

## MRA super-wavelets

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ABSTRACT. We construct a multiresolution theory for  $L^2(\mathbb{R}) \oplus \cdots \oplus L^2(\mathbb{R})$ . For a good choice of the dilation and translation operators on these larger spaces, it is possible to build singly generated wavelet bases, thus obtaining multiresolution super-wavelets. We give a characterization of super-scaling function, we analyze the convergence of the cascade algorithms and give examples of super-wavelets. Our analysis provides also more insight into the Cohen and Lawton condition for the orthogonality of the scaling function in the classical case on  $L^2(\mathbb{R})$ .

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## 1. Introduction

The applications of wavelet theory to signal processing and image processing are now well-known. Probably the main reason for the success of the wavelet theory was the introduction of the concept of multiresolution analysis (MRA), which provided the right framework to construct orthogonal wavelet bases with good localization properties.

One of the problems in networking is *multiplexing*, which consists of sending multiple signals or streams of information on a carrier at the same time in the form of a single, complex signal and then recovering the separate signals at the receiving end.

In [HL], Deguang Han and David Larson have shown that the technique of multiresolution analysis breaks down, when multiplexing is required, if one just

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amplifies (in the operator theory sense) the steps used in the construction of MRA-wavelets. We will state this more precisely in a moment.

In this paper we will show how multiresolution constructions *can* be realized for multiple signals, provided some slight modifications are done to the usual dilation and translation operators. We believe that our constructions have potential for applications to multiplexing problems.

In [Dut2] and [Dut3], the second named author introduced a certain affine structure on the space  $L^2(\mathbb{R}) \oplus \cdots \oplus L^2(\mathbb{R})$  which was shown to admit multiresolution wavelet bases. In this paper, we will further analyze this affine structure, give characterizations of its scaling vectors (Theorem 3.4, Corollary 3.7). Then, the well-known conditions for the orthogonality of the scaling functions due to A. Cohen [Co90] and W. Lawton [Law91a] are extended to this larger space in Theorem 3.9. We study the convergence of the cascade operator which provides numerical approximations for the scaling function. Finally, in Section 5, we construct several examples of super-scaling functions and super-wavelets and we prove that all these structures admit super-wavelet bases.

In this introduction we recall several fundamental ideas and notions of the wavelet theory. We refer to [Dau92], [BraJo] for more information on the topic.

A *wavelet* is a function  $\psi \in L^2(\mathbb{R})$  with the property that

$$\{U^m T^n \psi \mid m, n \in \mathbb{Z}\}$$

is an orthonormal basis for  $L^2(\mathbb{R})$ , where  $U$  is the dilation operator

$$Uf(x) = \frac{1}{\sqrt{2}} f\left(\frac{x}{2}\right), \quad (f \in L^2(\mathbb{R}), x \in \mathbb{R}),$$

and  $T$  is the translation operator

$$Tf(x) = f(x - 1), \quad (f \in L^2(\mathbb{R}), x \in \mathbb{R}).$$

A *multiresolution analysis* is an increasing nest of closed subspaces  $(V_n)_{n \in \mathbb{Z}}$  of  $L^2(\mathbb{R})$  which has a dense union, a zero intersection,  $UV_n = V_{n-1}$ , and there is a vector  $\varphi$  in  $V_0$  such that

$$\{T^k \varphi \mid k \in \mathbb{Z}\}$$

is an orthonormal basis for  $V_0$ . This  $\varphi$  is then called an *orthogonal scaling function*.

The wavelets  $\psi$  are then constructed from the scaling function in such a way that

$$\{T^k \psi \mid k \in \mathbb{Z}\}$$

is an orthonormal basis for  $W_0 := V_1 \ominus V_0$ , the relative orthocomplement.

There are generalizations of the multiresolution analyses some of which we will turn to later. One of the directions for these generalizations (see [BCMO]) is to replace the Hilbert space  $L^2(\mathbb{R})$  by an abstract one  $H$  and the dilation and translation operators  $U$  and  $T$  by some abstract unitaries that verify a commutation relation such as

$$UTU^{-1} = T^2.$$

The two unitaries will generate what we call a wavelet representation. We adopt this representation theoretic view point, and our approach emphasizes the strong connections between wavelet theory and the spectral properties of a certain transfer operator (Definition 1.1).

We maintain an abstract flavor throughout the paper, but we concentrate on some concrete examples of wavelet representations on the Hilbert space  $L^2(\mathbb{R}) \oplus \cdots \oplus L^2(\mathbb{R})$  (finite sum), which we will describe in a moment.

It is known (see [HL], Proposition 5.16), that no orthogonal scaling function can be constructed for  $L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$  with the dilation  $U \oplus U$  and the translation  $T \oplus T$ , therefore multiplexing can not be obtained in this way. However, we will see that a multiresolution theory can be developed on  $L^2(\mathbb{R}) \oplus \cdots \oplus L^2(\mathbb{R})$  with plenty of examples of orthogonal scaling functions and wavelets, if some slight modifications are done to these operators.

The dilation and translation operators are constructed as follows: first consider a fixed cycle, that is, a periodic orbit for the map  $z \mapsto z^2$  on the unit circle. Let  $C := \{z_1, z_2, \dots, z_p\}$  be this cycle,  $z_1^2 = z_2, z_2^2 = z_3, \dots, z_{p-1}^2 = z_p, z_p^2 = z_1$ . The Hilbert space is

$$H_C := \underbrace{L^2(\mathbb{R}) \oplus \cdots \oplus L^2(\mathbb{R})}_{p \text{ times}},$$

the translation operator  $T_C$  on  $H_C$  is given by

$$T_C(\xi_1, \dots, \xi_p) = (z_1 T \xi_1, \dots, z_p T \xi_p), \quad (\xi_1, \dots, \xi_p \in L^2(\mathbb{R})),$$

the dilation operator  $U_C$  on  $H_C$  is defined by

$$U_C(\xi_1, \dots, \xi_p) = (U \xi_2, U \xi_3, \dots, U \xi_p, U \xi_1), \quad (\xi_1, \dots, \xi_p \in L^2(\mathbb{R})),$$

where  $T$  and  $U$  are the translation and dilation operators on  $L^2(\mathbb{R})$  defined before.

It was shown in [Dut2, Proposition 2.13] that if a filter  $m_0 \in L^\infty(\mathbb{T})$  satisfies the condition

$$|m_0(z_i)| = \sqrt{N}, \quad (i \in \{1, \dots, p\}),$$

for some cycle  $C = \{z_1, \dots, z_p\}$  then there is a scaling vector  $\varphi \in H_C$ . Such a cycle is called an  $m_0$ -cycle.

With the operators  $U_C, T_C$ , multiresolutions can be constructed and super-wavelets (meaning wavelets in spaces larger than  $L^2(\mathbb{R})$ ) are obtained. The representations that correspond to different cycles are disjoint so that we will see that scaling functions and multiresolutions can be constructed also for the direct sum of these representations. Theorem 5.3 in [Dut3] shows that, gathering *all*  $m_0$ -cycles, one obtains an orthogonal scaling vector in the orthogonal sum  $\oplus_C H_C$ .

We prove here that the scaling vectors constructed out of some  $m_0$ -cycles are orthogonal only when all the  $m_0$ -cycles are taken into consideration, and when this is not the case, the super-wavelets form a normalized tight frame (Theorem 3.9). In particular, one of the consequences of our analysis is that the MRA normalized tight frame wavelets, which are known to occur on  $L^2(\mathbb{R})$  when the low-pass filter is not judiciously chosen, are in fact projections of good, orthogonal MRA super-wavelets. This is due to the fact that the only cycle considered in the construction of wavelets on  $L^2(\mathbb{R})$  is the trivial cycle  $\{1\}$ , while the filter  $m_0$  might have other cycles. The direct relation to Cohen's orthogonality condition [Co90] is now clear.

Our paper is structured as follows: in Section 2 we define the term of wavelet representation (Definition 2.1) and present the main examples that we will work with (especially Examples 2.5 and 2.6).

In Section 3 we define the notions of multiresolution analysis and scaling vector (Definition 3.1), imitating the one existent for  $L^2(\mathbb{R})$  and give a characterization

theorem for scaling vectors (Theorem 3.4). The theorem is then applied to our main examples, and we obtain in this way conditions which characterize scaling vectors in  $L^2(\mathbb{R}) \oplus \cdots \oplus L^2(\mathbb{R})$  (Corollary 3.7). Once a scaling vector is found, the construction of wavelets can be done in exactly the same fashion as for  $L^2(\mathbb{R})$  (see Proposition 3.8).

In addition, we will see how one can obtain *super-scaling functions* and *super-wavelets* from a trigonometric polynomial filter. Each component of the super-scaling function will be compactly supported. As in the  $L^2(\mathbb{R})$  case, the super-wavelet usually generates a normalized tight frame and, to get an orthogonal basis, some extra conditions must be imposed on the initial low-pass filter from which the super-wavelet is constructed (such as the Cohen condition or Lawton's condition, see Theorem 3.9).

The scaling vector is approximated by the so-called *cascade algorithm*. One starts with a well chosen function and then the *cascade operator* is applied successively to it. In this way one obtains a sequence which approaches the scaling vector in norm. We will see in Section 4 how the initial function must be chosen so that the algorithm is convergent.

Abstract or geometric constructions in wavelet theory must be tested against examples and explicit algorithms. We end our paper with Section 5, which contains several examples showing that plenty of multiresolution super-wavelets can be constructed.

Some notations that we will use in this paper: the Fourier transform of  $f \in L^1(\mathbb{R})$  is given by

$$\widehat{f}(\xi) = \int_{\mathbb{R}} f(x) e^{-i\xi x} dx.$$

We denote by  $\mathbb{T}$  the unit circle  $\{z \in \mathbb{C} \mid |z| = 1\}$  and by  $\mu$ , the normalized Haar measure on  $\mathbb{T}$ . We often identify functions  $f$  on  $\mathbb{T}$  with  $2\pi$ -periodic functions on  $\mathbb{R}$  or with functions on the interval  $[-\pi, \pi]$ . The identification is given by

$$f(z) \leftrightarrow f(\theta) \text{ where } z = e^{-i\theta}.$$

Many key properties of the scaling vectors are encoded in the spectral properties of a certain transfer operator. This is defined as follows:

**Definition 1.1.** Let  $N \geq 2$  be an integer. The *transfer operator* is associated to a function  $m_0 \in L^\infty(\mathbb{T})$  and is defined by

$$R_{m_0} f(z) = \frac{1}{N} \sum_{w^N=z} |m_0(w)|^2 f(w), \quad (z \in \mathbb{T}, f \in L^1(\mathbb{T})).$$

A function  $h \in L^1(\mathbb{T})$  is called *harmonic* with respect to  $R_{m_0}$  if

$$R_{m_0} h = h.$$

## 2. Wavelet representations

Fix an integer  $N \geq 2$  called the *scale*. Wavelet representations are an abstract version of the situation existent on  $L^2(\mathbb{R})$  as explained in the introduction. The Hilbert space  $L^2(\mathbb{R})$  is replaced by an abstract one  $H$  and the dilation and translation operators are replaced by two unitaries  $U$  and  $T$  satisfying the following commutation relation

$$UTU^{-1} = T^N.$$

The translation generates a representation of  $L^\infty(\mathbb{T})$  by the Borel functional calculus. We give here the proper definition of a wavelet representation and present some examples.

**Definition 2.1.** A *wavelet representation* is a triple  $\tilde{\pi} := (H, U, \pi)$  where  $H$  is a Hilbert space,  $U$  is a unitary on  $H$  and  $\pi$  is a representation of  $L^\infty(\mathbb{T})$  on  $H$  such that

$$(2.1) \quad U\pi(f)U^{-1} = \pi(f(z^N)), \quad (f \in L^\infty(\mathbb{T}))$$

(here, by  $f(z^N)$  we mean the map  $z \mapsto f(z^N)$ ).

A wavelet representation is called *normal* if for any sequence  $(f_n)_{n \in \mathbb{N}}$  which converges pointwise a.e. to a function  $f \in L^\infty(\mathbb{T})$  and such that  $\|f_n\|_\infty \leq M$ ,  $n \in \mathbb{N}$  for some  $M > 0$ , the sequence  $\{\pi(f_n)\}$  converges to  $\pi(f)$  in the strong operator topology.

Sometimes we call  $U$  the dilation and  $T := \pi(z)$  the translation of the wavelet representation (here  $z$  indicates the identity function on  $\mathbb{T}$ ,  $z \mapsto z$ ).

**Example 2.2.** The main example of a wavelet representation is the classical one:  $H = L^2(\mathbb{R})$ ,

$$U\xi(x) = \frac{1}{\sqrt{N}}\xi\left(\frac{x}{N}\right), \quad (\xi \in L^2(\mathbb{R})),$$

and  $\pi$  is defined by its Fourier transform

$$\widehat{\pi}(f)(\xi) = f\xi, \quad (f \in L^\infty(\mathbb{T}), \xi \in L^2(\mathbb{R})).$$

In particular

$$\widehat{T}\xi(x) = e^{-ix}\xi(x), \text{ so } T(\xi)(x) = \xi(x-1), \quad (\xi \in L^2(\mathbb{R}), x \in \mathbb{R}).$$

We denote this normal wavelet representation by  $\mathfrak{R}_0$ .

**Example 2.3.** If  $(H_i, U_i, \pi_i)$  are (normal) wavelet representations,  $i \in \{1, \dots, n\}$ , then  $(\oplus_{i=1}^n H_i, \oplus_{i=1}^n U_i, \oplus_{i=1}^n \pi_i)$  is a (normal) wavelet representation called the direct sum of the given wavelet representations.

**Example 2.4.** We call *cycle* a set  $\{z_1, \dots, z_p\}$  of distinct points in  $\mathbb{T}$ , such that  $z_1^N = z_2$ ,  $z_2^N = z_3$ ,  $\dots$ ,  $z_p^N = z_1$ .  $p$  is called the length of the cycle.  $\{1\}$  is called the trivial cycle.

Let  $(H, U, \pi)$  be a (normal) wavelet representation. Let  $C := \{z_1, \dots, z_p\}$  be a cycle and  $\alpha_1, \dots, \alpha_p \in \mathbb{T}$ . Define

$$H_{C,\alpha} := \underbrace{H \oplus H \oplus \dots \oplus H}_{p \text{ times}}$$

and, for  $f \in L^\infty(\mathbb{T})$ ,  $\xi_1, \dots, \xi_p \in H$ ,

$$\begin{aligned} U_{C,\alpha}(\xi_1, \dots, \xi_p) &:= (\alpha_1 U\xi_2, \alpha_2 U\xi_3, \dots, \alpha_{p-1} U\xi_p, \alpha_p U\xi_1), \\ \pi_{C,\alpha}(f)(\xi_1, \dots, \xi_p) &:= (\pi(f(z_1 z))\xi_1, \pi(f(z_2 z))\xi_2, \dots, \pi(f(z_p z))\xi_p). \end{aligned}$$

Then  $\tilde{\pi}_{C,\alpha} := (H_{C,\alpha}, U_{C,\alpha}, \pi_{C,\alpha})$  is a (normal) wavelet representation which we call the *cyclic amplification* of  $\tilde{\pi}$  with cycle  $C$  and modulation  $\alpha$ . We leave it to the reader to check that this is indeed a (normal) wavelet representation (see also [Dut2]).

Note that the cyclic amplification with the trivial cycle and  $\alpha_1 = 1$  is the initial wavelet representation.

When  $\alpha_1 = \dots = \alpha_p = 1$  we will use also the notation  $\tilde{\pi}_C := \tilde{\pi}_{C,\alpha}$ .

**Example 2.5.** If  $C$  is a cycle of length  $p$  and  $\alpha_1, \dots, \alpha_p$  are in  $\mathbb{T}$ , we denote by

$$\mathfrak{R}_{C,\alpha} = (L^2(\mathbb{R})_{C,\alpha}, U_{C,\alpha}, \pi_{C,\alpha})$$

the cyclic amplification  $(\mathfrak{R}_0)_{C,\alpha}$  of the main representation  $\mathfrak{R}_0$ . When all  $\alpha_i$  are 1, we use the shorter notation

$$\mathfrak{R}_C = (L^2(\mathbb{R})_C, U_C, \pi_C).$$

**Example 2.6.** The wavelet representation which is of main importance to us in this paper is the direct sum of wavelet representations associated to several cycles. That is, if  $C_1, \dots, C_n$  are distinct cycles and  $\alpha_1, \dots, \alpha_n$  are some finite sets of numbers in  $\mathbb{T}$ , then let  $C := C_1 \cup \dots \cup C_n$ ,  $\alpha = (\alpha_1, \dots, \alpha_n)$  and define

$$\mathfrak{R}_{C,\alpha} := \mathfrak{R}_{C_1,\alpha_1} \oplus \dots \oplus \mathfrak{R}_{C_n,\alpha_n},$$

which we will call the wavelet representation associated to the cycles  $C_1, \dots, C_n$  and the numbers  $\{\alpha_i\}$ .

### 3. Scaling vectors and compactly supported super-wavelets

We define now the key concepts of a multiresolution analysis and scaling vectors. The definition generalize the existent ones on  $L^2(\mathbb{R})$  (see [Dau92]).

**Definition 3.1.** Let  $(H, U, \pi)$  be a wavelet representation. A *multiresolution analysis* (MRA) is a sequence  $(V_n)_{n \in \mathbb{Z}}$  of closed subspaces of  $H$  with the properties:

- (i)  $V_n \subset V_{n+1}$ .
- (ii)  $\overline{\bigcup_{n \in \mathbb{Z}} V_n} = H$ .
- (iii)  $\bigcap_{n \in \mathbb{Z}} V_n = \{0\}$ .
- (iv)  $U(V_n) = V_{n-1}$ .
- (v) There is a  $\varphi \in V_0$  such that  $\{T^k \varphi \mid k \in \mathbb{Z}\}$  is an orthonormal basis for  $V_0$ .

A vector  $\varphi \in H$  for which there exists a MRA such that (i)–(v) hold, with  $\varphi$  as in (v), is called an *orthogonal scaling vector*.

Before we give a characterization for orthogonal scaling vectors, we need the following:

**Definition 3.2.** Let  $\tilde{\pi} = (H, U, \pi)$  be a normal wavelet representation and  $v_1, v_2 \in H$ . The Radon-Nikodym derivative  $h_{v_1, v_2}$  of the linear functional

$$f \mapsto \langle \pi(f)v_1 \mid v_2 \rangle \quad (f \in L^\infty(\mathbb{T})),$$

with respect to the Haar measure  $\mu$  on  $\mathbb{T}$ , is called the *correlation function* of  $v_1$  and  $v_2$ . We also use the notation  $h_v := h_{v,v}$ . We can rewrite this as

$$\langle \pi(f)v_1 \mid v_2 \rangle = \int_{\mathbb{T}} f h_{v_1, v_2} d\mu, \quad (f \in L^\infty(\mathbb{T})).$$

The existence of the correlation function is guaranteed by the fact that the representation  $\pi$  is normal.

If  $\mathcal{S}$  is a set of operators on a Hilbert space (such as  $U$  plus  $\pi(f)$  in the case of a wavelet representation), then we denote by  $\mathcal{S}'$  its commutant (i.e., the set of all operators that commute with all operators in  $\mathcal{S}$ ).

We proceed towards the theorem that gives a characterization of orthogonal scaling vectors. The scaling vector will satisfy a scaling equation ((ii) in the next theorem), which relates the scaling vector to a *low-pass filter*  $m_0 \in L^\infty(\mathbb{T})$ . In order to obtain a MRA, some nondegeneracy conditions must be imposed on  $m_0$ . When  $m_0$  is degenerate, a residual subspace appears as the intersection of the multiresolution subspaces  $V_n$  (see [BraJo97]).

**Definition 3.3.** A function  $m_0 \in L^\infty(\mathbb{T})$  is called *degenerate* if  $|m_0(z)| = 1$  for a.e.  $z \in \mathbb{T}$  and there exists a measurable function  $\xi : \mathbb{T} \rightarrow \mathbb{T}$  and a  $\lambda \in \mathbb{T}$  such that

$$m_0(z)\xi(z^N) = \lambda\xi(z), \quad (z \in \mathbb{T}).$$

**Theorem 3.4.** Let  $\tilde{\pi} = (H, U, \pi)$  be a wavelet representation and  $\varphi \in H$ . Then  $\varphi$  is an orthogonal scaling vector if and only if the following conditions are satisfied:

- (i) [Orthogonality] The correlation function of  $\varphi$  is  $h_\varphi = 1$  a.e.
- (ii) [Scaling equation] There exists a nondegenerate  $m_0 \in L^\infty(\mathbb{T})$  such that  $U\varphi = \pi(m_0)\varphi$ .
- (iii) [Cyclicity] There is no proper projection  $p$  in  $\tilde{\pi}'$  such that  $p\varphi = \varphi$ .

**Proof.** The ideas of the proof are similar to the case of  $L^2(\mathbb{R})$ , so we will only sketch them and refer also to [Dau92], [HeWe] and [BraJo] for more details.

The orthogonality of the translates is equivalent to the fact that the correlation function of  $\varphi$  has Fourier coefficients  $\delta_0$ , i.e.,  $h_\varphi$  is constant 1. The scaling equation is equivalent to the fact that  $UV_0 \subset V_0$ ; the nondegeneracy of  $m_0$  is equivalent to  $\cap V_n = \{0\}$  (see [BraJo97] and [Jor01, Theorem 5.6]).  $\cup V_n$  is dense if and only if  $\varphi$  is cyclic for  $\{U, \pi\}$  which is in turn equivalent to condition (iii).  $\square$

**Definition 3.5.** Let  $(H, \pi, U)$  be a wavelet representation. A vector  $\varphi \in H$  is called a *scaling vector* with filter  $m_0$  if it satisfies conditions (ii) and (iii) in Theorem 3.4, but here we allow  $m_0$  to be degenerate.

**Remark 3.6.** Note that condition (iii) is equivalent to

$$\{U^{-m}\pi(f)\varphi \mid m \geq 0, f \in L^\infty(\mathbb{T})\} \text{ is dense in } H,$$

since the scaling equation (ii) in 3.4 and the covariance relation (2.1) are satisfied.

We will apply the characterization theorem to the instance described in Example 2.6. When applied to the classical wavelet representation on  $L^2(\mathbb{R})$  we obtain a theorem similar to the one in [HeWe], Chapter 7.

**Corollary 3.7.** Let  $\mathfrak{R}_{C,\alpha}$  be the wavelet representation in Example 2.6. Denote by  $e^{-i\theta_{i,j}}$  the  $j$ -th point of the cycle  $C_i$ . Then  $\varphi_C = \varphi_{C_1} \oplus \varphi_{C_2} \oplus \cdots \oplus \varphi_{C_n}$  is an orthogonal scaling function for this wavelet representation if and only if the following conditions are satisfied:

- (i)

$$\sum_{i=1}^n \sum_{k=1}^{p_i} (\text{Per}(|\widehat{\varphi}_{C_i,k}|^2))(\xi - \theta_{i,k}) = 1 \quad \text{for a.e. } \xi \in \mathbb{R}.$$

- (ii) *There exists a function  $m_0 \in L^\infty(\mathbb{T})$  such that for a.e.  $\xi \in \mathbb{R}$  and for all  $i \in \{1, \dots, n\}$ :*

$$\alpha_{i,1} \sqrt{N} \widehat{\varphi}_{C_i,2}(N\xi) = m_0(\theta_{i,1} + \xi) \widehat{\varphi}_{C_i,1}(\xi),$$

$$\alpha_{i,2} \sqrt{N} \widehat{\varphi}_{C_i,3}(N\xi) = m_0(\theta_{i,2} + \xi) \widehat{\varphi}_{C_i,2}(\xi),$$

$$\vdots$$

$$\alpha_{i,p_i} \sqrt{N} \widehat{\varphi}_{C_i,1}(N\xi) = m_0(\theta_{i,p_i} + \xi) \widehat{\varphi}_{C_i,p_i}(\xi).$$

- (iii) *For each  $i \in \{1, \dots, n\}, j \in \{1, \dots, p_i\}$ ,  $\widehat{\varphi}_{C_i}$  does not vanish on any subset  $E$  of  $\mathbb{R}$  invariant under dilations by  $N^{p_i}$  (i.e.,  $N^{p_i}E = E$ ) of positive measure.*

**Proof.** The formula in (i) is just the correlaton function for  $h_\varphi$ . Applying the Fourier transform to the scaling equation, one obtains (ii). The projections in the commutant of this representation are given by sets which are invariant under multiplication by  $N^{p_i}$  (see [Dut2], lemma 2.14). The fact that  $m_0$  is nondegenerate is automatic. Indeed, suppose not, then  $|m_0(z)| = 1$  for a.e.  $z \in \mathbb{T}$  so, for all  $i \in \{1, \dots, n\}$ ,

$$\sqrt{N^{p_i}} |\widehat{\varphi}_{C_i,1}(N^{p_i}\xi)| = |\widehat{\varphi}_{C_i,1}(\xi)|, \quad (\xi \in \mathbb{R}).$$

We will conclude that  $\widehat{\varphi}_{C_i,1}$  must be 0. If for some  $a > 0$  there is a subset  $E$  of  $[-N^{p_i}, -1] \cup [1, N^{p_i}]$  of positive measure such that  $|\widehat{\varphi}_{C_i,1}(\xi)| \geq a$  for  $\xi \in E$ , then

$$|\widehat{\varphi}_{C_i,1}(\xi)| \geq \frac{a}{\sqrt{N^{kp_i}}}, \quad \text{for } \xi \in N^{kp_i}E, k \in \mathbb{Z}.$$

But then this contradicts the integrability of  $\widehat{\varphi}_{C_i,1}$ :

$$\int_{\mathbb{R}} |\widehat{\varphi}_{C_i,1}(\xi)|^2 d\xi \geq \sum_{k \in \mathbb{Z}} \frac{a^2}{N^{kp_i}} N^{kp_i} \lambda(E) = \infty. \quad \square$$

As soon as a scaling function is given for a wavelet representation, the construction of wavelets follows the procedure described for the  $L^2(\mathbb{R})$ -case in [Dau92]. We present below the required ingredients in an abstract version. For a proof look in [Dut3, Proposition 5.1].

**Proposition 3.8.** *Let  $\tilde{\pi}$  be a normal wavelet representation having an orthogonal scaling function  $\varphi$  with nondegenerate filter  $m_0$ . Denote by  $(V_n)_{n \in \mathbb{Z}}$  the associated MRA. Assume that there are given the “high-pass filters”  $m_1, \dots, m_{N-1} \in L^\infty(\mathbb{T})$  satisfying*

$$(3.1) \quad \frac{1}{\sqrt{N}} \begin{pmatrix} m_0(z) & m_0(\rho z) & \dots & m_0(\rho^{N-1}z) \\ m_1(z) & m_1(\rho z) & \dots & m_1(\rho^{N-1}z) \\ \vdots & \vdots & \ddots & \vdots \\ m_{N-1}(z) & m_{N-1}(\rho z) & \dots & m_{N-1}(\rho^{N-1}z) \end{pmatrix} \text{ is unitary for a.e. } z \in \mathbb{T},$$

( $\rho = e^{\frac{2\pi i}{N}}$ ), and define  $\psi_i \in H$  by

$$(3.2) \quad \psi_i =: U^{-1} \pi(m_i) \varphi, \quad (i \in \{1, \dots, N-1\}).$$

Then

$$(3.3) \quad \{T^k \psi_i \mid k \in \mathbb{Z}, i \in \{1, \dots, N-1\}\} \text{ is an orthonormal basis for } V_1 \ominus V_0,$$

$$(3.4) \quad \{U^m T^n \psi_i \mid m, n \in \mathbb{Z}, i \in \{1, \dots, N-1\}\} \text{ is an orthonormal basis for } H.$$



Consider a Lipschitz function  $m_0$  on  $\mathbb{T}$  that satisfies the following conditions:

$$(3.5) \quad R_{m_0}1 = 1,$$

$$(3.6) \quad m_0 \text{ has a finite number of zeros.}$$

We call a cycle  $C = \{z_1 = e^{-i\theta_1}, \dots, z_p = e^{-i\theta_p}\}$  an  $m_0$ -cycle if  $|m_0(z_k)| = \sqrt{N}$  for all  $k \in \{1, \dots, p\}$ .

We assume in addition that:

$$(3.7) \quad \text{There is at least one } m_0\text{-cycle.}$$

We have shown in [Dut2], Proposition 2.13 that for each  $m_0$ -cycle one can construct a scaling vector with filter  $m_0$  in the wavelet representation  $\mathfrak{R}_{C,\alpha}$  where  $\alpha_k = m_0(z_k)/\sqrt{N}$ , ( $k \in \{1, \dots, p\}$ ). The scaling vector is defined in the Fourier space as an infinite product:

$$(3.8) \quad \widehat{\varphi}_{C,k}(x) := \prod_{l=1}^{\infty} \frac{\overline{\alpha}_{k-l} m_0\left(\frac{x}{N^l} + \theta_{k-l}\right)}{\sqrt{N}}, \quad (x \in \mathbb{R}), \varphi_C := (\varphi_{C,1}, \dots, \varphi_{C,p}),$$

(here the subscripts of  $\theta$  are considered modulo  $p$ , i.e.,  $\theta_0 = \theta_p$ ,  $\theta_{-1} = \theta_{p-1}$ ,  $z_{p+2} = z_2$ , etc.). When  $m_0$  is a trigonometric polynomial each component of the scaling vector is compactly supported (see the argument used in [Dau92], Lemma 6.2.2).

As explained in [Dut2], Proposition 2.13, the function

$$h_C(\theta) := h_{\varphi_C}(\theta) = \sum_{k=1}^p \text{Per} |\widehat{\varphi}_{C,k}|^2(\theta - \theta_k), \quad (\theta \in [-\pi, \pi]),$$

is nonnegative, harmonic for  $R_{m_0}$  and Lipschitz (trigonometric polynomial when  $m_0$  is one).

Moreover  $h_C$  is constant 1 on the  $m_0$ -cycle  $C$  and it is constant 0 on every other  $m_0$ -cycle. This makes  $h_C$  linearly independent for different  $m_0$ -cycles.

By Remark 5.2.4 in [BraJo], the dimension of the eigenspace

$$\{h \in C(\mathbb{T}) \mid R_{m_0}h = h\}$$

is equal to the number of  $m_0$ -cycles so that

$$\{h_C \mid C \text{ is an } m_0\text{-cycle}\}$$

is a basis for this eigenspace. This shows that (ii) and (iii) in the next theorem are equivalent. Using Theorem 2.16 in [Dut2] we obtain:

**Theorem 3.9.** *Let  $m_0$  be a Lipschitz function satisfying (3.5), (3.6) and (3.7) and let  $C_1, \dots, C_n$  be distinct  $m_0$ -cycles. Denote by  $\alpha_{ij}$  the coefficients  $m_0(z_{ij})/\sqrt{N}$  where for each fixed  $i$  the numbers  $z_{ij}$  run through the cycle  $C_i$ . Define  $\varphi_{C_i}$  as in (3.8) ( $i \in \{1, \dots, n\}$ ). Then  $\varphi_{C,\alpha} := \varphi_{C_1} \oplus \dots \oplus \varphi_{C_n}$  is a scaling vector for the wavelet representation  $\mathfrak{R}_{C,\alpha} := \mathfrak{R}_{C_1,\alpha_1} \oplus \dots \oplus \mathfrak{R}_{C_n,\alpha_n}$ . Moreover, the following affirmations are equivalent:*

- (i)  $\varphi_{C,\alpha}$  is an orthogonal scaling vector.
- (ii) [Cohen's condition] The number of  $m_0$ -cycles is  $n$ .
- (iii) [Lawton's condition] The dimension of the eigenspace

$$\{h \in C(\mathbb{T}) \mid R_{m_0}h = h\}$$

is  $n$ .

If  $n$  is smaller than the number of  $m_0$ -cycles and  $\psi_1, \dots, \psi_{N-1}$  are defined as in Proposition 3.8, then

$$(3.9) \quad \{U^j T^k \psi_i \mid i \in \{1, \dots, N-1\}, j, k \in \mathbb{Z}\},$$

is a normalized tight frame for  $L^2(\mathbb{R})^n$ .

**Proof.** For the last statement we use the following argument: if  $n$  is equal to the number of cycles then  $\varphi$  is an orthogonal scaling function so the family in (3.9) is an orthonormal basis. If we project this orthonormal basis onto some of the components corresponding to a choice of a subset of  $m_0$ -cycles, we get the normalized tight frame that corresponds to the case when  $n$  is smaller than the number of cycles.  $\square$

**Remark 3.10.** Theorem 3.9 generalizes some well-known results of A. Cohen and W. Lawton for the classical wavelet representation  $\mathfrak{R}_0$  (see [Co90, Law91a, Dau92]). However much more is true: each normalized tight frame wavelet obtained in the way described before is in fact a projection of an orthonormal super-wavelet, the one which resides in the larger space of the larger wavelet representation

$$\mathfrak{R}_{C_1, \alpha_1} \oplus \dots \oplus \mathfrak{R}_{C_m, \alpha_m}$$

which takes into consideration *all* the  $m_0$ -cycles.

Of course, it is clear that this projection lies in the commutant of the larger representation and so all operators are intertwined.

## 4. Convergence of the cascade algorithm

We saw that the orthogonal scaling vector must satisfy the scaling equation

$$U\varphi = \pi(m_0)\varphi.$$

Generically, there is no closed formula for the scaling function/vector. For some choices of the filter  $m_0$ , the scaling function can have a fractal nature (see e.g., [BraJo]). In applications, numerical values are needed. The cascade algorithm provides approximates of the scaling vector  $\varphi$  in the  $L^2$ -norm. It starts with a well chosen function  $\psi^{(0)}$ , and, by iteration of the refinement (or cascade) operator

$$\begin{aligned} M &:= U^{-1}\pi(m_0), \\ \psi^{(n+1)} &:= M\psi^{(n)}, \end{aligned}$$

it produces a sequence which converges towards the scaling vector  $\varphi$ :

$$(4.1) \quad \lim_{n \rightarrow \infty} \|\varphi - \psi^{(n)}\| = 0.$$

The question here is: how should  $\psi^{(0)}$  be chosen so that the algorithm is convergent to the scaling vector? We will answer this question in this section. The result is related to spectral properties of the transfer operator  $R_{m_0}$ . For a treatment of the  $L^2(\mathbb{R})$  case we refer to [BraJo99].

We will consider  $m_0$ , a Lipschitz function on  $\mathbb{T}$  satisfying (3.5), (3.6) and (3.7). Let  $C_1, \dots, C_n$  be all the  $m_0$ -cycles and define the wavelet representation  $\mathfrak{R}_{C, \alpha}$  and the orthogonal scaling vector  $\varphi_{C, \alpha}$  as in Theorem 3.9.

We will use the notation  $\{z_{i1} = e^{-i\theta_{i1}}, z_{i2} = e^{-i\theta_{i2}}, \dots, z_{ip_i} = e^{-i\theta_{ip_i}}\}$  for the points in the  $m_0$ -cycle, and the vectors in the Hilbert space of this representation are of the form

$$(\xi_{ij})_{ij}, \quad \text{with } \xi_{ij} \in L^2(\mathbb{R}), i \in \{1, \dots, n\}, j \in \{1, \dots, p_i\}.$$

Consider a starting vector  $\psi^{(0)}$  in  $L^2(\mathbb{R})_{C,\alpha}$  and define inductively

$$\psi^{(n+1)} := U_{C,\alpha}^{-1} \pi_{C,\alpha}(m_0) \psi^{(n)}.$$

The result is:

**Theorem 4.1.** *If*

$$(4.2) \quad \text{Per } |\widehat{\psi}_{ij}^{(0)}|^2(\theta_{ij} - \theta_{i'j'}) = \delta_{ii'} \delta_{jj'},$$

for all  $i, i' \in \{1, \dots, n\}, j \in \{1, \dots, p_i\}, j' \in \{1, \dots, p_{i'}\}$ , and

$$(4.3) \quad \widehat{\psi}_{ij}^{(0)}(2k\pi) = \delta_k, \quad (i \in \{1, \dots, n\}, j \in \{1, \dots, p_i\}, k \in \mathbb{Z}),$$

then

$$(4.4) \quad \lim_{n \rightarrow \infty} \|\varphi - \psi^{(n)}\| = 0.$$

A simple choice of such a  $\psi^{(0)}$  is

$$(4.5) \quad \psi_{ij} = \frac{1}{L} \chi_{[0,L)}, \quad (i \in \{1, \dots, n\}, j \in \{1, \dots, p_i\}),$$

where  $L$  is a positive integer with the property that  $z_{ij}^L = 1$  for all  $i, j$ .

The proof of the theorem will involve the use of some spectral properties of the transfer operator  $R_{m_0}$  regarded as an operator on  $C(\mathbb{T})$ .

By Theorem 3.4.4. in [BraJo],  $R_{m_0}$  has a finite number of eigenvalues  $\lambda_1, \dots, \lambda_p$  of modulus 1 and  $R_{m_0}$  has a decomposition

$$(4.6) \quad R_{m_0} = \sum_{i=1}^p \lambda_i T_{\lambda_i} + S,$$

where  $T_{\lambda_i}$  and  $S$  are bounded operators on  $C(\mathbb{T})$  such that

$$(4.7) \quad T_{\lambda_i}^2 = T_{\lambda_i}, \quad T_{\lambda_i} T_{\lambda_j} = 0, \text{ for } i \neq j, \quad T_{\lambda_i} S = S T_{\lambda_i} = 0.$$

There is a constant  $M > 0$  such that

$$(4.8) \quad \|S^n\| \leq M, \quad (n \in \mathbb{N}).$$

By Proposition 4.4.4 in [BraJo], the following conditions (4.9) and (4.10) are equivalent:

$$(4.9) \quad \lim_{n \rightarrow \infty} R_{m_0}^n(g) = T_1(g)$$

$$(4.10) \quad T_{\lambda_i}(g) = 0, \quad \text{for all } \lambda_i \neq 1.$$

By Theorem 2.17 and Corollary 2.18 in [Dut2], a  $\lambda$  with  $|\lambda| = 1$  is an eigenvalue for  $R_{m_0}$  if and only if there is an  $i \in \{1, \dots, n\}$  such that  $\lambda^{p_i} = 1$ .

For such a  $\lambda$ , the eigenspace is described as follows: for each  $i \in \{1, \dots, n\}$  with the property that  $\lambda^{p_i} = 1$ , one can define a continuous function  $h_i^\lambda$  with

$$R_{m_0} h_i^\lambda = \lambda h_i^\lambda,$$

so that for different indices  $i$  the functions  $h_i^\lambda$  are linearly independent and, in fact, they form a basis for the eigenspace

$$\{h \in C(\mathbb{T}) \mid R_{m_0} h = \lambda h\}.$$

Define the discrete measures

$$(4.11) \quad \nu_i^\lambda := \frac{1}{p_i} \sum_{j=1}^{p_i} \lambda^{j-1} \delta_{z_{ij}}, \quad (i \in \{1, \dots, n\}, \lambda \in \mathbb{T}, \lambda^{p_i} = 1),$$

where  $\delta_z$  is the Dirac measure at  $z$ . Then

$$(4.12) \quad T_\lambda(f) = \sum_{i \in \{1, \dots, n\} \text{ with } \lambda^{p_i} = 1} \nu_i^\lambda(f) h_i^\lambda.$$

**Proof of Theorem 4.1.** We will omit the subscript  $C, \alpha$  to simplify the notation. We have the following relation between successive approximations: for  $f \in L^\infty(\mathbb{T})$

$$\begin{aligned} \int_{\mathbb{T}} f h_{\varphi - \psi^{(n+1)}} d\mu &= \langle \pi(f)(\varphi - \psi^{(n+1)}) \mid \varphi - \psi^{(n+1)} \rangle \\ &= \langle \pi(f) U^{-1} \pi(m_0)(\varphi - \psi^{(n)}) \mid U^{-1} \pi(m_0)(\varphi - \psi^{(n)}) \rangle \\ &= \langle \pi(f(z^N) m_0)(\varphi - \psi^{(n)}) \mid \pi(m_0)(\varphi - \psi^{(n)}) \rangle \\ &= \int_{\mathbb{T}} f(z^N) |m_0|^2 h_{\varphi - \psi^{(n)}} d\mu \\ &= \int_{\mathbb{T}} f(z) R_{m_0} h_{\varphi - \psi^{(n)}} d\mu. \end{aligned}$$

Thus, as  $f$  is arbitrary,

$$h_{\varphi - \psi^{(n+1)}} = R_{m_0} h_{\varphi - \psi^{(n)}}.$$

This implies by induction that

$$\begin{aligned} \langle \varphi - \psi^{(n)} \mid \varphi - \psi^{(n)} \rangle &= \int_{\mathbb{T}} h_{\varphi - \psi^{(n)}} d\mu \\ &= \int_{\mathbb{T}} R_{m_0} h_{\varphi - \psi^{(n-1)}} d\mu = \dots = \int_{\mathbb{T}} R_{m_0}^n (h_{\varphi - \psi^{(0)}}) d\mu. \end{aligned}$$

So

$$(4.13) \quad \|\varphi - \psi^{(n)}\|^2 = \int_{\mathbb{T}} R_{m_0}^n (h_{\varphi - \psi^{(0)}}) d\mu.$$

Note that

$$h_{\varphi - \psi^{(0)}} = h_\varphi - 2 \operatorname{Re} h_{\varphi, \psi^{(0)}} + h_{\psi^{(0)}},$$

because, for all real-valued  $f \in L^\infty(\mathbb{T})$ ,

$$\langle \pi(f)(\varphi - \psi^{(0)}) \mid \varphi - \psi^{(0)} \rangle = \langle \pi(f)\varphi \mid \varphi \rangle - 2 \operatorname{Re} \langle \pi(f)\varphi \mid \psi^{(0)} \rangle + \langle \pi(f)\psi^{(0)} \mid \psi^{(0)} \rangle.$$

We know that  $h_\varphi = 1$  because  $\varphi$  is an orthogonal scaling vector.

We see that the correlation functions are

$$(4.14) \quad h_{\varphi, \psi^{(0)}}(\theta) = \sum_{i=1}^n \sum_{j=1}^{p_i} \operatorname{Per} \left( \widehat{\varphi}_{ij} \overline{\widehat{\psi}_{ij}^{(0)}} \right) (\theta - \theta_{ij}),$$

$$(4.15) \quad h_{\psi^{(0)}}(\theta) = \sum_{i=1}^n \sum_{j=1}^{p_i} \operatorname{Per} |\widehat{\psi}_{ij}^{(0)}|^2 (\theta - \theta_{ij}).$$

We have to compute  $T_\lambda(h_{\varphi, \psi^{(0)}})$  and  $T_\lambda(h_{\psi^{(0)}})$  which amounts to computing the values of  $h_{\varphi, \psi^{(0)}}$  and  $h_{\psi^{(0)}}$  on the  $m_0$ -cycles. For this we will use the inequality

$$(4.16) \quad \left| \text{Per} \left( \widehat{\varphi}_{ij} \overline{\widehat{\psi}_{ij}^{(0)}} \right) (\theta) \right|^2 \leq \text{Per} |\widehat{\varphi}_{ij}|^2(\theta) \text{Per} |\widehat{\psi}_{ij}^{(0)}|^2(\theta).$$

Also, we know (see [Dut2], Proposition 2.13 and its proof) that the function

$$g_{ij}(\theta) := \text{Per} |\widehat{\varphi}_{ij}|^2(\theta - \theta_{ij}), \quad (i \in \{1, \dots, n\}, j \in \{1, \dots, p_i\}),$$

has

$$(4.17) \quad g_{ij}(\theta_{kl}) = \delta_{ik} \delta_{jl},$$

and also

$$(4.18) \quad \widehat{\varphi}_{ij}(2k\pi) = \delta_k \quad (k \in \mathbb{Z}).$$

Then, inserting (4.2), (4.3) in (4.15), we get

$$(4.19) \quad h_{\psi^{(0)}}(\theta_{kl}) = \text{Per} |\widehat{\psi}_{kl}^{(0)}|^2(0) = 1.$$

Using in (4.14) the relations (4.16), (4.17) and then (4.3) and (4.18), we have

$$(4.20) \quad h_{\varphi, \psi^{(0)}}(\theta_{kl}) = \text{Per} \left( \widehat{\varphi}_{kl} \overline{\widehat{\psi}_{kl}^{(0)}} \right) (0) = 1.$$

With (4.19) and (4.20) in (4.12) and (4.11), it follows that

$$T_\lambda(h_{\psi^{(0)}}) = \delta_{\lambda,1} \cdot 1 \quad T_\lambda(\text{Re } h_{\varphi, \psi^{(0)}}) = \delta_{\lambda,1} \cdot 1.$$

Therefore (4.10) is satisfied and

$$\lim_{n \rightarrow \infty} R_{m_0}^n(h_{\psi^{(0)}}) = \lim_{n \rightarrow \infty} R_{m_0}^n(\text{Re } h_{\varphi, \psi^{(0)}}) = 1$$

which implies, with (4.13), that

$$\lim_{n \rightarrow \infty} \|\varphi - \psi^{(n)}\|^2 = 0.$$

It only remains to check that the vector  $\psi^{(0)}$  given in Equation (4.5) satisfies (4.2) and (4.3).

$$\widehat{\psi}_{ij}^{(0)}(\xi) = e^{-iL\xi/2} \frac{\sin(L\xi/2)}{L\xi/2},$$

hence

$$\widehat{\psi}_{ij}^{(0)}\left(\frac{2k\pi}{L}\right) = \delta_k, \quad (k \in \mathbb{Z}).$$

Since  $z_{ij}^L = 1$ , we see that  $e^{-iL(\theta_{ij} - \theta_{i'j'})} = 1$  so  $\theta_{ij} - \theta_{i'j'} = \frac{2k_0\pi}{L}$  for some  $k_0 \in \mathbb{Z}$ . Therefore, if  $(i, j) \neq (i', j')$  then  $k_0 \neq 0 \pmod L$  and

$$\text{Per} |\widehat{\psi}_{ij}^{(0)}|^2(\theta_{ij} - \theta_{i'j'}) = \sum_{k \in \mathbb{Z}} |\widehat{\psi}_{ij}^{(0)}|^2 \left( \frac{2k_0\pi + 2kL\pi}{L} \right) = 0.$$

Also

$$\text{Per} |\widehat{\psi}_{ij}^{(0)}|^2(0) = 1$$

and

$$\widehat{\psi}_{ij}^{(0)}(2k\pi) = \delta_k, \quad (k \in \mathbb{Z}). \quad \square$$

## 5. Examples

**Example 5.1.** Consider the wavelet representation  $\mathfrak{R}_0$  and suppose  $m_0 \in L^\infty(\mathbb{T})$  is a filter that has an orthogonal scaling function  $\varphi \in L^2(\mathbb{R})$  for this representation.

Define a new filter  $\tilde{m}_0 \in L^\infty(\mathbb{T})$  as follows: let  $p$  be a positive integer which is prime with  $N$ , and define

$$\tilde{m}_0(z) = m_0(z^p), \quad (z \in \mathbb{T}).$$

Because  $p$  and  $N$  are mutually prime, the  $p$ -th roots of unity,  $\{z \in \mathbb{T} \mid z^p = 1\}$  split into several disjoint cycles (the map  $z \mapsto z^N$  is a bijection on  $\{z \in \mathbb{T} \mid z^p = 1\}$ ); for example, if  $N = 2$ ,  $p = 9$  and  $\rho_k = e^{-i\frac{2k\pi}{p}}$ ,  $i \in \{0, \dots, 8\}$  are the 9-th roots of 1, then the cycles are

$$\{\rho_0\}, \quad \{\rho_1, \rho_2, \rho_4, \rho_8, \rho_7, \rho_5\}, \quad \{\rho_3, \rho_6\}.$$

Let  $C_1, \dots, C_n$  be these cycles

$$C_1 \cup \dots \cup C_n = \{z \in \mathbb{T} \mid z^p = 1\}.$$

We show that  $\tilde{m}_0$  has an orthogonal scaling vector for the wavelet representation

$$\mathfrak{R}_{C_1} \oplus \dots \oplus \mathfrak{R}_{C_n},$$

namely

$$\tilde{\varphi} := (\tilde{\varphi}_0, \dots, \tilde{\varphi}_p), \quad \tilde{\varphi}_i(x) = \frac{1}{p} \varphi\left(\frac{x}{p}\right), \quad (x \in \mathbb{R}, i \in \{0, \dots, p-1\}).$$

We have to verify the conditions of Corollary 3.7. For this we first write the conditions that are satisfied by the scaling function  $\varphi$ :

$$(5.1) \quad \text{Per} |\hat{\varphi}|^2(x) = 1, \quad (x \in \mathbb{R}).$$

$$(5.2) \quad \sqrt{N} \hat{\varphi}(Nx) = m_0(x) \hat{\varphi}(x), \quad (x \in \mathbb{R}).$$

(5.3) There is no  $N$ -invariant set of positive measure such that  $\hat{\varphi}$  vanishes on it.

Having these, we check the conditions for  $\tilde{\varphi}$ .

The orthogonality condition can be restated in this case as

$$\sum_{j=0}^{p-1} \text{Per} |\hat{\tilde{\varphi}}|^2\left(x - \frac{2j\pi}{p}\right) = 1 \quad (x \in \mathbb{R}).$$

This holds because

$$\begin{aligned} \sum_{j=0}^{p-1} \text{Per} |\hat{\tilde{\varphi}}|^2\left(x - \frac{2j\pi}{p}\right) &= \sum_{i=0}^{p-1} \sum_{k \in \mathbb{Z}} |\hat{\varphi}|^2(px - 2j\pi + 2k\pi p) \\ &= \sum_{k \in \mathbb{Z}} |\hat{\varphi}|^2(px + 2k\pi) = \text{Per} |\hat{\varphi}|^2(px) = 1 \quad (x \in \mathbb{R}), \end{aligned}$$

where we used (5.1) in the last equality.

For the scaling equation we have to check that

$$\sqrt{N} \hat{\varphi}(pNx) = m_0\left(p\left(\frac{2j\pi}{p} + x\right)\right) \hat{\varphi}(px), \quad (x \in \mathbb{R}, j \in \{0, \dots, p-1\}),$$

which is clear from (5.2).

Suppose now that the cyclicity condition is not satisfied. Then some of the components of  $\tilde{\varphi}$  that correspond to one of the cycles do not satisfy the cyclicity condition. Let  $l$  be the length of this cycle. Then there is an  $N^l$ -invariant set of positive measure, call it  $E$ , such that  $\widehat{\varphi}(px)$  vanishes on  $E$ .

Since  $\tilde{\varphi}$  satisfies the scaling equation, it follows that  $\widehat{\varphi}(px)$  vanishes also on  $NE, N^2E, \dots, N^{l-1}E$ .

Take

$$A = \frac{1}{p}(E \cup NE \cup \dots \cup N^{l-1}E).$$

Then  $A$  is  $N$ -invariant, of positive measure, and  $\widehat{\varphi}(x)$  vanishes on  $A$ . This contradicts (5.3).

In conclusion,  $\tilde{\varphi}$  is indeed an orthogonal scaling vector with filter  $\tilde{m}_0$  for the wavelet representation

$$\mathfrak{R}_{C_1} \oplus \dots \oplus \mathfrak{R}_{C_n}.$$

The next example shows that for the scale  $N = 2$ , no matter how we choose the cycles  $C = C_1 \cup C_2 \cup \dots \cup C_p$ , there exists a MRA, orthogonal super-wavelet for the representation  $\mathfrak{R}_C$ .

**Example 5.2.** Let  $C_1, C_2, \dots, C_p$  be 2-cycles and let  $C = C_1 \cup C_2 \cup \dots \cup C_p$ . We will construct an orthogonal scaling vector  $\varphi$  for the wavelet representation

$$\mathfrak{R}_{C_1} \oplus \dots \oplus \mathfrak{R}_{C_p}.$$

The following definitions for  $x \in \mathbb{R}$  will be used in this example:

(i)  $x$  is called a cycle point if there is  $c$  in  $C$  such that

$$x \equiv \theta \pmod{2\pi}, \text{ where } e^{-i\theta} = c.$$

(ii)  $x$  is called a supplement if  $x - \pi$  is a cycle point.

(iii)  $x$  is called a main point if it is a cycle point or a supplement.

(iv)  $x$  is called mid-point if  $x = \frac{a+b}{2}$ , with  $a, b$  consecutive main points.

(v)  $x$  is called a cycle midpoint if  $x = \frac{a+b}{2}$  with  $a, b$  consecutive cycle points.

(Here, when we say ‘‘consecutive’’, we refer to the order on the real line ).

For  $z \in C, z = e^{-i\theta_0}, \theta_0 \in [-\pi, \pi]$ , define

$$\widehat{\varphi}_z(\theta) = \chi_{\left[\frac{a(\theta_0)+\theta_0}{2}, \frac{\theta_0+b(\theta_0)}{2}\right]}(\theta + \theta_0),$$

where  $a(\theta_0), \theta_0, b(\theta_0)$  are consecutive cycle points. It is easy to check that

$$\sum_{z=e^{-i\theta_0} \in C} \text{Per} |\widehat{\varphi}_z|^2(\theta - \theta_0) = 1, \quad (\theta \in \mathbb{R}).$$

Hence  $\varphi$  defined by its Fourier transform

$$\widehat{\varphi} = \oplus_{z \in C} \widehat{\varphi}_z = \oplus_{i=1}^p \oplus_{z \in C_i} \widehat{\varphi}_z =: \oplus_{i=1}^p \widehat{\varphi}_{C_i}$$

is a good candidate for an orthogonal scaling vector corresponding to  $C$  built out of orthogonal scaling vectors corresponding to each  $C_i$ . Next let us define the filter  $m_0$ :

$$m_0 = \sqrt{2} \sum_{\theta_0 \in [-\pi, \pi], \text{ cycle point}} \chi_{\left(\left[\frac{c(\theta_0)+\theta_0}{2}, \frac{\theta_0+d(\theta_0)}{2}\right] \cap [-\pi, \pi]\right)},$$

where for the cycle point  $\theta_0 \in [-\pi, \pi]$ ,  $c(\theta_0), \theta_0, d(\theta_0)$  are consecutive main points. We will now check the scaling equation for the above defined  $\varphi$  and filter  $m_0$ . It

suffices to show that for two consecutive elements  $z_0 = e^{i\theta_0}, z_1 = e^{i\theta_1}$  of a cycle  $C_i$  (i.e.,  $z_0^2 = z_1$ , or equivalently  $2\theta_0 \equiv \theta_1 \pmod{2\pi}$ ) the following holds:

$$\sqrt{2}\widehat{\varphi}_{z_1}(2\theta) = m_0(\theta + \theta_0)\widehat{\varphi}_{z_0}(\theta), \text{ a.e. } \theta \in \mathbb{R}$$

or, equivalently

$$(5.4) \quad \sqrt{2}\widehat{\varphi}_{z_1}(2\theta - 2\theta_0) = m_0(\theta)\widehat{\varphi}_{z_0}(\theta - \theta_0), \text{ a.e. } \theta \in \mathbb{R}.$$

Suppose  $\widehat{\varphi}_{z_0}(\theta - \theta_0) = \chi_{[\frac{\alpha+\theta_0}{2}, \frac{\beta+\theta_0}{2}]}(\theta)$ , where  $\alpha < \theta_0 < \beta$  are consecutive cycle points. By the definition of  $m_0$ , it follows that there are consecutive main points  $a < \theta_0 < b$  such that

$$(5.5) \quad m_0(\theta)\widehat{\varphi}_{z_0}(\theta - \theta_0) = \sqrt{2}\chi_{[\frac{a+\theta_0}{2}, \frac{b+\theta_0}{2}]}(\theta).$$

Actually  $a$  is either  $\alpha$  or the first supplement on the left of  $\theta_0$  and  $b$  is either  $\beta$  or the first supplement on the right of  $\theta_0$ . Suppose  $\widehat{\varphi}_{z_1}(\theta - \theta_1) = \chi_{[\frac{a_1+\theta_1}{2}, \frac{b_1+\theta_1}{2}]}(\theta)$ , with  $a_1 < \theta_1 < b_1$  consecutive cycle points. Then

$$\widehat{\varphi}_{z_1}(\theta) = \chi_{[\frac{a_1-\theta_1}{2}, \frac{b_1-\theta_1}{2}]}(\theta)$$

and

$$(5.6) \quad \widehat{\varphi}_{z_1}(2\theta - 2\theta_0) = \chi_{[\frac{a_1-\theta_1}{2}, \frac{b_1-\theta_1}{2}]}(2\theta - 2\theta_0).$$

Note that, under the map  $x \mapsto 2x$ , the consecutive main points  $a < \theta_0 < b$  are mapped into consecutive cycle points. Since  $2\theta_0 \equiv \theta_1 \pmod{2\pi}$ , and because a translation by an integer multiple of  $2\pi$  maps consecutive cycle points to consecutive cycle points, it follows in particular that there exists  $k \in \mathbb{Z}$  such that  $a_1 = 2a + 2k\pi$ ,  $\theta_1 = 2\theta_0 + 2k\pi$  and  $b_1 = 2b + 2k\pi$ . We have:

$$\begin{aligned} \frac{a_1 - \theta_1}{2} \leq 2\theta - 2\theta_0 \leq \frac{b_1 - \theta_1}{2} &\Leftrightarrow \\ \frac{2a + 2k\pi - (2\theta_0 + 2k\pi)}{2} \leq 2\theta - 2\theta_0 \leq \frac{2b + 2k\pi - (2\theta_0 + 2k\pi)}{2} &\Leftrightarrow \\ \frac{a + \theta_0}{2} \leq \theta \leq \frac{b + \theta_0}{2}, & \end{aligned}$$

and, with (5.5), (5.6), the relation (5.4) is obtained.

The cyclicity condition is automatically satisfied because all  $\widehat{\varphi}_z$  contain a neighborhood of 0. Consequently  $\varphi$  is an orthogonal scaling vector with filter  $m_0$ .

We can use Example 5.2 to obtain some information about the distribution of cycles. The idea is that when  $m_0$  is the characteristic function of some intervals, we can obtain a Cohen condition as in Theorem 3.9.

**Proposition 5.3.** *Suppose  $m_0$  is of the form*

$$m_0 = \sqrt{N}\chi_E,$$

where  $E \subset [-\pi, \pi]$  is a union of intervals such that none of the endpoints of these intervals lies on a cycle. Assume moreover that for some distinct cycles  $C_1, \dots, C_n$ , the wavelet representation

$$\mathfrak{R}_C := \mathfrak{R}_{C_1} \oplus \dots \oplus \mathfrak{R}_{C_n}$$

has an orthogonal scaling vector with filter  $m_0$ . Then every cycle  $D$  disjoint from  $C_1 \cup \dots \cup C_n$  must have a point in  $[-\pi, \pi] \setminus E$ .



**Proof.** Since there is an orthogonal scaling vector with filter  $m_0$ , the condition

$$R_{m_0}1 = 1$$

must be satisfied.

Suppose there is a cycle  $D = \{e^{-i\theta_1}, \dots, e^{-i\theta_p}\}$ , different from the given ones such that  $D$  is contained in  $E$ . Because the endpoint of the intervals of  $E$  are not on cycles, each point of  $D$  lies in the interior of  $E$ . Then one can construct a scaling vector with filter  $m_0$  for the wavelet representation  $\mathfrak{R}_D$ ,  $\widehat{\varphi}_{D,k}$  defined as in (3.8). The fact that  $\widehat{\varphi}_{D,k}$  is a well-defined  $L^2(\mathbb{R})$  function can be proved using the arguments in [Dut2], Proposition 2.13 and [Dau92], Lemma 6.2.1.

The scaling equation can be verified instantly and, since each point of  $D$  lies in the interior of one of the intervals of  $m_0$ , it follows that  $\widehat{\varphi}_{D,k}$  is 1 in a neighborhood of 0 so the cyclicity condition of Corollary 3.7 is also clear.

But then, having a scaling vector with filter  $m_0$  in the wavelet representation  $\mathfrak{R}_D$  means that  $\mathfrak{R}_D$  is the wavelet representation associated to the harmonic function  $h_{\widehat{\varphi}_D}$ , the correlation function of  $\varphi_D$  ([Jor01, Theorem 2.4]).

$h_{\widehat{\varphi}_{D,k}}$  must be bounded for the following reason: clearly  $\widehat{\varphi}_{D,k}$  is either 1 or 0. Also, it is impossible to have an  $x \in \mathbb{R}$  with  $\widehat{\varphi}_{D,k}(x) = 1$  and  $\widehat{\varphi}_{D,k}(2l_0\pi) = 1$  for some integer  $l_0 \neq 0$ . Indeed, otherwise we would have (from (3.8)):

$$m_0\left(\frac{x}{N^l} + \theta_{k-l}\right) = \sqrt{N}, \quad m_0\left(\frac{x + 2l_0\pi}{N^l} + \theta_{k-l}\right) = \sqrt{N}, \quad (l \in \mathbb{Z}, l \geq 1).$$

Write  $l_0 = N^r q$  with  $r, q \in \mathbb{N}$ ,  $q$  not divisible by  $N$ . Then

$$m_0\left(\frac{x}{N^{r+1}} + \theta_{k-l}\right) = \sqrt{N}, \quad m_0\left(\frac{x}{N^{r+1}} + \theta_{k-l} + \frac{2q\pi}{N}\right) = \sqrt{N},$$

which contradicts  $R_{m_0}1 = 1$ .

Thus, if  $\widehat{\varphi}_{D,k}(x) = 1$  then  $\widehat{\varphi}_{D,k}(x + 2k\pi) = 0$  for all integers  $k \neq 0$ . This implies that  $\text{Per} |\widehat{\varphi}_{D,k}|^2(x) \leq 1$  so

$$h_{\widehat{\varphi}_{D,k}}(x) = \sum \text{Per} |\widehat{\varphi}_{D,k}|^2(x - \theta_k) \leq p.$$

Having these, with [Dut1, Theorem 2.4], there exists some positive operator  $S$  that commutes with  $U_C$  and  $\pi_C$  such that the correlation function  $h_{S\varphi_C, \varphi_C}$  is  $h_{\widehat{\varphi}_{D,k}}$ . Then the correlation function of the vector  $S^{1/2}\varphi_C$  is  $h_{\widehat{\varphi}_{D,k}}$ . Mapping  $U_D^{-n}\pi_D(f)\varphi_D$  to  $U_C^{-n}\pi_C(f)S^{1/2}\varphi_C$ , we get a nontrivial operator which intertwines the representations associated to  $D$  and  $C$ . But this is impossible because the representations are disjoint [Dut2, Lemma 2.14].

In conclusion the assumption was erroneous so the cycle  $C$  must intersect the complement of  $E$ .  $\square$

We illustrate now some particular cases of the filters, orthogonal scaling vectors, and wavelets given in Example 5.2. Here the scale is  $N = 2$ . Since the low-pass filter  $m_0$  is just a characteristic function, the corresponding high-pass filter  $m_1$  can be chosen as the characteristic function of the complement of the set that gives  $m_0$ , and (3.1) holds. Then the wavelet is defined as in (3.2).

**Example 5.4.** Consider the cycle

$$\begin{aligned} C &= \{e^{-\frac{2\pi i}{7}}, e^{-\frac{4\pi i}{7}}, e^{-\frac{6\pi i}{7}}\}, \\ m_0 &= \sqrt{2}\chi_{[-\pi, -\frac{11\pi}{14}] \cup [\frac{3\pi}{14}, \pi]}, \\ \widehat{\varphi}_1 &= \chi_{[-\frac{4\pi}{7}, \frac{\pi}{7}]}, \quad \widehat{\varphi}_2 = \chi_{[-\frac{\pi}{7}, \frac{2\pi}{7}]}, \quad \widehat{\varphi}_3 = \chi_{[-\frac{2\pi}{7}, \frac{4\pi}{7}]}. \end{aligned}$$

Then  $(\varphi_1, \varphi_2, \varphi_3)$  is an orthogonal scaling vector for the wavelet representation  $\mathfrak{R}_C$  with filter  $m_0$ .

The high-pass filter is

$$m_1 = \sqrt{2}\chi_{[-\frac{11\pi}{14}, \frac{3\pi}{14}]},$$

and the orthogonal wavelet is  $(\psi_1, \psi_2, \psi_3)$  with

$$\widehat{\psi}_1 = \chi_{[\frac{\pi}{7}, \frac{8\pi}{7}]}, \quad \widehat{\psi}_2 = \chi_{[-\frac{8\pi}{7}, -\frac{\pi}{7}]}, \quad \widehat{\psi}_3 = 0.$$

**Example 5.5.** Consider the cycles

$$\begin{aligned} C_1 &= \{e^{-\frac{2\pi i}{5}}, e^{-\frac{4\pi i}{5}}, e^{-\frac{6\pi i}{5}}, e^{-\frac{8\pi i}{5}}\}, \quad C_2 = \{e^{-\frac{2\pi i}{3}}, e^{-\frac{4\pi i}{3}}\}, \\ m_0 &= \sqrt{2}\chi_E, \end{aligned}$$

where

$$\begin{aligned} E &:= \left[-\pi, -\frac{19\pi}{30}\right] \cup \left[-\frac{15\pi}{30}, -\frac{11\pi}{30}\right] \cup \left[\frac{11\pi}{30}, \frac{15\pi}{30}\right] \cup \left[\frac{19\pi}{30}, \pi\right], \\ \widehat{\varphi}_1^1 &= \chi_{[-\frac{6\pi}{15}, \frac{2\pi}{15}]}, \quad \widehat{\varphi}_2^1 = \chi_{[-\frac{\pi}{15}, \frac{3\pi}{15}]}, \quad \widehat{\varphi}_3^1 = \chi_{[-\frac{2\pi}{15}, \frac{6\pi}{15}]}, \quad \widehat{\varphi}_4^1 = \chi_{[-\frac{3\pi}{15}, \frac{\pi}{15}]}, \\ \widehat{\varphi}_1^2 &= \chi_{[-\frac{2\pi}{15}, \frac{\pi}{15}]}, \quad \widehat{\varphi}_2^2 = \chi_{[-\frac{\pi}{15}, \frac{2\pi}{15}]}. \end{aligned}$$

Then  $((\varphi_1^1, \dots, \varphi_4^1), (\varphi_1^2, \varphi_2^2))$  is an orthogonal scaling vector for the wavelet representation  $\mathfrak{R}_{C_1} \oplus \mathfrak{R}_{C_2}$  with filter  $m_0$ .

The high-pass filter is

$$m_1 = \sqrt{2}\chi_{E_1},$$

where

$$E_1 := \left[-\frac{19\pi}{30}, -\frac{15\pi}{30}\right] \cup \left[-\frac{11\pi}{30}, \frac{11\pi}{30}\right] \cup \left[\frac{15\pi}{30}, \frac{19\pi}{30}\right]$$

and the orthogonal wavelet is  $((\psi_1^1, \dots, \psi_4^1), (\psi_1^2, \psi_2^2))$  with

$$\begin{aligned} \widehat{\psi}_1^1 &= \widehat{\psi}_3^1 = 0, \quad \widehat{\psi}_2^1 = \chi_{[-\frac{12\pi}{15}, -\frac{\pi}{15}] \cup [\frac{3\pi}{15}, \frac{4\pi}{15}]}, \quad \widehat{\psi}_4^1 = \chi_{[-\frac{4\pi}{15}, -\frac{3\pi}{15}] \cup [\frac{\pi}{15}, \frac{12\pi}{15}]}, \\ \widehat{\psi}_1^2 &= \chi_{[\frac{\pi}{15}, \frac{4\pi}{15}]}, \quad \widehat{\psi}_2^2 = \chi_{[-\frac{4\pi}{15}, -\frac{\pi}{15}]}. \end{aligned}$$

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