(T') + DY_R \\ &= \left(C\mathbf{Q}GY_R + DY_R\right)^{\wedge R}(T') \end{aligned}$$

which coincides with the second equality in (1.14), due to the second representation in (4.16).

Thus, S belongs to $\mathcal{SA}_{\mathbf{Q}}(\mathcal{E}, \mathcal{E}_*)$ as the characteristic function of a unitary $\mathbf{Q}$-colligation and satisfies the first-order interpolation conditions (1.14). It remains to show that it satisfies also conditions (1.26)–(1.28). But it follows from (4.45), (4.47) and (4.35) that

$$T_{L,j}^* = \left(X_LH_j\right)^{\wedge L}(T) \quad \text{and} \quad T_{R,\ell} = \left(G_\ell Y_R\right)^{\wedge L}(T')$$

for $j = 1, \ldots, p$ and $\ell = 1, \ldots, q$. These last equalities together with factorization (4.34) imply (1.26)–(1.28). ☐

5. The universal unitary colligation associated with the interpolation problem

A general result of Arov and Grossman (see [11], [12]) describes how to parametrize the set of all unitary extensions of a given partially defined isometry $\mathbf{V}$. Their result has been extended to the multivariable case in [22] (for the case of the polydisk) and in [23] (for the case of the unit ball) and will be extended in this section to the case of $\mathbf{Q}$-colligations.

Let $\mathbf{V}: \mathcal{D}_{\mathbf{V}} \to \mathcal{R}_{\mathbf{V}}$ be the isometry given in (4.12) with $\mathcal{D}_{\mathbf{V}}$ and $\mathcal{R}_{\mathbf{V}}$ given in (4.13) and (4.14). Introduce the defect spaces

$$\Delta = \begin{bmatrix} \mathbb{C}^p \otimes \widehat{\mathcal{H}} \\ \mathcal{E} \end{bmatrix} \ominus \mathcal{D}_{\mathbf{V}} \quad \text{and} \quad \Delta_* = \begin{bmatrix} \mathbb{C}^q \otimes \widehat{\mathcal{H}} \\ \mathcal{E}_* \end{bmatrix} \ominus \mathcal{R}_{\mathbf{V}}$$

and let $\widetilde{\Delta}$ to be another copy of Δ and $\widetilde{\Delta}_*$ to be another copy of Δ_* with unitary identification maps

$$i: \ \Delta \to \widetilde{\Delta} \quad \text{and} \quad i_*: \ \Delta_* \to \widetilde{\Delta}_*.$$

Define a unitary operator $\mathbf{U}_0$ from $\mathcal{D}_{\mathbf{V}} \oplus \Delta \oplus \widetilde{\Delta}_*$ onto $\mathcal{R}_{\mathbf{V}} \oplus \Delta_* \oplus \widetilde{\Delta}$ by the rule

$$\mathbf{U}_0 x = \begin{cases} \mathbf{V}x, & \text{if } x \in \mathcal{D}_{\mathbf{V}}, \\ i(x) & \text{if } x \in \Delta, \\ i_*^{-1}(x) & \text{if } x \in \widetilde{\Delta}_*. \end{cases} \tag{5.1}$$

Identifying $\begin{bmatrix} \mathcal{D}_{\mathbf{V}} \\ \Delta \end{bmatrix}$ with $\begin{bmatrix} \mathbb{C}^p \otimes \widehat{\mathcal{H}} \\ \mathcal{E} \end{bmatrix}$ and $\begin{bmatrix} \mathcal{R}_{\mathbf{V}} \\ \Delta_* \end{bmatrix}$ with $\begin{bmatrix} \mathbb{C}^q \otimes \widehat{\mathcal{H}} \\ \mathcal{E}_* \end{bmatrix}$, we decompose $\mathbf{U}_0$ defined by (5.1) according to

$$\mathbf{U}_0 = \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ U_{21} & U_{22} & U_{23} \\ U_{31} & U_{32} & 0 \end{bmatrix} : \quad \begin{bmatrix} \mathbb{C}^p \otimes \widehat{\mathcal{H}} \\ \mathcal{E} \\ \widetilde{\Delta}_* \end{bmatrix} \to \begin{bmatrix} \mathbb{C}^q \otimes \widehat{\mathcal{H}} \\ \mathcal{E}_* \\ \widetilde{\Delta} \end{bmatrix}. \tag{5.2}$$

The $(3,3)$ block in this decomposition is zero, since (by definition (5.1)), for every $x \in \widetilde{\Delta}_*$, the vector $\mathbf{U}_0 x$ belongs to Δ, which is a subspace of $\begin{bmatrix} \widehat{\mathcal{H}} \\ \mathcal{E}_* \end{bmatrix}$ and therefore, is orthogonal to $\widetilde{\Delta}$ (in other words $\mathbf{P}_{\widetilde{\Delta}} \mathbf{U}_0|_{\widetilde{\Delta}_*} = 0$, where $\mathbf{P}_{\widetilde{\Delta}}$ stands for the orthogonal projection of $\mathcal{R}_{\mathbf{V}} \oplus \Delta_* \oplus \widetilde{\Delta}$ onto $\widetilde{\Delta}$).

The unitary operator $\mathbf{U}_0$ is the connecting operator of the unitary colligation

$$\mathcal{C}_0 = \left\{ \widehat{\mathcal{H}}, \ \begin{bmatrix} \mathcal{E} \\ \widetilde{\Delta}_* \end{bmatrix}, \ \begin{bmatrix} \mathcal{E}_* \\ \widetilde{\Delta} \end{bmatrix}, \ \mathbf{U}_0 \right\}, \tag{5.3}$$

which is called *the universal unitary colligation* associated with the interpolation problem.

Let $\widetilde{\mathcal{C}}$ be any $\mathbf{Q}$-colligation of the form

$$\widetilde{\mathcal{C}} = \left\{ \widetilde{\mathcal{H}}, \ \widetilde{\Delta}, \ \widetilde{\Delta}_*, \ \widetilde{\mathbf{U}} \right\}. \tag{5.4}$$

We define another $\mathbf{Q}$-colligation $\mathcal{F}_{\mathcal{C}_0}[\widetilde{\mathcal{C}}]$, called the *coupling* of $\mathcal{C}_0$ and $\widetilde{\mathcal{C}}$, to be the $\mathbf{Q}$-colligation of the form

$$\mathcal{F}_{\mathcal{C}_0}[\widetilde{\mathcal{C}}] = \left\{ \widehat{\mathcal{H}} \oplus \widetilde{\mathcal{H}}, \ \mathcal{E}, \ \mathcal{E}_*, \ \mathcal{F}_{\mathbf{U}_0}[\widetilde{\mathbf{U}}] \right\}$$

with the connecting operator $\mathcal{F}_{\mathbf{U}_0}[\widetilde{\mathbf{U}}]$ defined as follows:

$$\mathcal{F}_{\mathbf{U}_0}[\widetilde{\mathbf{U}}]:\ \begin{bmatrix} c \\ h \\ e \end{bmatrix} \to \begin{bmatrix} c' \\ h' \\ e_* \end{bmatrix} \tag{5.5}$$

if the system of equations

$$\mathbf{U}_0:\ \begin{bmatrix} c \\ e \\ \widetilde{d}_* \end{bmatrix} \to \begin{bmatrix} c' \\ e_* \\ \widetilde{d} \end{bmatrix} \quad \text{and} \quad \widetilde{\mathbf{U}}:\ \begin{bmatrix} h \\ \widetilde{d} \end{bmatrix} \to \begin{bmatrix} h' \\ \widetilde{d}_* \end{bmatrix} \tag{5.6}$$

is satisfied for some choice of $\widetilde{d} \in \widetilde{\Delta}$ and $\widetilde{d}_* \in \widetilde{\Delta}_*$. To show that the operator $\mathcal{F}_{\mathbf{U}_0}[\widetilde{\mathbf{U}}]$ is well-defined, (i.e., that for every triple $(c,\, h,\, e)$, there exist $\widetilde{d}$ and $\widetilde{d}_*$ for which the system (5.6) is consistent and the resulting triple $(c',\, h',\, e_*)$ does not depend on the choice of $\widetilde{d}$ and $\widetilde{d}_*$) we note first that on account of (5.1) and (5.2), the first equation in (5.6) determines $\widetilde{d}$ uniquely by

$$\widetilde{d} = \mathbf{P}_{\widetilde{\Delta}}\left(\mathbf{V}\mathbf{P}_{\mathcal{D}_{\mathbf{V}}} + i\mathbf{P}_{\Delta}\right)\begin{bmatrix} c \\ e \end{bmatrix} = i\mathbf{P}_{\Delta}\begin{bmatrix} c \\ e \end{bmatrix}.$$

With this $\widetilde{d}$, the second equation in (5.6) determines uniquely $\widetilde{d}_*$ and h'. Using $\widetilde{d}_*$ one can recover now c' and e_* from the first equation in (5.6).

Since operators $\mathbf{U}_0$ and $\widetilde{\mathbf{U}}$ are unitary, it follows from (5.6) that

$$\|c\|^2 + \|e\|^2 + \|\widetilde{d}_*\|^2 = \|c'\|^2 + \|e_*\|^2 + \|\widetilde{d}\|^2,$$
$$\|h\|^2 + \|\widetilde{d}\|^2 = \|h'\|^2 + \|\widetilde{d}_*\|^2$$

and therefore, that

$$\|c\|^2 + \|e\|^2 + \|h\|^2 = \|c'\|^2 + \|e_*\|^2 + \|h'\|^2,$$

which means that the coupling operator $\mathcal{F}_{\mathbf{U}_0}[\widetilde{\mathbf{U}}]$ is isometric. A similar argument can be made with the adjoints of $\mathbf{U}_0$, $\widetilde{\mathbf{U}}$ and $\mathcal{F}_{\mathbf{U}_0}[\widetilde{\mathbf{U}}]$, and hence $\mathcal{F}_{\mathbf{U}_0}[\widetilde{\mathbf{U}}]$ is unitary. Furthermore, by (5.5) and (5.6),

$$\mathcal{F}_{\mathbf{U}_0}[\widetilde{\mathbf{U}}]|_{(\mathbb{C}^p\otimes\widehat{\mathcal{H}})\oplus\mathcal{E}} = \mathbf{U}_0|_{(\mathbb{C}^p\otimes\widehat{\mathcal{H}})\oplus\mathcal{E}}$$

and since $\mathcal{D}_{\mathbf{V}} \subset (\mathbb{C}^p \otimes \widehat{\mathcal{H}}) \oplus \mathcal{E}$, it follows that

$$\mathcal{F}_{\mathbf{U}_0}[\widetilde{\mathbf{U}}]|_{\mathcal{D}_{\mathbf{V}}} = \mathbf{U}_0|_{\mathcal{D}_{\mathbf{V}}} = \mathbf{V}. \tag{5.7}$$

Thus, the coupling of the connecting operator $\mathbf{U}_0$ of the universal unitary colligation associated with Problem 1.6 and any other unitary operator is a unitary extension of the isometry $\mathbf{V}$ defined in (4.12). Conversely for every unitary $\mathbf{Q}$-colligation $\mathcal{C} = \{\widehat{\mathcal{H}} \oplus \widetilde{\mathcal{H}},\ \mathcal{E},\ \mathcal{E}_*,\ \mathbf{U}\}$ with the connecting operator being a unitary extension of $\mathbf{V}$, there exists a unitary $\mathbf{Q}$-colligation $\widetilde{\mathcal{C}}$ of the form (5.4) such that $\mathcal{C} = \mathcal{F}_{\mathcal{C}_0}[\widetilde{\mathcal{C}}]$ (the proof is the same as in [22, Theorem 6.2]). Thus, all unitary extensions $\mathbf{U}$ of the isometry $\mathbf{V}$ defined in (4.12) are parametrized by the formula

$$\mathbf{U} = \mathcal{F}_{\mathbf{U}_0}[\widetilde{\mathbf{U}}], \qquad \widetilde{\mathbf{U}}:\ (\mathbb{C}^p \otimes \widetilde{\mathcal{H}}) \oplus \widetilde{\Delta} \to (\mathbb{C}^q \otimes \widetilde{\mathcal{H}}) \oplus \widetilde{\Delta}_* \tag{5.8}$$

and $\widetilde{\mathcal{H}}$ is an auxiliary Hilbert space.

According to (4.5), the characteristic function of the $\mathbf{Q}$-colligation $\mathcal{C}_0$ defined in (5.3) with the connecting operator $\mathbf{U}_0$ partitioned as in (5.2), is given by

$$\begin{aligned}\Sigma(z) &= \begin{bmatrix} \Sigma_{11}(z) & \Sigma_{12}(z) \\ \Sigma_{21}(z) & \Sigma_{22}(z) \end{bmatrix} \\ &= \begin{bmatrix} U_{22} & U_{23} \\ U_{32} & 0 \end{bmatrix} + \begin{bmatrix} U_{21} \\ U_{31} \end{bmatrix} (I - \mathbf{Q}(z)U_{11})^{-1}\mathbf{Q}(z)\begin{bmatrix} U_{12} & U_{13} \end{bmatrix}\end{aligned} \tag{5.9}$$

and belongs to the class $\mathcal{SA}_{\mathbf{Q}}(\mathcal{E}\oplus\widetilde{\Delta}_*,\ \mathcal{E}_*\oplus\widetilde{\Delta})$, by Theorem 1.1.

Theorem 5.1. *Let $\mathbf{V}$ be the isometry defined in (4.12), let Σ be the function constructed as above and let S be a $\mathcal{L}(\mathcal{E},\mathcal{E}_*)$-valued function. Then the following are equivalent:*

(1) *S is a solution of Problem 1.6.*
(2) *S is a characteristic function of a $\mathbf{Q}$-colligation $\mathcal{C} = \{\widehat{\mathcal{H}}\oplus\widetilde{\mathcal{H}},\ \mathcal{E},\ \mathcal{E}_*,\ \mathbf{U}\}$ with the connecting operator $\mathbf{U}$ being a unitary extension of $\mathbf{V}$.*
(3) *S is of the form*

$$S(z) = \Sigma_{11}(z) + \Sigma_{12}(z)\left(I_{\widetilde{\Delta}_*} - \mathcal{T}(z)\Sigma_{22}(z)\right)^{-1}\mathcal{T}(z)\Sigma_{21}(z) \tag{5.10}$$

where $\mathcal{T}$ is a function from the class $\mathcal{SA}_{\mathbf{Q}}(\widetilde{\Delta},\ \widetilde{\Delta}_*)$.

Proof. The equivalence **1** $\Longleftrightarrow$ **2** follows by Lemmas 4.4 and 4.5.

2 $\Longrightarrow$ **3**. By the preceding analysis, the colligation $\mathcal{C}$ is the coupling of the universal colligation $\mathcal{C}_0$ defined in (5.3) and some unitary $\mathbf{Q}$-colligation $\widetilde{\mathcal{C}}$ of the form (5.4). The connecting operators $\mathbf{U}$, $\mathbf{U}_0$ and $\widetilde{\mathbf{U}}$ of these colligations are related as in (5.8). Let S, Σ and $\mathcal{T}$ be characteristic functions of $\mathcal{C}$, $\mathcal{C}_0$ and $\widetilde{\mathcal{C}}$, respectively. Applying Remark 4.2 to (5.5) and (5.6), we get

$$S(z)e = e_*, \quad \Sigma(z)\begin{bmatrix} e \\ \widetilde{d}_* \end{bmatrix} = \begin{bmatrix} e_* \\ \widetilde{d} \end{bmatrix}, \quad \mathcal{T}(z)\widetilde{d} = \widetilde{d}_*. \tag{5.11}$$

Substituting the third relation in (5.11) into the second we get

$$\Sigma(z)\begin{bmatrix} e \\ \mathcal{T}(z)\widetilde{d} \end{bmatrix} = \begin{bmatrix} e_* \\ \widetilde{d} \end{bmatrix},$$

which in view of the block decomposition (5.9) of Σ splits into

$$\Sigma_{11}(z)e + \Sigma_{12}(z)\mathcal{T}(z)\widetilde{d} = e_* \quad \text{and} \quad \Sigma_{21}(z)e + \Sigma_{22}(z)\mathcal{T}(z)\widetilde{d} = \widetilde{d}.$$

The second from the two last equalities gives

$$\widetilde{d} = (I - \Sigma_{22}(z)\mathcal{T}(z))^{-1}\Sigma_{21}(z)e$$

which, being substituted into the first equality, implies

$$\left(\Sigma_{11}(z) + \Sigma_{12}(z)\mathcal{T}(z)\left(I - \Sigma_{22}(z)\mathcal{T}(z)\right)^{-1}\Sigma_{21}(z)\right)e = e_*.$$

The latter is equivalent to

$$\left(\Sigma_{11}(z) + \Sigma_{12}(z)\left(I - \mathcal{T}(z)\Sigma_{22}(z)\right)^{-1}\mathcal{T}(z)\Sigma_{21}(z)\right)e = e_*$$

and the comparison of the last equality with the first relation in (5.11) leads to representation (5.10) of S, since a vector $e\in\mathcal{E}$ is arbitrary.

3 $\Longrightarrow$ **2**. Let S be of the form (5.10) for some $\mathcal{T} \in \mathcal{SA}_{\mathbf{P}}(\widetilde{\Delta}, \ \widetilde{\Delta}_*)$. By Theorem 1.1, $\mathcal{T}$ is the characteristic function of a unitary $\mathbf{Q}$-colligation $\widetilde{\mathcal{C}}$ of the form (5.4). Let $\mathcal{C}$ be the unitary $\mathbf{Q}$-colligation defined by $\mathcal{C} = \mathcal{F}_{\mathcal{C}_0}[\widetilde{\mathcal{C}}]$. By the preceding "**2** $\Longrightarrow$ **3**" part, S of the form (5.10) is the characteristic function of $\mathcal{C}$. It remains to note that the colligation $\mathcal{C}$ is of required the form: its input and output spaces coincide with $\mathcal{E}$ and $\mathcal{E}_*$, respectively (by the definition of coupling) and its connecting operator is an expansion of $\mathbf{V}$, by (5.7). $\square$

As a corollary we obtain the sufficiency part of Theorem 1.4, including the parametrization of the set of all solutions: *under the assumption that P is positive semidefinite and satisfies the Stein identity* (1.23)*, the set of all solutions of Problem* 1.6 *is parametrized by formula* (5.10) *and is nonempty.*

6. Explicit formulas

In the case when the spaces $\mathcal{K}$ and $\mathcal{K}'$ are finite-dimensional,

$$\dim \mathcal{K} < \infty, \qquad \dim \mathcal{K}' < \infty, \tag{6.1}$$

it is possible to get more explicit formulas for coefficients $\Sigma_{ij}(z)$ of the linear fractional transformation (5.10) parametrizing all the solutions of Problem 1.6 in terms of interpolation data. We now explain this point in detail.

Setting $\mathcal{H}_0 = (\mathbb{C}^p \otimes \mathcal{K}) \oplus (\mathbb{C}^q \otimes \mathcal{K}')$ (as in (4.9)), we define operators $W_1 \in \mathcal{L}(\mathcal{K} \oplus \mathcal{K}', \, \mathbb{C}^p \otimes \mathcal{H}_0)$ and $W_2 \in \mathcal{L}(\mathcal{K} \oplus \mathcal{K}', \, \mathbb{C}^q \otimes \mathcal{H}_0)$ as follows:

$$W_1 = \begin{bmatrix} P^{\frac{1}{2}} M_1 \\ \vdots \\ P^{\frac{1}{2}} M_p \end{bmatrix} \quad \text{and} \quad W_2 = \begin{bmatrix} P^{\frac{1}{2}} N_1 \\ \vdots \\ P^{\frac{1}{2}} N_q \end{bmatrix}, \tag{6.2}$$

The Stein equation (4.10), which is assumed to be in force, can be written in terms of the matrices (6.2) as

$$W_1^* W_1 + Y^* Y = W_2^* W_2 + X^* X \tag{6.3}$$

and we let $\mathbf{T}$ be the operator on the space $\mathcal{K} \oplus \mathcal{K}'$ defined by

$$\mathbf{T} := W_1^* W_1 + Y^* Y = W_2^* W_2 + X^* X. \tag{6.4}$$

Due to assumptions (6.1), $\operatorname{Ran} \mathbf{T}$ is a (finite dimensional) subspace of $\mathcal{K} \oplus \mathcal{K}'$. Equality (6.3) guarantees that the linear map

$$\mathbf{V} : \begin{bmatrix} W_1 \\ Y \end{bmatrix} f \longrightarrow \begin{bmatrix} W_2 \\ X \end{bmatrix} f \qquad (f \in \mathcal{K} \oplus \mathcal{K}') \tag{6.5}$$

is an isometry from

$$\mathcal{D}_{\mathbf{V}} = \operatorname{Ran} \begin{bmatrix} W_1 \\ Y \end{bmatrix} \subset \begin{bmatrix} \mathbb{C}^p \otimes \mathcal{H}_0 \\ \mathcal{E} \end{bmatrix} \quad \text{onto} \quad \mathcal{R}_{\mathbf{V}} = \operatorname{Ran} \begin{bmatrix} W_2 \\ X \end{bmatrix} \subset \begin{bmatrix} \mathbb{C}^q \otimes \mathcal{H}_0 \\ \mathcal{E}_* \end{bmatrix}.$$

The isometry in (6.5) would coincide with that in (4.12) if we took the factor–spaces of $\mathbb{C}^p \otimes \mathcal{H}_0$ and $\mathbb{C}^q \otimes \mathcal{H}_0$ over the kernel of P instead of $\mathbb{C}^p \otimes \mathcal{H}_0$ and $\mathbb{C}^q \otimes \mathcal{H}_0$ themselves. In this case formulas would become less explicit. The defect spaces now take the form

$$\Delta = \begin{bmatrix} \mathbb{C}^p \otimes \mathcal{H}_0 \\ \mathcal{E} \end{bmatrix} \ominus \mathcal{D}_{\mathbf{V}} \quad \text{and} \quad \Delta_* = \begin{bmatrix} \mathbb{C}^q \otimes \mathcal{H}_0 \\ \mathcal{E}^* \end{bmatrix} \ominus \mathcal{R}_{\mathbf{V}}.$$

As in the general case we let $\widetilde{\Delta}$ to be another copy of Δ and $\widetilde{\Delta}_*$ to be another copy of Δ_* with unitary identification maps

$$i: \ \Delta \to \widetilde{\Delta} \quad \text{and} \quad i_*: \ \Delta_* \to \widetilde{\Delta}_*. \tag{6.6}$$

Let the operator $\begin{bmatrix} \alpha \\ \beta \end{bmatrix} : \widetilde{\Delta} \to \begin{bmatrix} \mathbb{C}^p \otimes \mathcal{H}_0 \\ \mathcal{E} \end{bmatrix}$ be defined via

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix} \widetilde{\delta} = i^{-1}\widetilde{\delta} \quad (\widetilde{\delta} \in \widetilde{\Delta}).$$

Then

$$\alpha^*\alpha + \beta^*\beta = I_{\widetilde{\Delta}} \tag{6.7}$$

and since $\begin{bmatrix} \alpha \\ \beta \end{bmatrix} \widetilde{\delta}$ belongs to Δ which is orthogonal to $\mathcal{D}_{\mathbf{V}}$, it follows that

$$\left\langle \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \widetilde{\delta}, \ \begin{bmatrix} W_1 \\ Y \end{bmatrix} d \right\rangle = 0 \quad \text{for every} \ \ \widetilde{\delta} \in \widetilde{\Delta} \ \ \text{and} \ \ d \in \mathcal{K} \oplus \mathcal{K}'$$

which can be written in operator form as

$$\alpha^* W_1 + \beta^* Y = 0. \tag{6.8}$$

Furthermore, we introduce the operator $\begin{bmatrix} \gamma \\ \pi \end{bmatrix} : \widetilde{\Delta}_* \mapsto \begin{bmatrix} \mathbb{C}^q \otimes \mathcal{H}_0 \\ \mathcal{E}_* \end{bmatrix}$ by the rule

$$\begin{bmatrix} \gamma \\ \pi \end{bmatrix} \widetilde{\delta}_* = i_*^{-1}\widetilde{\delta}_* \quad (\widetilde{\delta}_* \in \widetilde{\Delta}_*)$$

and conclude, that similarly to (6.7) and (6.8),

$$\gamma^*\gamma + \pi^*\pi = I_{\widetilde{\Delta}_*} \quad \text{and} \quad \gamma^* W_2 + \pi^* X = 0. \tag{6.9}$$

The second relation in (6.9) is a consequence of the fact that $\begin{bmatrix} \gamma \\ \pi \end{bmatrix} \widetilde{\delta}_*$ belongs to Δ_* which is orthogonal to $\mathcal{R}_{\mathbf{V}}$. Now let

$$\mathbb{A} := \begin{bmatrix} W_1 & \alpha & 0 \\ Y & \beta & 0 \\ 0 & 0 & I_{\widetilde{\Delta}_*} \end{bmatrix} : \begin{bmatrix} \operatorname{Ran} \mathbf{T} \\ \widetilde{\Delta} \\ \widetilde{\Delta}_* \end{bmatrix} \to \begin{bmatrix} \mathbb{C}^p \otimes \mathcal{H}_0 \\ \mathcal{E} \\ \widetilde{\Delta}_* \end{bmatrix} \tag{6.10}$$

and

$$\mathbb{B} := \begin{bmatrix} W_2 & 0 & \gamma \\ X & 0 & \pi \\ 0 & I_{\widetilde{\Delta}} & 0 \end{bmatrix} : \begin{bmatrix} \operatorname{Ran} \mathbf{T} \\ \widetilde{\Delta} \\ \widetilde{\Delta}_* \end{bmatrix} \to \begin{bmatrix} \mathbb{C}^q \otimes \mathcal{H}_0 \\ \mathcal{E}_* \\ \widetilde{\Delta} \end{bmatrix}. \tag{6.11}$$

Then the definition (5.1) of the universal unitary colligation $\mathbf{U}_0$ can be written equivalently in terms of the operators $\mathbb{A}$ and $\mathbb{B}$ as

$$\mathbf{U}_0\mathbb{A} = \mathbb{B}. \tag{6.12}$$

Note that by (6.4), (6.7), (6.8) and (6.9),

$$\mathbb{A}^*\mathbb{A} = \mathbb{B}^*\mathbb{B} = \begin{bmatrix} \mathbf{T} & 0 & 0 \\ 0 & I_{\widetilde{\Delta}} & 0 \\ 0 & 0 & I_{\widetilde{\Delta}_*} \end{bmatrix} =: \widehat{\mathbf{T}} \tag{6.13}$$

and the operator $\widehat{\mathbf{T}}$ is invertible (with bounded inverse) on $\operatorname{Ran}\mathbf{T}\oplus\widetilde{\Delta}\oplus\widetilde{\Delta}_*$. Since $\mathbb{A}$ is onto, it follows from (6.12) that

$$\mathbf{U}_0=\mathbb{B}\widehat{\mathbf{T}}^{-1}\mathbb{A}^*. \tag{6.14}$$

According to the formula (4.8), the characteristic function of the universal unitary colligation is given by

$$\Sigma(z)=\mathbf{P}_{\mathcal{E}_*\oplus\tilde{\Delta}}\mathbf{U}_0\left(I-\mathbf{P}_{\mathbb{C}^p\otimes\mathcal{H}}\mathbf{Q}(z)\mathbf{P}_{\mathbb{C}^q\otimes\mathcal{H}}\mathbf{U}_0\right)^{-1}|_{\mathcal{E}\oplus\widetilde{\Delta}_*}.$$

Substituting (6.14) into the latter equality we get

$$\begin{aligned}\Sigma(z)&=\mathbf{P}_{\mathcal{E}_*\oplus\tilde{\Delta}}\mathbb{B}\widehat{\mathbf{T}}^{-1}\mathbb{A}^*\left(I-\mathbf{P}_{\mathbb{C}^p\otimes\mathcal{H}}\mathbf{Q}(z)\mathbf{P}_{\mathbb{C}^q\otimes\mathcal{H}}\mathbb{B}\widehat{\mathbf{T}}^{-1}\mathbb{A}^*\right)^{-1}|_{\mathcal{E}\oplus\widetilde{\Delta}_*}\\&=\mathbf{P}_{\mathcal{E}_*\oplus\tilde{\Delta}}\mathbb{B}\widehat{\mathbf{T}}^{-1}\left(I-\mathbb{A}^*\mathbf{P}_{\mathbb{C}^p\otimes\mathcal{H}}\mathbf{Q}(z)\mathbf{P}_{\mathbb{C}^q\otimes\mathcal{H}}\mathbb{B}\widehat{\mathbf{T}}^{-1}\right)^{-1}\mathbb{A}^*|_{\mathcal{E}\oplus\widetilde{\Delta}_*}\\&=\mathbf{P}_{\mathcal{E}_*\oplus\tilde{\Delta}}\mathbb{B}\left(\widehat{\mathbf{T}}-\mathbb{A}^*\mathbf{P}_{\mathbb{C}^p\otimes\mathcal{H}}\mathbf{Q}(z)\mathbf{P}_{\mathbb{C}^q\otimes\mathcal{H}}\mathbb{B}\right)^{-1}\mathbb{A}^*|_{\mathcal{E}\oplus\widetilde{\Delta}_*}\end{aligned}$$

which, on account of (6.10) and (6.11), can be written as

$$\Sigma(z)=\begin{bmatrix}X&0&\pi\\0&I_{\widetilde{\Delta}}&0\end{bmatrix}\left(\widehat{\mathbf{T}}-\begin{bmatrix}W_1^*\\\alpha^*\\0\end{bmatrix}\mathbf{Q}(z)\begin{bmatrix}W_2&0&\gamma\end{bmatrix}\right)^{-1}\begin{bmatrix}Y^*&0\\\beta^*&0\\0&I_{\widetilde{\Delta}_*}\end{bmatrix}. \tag{6.15}$$

The first inverse in this chain of equalities is invertible since $\mathbf{U}_0$ is unitary and $\|\mathbf{Q}(z)\|<1$, all the other inverses exist since the first one does. By (6.13) and (6.4),

$$\widehat{\mathbf{T}}-\begin{bmatrix}W_1^*\\\alpha^*\\0\end{bmatrix}\mathbf{Q}(z)\begin{bmatrix}W_2&0&\gamma\end{bmatrix}=\begin{bmatrix}\mathbf{D}(z)&0&-W_1^*\mathbf{Q}(z)\gamma\\-\alpha^*\mathbf{Q}(z)W_2&I_{\widetilde{\Delta}}&-\alpha^*\mathbf{Q}(z)\gamma\\0&0&I_{\widetilde{\Delta}_*}\end{bmatrix}$$

where

$$\mathbf{D}(z):=\mathbf{T}-W_1^*\mathbf{Q}(z)W_2=W_1^*(W_1-\mathbf{Q}(z)W_2)+Y^*Y. \tag{6.16}$$

Inverting the latter operator gives

$$\begin{bmatrix}\mathbf{D}(z)^{-1}&0&\mathbf{D}(z)^{-1}W_1^*\mathbf{Q}(z)\gamma\\\alpha^*\mathbf{Q}(z)W_2&I_{\widetilde{\Delta}}&\alpha^*\mathbf{Q}(z)\left[I+W_2\mathbf{D}(z)^{-1}W_1^*\mathbf{Q}(z)\right]\gamma\\0&0&I_{\widetilde{\Delta}_*}\end{bmatrix}$$

which, being substituted into (6.15), leads us to

$$\Sigma(z)=\begin{bmatrix}X\mathbf{D}(z)^{-1}Y^*&\pi+X\mathbf{D}(z)^{-1}W_1^*\mathbf{Q}(z)\gamma\\\beta^*+\alpha^*\mathbf{Q}(z)W_2\mathbf{D}(z)^{-1}Y^*&\alpha^*\mathbf{Q}(z)\left[I+W_2\mathbf{D}(z)^{-1}W_1^*\mathbf{Q}(z)\right]\gamma\end{bmatrix}. \tag{6.17}$$

This is the formula we desired to get. Note that $\mathbf{D}(z)$ is considered as a function taking values in $\mathcal{L}(\operatorname{Ran}\mathbf{T})$. For every $z\in\mathcal{D}_{\mathbf{Q}}$, $\mathbf{D}(z)$ is invertible. In the case when $\mathbf{T}$ is positive definite, $\mathbf{D}(z)$ is considered as an operator on $\mathcal{K}\otimes\mathcal{K}'$ and is invertible at every $z\in\mathcal{D}_{\mathbf{Q}}$.

7. Nevanlinna–Pick interpolation problem

In [17] we considered the following bitangential Nevanlinna–Pick interpolation problem whose data set consists of two subsets Ω_L and Ω_R of $\mathcal{D}_{\mathbf{Q}}$, two Hilbert spaces $\mathcal{E}_L$ and $\mathcal{E}_R$ and four operator-valued functions

$$\mathbf{a}\colon \Omega_L \mapsto \mathcal{L}(\mathcal{E}_*, \mathcal{E}_L), \qquad \mathbf{c}\colon \Omega_L \mapsto \mathcal{L}(\mathcal{E}, \mathcal{E}_L), \tag{7.1}$$

$$\mathbf{b}\colon \Omega_R \mapsto \mathcal{L}(\mathcal{E}_R, \mathcal{E}), \qquad \mathbf{d}\colon \Omega_R \mapsto \mathcal{L}(\mathcal{E}_R, \mathcal{E}_*). \tag{7.2}$$

Problem 7.1. *Find all functions $S \in \mathcal{SA}_{\mathbf{Q}}(\mathcal{E}, \mathcal{E}_*)$ such that S satisfies the interpolation conditions*

$$\mathbf{a}(\zeta)S(\zeta) = \mathbf{c}(\zeta) \quad \textit{for all} \quad \xi \in \Omega_L \tag{7.3}$$

$$S(\xi)\mathbf{b}(\xi) = \mathbf{d}(\xi) \quad \textit{for all} \quad \xi \in \Omega_R. \tag{7.4}$$

If $S \in \mathcal{SA}_{\mathbf{Q}}(\mathcal{E}, \mathcal{E}_*)$ meets conditions (7.3) and (7.4), then, as its values are contractive operators, we have necessarily $\|\mathbf{c}(\zeta)\| \le \|\mathbf{a}(\zeta)\|$ and $\|\mathbf{d}(\xi)\| \le \|\mathbf{b}(\xi)\|$ for every $\zeta \in \Omega_L$ and every $\xi \in \Omega_R$. Furthermore, without loss of generality we can normalize $\mathbf{a}$ and $\mathbf{b}$ pointwise and to assume that

$$\|\mathbf{c}(\zeta)\| \le \|\mathbf{a}(\zeta)\| = 1 \quad \text{and} \quad \|\mathbf{d}(\xi)\| \le \|\mathbf{b}(\xi)\| = 1 \quad (\zeta \in \Omega_L, \ \xi \in \Omega_R). \tag{7.5}$$

In [17] we also considered a modified interpolation problem with the extended data set including also the functions

$$\begin{cases} \widetilde{\Psi}_{j\ell}(\xi,\mu): \ \Omega_L \times \Omega_L \to \mathcal{L}(\mathcal{E}_L) & (j,\ell = 1,\dots p), \\ \widetilde{\Lambda}_{j\ell}(\xi,\mu): \ \Omega_L \times \Omega_R \to \mathcal{L}(\mathcal{E}_R, \mathcal{E}_L) & (j = 1,\dots p; \ \ell = 1,\dots q), \\ \widetilde{\Phi}_{j\ell}(\xi,\mu): \ \Omega_R \times \Omega_R \to \mathcal{L}(\mathcal{E}_R) & (j,\ell = 1,\dots q). \end{cases} \tag{7.6}$$

Problem 7.2. *Given four functions $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$ as in (7.1) and (7.2) and given p^2+pq+q^2 functions $\widetilde{\Psi}_{j\ell}$, $\widetilde{\Lambda}_{j\ell}$, $\widetilde{\Phi}_{j\ell}$ as in (7.6), find all functions $S \in \mathcal{SA}_{\mathbf{Q}}(\mathcal{E}, \mathcal{E}_*)$ such that the interpolation conditions (7.3), (7.4) are satisfied, and in addition, there exists a choice of functions $H(z)$ and $G(z)$ of the form (1.3) and (1.5), respectively, and associated with S as in representations (1.4), (1.6) and (1.7), which satisfy equalities*

$$\mathbf{a}(\xi)H_j(\xi)H_\ell(\mu)^*\mathbf{a}(\mu)^* = \widetilde{\Psi}_{j\ell}(\xi,\mu) \quad (\xi,\mu \in \Omega_L; \ j,\ell = 1,\dots,p), \tag{7.7}$$

$$\mathbf{a}(\xi)H_j(\xi)G_\ell(\mu)\mathbf{b}(\mu) = \widetilde{\Lambda}_{j\ell}(\xi,\mu) \quad (\xi \in \Omega_L, \ \mu \in \Omega_R; \quad j = 1,\dots,p; \ \ell = 1,\dots,q), \tag{7.8}$$

$$\mathbf{b}(\xi)^*G_j(\xi)^*G_\ell(\mu)\mathbf{b}(\mu) = \widetilde{\Phi}_{j\ell}(\xi,\mu) \quad (\xi,\mu \in \Omega_R; \ j,\ell = 1,\dots,q). \tag{7.9}$$

We now show that Problems 7.1 and 7.2, at least for the case where Ω_L and Ω_R are contained in compact subsets of $\mathcal{D}_{\mathbf{Q}}$, are particular cases of Problems 1.3 and 1.6, respectively. To this end, let $\mathcal{K}_0$ be the set of all $\mathcal{E}_L$-valued functions on Ω_L which take nonzero values at at most finitely many points, with inner product

$$\langle g_1, \ g_2\rangle_{\mathcal{K}_0} = \sum_{\zeta\in\Omega_L} \langle g_1(\zeta), \ g_2(\zeta)\rangle_{\mathcal{E}_L}$$

and let $\mathcal{K} = \mathcal{E}_L \otimes \ell^2(\Omega_L)$ be the completion of $\mathcal{K}_0$ with respect to this inner product. Similarly, let $\mathcal{K}_0'$ be the set of $\mathcal{E}_R$-valued functions defined on Ω_R and vanishing everywhere but at at most finitely many points, with inner product

$$\langle f_1,\ f_2\rangle_{\mathcal{K}_0'} = \sum_{\zeta\in\Omega_R} \langle f_1(\zeta),\ f_2(\zeta)\rangle_{\mathcal{E}_R}$$

and let $\mathcal{K}' = \mathcal{E}_L \otimes \ell^2(\Omega_R)$ be the completion of $\mathcal{K}_0'$ with respect to this inner product.

The spaces $\mathcal{K}$ and $\mathcal{K}'$ are invariant with respect to multiplication by each of the coordinate functions $z \mapsto z_j$ for $j = 1, \ldots, n$. Let T_j and T_j' be equal to the operator $\mathbf{M}_{z_j}$ of multiplication by z_j restricted to $\mathcal{K}$ and $\mathcal{K}'$, respectively. Then the n-tuples $T = (T_1, \ldots, T_n) \in \mathcal{L}(\mathcal{K})^n$ and $T' = (T_1', \ldots, T_n') \in \mathcal{L}(\mathcal{K}')^n$ are commutative. By the diagonal form of T and T', we see that $\sigma_{\text{Taylor}}(T) = \overline{\Omega_L}$ and $\sigma_{\text{Taylor}}(T') = \overline{\Omega_R}$. Furthermore, we introduce the operators

$$X_R\colon \mathcal{K}' \to \mathcal{E}_*, \quad Y_R\colon \mathcal{K}' \to \mathcal{E}, \quad X_L\colon \mathcal{K} \to \mathcal{E}_*, \quad Y_L\colon \mathcal{K} \to \mathcal{E}$$

defined first by equalities

$$X_R f = \sum_{\zeta\in\Omega_R} \mathbf{d}(\zeta) f(\zeta), \qquad Y_R f = \sum_{\zeta\in\Omega_R} \mathbf{b}(\zeta) f(\zeta), \tag{7.10}$$

$$X_L g = \sum_{\zeta\in\Omega_L} \mathbf{a}(\zeta) g(\zeta), \qquad Y_L g = \sum_{\zeta\in\Omega_L} \mathbf{c}(\zeta) g(\zeta) \tag{7.11}$$

for $f \in \mathcal{K}_0'$ and $g \in \mathcal{K}_0$ and then extended to all of $\mathcal{K}'$ and $\mathcal{K}$ by continuity (as is possible due to (7.5)). Furthermore, we make use of the functions (7.6) to define operators $\Psi_{j\ell}$, $\Phi_{j\ell}$ and $\Lambda_{j\ell}$ (acting as in (1.25)) by the rules

$$\begin{cases} \Psi_{j\ell} g = \sum_{\zeta\in\Omega_L} \widetilde{\Psi}_{j\ell}(z,\zeta) g(\zeta) & (g \in \mathcal{K}_0,\ z \in \Omega_L), \\ \Phi_{j\ell} f = \sum_{\zeta\in\Omega_R} \widetilde{\Phi}_{j\ell}(z,\zeta) f(\zeta) & (f \in \mathcal{K}_0',\ z \in \Omega_R), \\ \Lambda_{j\ell} f = \sum_{\zeta\in\Omega_L} \widetilde{\Lambda}_{j\ell}(z,\zeta) f(\zeta) & (f \in \mathcal{K}_0',\ \zeta \in \Omega_R) \end{cases} \tag{7.12}$$

extended to $\mathcal{K}$ and $\mathcal{K}'$ by continuity.

Now we will show that for the above choice of T, T', X_L, Y_L, X_R and Y_R, conditions (1.14) coincide with conditions (7.3), (7.4). To this end, we pick a point $\zeta \in \Omega_R$ and a vector $e_R \in \mathcal{E}_R$ and let

$$f(z) := \begin{cases} e_R, & \text{if } z = \xi, \\ 0, & \text{otherwise.} \end{cases} \tag{7.13}$$

Then

$$(SY_R)^{\wedge R}(T') f = S(\zeta)\mathbf{b}(\zeta) e_R \quad \text{and} \quad X_R f = \mathbf{d}(\zeta) e_R. \tag{7.14}$$

The second relation in (7.14) follows immediately from definitions of X_R, T' and f; the first equality follows from definitions of Y_R, T' and f by (2.4):

$$\begin{aligned} (SY_R)^{\wedge R}(T') f &= \frac{1}{(2\pi i)^n} \int_{\partial\Omega'} S(z) Y_R \cdot M(z - T') f \wedge dz \\ &= \frac{1}{(2\pi i)^n} \int_{\partial\Omega'} S(z) \cdot M(z - \xi) \mathbf{b}(\xi) e_R \wedge dz \\ &= \frac{1}{(2\pi i)^n} \int_{\partial\Omega'} S(z) \cdot M(z - \xi) \wedge dz \ \ \mathbf{b}(\xi) e_R \\ &= S(\xi)\mathbf{b}(\xi) e_R \end{aligned}$$

where we have chosen Ω' to be a domain with smooth boundary $\partial\Omega'$ such that the closure $\overline{\Omega'}$ of Ω' is compact and $\Omega' \subset \overline{\Omega'} \subset \mathcal{D}_{\mathbf{Q}}$. Since $e_R \in \mathcal{E}_R$ and $\xi \in \Omega_R$ were picked arbitrarily, it follows from (7.14) that the second condition in (1.14) implies (7.4). On the other hand, (7.4) implies (again due to equalities (7.14)) that

$$(SY_R)^{\wedge R}(T')f = X_R f$$

for every function $f \in \mathcal{K}'$ of the form (7.13). By linearity and continuity, the latter equality holds for every $f \in \mathcal{K}'$ and thus, (7.4) implies (and therefore, is equivalent to) the second condition in (1.14).

Similarly, taking a point $\zeta \in \Omega_L$ and a vector $e_L \in \mathcal{E}_L$ we consider a function

$$g(z) := \begin{cases} e_L, & \text{if } z = \zeta, \\ 0, & \text{otherwise.} \end{cases} \tag{7.15}$$

Then, similarly to (7.14) we have

$$(X_L S)^{\wedge L}(T)g = \mathbf{a}(\zeta)S(\zeta)e_L \quad \text{and} \quad Y_L g = \mathbf{c}(\zeta)e_L \tag{7.16}$$

which allows us to conclude that (7.3) is equivalent to the first condition in (1.14). Furthermore, replacing S by G_ℓ in (7.14) and by H_j in (7.16) leads us to equalities

$$(G_\ell Y_R)^{\wedge R}(T')f = G_\ell(\xi)\mathbf{b}(\xi)e_R \quad \text{for } \ell = 1, \dots, q, \tag{7.17}$$

$$(X_L H)^{\wedge L}(T)g = \mathbf{a}(\zeta)H_j(\zeta)e_R \quad \text{for } j = 1, \dots, p, \tag{7.18}$$

holding for functions f and g defined via formulas (7.13) and (7.15) from arbitrarily chosen $\zeta \in \Omega_L$, $\xi \in \Omega_R$, $e_L \in \mathcal{E}_L$, $e_R \in \mathcal{E}_R$, $e_L \in \mathcal{E}_L$. Take the function f defined in (7.13) and the function $\widetilde{f}$ of the same form based on a point $\mu \in \Omega_R$ and a vector $\widetilde{e}_R \in \mathcal{E}_R$. By (7.17),

$$\left\langle (G_j Y_R)^{\wedge R}(T')\widetilde{f},\ (G_\ell Y_R)^{\wedge R}(T')f \right\rangle_{\mathcal{H}} = \langle G_j(\mu)\mathbf{b}(\mu)\widetilde{e}_R,\ G_\ell(\xi)\mathbf{b}(\xi)e_R\rangle_{\mathcal{H}}. \tag{7.19}$$

On the other hand, by the definition (7.12) of the operator $\Phi_{j\ell}$,

$$\langle \Phi_{j\ell}\widetilde{f},\ f\rangle_{\mathcal{K}'} = \langle \widetilde{\Phi}_{j\ell}(\xi, \mu)\widetilde{e}_R,\ e_R\rangle_{\mathcal{E}_R}. \tag{7.20}$$

Since $e_R, \widetilde{e}_R \in \mathcal{E}_R$ and $\xi, \mu \in \Omega_R$ were picked arbitrarily, it follows from (7.19) and (7.20) that conditions (1.28) imply (7.9). On the other hand, (7.9) implies (again due to equalities (7.19), (7.20)) that

$$\left\langle (G_j Y_R)^{\wedge R}(T')\widetilde{f},\ (G_\ell Y_R)^{\wedge R}(T')f \right\rangle_{\mathcal{H}} = \langle \Phi_{j\ell}\widetilde{f},\ f\rangle_{\mathcal{K}'}$$

for any functions $f, \widetilde{f} \in \mathcal{K}'$ of the form (7.13). By linearity and continuity, the latter equality holds for every choice $f, \widetilde{f} \in \mathcal{K}'$ and thus, (7.9) is equivalent to (1.28). Using much the same arguments one can check that conditions (1.26) and (1.27) are equivalent to conditions (7.7) and (7.8), respectively. Therefore, Problems 7.1 and 7.2 are particular cases of Problems 1.3 and 1.6, respectively. Therefore, Theorems 1.4 and 1.7 give necessary and sufficient conditions for Problems 7.1 and 7.2 to have a solution. These conditions were presented in [17] in a slightly different form. Now we have more: Theorem 5.1 gives a description of all solutions to Problem 7.2 in terms of a linear fractional transformation.

Let us display in some more detail the case when the number of interpolation conditions is finite (i.e., when the sets Ω_L and Ω_R of interpolation nodes are finite). Let

$$\Omega_L = \{z^{(1)}, \dots, z^{(k)}\} \quad \text{and} \quad \Omega_R = \{\omega^{(1)}, \dots, \omega^{(m)}\} \tag{7.21}$$

$z^{(j)} = (z_1^{(j)}, \dots, z_n^{(j)}) \in \mathcal{D}_{\mathbf{Q}}$ and $\omega^{(i)} = (\omega_1^{(i)}, \dots, \omega_n^{(i)}) \in \mathcal{D}_{\mathbf{Q}}$. Now the functions $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$ in (7.1), (7.2) are completely defined by their values

$$x_j := \mathbf{a}(z^{(j)}), \quad y_j := \mathbf{c}(z^{(j)}), \quad u_i := \mathbf{b}(\omega^{(i)}), \quad v_i := \mathbf{d}(\omega^{(i)})$$

for $j = 1, \dots, k$ and $i = 1, \dots, m$ which can be considered as part of interpolation data instead of the original functions. We assume for simplicity that $\mathcal{E}_L = \mathcal{E}_R = \mathbb{C}$ so that

$$v_1, \dots, v_m \in \mathcal{E}_* \quad \text{and} \quad u_1, \dots, u_m \in \mathcal{E}. \tag{7.22}$$

whereas x_j and y_j are functionals on the spaces $\mathcal{E}_*$ and $\mathcal{E}$, respectively:

$$x_1, \dots, x_k \in \mathcal{E}_*^* \quad \text{and} \quad y_1, \dots, y_k \in \mathcal{E}^*. \tag{7.23}$$

The functions (7.6) now are scalar-valued and also can be replaced by their values at interpolating nodes

$$\begin{cases} \psi_{ri}^{j\ell} := \widetilde{\Psi}_{j\ell}(z^{(r)}, z^{(i)}) & \text{for } j, \ell = 1, \dots p;\ r, i = 1, \dots, k, \\ \lambda_{ri}^{j\ell} := \widetilde{\Lambda}_{j\ell}(z^{(r)}, \omega^{(i)}) & \text{for } j = 1, \dots, p;\ \ell = 1, \dots, q; \\ & \qquad r = 1, \dots, k;\ i = 1, \dots, m, \\ \phi_{ri}^{j\ell} := \widetilde{\Phi}_{j\ell}(\omega^{(r)}, \omega^{(i)}) & \text{for } j, \ell = 1, \dots, q;\ r, i = 1, \dots, m. \end{cases} \tag{7.24}$$

In this special context, Problem 7.2 reads:

Problem 7.3. *Given interpolation data* (7.21)–(7.24), *find all functions S in the class $\mathcal{SA}_{\mathbf{Q}}(\mathcal{E}, \mathcal{E}_*)$ satisfying conditions*

$$x_r S(z^{(r)}) = y_r \ \ (r = 1, \dots, k), \qquad S(\omega^{(i)}) u_i = v_i \ \ (i = 1, \dots, m) \tag{7.25}$$

and such that, for some choice of functions H and G of the form (1.3) *and* (1.5) *associated with S via representations* (1.4), (1.6) *and* (1.7), *it holds that*

$$\begin{cases} x_r H_j(z^{(r)}) H_\ell(z^{(i)})^* x_i^* = \psi_{ri}^{j\ell} & \textit{for } j, \ell = 1, \dots, p;\ r, i = 1, \dots, k, \\ x_r H_j(z^{(r)}) G_\ell(\omega^{(i)}) u_i = \lambda_{ri}^{j\ell} & \textit{for } j = 1, \dots, p;\ \ell = 1, \dots, q; \\ & \qquad r = 1, \dots, k;\ i = 1, \dots, m, \\ u_r^* G_j(\omega^{(r)})^* G_\ell(\omega^{(i)}) u_i = \phi_{ri}^{j\ell} & \textit{for } j, \ell = 1, \dots, q;\ r, i = 1, \dots, m. \end{cases} \tag{7.26}$$

The first condition in (7.26) is understood in the sense that $x_r(S(z^{(r)})e) = y_r(e)$ for every vector $e \in \mathcal{E}$ and every $r \in \{1, \dots, k\}$. In (7.26), x_i^* is the vector in $\mathcal{E}_*$ uniquely defined by $x_i = \langle \cdot\,, x_i^* \rangle_{\mathcal{E}_*}$, whereas $u_r^* := \langle \cdot\,, u_r \rangle_{\mathcal{E}}$.

The latter problem can be derived directly from Problem 1.6 upon taking $\mathcal{K} = \mathbb{C}^k$, $\mathcal{K}' = \mathbb{C}^m$ and setting

$$T_j = \begin{bmatrix} z_j^{(1)} & & \\ & \ddots & \\ & & z_j^{(k)} \end{bmatrix}, \quad T_j' = \begin{bmatrix} \omega_j^{(1)} & & \\ & \ddots & \\ & & \omega_j^{(k)} \end{bmatrix} \quad (j = 1, \dots, n), \tag{7.27}$$

$$X_L = \begin{bmatrix} x_1 \\ \vdots \\ x_k \end{bmatrix}, \quad Y_L = \begin{bmatrix} y_1 \\ \vdots \\ y_k \end{bmatrix}, \tag{7.28}$$

$$X_R = \begin{bmatrix} v_1 & \dots & v_m \end{bmatrix}, \quad Y_R = \begin{bmatrix} u_1 & \dots & u_m \end{bmatrix}, \tag{7.29}$$

and

$$\Psi_{j\ell} = \left[\psi_{ri}^{j\ell}\right] \in \mathbb{C}^{k\times k}, \quad \Lambda_{j\ell} = \left[\lambda_{ri}^{j\ell}\right] \in \mathbb{C}^{k\times m}, \quad \Phi_{j\ell} = \left[\phi_{ri}^{j\ell}\right] \in \mathbb{C}^{m\times m}.$$

Indeed, by definitions of left and right evaluation maps in (2.3) and (2.4) and in view of diagonal structure of matrices in (7.25),

$$\begin{aligned} (X_L S)^{\wedge L}(T) &= \begin{bmatrix} x_1 S(z^{(1)}) \\ \vdots \\ x_k S(z^{(k)}) \end{bmatrix}, \\ (SY_R)^{\wedge R}(T') &= \begin{bmatrix} S(\omega^{(1)})u_1 & \dots & S(\omega^{(m)})u_m \end{bmatrix} \end{aligned}$$

and thus, due to the choice of Y_L and Y_R in (7.28) and (7.29), interpolation conditions (1.14) reduce to Nevanlinna–Pick interpolation conditions (7.25). Similarly,

$$\begin{aligned} (X_L H_j)^{\wedge L}(T) &= \begin{bmatrix} x_1 H_j(z^{(1)}) \\ \vdots \\ x_k H_j(z^{(k)}) \end{bmatrix} \quad (j = 1, \dots, p), \\ (G_\ell Y_R)^{\wedge R}(T') &= \begin{bmatrix} G_\ell(\omega^{(1)})u_1 & \dots & G_\ell(\omega^{(m)})u_m \end{bmatrix} \quad (\ell = 1, \dots, q) \end{aligned}$$

and conditions (1.26)–(1.28) take the form

$$\begin{bmatrix} x_1 H_j(z^{(1)}) \\ \vdots \\ x_k H_j(z^{(k)}) \end{bmatrix} \begin{bmatrix} H_\ell(z^{(1)})x_1^* & \dots & H_\ell(z^{(k)})x_k^* \end{bmatrix} = \Psi_{j\ell},$$

$$\begin{bmatrix} x_1 H_j(z^{(1)}) \\ \vdots \\ x_k H_j(z^{(k)}) \end{bmatrix} \begin{bmatrix} G_\ell(\omega^{(1)})u_1 & \dots & G_\ell(\omega^{(m)})u_m \end{bmatrix} = \Lambda_{j\ell},$$

$$\begin{bmatrix} u_1^* G_j(\omega^{(1)}) \\ \vdots \\ u_m^* G_j(\omega^{(m)}) \end{bmatrix} \begin{bmatrix} G_\ell(\omega^{(1)})u_1 & \dots & G_\ell(\omega^{(m)})u_m \end{bmatrix} = \Phi_{j\ell},$$

which are equivalent to (7.26). The advantage of the "finite" case is that the coefficients of the linear fractional transformation parametrizing the solution set can be written down explicitly. We illustrate this possibility by a numerical example. For simplicity we consider the scalar-valued case with one interpolation node.

Example 7.4. Let $\mathbf{Q}(z) = \begin{bmatrix} z_1 & z_2 \\ z_3 & z_4 \end{bmatrix}$ so that $\mathcal{D}_{\mathbf{Q}}$ is a Cartan domain of type I in $\mathbb{C}^4$. Thus, $p = q = 2$ and $n = 4$. By Theorem 1.1, a scalar-valued function

S belongs to the corresponding class $\mathcal{SA}_{\mathbf{Q}}$ if and only if there exist an auxiliary Hilbert space $\mathcal{H}$ and a function

$$H(z) = \begin{bmatrix} H_1(z) & H_2(z) \end{bmatrix} \tag{7.30}$$

analytic on $\mathcal{D}_{\mathbf{Q}}$ with values in $\mathcal{L}(\mathbb{C}^2 \otimes \mathcal{H}, \mathbb{C})$ so that

$$1 - S(z)S(w)^* = H(z)\left(I_{\mathcal{H}\oplus\mathcal{H}} - \mathbf{Q}(z)\mathbf{Q}(w)^*\right)H(w)^*. \tag{7.31}$$

We consider the following interpolation problem: *find all functions $S \in \mathcal{SA}_{\mathbf{Q}}$ satisfying condition*

$$S(0) = \frac{1}{2}. \tag{7.32}$$

The latter problem can be viewed as a very particular case of Problem 1.3 with $\mathcal{K} = \mathbb{C}$, $\mathcal{K}' = \{0\}$, $\mathcal{E} = \mathcal{E}_* = \mathbb{C}$,

$$T_j = 0 \ \ (j = 1,2,3,4), \quad X = X_L^* = 1, \quad Y = Y_L^* = \frac{1}{2}, \tag{7.33}$$

and the formulas (1.18) and (1.19) now take the form

$$M_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad M_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad N_1 = N_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{7.34}$$

By Theorem 1.4, the above problem has a solution if and only if there exists a positive semidefinite 2×2 matrix $P = P_L = \begin{bmatrix} p_{11} & p_{12} \\ p_{12}^* & p_{22} \end{bmatrix}$ satisfying the Stein identity (1.20):

$$M_1^*PM_1 + M_2^*PM_2 - N_1^*PN_1 + N_2^*PN_2 = \frac{3}{4} = X^*X - Y^*Y.$$

By the choice of (7.33) and (7.34), this identity simplifies to

$$p_{11} + p_{22} = \frac{3}{4}. \tag{7.35}$$

Now positivity of P is equivalent to

$$0 \le p_{11} \le \frac{3}{4} \quad \text{and} \quad |p_{12}|^2 \le \frac{3}{4}p_{11} - p_{11}^2. \tag{7.36}$$

Theorem 1.7 allows us to make the following three conclusions. First, there are functions $S \in \mathcal{SA}_{\mathbf{Q}}$ satisfying condition (7.32). Secondly, for every such function and for each of its representations (7.31), it holds that

$$H_1(0)H_1(0)^* + H_2(0)H_2(0)^* = \frac{3}{4}.$$

Finally, for every choice of the numbers p_{11}, p_{12} and p_{22} meeting the requirements (7.35), (7.36), there is a solution S of problem (7.32) with the additional property that, for some choice of H of the form (7.30) in the representation (7.31), it holds that

$$H_1(0)H_1(0)^* = p_{11} \quad \text{and} \quad H_1(0)H_2(0)^* = p_{12} \tag{7.37}$$

and there is a linear fractional parametrization of all S satisfying (7.32) and (7.37). We consider a particular choice of p_{11} and p_{12} to demonstrate how the explicit formula (6.17) works.

Let $p_{11} = \frac{1}{2}$ and $p_{12} = \frac{1}{4}$. Then conditions (7.35), (7.36) are satisfied with $p_{22} = \frac{1}{4}$. Thus,

$$P = \frac{1}{4} \cdot \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}, \quad P^{\frac{1}{2}} = \frac{1}{2\sqrt{5}} \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}$$

and by formulas (6.2),

$$W_1 = \frac{1}{2\sqrt{5}} \begin{bmatrix} 3 \\ 1 \\ 1 \\ 2 \end{bmatrix} \quad \text{and} \quad W_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

The formula (6.16) now gives $\mathbf{D}(z) \equiv 1$ and we get from (6.17)

$$\Sigma(z) = \begin{bmatrix} \frac{1}{2} & \pi + W_1^* \mathbf{Q}(z)\gamma \\ \beta^* & \alpha^* \mathbf{Q}(z)\gamma \end{bmatrix}. \tag{7.38}$$

By (6.7)–(6.9), $\begin{bmatrix} \alpha \\ \beta \end{bmatrix}$ and $\begin{bmatrix} \gamma \\ \pi \end{bmatrix}$ are 5×4 isometric matrices such that

$$\begin{bmatrix} W_1^* & Y^* \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} W_2^* & X^* \end{bmatrix} \begin{bmatrix} \gamma \\ \pi \end{bmatrix} = 0.$$

One can take for example,

$$\alpha = \begin{bmatrix} 0 & 2 & 6 & 3 \\ -1 & -3 & 2 & 1 \\ 1 & -3 & 2 & 1 \\ 0 & 0 & -11 & 2 \\ 0 & 0 & 0 & -3\sqrt{5} \end{bmatrix} \cdot \operatorname{diag}\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{22}}, \frac{1}{\sqrt{165}}, \frac{1}{2\sqrt{15}} \right),$$

$$\beta = \begin{bmatrix} 0 & 0 & 0 & -\frac{\sqrt{3}}{2} \end{bmatrix}, \quad \gamma = I_4 \quad \pi = 0.$$

Substituting the latter entries into (7.38) we conclude by Theorem 6.6 that a function S belongs to the class $\mathcal{SA}_{\mathbf{Q}}$ and satisfies conditions (7.33) and (7.37) (with $p_{11} = \frac{1}{2}$ and $p_{12} = \frac{1}{4}$) if and only if it is of the form

$$S(z) = \frac{1}{2} + \frac{1}{2\sqrt{5}} \begin{bmatrix} 3 & 1 & 1 & 2 \end{bmatrix} \mathbf{Q}(z) \left(I_4 - \mathcal{T}(z)\alpha^* \mathbf{Q}(z) \right)^{-1} \mathcal{T}(z)\beta^*,$$

where α and β are as above, where $\mathbf{Q}(z) = \begin{bmatrix} z_1 I_2 & z_2 I_2 \\ z_3 I_2 & z_4 I_2 \end{bmatrix}$ and $\mathcal{T}$ is any 4×4 matrix valued function of the class $\mathcal{SA}_{\mathbf{Q}}$.

Remark 7.5. The hypotheses of Theorems 1.4 and 1.7 can be weakened as follows. Given the interpolation data set $\mathcal{D}$ as in (1.13) or (1.29), rather than assuming that T and T' have Taylor spectrum equal to a compact subset of $\mathcal{D}_{\mathbf{Q}}$, assume instead that T and T' have diagonal direct sum decompositions

$$T_j = \operatorname{diag}_{\omega \in \Omega} T_{j,\omega}, \qquad T'_j = \operatorname{diag}_{\omega \in \Omega'} T'_{j,\omega'}$$

for some index sets Ω and Ω', where the operator d-tules $T_\omega = (T_{1,\omega}, \dots, T_{n,\omega})$ and $T'_{\omega'} = (T'_{1,\omega'}, \dots, T'_{n,\omega'})$ have Taylor spectrum inside $\mathcal{D}_{\mathbf{Q}}$ for each $\omega \in \Omega$ and $\omega' \in \Omega'$. Thus the space $\mathcal{K}$ is expressed as $\mathcal{K} = \oplus_{\omega \in \Omega} \mathcal{K}_\omega$ and $\mathcal{K}' = \oplus_{\omega \in \Omega'} \mathcal{K}'_{\omega'}$ for some Hilbert spaces $\mathcal{K}_\omega$ and $\mathcal{K}'_{\omega'}$ and T_ω and $T'_{\omega'}$ are operators on $\mathcal{K}_\omega$ and $\mathcal{K}_{\omega'}$, respectively, for each $\omega \in \Omega$ and $\omega' \in \Omega'$. The case considered in [17] is exactly this situation with $\Omega = \Omega_L \subset \mathcal{D}_{\mathbf{Q}}$ and $\Omega' = \Omega_R \subset \mathcal{D}_{\mathbf{Q}}$, with $\mathcal{K}_\omega = \mathcal{E}_L$ and $\mathcal{K}'_{\omega'} = \mathcal{E}_R$

for some fixed auxiliary Hilbert spaces $\mathcal{E}_L$ and $\mathcal{E}_R$, and with $T_{j,\omega} = \omega_j I_{\mathcal{E}_L}$ and $T'_{j,\omega'} = \omega'_j I_{\mathcal{E}_R}$ for $\omega \in \Omega_L$, $\omega' \in \Omega_R$ and $j = 1, \dots, d$. By using the results of the present paper (for the case where T and T' have Taylor spectrum contained compactly in $\mathcal{D}_{\mathbf{Q}}$), one can see that the construction in [17] can be extended to the more general setting described in the present Remark to arrive at the analogues of Theorems 1.4 and 1.7 for this more general situation. These more general versions of Theorem 1.4 and 1.7 then contain the results of [17] in full generality — without the special assumption that Ω_L and Ω_R are contained in compact subsets of $\mathcal{D}_{\mathbf{Q}}$ as was required in the discussion above.

8. Some further examples

The operator argument formulation of Problems 1.3 and 1.6 can be viewed as a way to treat various interpolation problems in a unified way. A natural question would be to clarify what specific problems can be included into this general scheme. In the previous section we showed that the case when the n-tuples T and T' consist of diagonal (and therefore commuting) matrices, Problem 1.6 reduces to Problem 7.3 of Nevanlinna–Pick type. Thus, the question is to classify what problems arise in this way from some choice of commuting (possibly nondiagonal) n-tuples of operators. We will focus on the left-sided (the first) condition in (1.14); the right-sided condition in (1.14) and supplementary two-sided conditions (1.26)–(1.28) can be treated quite similarly.

Example 8.1. Let us consider the classical case $p = q = n = 1$ and $\mathbf{Q}(z) = z$. Let us also suppose that $\mathcal{K}$, $\mathcal{K}'$, $\mathcal{E}$ and $\mathcal{E}_*$ are all finite-dimensional Hilbert spaces, and that (T, X_L, Y_L) is a data set for a left operator-argument interpolation condition. It is easily seen that $(\widetilde{T}, \widetilde{X}_L, \widetilde{Y}_L) = (STS^{-1}, SX_L, SY_L)$ is also a left interpolation data set which generates the same aggregate of interpolation conditions:

$$(\widetilde{X}_L F)^{\wedge L}(\widetilde{T}) = \widetilde{Y}_L \iff (X_L F)^{\wedge L}(T) = Y_L.$$

Thus, without loss of generality, we may assume that T is in Jordan form. Consider first the case where T is a Jordan block:

$$T = \begin{bmatrix} \omega & & & \\ 1 & \omega & & \\ & \ddots & \ddots & \\ & & 1 & \omega \end{bmatrix}, \qquad X_L = \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_m \end{bmatrix}, \qquad Y_L = \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_m \end{bmatrix}.$$

Then the interpolation condition $(X_L F)^{\wedge L}(T) = Y_L$ amounts to the aggregate of conditions

$$\frac{d^j}{dz^j}(x(z)F(z))|_{z=\omega} = \frac{d^j}{dz^j} y(z)|_{z=\omega} \text{ for } j = 0, 1, \dots, m \tag{8.1}$$

where we have set

$$x(z) = \sum_{j=0}^{m} x_j (z - \omega)^j, \qquad y(z) = \sum_{j=0}^{m} y_j (z - \omega)^j.$$

When T has several Jordan blocks, one then gets a finite collection of sets of interpolation conditions of this type at each eigenvalue ω of T. Since any finite matrix

T can be brought to Jordan form via a similarity transformation, by the discussion above we see that such sets of interpolation conditions give the most general such expressible through the left-tangential operator-argument formalism presented here (for the finite-dimensional, classical case). Complete details can be found in Chapter 16 of [18].

Just as in the single variable case, two left-sided conditions

$$(X_{L,1}S)^{\wedge L}(T_1) = Y_{L,1} \quad \text{and} \quad (X_{L,2}S)^{\wedge L}(T_2) = Y_{L,2}$$

imposed for the same interpolant S can be written as just one condition

$$(X_L S)^{\wedge L}(T) = Y_L \text{ with } T = \begin{bmatrix} T_1 & 0 \\ 0 & T_2 \end{bmatrix}, \ X_L = \begin{bmatrix} X_{L,1} \\ X_{L,2} \end{bmatrix}, \ Y_L = \begin{bmatrix} Y_{L,1} \\ Y_{L,2} \end{bmatrix}. \tag{8.2}$$

Thus, if certain conditions can be included into the general scheme of Problem 1.3, then the problem with several conditions of this type can be also included into the scheme. Moreover, if (T, X_L, Y_L) is the (multivariable) data set for a left operator-argument interpolation condition $(X_L F)^{\wedge L}(T) = Y_L$, then, as in the one-variable case, if S is any invertible bounded linear operator on $\mathcal{K}$, we have that $(\widetilde{T}, \widetilde{X}_L, \widetilde{Y}_L) = (STS^{-1}, SX_L, SY_L)$ is the data set for the very same left interpolation condition, i.e.:

$$(\widetilde{X}_L F)^{\wedge L}(\widetilde{T}) = \widetilde{Y}_L \Longleftrightarrow (X_L F)^{\wedge L}(T) = Y_L.$$

Here $\widetilde{T} = STS^{-1}$ refers to the operator-tuple

$$STS^{-1} = (ST_1S^{-1}, \dots, ST_nS^{-1}).$$

However, in case $n > 1$, there is no canonical form for equivalence of operator-tuples up to similarity, even in the finite-dimensional case. We therefore are content here to discuss a couple of possible generalizations of the chain of interpolation conditions (8.1) to the multivariable situation. We focus on the case where $\dim \mathcal{K} < \infty$ and the joint spectrum of T consists of a single point $\omega = (\omega_1, \dots, \omega_n) \in \mathcal{D}_{\mathbf{Q}}$, as the general case is a direct sum of cases of this form.

Example 8.2. Let us take

$$T_j = \begin{bmatrix} \omega_j & & & \\ 1 & \omega_j & & \\ & \ddots & \ddots & \\ & & 1 & \omega_j \end{bmatrix} \text{ for } j = 1, \dots, n, \qquad X_L = \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_m \end{bmatrix}, \qquad Y_L = \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_m \end{bmatrix}.$$

Then the left tangential interpolation condition with operator argument

$$(X_L S)^{\wedge L}(T) = Y$$

assumes the form

$$\begin{aligned} x_0 \left(\sum_{\mathbf{j}\colon\, |\mathbf{j}|=i} \frac{1}{i!} \frac{\partial^i S}{\partial^{\mathbf{j}} z}(\omega) \right) + x_1 \left(\sum_{\mathbf{j}\colon\, |\mathbf{j}|=i-1} \frac{1}{(i-1)!} \frac{\partial^{i-1} S}{\partial^{\mathbf{j}} z}(\omega) \right) + \dots + x_i S(\omega) \\ = y_i \text{ for } i = 0, 1, \dots, m. \end{aligned} \tag{8.3}$$

Here we use the standard multivariable notation

$$|\mathbf{j}| = j_1 + \dots + j_n \text{ if } \mathbf{j} = (j_1, \dots, j_n), \qquad \frac{\partial^{|\mathbf{j}|} S}{\partial z^{\mathbf{j}}} = \frac{\partial^{|\mathbf{j}|} S}{\partial z_1^{j_1} \partial z_2^{j_2} \cdots \partial z_n^{j_n}}.$$

Note that in the one variable case this collection of interpolation conditions reduces to (8.1).

Example 8.3. Let $E \subset \mathbb{Z}_+^n$ be a subset of indices in $\mathbb{Z}_+^n$ (n-tuples of nonnegative integers) which is *lower inclusive*, i.e.: whenever $\mathbf{n} \in E$ and $\mathbf{n} - \mathbf{e}_i \in \mathbb{Z}_+^d$ (where $\mathbf{e}_i$ is the unit vector with i-th component equal to 1 and all other components equal to 0), then it is the case that also $\mathbf{n} - \mathbf{e}_i \in E$. We assume that the space $\mathcal{K}$ has the form $\mathcal{K} = \widetilde{\mathcal{K}} \otimes \ell^2(E)$ for some other auxiliary Hilbert space $\widetilde{\mathcal{K}}$, i.e., vectors $k \in \mathcal{K}$ can be written as $k = \operatorname{col}_{\mathbf{n}\in E}[k_{\mathbf{n}}]$ where $k_{\mathbf{n}} \in \widetilde{\mathcal{K}}$ for each $\mathbf{n} \in E$. Define operators T_j on $\mathcal{K}$ for $j = 1, \dots, n$ via block matrices: $T_j = \omega_j I_{\mathcal{K}} + [t^j_{\mathbf{n}',\mathbf{n}}]$ where

$$t^j_{\mathbf{n}',\mathbf{n}} = \begin{cases} I_{\widetilde{\mathcal{K}}} & \text{if } \mathbf{n}' = \mathbf{n} + \mathbf{e}_j, \\ 0 & \text{otherwise} \end{cases}$$

for $\mathbf{n}', \mathbf{n} \in E$. Define operators $X_L \colon \mathcal{E}_* \to \mathcal{K}$ and $Y_L \colon \mathcal{E} \to \mathcal{K}$ by

$$X_L = \operatorname{col}_{\mathbf{n}\in E}[x_{\mathbf{n}}], \qquad Y_L = \operatorname{col}_{\mathbf{n}\in E}[y_{\mathbf{n}}]$$

where $x_{\mathbf{n}} \colon \mathcal{E}_* \to \widetilde{\mathcal{K}}$ and $y_{\mathbf{n}} \colon \mathcal{E} \to \widetilde{\mathcal{K}}$ are given operators for $\mathbf{n} \in E$. Then the left tangential interpolation condition with operator argument $(X_L S)^{\wedge L}(T) = Y_L$ in this case becomes the aggregate of interpolation conditions

$$\frac{\partial^{|\mathbf{n}|}}{\partial z^{\mathbf{n}}} \{x(z)S(z)\}\,|_{z=\omega} = \frac{\partial^{|\mathbf{n}|} y}{\partial z^{\mathbf{n}}}(\omega) \text{ for all } \mathbf{n} \in E \tag{8.4}$$

where we have set

$$x(z) = \sum_{\mathbf{n}\in E} x_{\mathbf{n}} z^{\mathbf{n}}, \qquad y(z) = \sum_{\mathbf{n}\in E} y_{\mathbf{n}} z^{\mathbf{n}}$$

and we use the standard multivariable notation

$$z^{\mathbf{n}} = z_1^{n_1} z_2^{n_2} \cdots z_n^{n_n} \text{ if } \mathbf{n} = (n_1, n_2, \dots, n_n).$$

Note that the set of interpolation conditions (8.4) also collapses to (8.1) in the one-variable case. The interpolation problem for the class $\mathcal{SA}_{\mathbf{Q}}(\mathcal{E}, \mathcal{E}_*)$ with interpolation conditions of this form (with $\widetilde{\mathcal{K}} = \mathcal{E}_*$ and $x(z) = I_{\mathcal{E}_*}$) was solved in [8] as an application of a commutant lifting theorem. We mention that a even more general commutant lifting theorem of a more operator-algebra flavor has recently appeared in [42]. The interpolation problem with interpolation conditions of the form (8.4) has been worked out earlier for various special settings: the Herglotz–Agler class on the polydisk in [53], contractive multipliers between general reproducing kernel Hilbert spaces in [25], and contractive multipliers of the Arveson space in [6].

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