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The graph traces of finite graphs and applications to tracial states of C^*-algebras

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Abstract. We determine the extreme points of the set of graph traces of norm one for any finite graph E satisfying Condition (K). We also describe an application to the space of tracial states on the graph C^*-algebra.

Contents

1. Introduction

If E is a directed graph, the graph algebra $C^*(E)$ is the universal C^*-algebra generated by a collection of partial isometries satisfying certain relations determined by E. This paper will focus on the space of tracial states on $C^*(E)$, denoted $T(C^*(E))$, which is the set of all positive, linear functionals $\tau : C^*(E) \to \mathbb{C}$ of norm one such that $\tau(ab) = \tau(ba)$ for all $a, b \in C^*(E)$. If E is a finite graph satisfying the so-called Condition (K), to be defined below, then $T(C^*(E))$ is isomorphic to the set of graph traces of norm one defined on E, denoted $T(E)$ [5]. Since $T(E)$ is a compact convex set in a certain locally convex topological vector space, the Krein–Millman Theorem implies that $T(E)$ is the closed convex hull of its extreme points [4]. The objective of this paper is to provide an effective way of computing the extreme points of $T(E)$ so that $T(E)$ and subsequently $T(C^*(E))$ will be completely and effectively described when E is a finite graph satisfying Condition (K).

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In more detail, we show that the set extreme points of $T(E)$, where E is a finite graph satisfying Condition (K), consists of a finite number of graph traces corresponding to certain sinks of the graph. These graph traces are easy to compute in general, and an example is provided to show how to find them. As a result, one can use these graph traces to determine the space of tracial states on $C^*(E)$.

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2. Preliminaries

A (directed) graph $E = (E^0, E^1, r, s)$ consists of a countable set E^0 of vertices, a countable set E^1 of edges, and maps $r, s : E^1 \to E^0$ identifying the range and source of each edge. While our focus will be on finite graphs, i.e., graphs where the sets of vertices and edges are finite, we shall make this assumption only when it is relevant to the proof of our main result, Theorem 4.4.

A vertex $v \in E^0$ is called a *sink* if $|s^{-1}(v)| = 0$, and v is called an *infinite emitter* if $|s^{-1}(v)| = \infty$. A *singular vertex* is a vertex which is either a sink or an infinite emitter. We will let $S_E \subseteq E^0$ denote the set of all of the sinks of E.

A *path* is a finite sequence of edges $\alpha = e_1 \dots e_n$ with $r(e_i) = s(e_{i+1})$ for $1 \leq i \leq n-1$; we let $|\alpha| = n$ denote the path's length. Let E^n be the set of all paths in E of length n and let $E^* := \bigcup_{n=0}^{\infty} E^n$. We extend r and s to E^* in the natural way: $s(e_1 \dots e_n) = s(e_1)$ and $r(e_1 \dots e_n) = r(e_n)$.

If v and w are vertices, we say that $v \geq w$ if there is a path $e_1 \dots e_n$ with $s(e_1) = v$ and $r(e_n) = w$. Let $n(w, v) := \#\{\alpha \in E^* : s(\alpha) = w \text{ and } r(\alpha) = v\}$ and $n(v) := \#\{\alpha \in E^* : r(\alpha) = v\}$. A *loop* is a path $e_1 \dots e_n$ with $s(e_1) = r(e_n)$ and we call $s(e_1)$ the *base point* of the loop. A loop $e_1 \dots e_n$ is *simple* if $s(e_i) \neq s(e_1)$ for all $i \in \{2, 3, \dots, n\}$.

Definition 2.1. A graph E satisfies Condition (K) if no vertex is the base point of exactly one simple loop.

Definition 2.2. If E is a graph, then a graph trace on E is a function $g : E^0 \to [0, \infty)$ with the following two properties:

(1) For any nonsingular vertex $v \in E^0$ we have $g(v) = \sum_{\{e \in E^1 : s(e) = v\}} g(r(e))$.
(2) For any infinite emitter $v \in E^0$ and any finite collection of edges $e_1, \dots, e_n \in s^{-1}(v)$ we have $g(v) \geq \sum_{i=1}^{n} g(r(e_i))$.

We define the norm of a graph trace g to be the (possibly infinite) value $||g|| := \sum_{v \in E^0} g(v)$. Let $T(E)$ denote the set of all graph traces of norm one. We will view $T(E)$ as a subset of the space $\mathbb{C}^{E^0}$ consisting of all $\mathbb{C}$-valued functions on E^0, endowed with the product topology. This space has the subbasis $\{N_{v,\epsilon}(g) : v \in E^0, \epsilon > 0, \text{ and } g \in T(E)\}$, where $N_{v,\epsilon}(g) := \{h \in T(E) : |h(v) - g(v)| < \epsilon\}$. Evidently, $T(E)$ is a convex subset of $\mathbb{C}^{E^0}$, and it is easy to see that $T(E)$ is compact in this topology. Indeed, $T(E)$ is a closed subset of $\overline{\mathbb{D}}^{E^0}$ (where $\overline{\mathbb{D}}$ is the closed unit disc in $\mathbb{C}$), which is a compact subset of $\mathbb{C}^{E^0}$.

Recall that an *extreme point* of a convex set X in a vector space is an element $x \in X$ such that if $x = tx_1 + (1-t)x_2$ for some points x_1 and x_2 in X and for some t, $0 < t < 1$, then either $x = x_1$ or $x = x_2$. As is customary, we write $\mathrm{ext}(X)$ to denote the set of all of the extreme points of X. From the Krein–Millman Theorem [4, pg. 70] we know that since $T(E)$ is a compact convex subset of a locally convex space it is the closed convex hull of its extreme points. Thus, if we know the extreme points of $T(E)$, then we know $T(E)$, especially when E is finite, since then any graph trace in $T(E)$ can be written as a convex combination of extreme points. Indeed, by Carathéodory's theorem, each point may be written as a convex combination of $|E^0| + 1$ or fewer extreme points in $T(E)$.

For a graph E, a *Cuntz–Krieger E-family* is a set of mutually orthogonal projections $\{p_v : v \in E^0\}$ and a set of partial isometries $\{s_e : e \in E^1\}$ with orthogonal ranges that satisfy the *Cuntz–Krieger relations*:

(1) $s_e^* s_e = p_{r(e)}$ for every $e \in E^1$;

(2) $s_e s_e^* \leq p_{s(e)}$ for every $e \in E^1$;

(3) $p_v = \sum_{\{e : s(e)=v\}} s_e s_e^*$ for every $v \in E^0$ that is not a singular vertex.

It is possible to associate a C^*-algebra with E; namely the C^*-algebra generated by a universal Cuntz–Krieger E-family, $C^*(E) := \{p_v, s_e : v \in E^0 \text{ and } e \in E^1\}$ [2, §1].

We have a map $\iota : T(C^*(E)) \to T(E)$ defined by

$$\iota(\tau)(v) = \tau(p_v) \qquad \text{where } v \in E^0 \text{ and } \tau \in T(C^*(E)).$$

It is shown in [5] that if E satisfies Condition (K) then ι is an isomorphism. In order to calculate $T(C^*(E))$ we use ι to map the extreme points of $T(E)$ to the extreme points of $T(C^*(E))$.

In this paper it will be shown that the extreme points of $T(E)$ consist of graph traces of the following type.

Definition 2.3. For a finite graph E with no loops and $v \in S_E$, define $g_v : E^0 \to [0, \infty)$ by

$$g_v(w) := \frac{n(w,v)}{n(v)} \quad \text{for } w \in E^0.$$

Note that if E is finite with no loops, then the term $n(w,v)$ is always finite and nonnegative and the term $n(v)$ is always finite and positive.

3. Finite graphs with no loops

Lemma 3.1. *If $v, w \in E^0$ with $v \geq w$, then $g(v) \geq g(w)$ for any graph trace g on E.*

Proof. If $v \geq w$ then there is a path $e_1 \dots e_n$ with $s(e_1) = v$ and $r(e_n) = w$. Since g is a graph trace it takes nonnegative values and thus

$$g(v) = \sum_{s(e)=v} g(r(e)) \geq g(r(e_1)) = \sum_{s(e)=r(e_1)} g(r(e)) \geq \cdots \geq g(r(e_n)) = g(w). \qquad \square$$

Lemma 3.2. *If $e_1 \dots e_n$ is a loop and g is a graph trace on E, then $g(s(e_1)) = g(s(e_i))$ for all $1 \leq i \leq n$.*

Proof. Since $e_1 \dots e_n$ is a loop we know that $s(e_1) = r(e_n)$ and that

$$s(e_1) \geq s(e_2) \geq \cdots \geq s(e_n) \geq r(e_n) = s(e_1).$$

Using Lemma 3.1 we have

$$g(s(e_1)) \geq g(s(e_2)) \geq \cdots \geq g(s(e_n)) \geq g(s(e_1))$$

from which it follows that $g(s(e_1)) = g(s(e_i))$ for all $1 \leq i \leq n$. □

Proposition 3.3. *Let $E = (E^0, E^1, r, s)$ be a graph that satisfies Condition* (K)*, let g be a graph trace on E, and let $e_1 \dots e_n$ be a loop in E. Then $g(s(e_i)) = 0$ for all $1 \leq i \leq n$.*

Proof. Let $v = s(e_i)$ for some $1 \leq i \leq n$. Since E satisfies Condition (K) and v is the base point of a loop, we know that v is the base point of two distinct loops $f_1 \dots f_m$ and $h_1 \dots h_l$. Because these two loops are distinct, there exists $1 \leq j \leq \min\{m, l\}$ for which $s(f_j) = s(h_j)$ but $f_j \neq h_j$.

Thus, using the definition of a graph trace and Lemma 3.2 we have

$$\begin{aligned}
g(v) &= g(s(f_j)) && \text{(by Lemma 3.2)}\\
&= \sum_{\{e \in E^1 : s(e) = s(f_j)\}} g(r(e)) && \text{(by Definition 2.2)}\\
&\geq g(r(f_j)) + g(r(h_j)) &&\\
&= 2g(v) && \text{(by Lemma 3.2).}
\end{aligned}$$

Since g has nonnegative values, this implies that $g(v) = 0$. □

Lemma 3.4. *Let E be a finite graph with no loops and let $S_E \subseteq E^0$ be the set of all sinks of E. If g_1 and g_2 are graph traces on E and $g_1(v) = g_2(v)$ for all $v \in S_E$, then $g_1 = g_2$.*

Proof. Assume that $g_1 \neq g_2$ and $g_1(v) = g_2(v)$ for all $v \in S_E$. Since $g_1 \neq g_2$ there exists $w \in E^0$ such that

$$g_1(w) \neq g_2(w).$$

Because $g_1(v) = g_2(v)$ for all $v \in S_E$ we know that $w \notin S_E$. By the definition of a graph trace we know that $g_i(w) = \sum_{\{e \in E^1 : s(e) = w\}} g_i(r(e))$ for $i = 1, 2$. Hence, there exists $e \in E^1$ with $s(e) = w$ and $g_1(r(e)) \neq g_2(r(e))$. Because E is a finite graph with no loops we are able to repeat this process and eventually end at a sink v_0 with $w \geq v_0$ and

$$g_1(v_0) \neq g_2(v_0).$$

Thus, it is not the case that $g_1(v) = g_2(v)$ for all $v \in S_E$ and we arrive at a contradiction. □

Lemma 3.5. *Let E be a finite graph with no loops. If $v \in S_E$, then $g_v \in T(E)$.*

Proof. If $w \in E^0$ and $w \neq v$ then $n(w, v) = \sum_{\{e \in E^1 : s(e) = w\}} n(r(e), v)$ since any path α from w to v must be of the form $e\beta$ for a unique $e \in s^{-1}(w)$ and a unique path β. Thus

$$\frac{n(w, v)}{n(v)} = \sum_{s(e) = w} \frac{n(r(e), v)}{n(v)}.$$

From which it follows that

$$g_v(w) = \sum_{s(e)=w} g_v(r(e))$$

and hence g_v is a graph trace on E. In order to show $g_v \in T(E)$ we first note that $n(v) = \sum_{w\in E^0} n(w,v)$. Dividing both sides by $n(v)$ gives

$$1 = \sum_{w\in E^0} \frac{n(w,v)}{n(v)} = \sum_{w\in E^0} g_v(w).$$

Therefore $g_v \in T(E)$ for all $v \in S_E$. □

Lemma 3.6. *Let E be a finite graph with no loops. If g is a graph trace on E, then $g(w) \geq n(w,v)g(v)$ for all $v, w \in E^0$.*

Proof. If there is no path from w to v then $n(w,v) = 0$ and the claim holds trivially. Thus we need only consider the case were $w \geq v$.
Let $w, v \in E^0$ with $w \geq v$. This will be a proof by induction on k, where

$$k := \max\ \{|\alpha| : \alpha \in E^* \text{ with } s(\alpha) = w \text{ and } r(\alpha) = v\ \}.$$

If $k = 0$, then $w = v$ and $n(w,v) = n(v,v) = 1$, and the lemma holds. For the inductive step, assume that the lemma holds for some particular $k \geq 0$. Choose $w \in E^0$ such that

$$\max\ \{|\alpha| : \alpha \in E^* \text{ with } s(\alpha) = w \text{ and } r(\alpha) = v\} = k + 1.$$

Let $\alpha_0 \in E^*$ with $s(\alpha_0) = w$, $r(\alpha_0) = v$, and $|\alpha_0| = k+1$. Write $\alpha_0 = e_0\beta_0$ for $e_0 \in E^1$ and $\beta_0 \in E^*$. Note that $|\beta_0| = k$.
Observe that

$$\max\ \{|\alpha| : \alpha \in E^* \text{ with } s(\alpha) = r(e_0) \text{ and } r(\alpha) = v\ \} = |\beta_0| = k.$$

By the inductive hypothesis we know that $g(r(e)) \geq n(r(e),v)g(v)$. Using the definition of a graph trace it follows that

$$g(w) = \sum_{s(e)=w} g(r(e)) \geq \sum_{s(e)=w} n(r(e),v)g(v) = g(v) \sum_{s(e)=w} n(r(e),v) \geq g(v)n(w,v).$$

By induction $g(w) \geq n(w,v)g(v)$ for all $v, w \in E^0$. □

Theorem 3.7. *If E is a finite graph with no loops, then*

$$\operatorname{ext}(T(E)) = \{g_v \in T(E) : v \in S_E\}.$$

Proof. Suppose that $v \in S_E$ and that $g_v \in T(E)$ is a graph trace such that $g_v = tg_1 + (1-t)g_2$ for $0 < t < 1$ and $g_1, g_2 \in T(E)$. In order to show that g_v is an extreme point we need only show that either $g_v = g_1$ or $g_v = g_2$. By Lemma 3.4 it suffices to show that the two graph traces agree at all of the sinks of E.

We first evaluate $g_v = tg_1 + (1-t)g_2$ at $v_0 \in S_E$ where $v_0 \neq v$. We know that since both v_0 and v are in S_E there is no path from v_0 to v, and hence $g_v(v_0) = 0$. So we now have

$$g_v(v_0) = 0 = tg_1(v_0) + (1-t)g_2(v_0).$$

Since both g_1 and g_2 are graph traces it follows that both $g_1(v_0)$ and $g_2(v_0)$ are nonnegative. Because $0 < t < 1$ it is clear that $g_1(v_0) = 0$ and $g_2(v_0) = 0$. Thus, $g_v(v_0) = g_1(v_0) = g_2(v_0)$ for all $v_0 \in S_E$ with $v_0 \neq v$.

Since $g_i \in T(E)$ for $i = 1, 2$ we know from Lemma 3.6 that

$$1 = \sum_{w \in E^0} g_i(w) \geq \sum_{\{w \in E^0 : w \geq v\}} g_i(w) \geq \sum_{\{w \in E^0 : w \geq v\}} n(w,v) g_i(v) = n(v) g_i(v),$$

from which it follows that

$$\frac{1}{n(v)} \geq g_i(v).$$

We shall now evaluate $g_v = tg_1 + (1-t)g_2$ at v. Using the definition of g_v we have

$$\begin{aligned}\frac{1}{n(v)} &= g_v(v) \\ &= tg_1(v) + (1-t)g_2(v) \\ &\leq tg_1(v) + (1-t)\frac{1}{n(v)} \\ &= tg_1(v) + \frac{1}{n(v)} - t\frac{1}{n(v)}.\end{aligned}$$

Subtracting $\frac{1}{n(v)}$ from both sides yields

$$\begin{aligned}0 &\leq tg_1(v) - t\frac{1}{n(v)} \\ \frac{1}{n(v)} &\leq g_1(v).\end{aligned}$$

So we now have

$$\frac{1}{n(v)} \leq g_1(v) \leq \frac{1}{n(v)}.$$

From this we conclude that $g_1(v) = \frac{1}{n(v)} = g_v(v)$. Therefore by Lemma 3.4, $g_1 = g_v$ because g_1 and g_v agree at each sink. Hence

$$\{g_v \in T(E) : v \in S_E\} \subseteq \text{ext}(T(E)).$$

To see the reverse inclusion, let $g \in T(E)$ and set $t_v = n(v)g(v)$ for $v \in S_E$. Then for any $w \in S_E$ we have

$$\begin{aligned}\left(\sum_{v \in S_E} t_v g_v\right)(w) &= \sum_{v \in S_E} t_v g_v(w) \\ &= t_w g_w(w) \\ &= t_w \frac{1}{n(w)} \\ &= g(w).\end{aligned}$$

Thus by Lemma 3.4 we have $g = \sum_{v \in S_E} t_v g_v$ and any element of $T(E)$ can be written as a convex combination of the g_v's. Hence

$$\text{ext}(T(E)) = \{g_v \in T(E) : v \in S_E\}. \qquad \square$$

4. Finite graphs with loops

In this section we extend the results of Section 3 to *all* finite graphs satisfying Condition (K).

If E is a (not necessarily finite) graph, then a subset $H \subseteq E^0$ is called *hereditary* if $s(e) \in H$ implies that $r(e) \in H$. If $H \subseteq E^0$ is hereditary, H is called *saturated* if whenever $v \in E^0$ with $0 < |s^{-1}(v)| < \infty$, then $r(s^{-1}(v)) \subseteq H$ implies that $v \in H$.

For a finite graph E, let $H := \{v \in E^0 : \; s(\alpha) \geq v$ for some loop $\alpha \in E^*\}$. It is clear that H is hereditary since if $s(e) \in H$ for some $e \in E^1$ then there exists a loop α such that $s(\alpha) \geq s(e)$. Thus by definition we know that $s(e) \geq r(e)$, and by transitivity we know that $s(\alpha) \geq r(e)$ which implies that $r(e) \in H$.

Let $H_0 = H$ and

$$H_n = H_{n-1} \cup \{v \in E^0 : \; 0 < |s^{-1}(v)| < \infty \text{ and } r(s^{-1}(v)) \subseteq H_{n-1}\}.$$

The set $\overline{H} = \bigcup_{n=0}^{\infty} H_n$ is called the saturation of H, and it is the smallest saturated hereditary set containing H. Let $E \backslash \overline{H}$ be the graph defined by

$$E \backslash \overline{H} := \left(E^0 \backslash \overline{H}, E^1 \backslash r^{-1}(\overline{H}), r|_{E^0 \backslash \overline{H}}, s|_{E^0 \backslash \overline{H}}\right).$$

Lemma 4.1. *Let E be a finite graph which satisfies Condition* (K) *and let g be a graph trace on E. Define the hereditary set*

$$H := \left\{v \in E^0 : s(\alpha) \geq v \textit{ for some loop } \alpha \in E^*\right\}.$$

If $\overline{H}$ is the saturation of H as defined above, then $g(w) = 0$ for all $w \in \overline{H}$.

Proof. Since $\overline{H} = \bigcup_{n=0}^{\infty} H_n$ it suffices to show that if $w \in H_n$ then $g(w) = 0$ for all nonnegative integers n. We shall prove this by induction on n.

If $w \in H_0$ then by definition there is a loop $\alpha \in E^*$ such that $s(\alpha) \geq w$. By Lemma 3.1 we know that $g(s(\alpha)) \geq g(w)$. But $g(s(\alpha)) = 0$ by Proposition 3.3 and it follows that $g(w) = 0$.

Assume that $g(w) = 0$ for all $w \in H_n$. Let $w \in H_{n+1}$; it follows that either $w \in H_n$ or $w \in \{v \in E^0 : \; 0 < |s^{-1}(v)| < \infty$ and $r(s^{-1}(v)) \subseteq H_n\}$. If $w \in H_n$ then by the inductive hypothesis $g(w) = 0$. If $w \in \{v \in E^0 : \; 0 < |s^{-1}(v)| < \infty$ and $r(s^{-1}(v)) \subseteq H_n\}$ then $r(s^{-1}(w)) \subseteq H_n$, from which it follows that that $g(r(e)) = 0$ for all $e \in s^{-1}(w) \subseteq H_n$. By the definition of a graph trace we have

$$g(w) = \sum_{\{e \in E^1 : s(e) = w\}} g(r(e)) = 0.$$

Therefore, $g(w) = 0$ for all $w \in H_{n+1}$. By induction we conclude that if $w \in H_n$ then $g(w) = 0$ for all nonnegative integers n. □

Definition 4.2. If E is a graph satisfying Condition (K) and $v \in S_{E \backslash \overline{H}}$, define $\widetilde{g}_v \in T(E)$ by

$$\widetilde{g}_v(w) = \begin{cases} g_v(w) & w \in E \backslash \overline{H} \\ 0 & w \in \overline{H} \end{cases}$$

where $g_v \in T(E \backslash \overline{H})$ is the graph trace of Definition 2.3. (Note that $\widetilde{g}_v$ is a graph trace since $\overline{H}$ is saturated and hereditary, and $||\widetilde{g}_v|| = ||g_v|| = 1$ so $\widetilde{g}_v \in T(E)$.)

We now prove that $T(E)$ may be identified with $T(E \backslash \overline{H})$.

Lemma 4.3. *Let E be a finite graph satisfying Condition* (K). *Define the hereditary set $H := \{v \in E^0 : \ s(\alpha) \geq v$ for some loop $\alpha \in E^*\}$. If $\overline{H}$ denotes its saturation, then $\phi : T(E) \to T(E\backslash\overline{H})$ defined by $\phi(g) = g|_{E^0\backslash\overline{H}}$ is a homeomorphism.*

Proof. We shall first show that ϕ is an injective function. For $g_1, g_2 \in T(E)$ assume that $\phi(g_1) = \phi(g_2)$. It follows that g_1 and g_2 agree on $E^0\backslash\overline{H}$. By Lemma 4.1 we know that any graph trace in $T(E)$ will be zero on $\overline{H}$. Thus, g_1 and g_2 agree on all of E and by definition $g_1 = g_2$. Therefore, ϕ is injective.

Given $g \in T(E\backslash\overline{H})$ define $\widetilde{g} \in T(E)$ by

$$\widetilde{g}(v) = \begin{cases} g(v) & v \in E\backslash\overline{H} \\ 0 & v \in \overline{H}. \end{cases}$$

It is clear that $\widetilde{g}$ is a graph trace and $||\widetilde{g}|| = ||g|| = 1$, so $\widetilde{g} \in T(E)$. Also $\phi(\widetilde{g}) = g$ so that ϕ is surjective. Since a continuous bijection between compact Hausdorff spaces is a homeomorphism, the result follows. □

Theorem 4.4. *If E is a finite graph satisfying Condition* (K) *then*

$$\text{ext}(T(E)) = \left\{\widetilde{g}_v \in T(E) : v \in S_{E\backslash\overline{H}}\right\}.$$

Proof. By Lemma 4.3 we have that $\phi : T(E) \to T(E\backslash\overline{H})$ is a homeomorphism. Thus, $\text{ext}(T(E)) = \phi^{-1}(\text{ext}(T(E\backslash\overline{H})))$. However,

$$\text{ext}(T(E\backslash\overline{H}) = \left\{g_v \in T(E) : v \in S_{E\backslash\overline{H}}\right\}$$

by Theorem 3.7. Hence

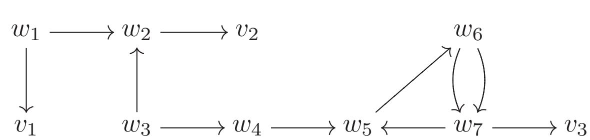
$$\begin{aligned}\phi^{-1}(\text{ext}(E\backslash\overline{H}))) &= \left\{\phi^{-1}(g_v) : v \in S_{E\backslash\overline{H}}\right\} \\ &= \left\{\widetilde{g}_v \in T(E) : v \in S_{E\backslash\overline{H}}\right\}.\end{aligned}$$

Therefore $\text{ext}(T(E)) = \left\{\widetilde{g}_v \in T(E) : v \in S_{E\backslash\overline{H}}\right\}$. □

From this it follows that $\text{ext}(T(C^*(E))) = \left\{\iota^{-1}(\widetilde{g}_v) \in T(C^*(E)) : v \in S_{E\backslash\overline{H}}\right\}$. And since $T(C^*(E))$ is a compact convex set we know by the Krein–Millman Theorem that every element in $T(C^*(E))$ is a convex combination of these extreme points.

5. An example

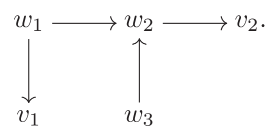
Let E be the graph shown below:



We see that E satisfies Condition (K) since no vertex is the base point of exactly one simple loop. Let $H := \{v \in E^0 : s(\alpha) \geq v$ for some loop $\alpha\}$. The set H is hereditary and we have

$$H = \{w_5, w_6, w_7, v_3\}.$$

Let $H_0 = H$ and $H_n = H_{n-1} \cup \{v \in E^0 : 0 < |s^{-1}(v)| < \infty \text{ and } r(s^{-1}(v)) \in H_{n-1}\}$. Using this definition we have $H_1 = H_0 \cup \{w_4\}$. Observe that $\overline{H} = H_0 \cup H_1$ is the saturation of H. By Lemma 4.3 we know that $T(E)$ is homeomorphic to $T(E\backslash\overline{H})$; thus we need only find the extreme points of $T(E\backslash\overline{H})$ which is shown here:



We shall now determine $\operatorname{ext}(T(E))$. We know from Theorem 4.4 that

$$\operatorname{ext}(T(E)) = \left\{\widetilde{g}_v \in T(E) : v \in S_{E\backslash\overline{H}}\right\} = \{\widetilde{g}_{v_1}, \widetilde{g}_{v_2}\}.$$

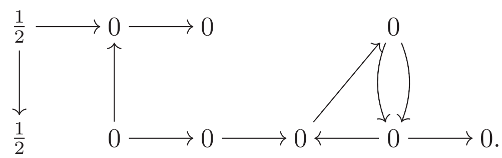
Thus, we need only compute $\widetilde{g}_{v_1}$ and $\widetilde{g}_{v_2}$. By the definition of $\widetilde{g}_{v_1}$ we know that

$$\widetilde{g}_{v_1}(w_1) = \frac{n(w_1, v_1)}{n(v_1)} = \frac{1}{2} \qquad \text{and} \qquad \widetilde{g}_{v_1}(v_1) = \frac{n(v_1, v_1)}{n(v_1)} = \frac{1}{2}.$$

And since there are no paths from $w_1, w_3,$ or v_2 to v_1 we have

$$\widetilde{g}_{v_1}(w_2) = \widetilde{g}_{v_1}(w_3) = \widetilde{g}_{v_1}(v_2) = 0.$$

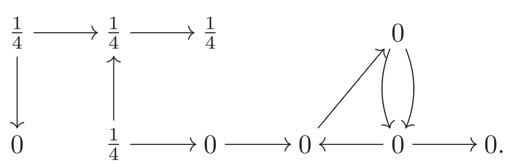
We represent $\widetilde{g}_{v_1}$ here:



In a similar way we use the definition of $\widetilde{g}_{v_2}$ to conclude that

$$\widetilde{g}_{v_2}(w_1) = \widetilde{g}_{v_2}(w_2) = \widetilde{g}_{v_2}(w_3) = \widetilde{g}_{v_2}(v_2) = \frac{1}{4}.$$

And since there are no paths from v_1 to v_2 we have $\widetilde{g}_{v_2}(v_1) = 0$. We represent $\widetilde{g}_{v_2}$ here:



Thus, the extreme points of E are $\widetilde{g}_{v_1}$ and $\widetilde{g}_{v_2}$, which are shown above, and any element of $T(E)$ is a convex combination of $\widetilde{g}_{v_1}$ and $\widetilde{g}_{v_2}$. Consequently, $T(E)$ may be viewed as a line segment between $\widetilde{g}_{v_1}$ and $\widetilde{g}_{v_2}$.

Since $T(E)$ has two extreme points we know that $T(C^*(E))$ does as well; namely $\iota^{-1}(\widetilde{g}_{v_1})$ and $\iota^{-1}(\widetilde{g}_{v_2})$, where $\iota : T(C^*(E)) \to T(E)$ is the map established in [5]. Consequently, every point $\tau \in T(C^*(E))$ is of the form

$$\tau = t\iota^{-1}(\widetilde{g}_{v_1}) + (1-t)\iota^{-1}(\widetilde{g}_{v_2})$$

where $0 \le t \le 1$.

References

[1] T. Bates, D. Pask, I. Raeburn and W. Szymański, *The C^*-algebras of row-finite graphs*, New York J. Math. **6** (2000), 307–324, MR1777234 (2001k:46084), Zbl 0976.46041.

[2] A. Kumjian, D. Pask, and I. Raeburn, *Cuntz–Krieger algebras of directed graphs*, Pacific J. Math. **184** (1998), 161–174, MR1626528 (99i:46049), Zbl 0917.46056.

[3] A. Kumjian, D. Pask, I. Raeburn, and J. Renault, *Graphs, groupoids, and Cuntz–Krieger algebras*, J. Funct. Anal. **144** (1997), 505–541, MR1432596 (98g:46083), Zbl 0929.46055.

[4] W. Rudin, *Functional analysis*, McGraw-Hill Series in Higher Mathematics, McGraw-Hill Book Co., New York-Dusseldorf-Johannesburg, 1973, MR0365062 (51 #1315), Zbl 0253.46001.

[5] M. Tomforde, *The ordered K_0-group of a graph C^*-algebra*, C.R. Math. Acad. Sci. Soc. R. Can. **25** (2003), 19–25, MR1962131 (2003m:46104), Zbl 1046.46052.

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