New York Journal of Mathematics

New York J. Math. **11** (2005) 345–349.

An ergodic sum related to the approximation by continued fractions

Andrew Haas

Abstract. To each irrational number x is associated an infinite sequence of rational fractions $\frac{p_n}{q_n}$, known as the convergents of x. Consider the functions $q_n|q_n x - p_n| = \theta_n(x)$. We shall primarily be concerned with the computation, for almost all real x, of the ergodic sum

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \log \theta_k(x) = -1 - \frac{1}{2}\log 2 \approx -1.34657.$$

Each irrational number x has a unique infinite, regular continued fraction expansion of the form

$$x = [a_1; a_2, a_3 \dots] = a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \dots}}$$

where the a_i are integers and $a_i > 0$ for $i > 1$. To x is associated an infinite sequence of rational fractions $\frac{p_n}{q_n} = [a_1; a_2, \dots, a_n]$, in lowest terms, known as the convergents of x. Define the functions $\theta_n(x)$ by the identity

$$\left| x - \frac{p_n}{q_n} \right| = \frac{\theta_n(x)}{q_n^2}.$$

Important metrical results on the $\theta_n(x)$ are proved in the papers [3],[5] and [7]. Since the convergents satisfy the following well-known inequality, usually attributed to Dirichlet,

$$\left| x - \frac{p_n}{q_n} \right| < \frac{1}{q_n^2},$$

we have $0 < \theta_n(x) < 1$.

It does not seem to have been observed that for almost all x

$$\lim_{n\to\infty} \frac{\log \theta_n(x)}{n} = 0. \tag{1}$$

To begin we supply a proof of this fact which was suggested by A. Rockett.

Received February 4, 2005.

Mathematics Subject Classification. 11J70, 11J83, 37E05.

Key words and phrases. Continued fractions, metric theory, interval maps.

ISSN 1076-9803/05

From the sequence of inequalities (see [9]),

$$\frac{1}{q_n(q_n+q_{n+1})} \le \left| x - \frac{p_n}{q_n} \right| \le \frac{1}{q_n q_{n+1}}$$

we get

$$\frac{1}{n}\log\left(\frac{q_n}{q_n+q_{n+1}}\right) \le \frac{1}{n}\log\theta_n(x) \le \frac{1}{n}\log\left(\frac{q_n}{q_{n+1}}\right).$$

The result then follows easily from the Khintchine–Lévy Theorem, which asserts that for almost all x

(2) $$\lim_{n\to\infty}\frac{1}{n}\log q_n = \frac{\pi^2}{12\log 2}, \qquad \text{(see [2] or [9]).}$$

Now we look at the limiting average of the functions $\log\theta_n(x)$. While this average resembles those, such as $\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\theta_k(x)$, computed in [3] and [5], its evaluation is complicated by the fact that $\log x$ is not continuous on the interval [0,1]. As a result, knowledge of the distribution function for $\theta_n(x)$ is not sufficient to prove the theorem. As in [3], we work with a form of the natural automorphic extension of the Gauss transform, derived from the extension originally given by Nakada [8].

Theorem 1. *For almost all* x

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\log\theta_k(x) = -1 - \frac{1}{2}\log 2 \approx -1.34657.$$

Let $\Lambda = ((0,1)\setminus\mathbb{Q})\times[0,1]$ and define the map $\widetilde{S}:\Lambda\to\Lambda$ by

$$\widetilde{S}(s,t) = \left(\frac{1}{s} - \left[\frac{1}{s}\right], \frac{1}{t+[\frac{1}{s}]}\right)$$

where $[x]$ is the greatest integer function. Let ν be the probability measure with density $m(s,t) = \frac{1}{\log 2}(1+st)^{-2}$. It was first observed by Nakada [8] that the dynamical system $(\Lambda, B, \nu, \widetilde{S})$ is ergodic. See also [1].

Consider the related self-mapping $\widetilde{T}$ of $\Omega = ((0,1)\setminus\mathbb{Q})\times[-\infty,-1]$ defined by

$$\widetilde{T}(x,y) = \left(\frac{1}{x} - \left[\frac{1}{x}\right], \frac{1}{y} - \left[\frac{1}{x}\right]\right).$$

Let $\varphi:\Lambda\to\Omega$ be the invertible function given by $\varphi(s,t) = (s, -\frac{1}{t})$. It is clear that φ maps Λ onto Ω and that $\widetilde{T} = \varphi\circ\widetilde{S}\circ\varphi^{-1}$.

The measure $\mu = \varphi^* \nu$ is defined by

$$\mu(D) = \frac{1}{\log} \int_{\varphi^{-1}(D)} \frac{1}{(1+st)^2} dsdt,$$

where D is a borel subset of Ω. It follows by an application of the chain rule that μ has the density $p(x,y) = \frac{1}{\log 2}(x-y)^{-2}$. As constructed, μ is invariant under the action of $\widetilde{T}$ and φ defines an isomorphism between the dynamical systems $(\Lambda, B, \nu, \widetilde{S})$ and $(\Omega, B, \mu, \widetilde{T})$. It follows that $(\Omega, B, \mu, \widetilde{T})$ is an ergodic dynamical system. The ergodicity of $\widetilde{T}$ is central to the proof of Theorem 1 that follows.

Let $T(x) = \frac{1}{x} - [\frac{1}{x}]$ be the classical Gauss map and let $\pi(x_1, x_2) = x_1$ be projection on the first factor. Then independent of y, $\pi \circ \widetilde{T}(x,y) = T(x)$ and $\pi^*(\mu)$ is the classical Gauss measure, which is an invariant measure for T, with density $g(x) = \frac{1}{\log 2}\frac{1}{1+x}$.

For irrational $x = [0; a_1, a_2 \dots]$, the Gauss map T acts as a shift on continued fraction expansions with $T(x) = [0; a_2, a_3 \dots]$. Even when x is rational, T acts as a shift on the finite continued fraction expansion and the iterates are defined until $T^n(x) = 0$. We assume henceforth that $x = [0; a_1, a_2 \dots]$ is irrational. The iterates $\widetilde{T}^n(x)$ are then defined for all positive integers n. If $y = -[a_{-1}; a_{-2}, \dots] \in (-\infty, -1]$, with a possibly finite continued fraction expansion and x is as above then $\widetilde{T}^n(x,y) = (\hat{x}, \hat{y})$ where $\hat{x} = [0; a_{n+1}, a_{n+2} \dots]$ and $\hat{y} = -[a_n; a_{n-1}, a_{n-2}, \dots, a_1, a_{-1} \dots]$.

Define the function

$$F(x,y) = \log\left(\frac{1}{x} - \frac{1}{y}\right) = \log\left(\frac{y-x}{xy}\right).$$

We show that F is integrable on $\Omega = (0,1) \times (-\infty, -1]$ with respect to the density p. It shall soon be clear that F is very useful for computing the quantity $\log \theta_n(x)$.

$$\begin{aligned}
&\int_\Omega \log\left(\frac{y-x}{xy}\right) p(x,y) \quad dxdy \\
&= \frac{1}{\log 2} \int_{-\infty}^{-1} \int_0^1 \frac{\log\left(\frac{y-x}{xy}\right)}{(x-y)^2} \quad dxdy \\
&= \frac{1}{\log 2} \int_{-\infty}^{-1} \frac{1 + \log(1-y) - \log(-y)}{y(y-1)} \quad dy \\
&= \frac{1}{\log 2} \lim_{h\to\infty} \Bigg[\log(-y) + \log(1-y) + \frac{1}{2}(\log(-y))^2 \\
&\quad + \frac{1}{2}(\log(1-y))^2 - \log(-y)\log(1-y) \Bigg] \quad \Bigg]\Bigg|_{-h}^{-1} \\
&= \frac{1}{\log 2} \Bigg[\left(\log 2 + \frac{1}{2}(\log 2)^2\right) \\
&\quad - \lim_{h\to\infty} \left(\frac{\frac{1}{2}\log(-y) - \frac{1}{2}\log(1-y) + 1}{\frac{1}{\log(-y)}} + \frac{\frac{1}{2}\log(1-y) - \frac{1}{2}\log(-y) + 1}{\frac{1}{\log(1-y)}} \right) \Bigg]
\end{aligned}$$

where the last limit is zero by L'Hospital's rule.

Since F is μ-integrable and $\widetilde{T}^n(x,y)$ is defined on a set of full measure for all $n \geq 0$, it is a direct consequence of the Birkhoff Ergodic Theorem (see [4] or [2]) that for almost all $(x,y) \in \Omega$

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} F(\widetilde{T}^i(x,y)) = \int_\Omega F(x,y)p(x,y) \quad dxdy. \tag{3}$$

This value was just computed to be $1 + \frac{1}{2}\log 2$.

It is known, see, e.g., [2], that if two numbers α, β have continued fractions expansions which agree in their first n digits, then $|\alpha - \beta| < 2^{-n+1}$. Thus for $y, y' \in [-\infty, -1]$, $\quad |\pi_2 \circ \widetilde{T}^n(x,y) - \pi_2 \circ \widetilde{T}^n(x,y')| < 2^{-n+1}$.

Our next task is to prove that if the equality (3) holds for a given (x,y) then it holds for (x,y') for all $y' \in [-\infty,-1]$. In essence, the equality is true for almost all x independent of y. Fix x and suppose that the equality (3) holds for $(x,y) \in \Omega$. Let $y' \in [-\infty,-1]$. To simplify the computation write $\widetilde{T^i}(x,y) = (x_i, y_i)$ and $\widetilde{T^i}(x,y') = (x_i, y_i')$. Keep in mind that y_i and y_i' are negative numbers. Then

$$\begin{aligned}
\frac{1}{n}\sum_{i=1}^{n}\left|F(\widetilde{T^i}(x,y)) - F(\widetilde{T^i}(x,y'))\right| &= \frac{1}{n}\sum_{i=1}^{n}\left|\log\left(\frac{y_i - x_i}{x_i y_i}\right) - \log\left(\frac{y_i' - x_i}{x_i y_i'}\right)\right| \\
&= \frac{1}{n}\sum_{i=1}^{n}\left|\Big(\log\big(x_i - y_i\big) - \log(x_i) - \log(-y_i)\Big)\right. \\
&\quad \left. - \Big(\log\big(x_i - y_i'\big) - \log(x_i) - \log(-y_i')\Big)\right| \\
&\leq \frac{1}{n}\sum_{i=1}^{n}\left|\log\frac{x_i - y_i}{x_i - y_i'}\right| + \frac{1}{n}\sum_{i=1}^{n}\left|\log\frac{y_i'}{y_i}\right|. \qquad (4)
\end{aligned}$$

There is no loss of generality in supposing that $x_i - y_i \geq x_i - y_i'$, since the absolute value of the log of the quotient in the first sum of (4) is the same either way the inequality goes. Then by an earlier observation

$$0 < (x_i - y_i) - (x_i - y_i') = y_i' - y_i < 2^{-i+1}.$$

Since $x_i - y_i' > 1$,

$$\frac{x_i - y_i}{x_i - y_i'} < 1 + 2^{-i+1}(x_i - y_i')^{-1} < 1 + 2^{-i+1}.$$

Now take logs and apply the standard estimate that comes from the alternating series for $\log x$ to get

$$\log\frac{x_i - y_i}{x_i - y_i'} < \log(1 + 2^{-i+1}) < 2^{-i+1}.$$

It follows that the first sum in (4) converges to zero as n goes to ∞. By a similar argument the same conclusion can be reached for the second sum in (4). This shows that if the equality (3) holds for some $(x,y) \in \Omega$ then it holds for any $(x,y') \in \Omega$.

We are now close to completing the the proof of Theorem 1. Two identities from the classical theory will link the above to our main theorem. If $x = [0; a_1, a_2 \ldots]$ then

$$[a_n; a_{n-1}, \ldots, a_1] = \frac{q_n}{q_{n-1}} \quad \text{(see [9])} \tag{5}$$

and

$$\theta_n(x) = \left(\frac{1}{T^n(x)} + \frac{q_{n-1}}{q_n}\right)^{-1} \quad \text{(see [6, p. 29, (11)]).} \tag{6}$$

Given $x \in (0,1)$, let $(x_0, y_0) = \widetilde{T}(x, \infty) = (T(x), -[1/x]) \in \Omega$. As above define $\widetilde{T^i}(x_0, y_0) = (x_i, y_i)$. If $x = [0; a_1, a_2 \dots]$ then for $i > 0$

$$(x_{i-1}, y_{i-1}) = ([0; a_{i+1}, a_{i+2} \dots], -[a_i; a_{i-1}, a_{i-2}, \dots, a_1]) = \left(T^i(x), -\frac{q_i}{q_{i-1}}\right)$$

where we have used (5) above. From (6) and the definition of F,

$$\begin{aligned} -F(\widetilde{T^i}(x_0, y_0)) &= -\log\left(\frac{1}{x_i} - \frac{1}{y_i}\right) \\ &= \log\left(\frac{1}{T^{i+1}(x)} + \frac{q_i}{q_{i+1}}\right)^{-1} \\ &= \log \theta_{i+1}(x). \end{aligned}$$

Therefore

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \log \theta_{i+1}(x) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} -F(\widetilde{T}^i(x_0, y_0))$$

converges to $-1 - \frac{1}{2}\log 2$ for almost all $x_0 \in (0,1)$, independent of y_0, and consequently for almost all $x \in (0,1)$. The proof of Theorem 1 is complete.

References

[1] R. L. Adler and L. Flatto, *Cross-section maps for geodesic flows.* I, Ergodic theory and dynamical systems, II (College Park, Md., 1979/1980), Progress in Math., **21**, Birkhäuser, Boston, 1980, 103–161, MR 0670077 (84h:58113), Zbl 0496.58009.

[2] P. Billingsley, *Ergodic theory and information*, J. Wiley & Sons, New York-London, 1965, MR 0192027 (33 #254), Zbl 0141.16702.

[3] W. Bosma, H. Jager and F. Wiedijk, *Some metrical observations on the approximation by continued factions*, Indag. Math. **45** (1983), 281–299, MR 0718069 (85f:11059), Zbl 0519.10043.

[4] I. P. Cornfeld, S. V. Fomin and Ya. G. Sinai, *Ergodic Theory*, Grundlehren der Mathematischen Wissenschaften, **245**, Springer-Verlag, Berlin-Heidelberg-New York, 1982, MR 0832433 (87f:28019), Zbl 0493.28007.

[5] H. Jager, *The distribution of certain sequences connected with the continued fraction*, Indag. Math. **89** (1986), 61–69, MR 0834320 (87g:11092), Zbl 0588.10061.

[6] J. F. Koksma, *Diophantische approximationen*, Julius Springer, Berlin 1936, MR 0344200 (49 #8940), Zbl 0012.39602.

[7] D. Knuth, *The distribution of continued fraction approximations*, J. Number Theory **19** (1984), 443–448, MR 0769794 (86d:11058), Zbl 0547.10030.

[8] H. Nakada, *Metrical theory for a class of continued fraction transformations and their natural extensions*, Tokyo J. Math. **4** (1981), 399–426, MR 0646050 (83k:10095), Zbl 0479.10029.

[9] A. Rockett and P. Szüsz, *Continued fractions*, World Sci. Pub., Singapore, 1992, MR 1188878 (93m:11060), Zbl 0925.11038.

DEPARTMENT OF MATHEMATICS, THE UNIVERSITY OF CONNECTICUT, STORRS, CT. 06269-3009
haas@math.uconn.edu

This paper is available via http://nyjm.albany.edu:8000/j/2005/11-17.html.