New York Journal of Mathematics

New York J. Math. **14** (2008) 379–392.

v_1-periodic homotopy groups of the Dwyer–Wilkerson space

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Abstract. The Dwyer–Wilkerson space DI(4) is the only exotic 2-compact group. We compute $v_1^{-1}\pi_*(\mathrm{DI}(4))$, its v_1-periodic homotopy groups.

Contents

1. Introduction

In [13], Dwyer and Wilkerson constructed a 2-complete space $B\mathrm{DI}(4)$, so named because its F_2-cohomology groups form an algebra isomorphic to the ring of rank-4 mod-2 Dickson invariants. Its loop space, called DI(4), has $H^*(\mathrm{DI}(4);F_2)$ finite. In [14], they then defined a p-compact group to be a pair (X, BX), such that $X = \Omega BX$ (hence X is redundant), BX is connected and p-complete, and $H^*(X;F_p)$ is finite. In [1], Andersen and Grodal proved that $(\mathrm{DI}(4), B\mathrm{DI}(4))$ is the only simple 2-compact group not arising as the 2-completion of a compact connected Lie group.

The p-primary v_1-periodic homotopy groups of a topological space X, defined in [12] and denoted $v_1^{-1}\pi_*(X)_{(p)}$ or just $v_1^{-1}\pi_*(X)$ if the prime is clear, are a first approximation to the p-primary homotopy groups. Roughly, they are a localization of the portion of the actual homotopy groups detected

Received June 7, 2007.

Mathematics Subject Classification. 55Q52, 57T20, 55N15.

Key words and phrases. v_1-periodic homotopy groups, p-compact groups, Adams operations, K-theory.

ISSN 1076-9803/08

by p-local K-theory. In [11], the second author completed a 13-year project, often in collaboration with the first author, of determining $v_1^{-1}\pi_*(X)_{(p)}$ for all compact simple Lie groups and all primes p.

In this paper, we determine the 2-primary groups $v_1^{-1}\pi_*(\mathrm{DI}(4))$. Here and throughout, $\nu(-)$ denotes the exponent of 2 in the prime factorization of the absolute value of an integer.

Theorem 1.1. *For any integer i, let $e_i = \min(21, 4 + \nu(i - 90627))$. Then*

$$v_1^{-1}\pi_{8i+d}(\mathrm{DI}(4)) \approx \begin{cases} \mathbb{Z}/2^{e_i} \oplus \mathbb{Z}/2 & d = 1 \\ \mathbb{Z}/2^{e_i} & d = 2 \\ 0 & d = 3, 4 \\ \mathbb{Z}/8 & d = 5 \\ \mathbb{Z}/8 \oplus \mathbb{Z}/2 & d = 6 \\ \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2 & d = 7, 8. \end{cases}$$

Since every v_1-periodic homotopy group is a subgroup of some actual homotopy group, this result implies that $\exp_2(\mathrm{DI}(4)) \geq 21$, i.e., some homotopy group of DI(4) has an element of order 2^{21}. It would be interesting to know whether this bound is sharp.

Our proof involves studying the spectrum $\Phi_1\mathrm{DI}(4)$ which satisfies

$$\pi_*(\Phi_1\mathrm{DI}(4)) \approx v_1^{-1}\pi_*(\mathrm{DI}(4)).$$

We will relate $\Phi_1\mathrm{DI}(4)$ to the 2-completed K-theoretic pseudosphere $T_{K/2}$ discussed in [8, 8.6]. We will prove the following surprising result, which was pointed out by Pete Bousfield.

Theorem 1.2. *There is an equivalence of spectra*

$$\Phi_1\mathrm{DI}(4) \simeq \Sigma^{725019} T_{K/2} \wedge M(2^{21}),$$

where $M(2^{21})$ is a mod 2^{21} Moore spectrum.

In Section 3, we will give the easy deduction of Theorem 1.1 from Theorem 1.2. As an immediate corollary of 1.2, we deduce that the 2^{21} bound on $\pi_*(\Phi_1\mathrm{DI}(4))$ is induced from a bound on the spectrum itself.

Corollary 1.3. *The exponent of the spectrum $\Phi_1\mathrm{DI}(4)$ is 2^{21}; i.e., $2^e 1_{\Phi_1\mathrm{DI}(4)}$ is null if and only if $e \geq 21$.*

In [5], Bousfield presented a framework that enables determination of the v_1-periodic homotopy groups of many simply-connected H-spaces X from their united K-theory groups and Adams operations. The intermediate step is $KO^*(\Phi_1 X)$. (All of our $K^*(-)$ and $KO^*(-)$-groups have coefficients in the 2-adic integers $\hat{\mathbb{Z}}_2$, which we omit from our notation.) Our first proof of Theorem 1.1 used Bousfield's exact sequence [5, 9.2] which relates $v_1^{-1}\pi_*(X)$ with $KO^*(\Phi_1 X)$, but the approach via the pseudosphere, which we present here, is stronger and more elegant. The insight for Theorem 1.2 is the observation that the two spectra have isomorphic Adams modules $KO^*(-)$.

In several earlier e-mails, Bousfield explained to the authors how the results of [5] should enable us to determine $KO^*(\Phi_1\mathrm{DI}(4))$. In Section 4, we present our account of these ideas of Bousfield. We thank him profusely for sharing his insights with us.

The other main input is the Adams operations in $K^*(B\mathrm{DI}(4))$. In [18], Osse and Suter showed that $K^*(B\mathrm{DI}(4))$ is a power series algebra on three specific generators, and gave some information toward the determination of the Adams operations. In private communication in 2005, Suter expanded on this to give explicit formulas for ψ^k in $K^*(B\mathrm{DI}(4))$. We are very grateful to him for sharing this information. In Section 2, we will explain these calculations and also how they then lead to the determination of $KO^*(\Phi_1\mathrm{DI}(4))$.

2. Adams operations

In this section, we present Suter's determination of ψ^k in $K^*(B\mathrm{DI}(4))$ and state a result, proved in Section 4, that allows us to determine $KO^*(\Phi_1\mathrm{DI}(4))$ from these Adams operations.

Our first result, communicated by Suter, is the following determination of Adams operations in $K^*(B\mathrm{DI}(4))$. An element of $K^*(X)$ is called *real* if it is in the image of $KO^*(X) \xrightarrow{c} K^*(X)$.

Theorem 2.1 (Suter)**.** *There is an isomorphism of algebras*

$$K^*(B\mathrm{DI}(4)) \approx \hat{\mathbb{Z}}_2[\![\xi_8, \xi_{12}, \xi_{24}]\!] \tag{2.2}$$

such that the generators are in $K^0(-)$ and are real, $\psi^{-1} = 1$, and the matrices of ψ^2 and ψ^3 on the three generators, mod decomposables, are

$$\Psi^2 \equiv \begin{pmatrix} 2^4 & 0 & 0 \\ -2 & 2^6 & 0 \\ 0 & -2 & 2^{14} \end{pmatrix}, \qquad \Psi^3 \equiv \begin{pmatrix} 3^4 & 0 & 0 \\ -3^3 & 3^6 & 0 \\ 36/527 & -3^5 \cdot 41/17 & 3^{14} \end{pmatrix}.$$

Proof. The subscripts of the generators indicate their "filtration," meaning the dimension of the smallest skeleton on which they are nontrivial. A standard property of Adams operations is that if ξ has filtration $2r$, then $\psi^k(\xi)$ equals $k^r\xi$ plus elements of higher filtration.

The isomorphism (2.2) is derived in [18, p. 184] along with the additional information that $4\xi_{24} - \xi_{12}^2$ has filtration 28, and

$$\begin{aligned} \xi_{12} &= \lambda^2(\xi_8) + 8\xi_8 \\ \xi_{24} &= \lambda^2(\xi_{12}) + 32\xi_{12} + c_1\xi_8^2 + c_2\xi_8^3 + c_3\xi_8\xi_{12}, \end{aligned} \tag{2.3}$$

for certain explicit even coefficients c_i.

The Atiyah–Hirzebruch spectral sequence easily shows that ξ_8 is real, since the 11-skeleton of $B\mathrm{DI}(4)$ equals S^8. Since $\lambda^2(c(\theta)) = c(\lambda^2(\theta))$, and products of real bundles are real, we deduce from (2.3) that ξ_{12} and ξ_{24} are also real. Since $tc = c$, where t denotes conjugation, which corresponds to ψ^{-1}, we obtain that the generators are invariant under ψ^{-1}, and hence so is all of $K^*(B\mathrm{DI}(4))$.

We compute Adams operations mod decomposables, writing $\equiv$ for equivalence mod decomposables. Because $4\xi_{24} - \xi_{12}^2$ has filtration 28, we obtain

$$\psi^k(\xi_{24}) \equiv k^{14}\xi_{24}. \tag{2.4}$$

Here we use, from [18, p. 183], that all elements of $K^*(B\mathrm{DI}(4))$ of filtration greater than 28 are decomposable. Equation (2.4) may seem surprising, since ξ_{24} has filtration 24, but there is a class ξ_{28} such that $4\xi_{24} - \xi_{12}^2 = \xi_{28}$, and we can have $\psi^k(\xi_{24}) \equiv k^{12}\xi_{24} + \alpha_k\xi_{28}$ consistently with (2.4).

Using (2.3) and that $\psi^2 \equiv -2\lambda^2$ mod decomposables, we obtain

$$\begin{aligned} \psi^2(\xi_8) &\equiv 2^4\xi_8 - 2\xi_{12} \\ \psi^2(\xi_{12}) &\equiv 2^6\xi_{12} - 2\xi_{24}, \end{aligned} \tag{2.5}$$

yielding the matrix Ψ^2 in the theorem.

Applying $\psi^2\psi^3 = \psi^3\psi^2$ to $\psi^3(\xi_{12}) \equiv 3^6\xi_{12} + \gamma\xi_{24}$ yields $-2 \cdot 3^6 + 2^{14}\gamma = 2^6\gamma - 2 \cdot 3^{14}$, from which we obtain $\gamma = -3^5 \cdot 41/17$. Applying the same relation to $\psi^3(\xi_8) = 3^4\xi_8 + \alpha\xi_{12} + \beta\xi_{24}$, coefficients of ξ_{12} yield $-2 \cdot 3^4 + \alpha \cdot 2^6 = 2^4\alpha - 2 \cdot 3^6$ and hence $\alpha = -3^3$. Now coefficients of ξ_{24} yield $-2\alpha + 2^{14}\beta = 2^4\beta - 2\gamma$ and hence $\beta = 36/527$. □

Let $\Phi_1(-)$ denote the functor from spaces to $K/2_*$-local spectra described in [5, 9.1], which satisfies $v_1^{-1}\pi_*X \approx \pi_*\tau_2\Phi_1X$, where $\tau_2\Phi_1X$ is the 2-torsion part of Φ_1X. In Section 4, we will use results of Bousfield in [5] to prove the following result. Aspects of Theorem 2.1, such as $K^*(B\mathrm{DI}(4))$ being a power series algebra on real generators, are also used in proving this theorem.

Recall that $KO^*(-)$ has period 8.

Theorem 2.6. *The groups $KO^i(\Phi_1\mathrm{DI}(4))$ are 0 if $i \equiv 0, 1, 2 \bmod 8$, and $K^0(\Phi_1\mathrm{DI}(4)) = 0$. Let M denote a free $\hat{\mathbb{Z}}_2$-module on three generators, acted on by ψ^2 and ψ^3 by the matrices of Theorem 2.1, with $\psi^{-1} = 1$. Let $\theta = \frac{1}{2}\psi^2$ act on M. Then there are exact sequences*

$$\begin{aligned} 0 &\to 2M \xrightarrow{\theta} 2M \to KO^3(\Phi_1\mathrm{DI}(4)) \to 0 \to 0 \to KO^4(\Phi_1\mathrm{DI}(4)) \to M/2 \\ &\xrightarrow{\theta} M/2 \to KO^5(\Phi_1\mathrm{DI}(4)) \to M/2 \xrightarrow{\theta} M/2 \to KO^6(\Phi_1\mathrm{DI}(4)) \to M \\ &\xrightarrow{\theta} M \to KO^7(\Phi_1\mathrm{DI}(4)) \to 0 \end{aligned}$$

and

$$0 \to M \xrightarrow{\theta} M \to K^1(\Phi\mathrm{DI}(4)) \to 0.$$

For $k = -1$ and 3, the action of ψ^k in $KO^{2j-1}(\Phi_1\mathrm{DI}(4))$, $KO^{2j-2}(\Phi_1\mathrm{DI}(4))$, and $K^{2j-1}(\Phi\mathrm{DI}(4))$ agrees with $k^{-j}\psi^k$ in adjacent M-terms.

In the remainder of this section, we use 2.1 and 2.6 to give explicit formulas for the Adams module $KO^i(\Phi_1\mathrm{DI}(4))$. A similar argument works for $K^*(\Phi_1\mathrm{DI}(4))$. If g_1, g_2, and g_3 denote the three generators of M, then the action of θ is given by

$$\begin{aligned}\theta(g_1) &= 8g_1 - g_2\\ \theta(g_2) &= 2^5 g_2 - g_3\\ \theta(g_3) &= 2^{13} g_3.\end{aligned}$$

Clearly θ is injective on M and $2M$. We have

$$KO^7(\Phi_1\mathrm{DI}(4)) \approx \mathrm{coker}(\theta|M) \approx \mathbb{Z}/2^{21}$$

with generator g_1; note that $g_2 = 2^3 g_1$ in this cokernel, and then $g_3 = 2^8 g_1$. Similarly $KO^3(\Phi_1\mathrm{DI}(4)) \approx \mathrm{coker}(\theta|2M) \approx \mathbb{Z}/2^{21}$. Also

$$KO^4(\Phi_1\mathrm{DI}(4)) \approx \ker(\theta|M/2) = \mathbb{Z}/2$$

with generator g_3, while $KO^6(\Phi_1\mathrm{DI}(4)) \approx \mathrm{coker}(\theta|M/2) = \mathbb{Z}/2$ with generator g_1. There is a short exact sequence

$$0 \to \mathrm{coker}(\theta|M/2) \to KO^5(\Phi_1\mathrm{DI}(4)) \to \ker(\theta|M/2) \to 0,$$

with the groups at either end being $\mathbb{Z}/2$ as before. To see that this short exact sequence is split, we use the map $S^7 \xrightarrow{f} \mathrm{DI}(4)$ which is inclusion of the bottom cell. The morphism f^* sends the first summand of $KO^5(\Phi_1\mathrm{DI}(4))$ to one of the two $\mathbb{Z}/2$-summands of $KO^5(\Phi_1 S^7)$, providing a splitting homomorphism. Thus we have proved the first part of the following result.

Theorem 2.7. *We have*

$$K^i(\Phi_1\mathrm{DI}(4)) \approx \begin{cases} 0 & i=0\\ \mathbb{Z}/2^{21} & i=1,\end{cases}$$

$$KO^i(\Phi_1\mathrm{DI}(4)) \approx \begin{cases} 0 & i=0,1,2\\ \mathbb{Z}/2^{21} & i=3,7\\ \mathbb{Z}/2 & i=4,6\\ \mathbb{Z}/2\oplus\mathbb{Z}/2 & i=5.\end{cases}$$

For $k=-1$ and 3, we have $\psi^k = 1$ on the $\mathbb{Z}/2$'s, and on $KO^{2j-1}(\Phi_1\mathrm{DI}(4))$ with j even and $K^{2j-1}(\Phi_1\mathrm{DI}(4))$, $\psi^{-1} = (-1)^j$ and

$$\psi^3 = 3^{-j}\big(3^4 - 3^3\cdot 2^3 + \tfrac{36}{527}2^8\big).$$

Completion of proof. To obtain ψ^3 on the $\mathbb{Z}/2$'s, we use the last part of Theorem 2.6 and the matrix Ψ^3 of Theorem 2.1. If ψ^3 is as in Ψ^3, then, mod 2, $\psi^3 - 1$ sends $g_1 \mapsto g_2$, $g_2 \mapsto g_3$, and $g_3 \mapsto 0$. Thus $\psi^3 - 1$ equals 0 on $KO^4(\Phi_1\mathrm{DI}(4))$ and $KO^6(\Phi_1\mathrm{DI}(4))$. Clearly $\psi^{-1} = 1$ on these groups.

To see that ψ^k-1 is 0 on $KO^5(\Phi_1\mathrm{DI}(4))$, we use the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z}/2 & \xrightarrow{i} & KO^5(\Phi_1\mathrm{DI}(4)) & \xrightarrow{\rho} & \mathbb{Z}/2 & \longrightarrow & 0 \\ & & \approx\downarrow & & f^*\downarrow & & 0\downarrow & & \\ 0 & \longrightarrow & \mathbb{Z}/2 & \longrightarrow & KO^5(\Phi_1 S^7) & \longrightarrow & \mathbb{Z}/2 & \longrightarrow & 0. \end{array}$$

We can choose generators G_1 and G_2 of $KO^5(\Phi_1\mathrm{DI}(4)) \approx \mathbb{Z}/2 \oplus \mathbb{Z}/2$ so that $G_1 \in \mathrm{im}(i)$, $\rho(G_2) \neq 0$, and $f^*(G_2) = 0$. Since $\psi^k - 1 = 0$ on the $\mathbb{Z}/2$'s on either side of $KO^5(\Phi_1\mathrm{DI}(4))$, the only way that $\psi^k - 1$ could be nonzero on $KO^5(\Phi_1\mathrm{DI}(4))$ is if $(\psi^k - 1)(G_2) = G_1$. However this yields the contradiction

$$0 = (\psi^k - 1)f^*G_2 = f^*(\psi^k - 1)G_2 = f^*(G_1) \neq 0.$$

On $KO^{2j-1}(\Phi_1\mathrm{DI}(4))$ with j even and $K^{2j-1}(\Phi_1\mathrm{DI}(4))$, ψ^3 sends the generator g_1 to

$$3^{-j}\big(3^4 g_1 - 3^3 g_2 + \tfrac{36}{527} g_3\big) = 3^{-j}\big(3^4 - 3^3 \cdot 2^3 + \tfrac{36}{527} 2^8\big) g_1,$$

and $\psi^{-1}(g_1) = (-1)^j g_1$ by Theorem 2.6. □

3. Relationship with pseudosphere

In this section, we prove Theorems 1.2 and 1.1.

Following [8, 8.6], we let $T = S^0 \cup_\eta e^2 \cup_2 e^3$, and consider its $K/2$-localization $T_{K/2}$. The groups $\pi_*(T_{K/2})$ are given in [8, 8.8], while the Adams module is given by

$$K^i(T_{K/2}) = \begin{cases} \hat{\mathbb{Z}}_2 & i \text{ even, with } \psi^k = k^{-i/2} \\ 0 & i \text{ odd;} \end{cases}$$

$$KO^i(T_{K/2}) = \begin{cases} \hat{\mathbb{Z}}_2 & i \equiv 0 \bmod 4, \text{ with } \psi^k = k^{-i/2} \\ \mathbb{Z}/2 & i = 2,3, \text{ with } \psi^k = 1 \\ 0 & i = 1,5,6,7. \end{cases}$$

Bousfield calls this the 2-completed K-theoretic pseudosphere. Closely related spectra have been also considered in [15] and [4].

Let $M(n) = S^{-1} \cup_n e^0$ denote the mod n Moore spectrum. Then, for $e > 1$ and k odd,

$$K^i(T_{K/2} \wedge M(2^e)) = \begin{cases} \mathbb{Z}/2^e & i \text{ even, with } \psi^k = k^{-i/2} \\ 0 & i \text{ odd;} \end{cases}$$

$$KO^i(T_{K/2} \wedge M(2^e)) = \begin{cases} \mathbb{Z}/2^e & i \equiv 0 \bmod 4, \text{ with } \psi^k = k^{-i/2} \\ \mathbb{Z}/2 & i = 1,3, \text{ with } \psi^k = 1 \\ \mathbb{Z}/2 \oplus \mathbb{Z}/2 & i = 2, \text{ with } \psi^k = 1 \\ 0 & i = 5,6,7. \end{cases} \quad (3.1)$$

Proof. Let $Y = T_{K/2} \wedge M(2^e)$. Most of (3.1) is immediate from the exact sequence

$$\xrightarrow{2^e} KO^i(T_{K/2}) \to KO^i(Y) \to KO^{i+1}(T_{K/2}) \xrightarrow{2^e} .$$

To see that $KO^2(Y) = \mathbb{Z}/2 \oplus \mathbb{Z}/2$ and not $\mathbb{Z}/4$, one can first note that

$$M(2^e) \wedge M(2) \simeq \Sigma^{-1} M(2) \vee M(2). \tag{3.2}$$

The exact sequence

$$KO^2(Y) \xrightarrow{2} KO^2(Y) \to KO^2(Y \wedge M(2)) \to KO^3(Y) \xrightarrow{2} \tag{3.3}$$

implies that if $KO^2(Y) = \mathbb{Z}/4$, then $|KO^2(Y \wedge M(2))| = 4$. However, by (3.2),

$$KO^2(Y \wedge M(2)) \approx KO^2(T_{K/2} \wedge M(2)) \oplus KO^3(T_{K/2} \wedge M(2)). \tag{3.4}$$

Also, there is a cofiber sequence

$$T \wedge M(2) \to \Sigma^{-1} A_1 \to \Sigma^5 M(2), \tag{3.5}$$

where $H^*(A_1; F_2)$ is isomorphic to the subalgebra of the mod 2 Steenrod algebra generated by Sq^1 and Sq^2, and satisfies $KO^*(A_1) = 0$. Thus

$$KO^i(T_{K/2} \wedge M(2)) \approx KO^i(\Sigma^4 M(2)) \approx \begin{cases} \mathbb{Z}/4 & i = 2 \\ \mathbb{Z}/2 & i = 3, \end{cases}$$

so that $|KO^2(Y \wedge M(2))| = 8$, contradicting a consequence of the hypothesis that $KO^2(Y) = \mathbb{Z}/4$.

We conclude the proof by showing that, for odd k, $\psi^k = 1$ on $KO^2(Y)$. First note that $\psi^k = 1$ on $KO^*(M(2))$. This follows immediately from the Adams operations on the sphere, except for ψ^k on $KO^{-2}(M(2)) \approx \mathbb{Z}/4$. This is isomorphic to $\widetilde{KO}(RP^2)$, where $\psi^k = 1$ is well-known. Now use (3.13) to deduce that $\psi^k = 1$ on $KO^*(T_{K/2} \wedge M(2))$, and then (3.4) to see that $\psi^k = 1$ on $KO^2(Y \wedge M(2))$. Finally, use (3.3) to deduce that $\psi^k = 1$ on $KO^2(Y)$. □

Comparison of 2.7 and (3.1) yields an isomorphism of graded abelian groups

$$KO^*(\Sigma^{8L+3} T_{K/2} \wedge M(2^{21})) \approx KO^*(\Phi_1 \mathrm{DI}(4)) \tag{3.6}$$

for any integer L. We will show that if $L = 90627$, then the Adams operations agree too. By [9, 6.4], it suffices to prove they agree for ψ^3 and ψ^{-1}.

Note that one way of distinguishing a K-theoretic pseudosphere from a sphere is that in $KO^*(\text{sphere})$ (resp. $KO^*(\text{pseudosphere})$) the $\mathbb{Z}/2$-groups are in dimensions 1 and 2 less than the dimensions in which $\psi^3 \equiv 1 \bmod 16$ (resp. $\psi^3 \equiv 9 \bmod 16$), and similarly after smashing with a mod 2^e Moore spectrum. Since $3^4 - 6^3 + \frac{36}{527} 2^8 \equiv 9 \bmod 16$, the $\mathbb{Z}/2$-groups in $KO^*(\Phi_1 \mathrm{DI}(4))$ occur in dimensions 1, 2, and 3 less that the dimension in which $\psi^3 \equiv 9 \bmod 16$, and so $\Phi_1\mathrm{DI}(4)$ should be identified with a suspension of $T_{K/2} \wedge M(2^{21})$ and not $S_{K/2} \wedge M(2^{21})$.

In $KO^{4t-1}(\Sigma^{8L+3}T_{K/2} \wedge M(2^{21}))$, $\psi^3 = 3^{-2(t-2L-1)}$ and $\psi^{-1} = 1$. Thus if L satisfies

$$3^{4L+2} \equiv 3^4 - 6^3 + \tfrac{36}{527}2^8 \bmod 2^{21}, \tag{3.7}$$

then $KO^*(\Sigma^{8L+3}T_{K/2} \wedge M(2^{21}))$ and $KO^*(\Phi_1\mathrm{DI}(4))$ will be isomorphic Adams modules. `Maple` easily verifies that (3.7) is satisfied for $L = 90627$.

A way in which this number L can be found begins with the mod 2^{18} equation

$$\sum_{i=1}^{6} \tbinom{2L-1}{i} 8^{i-1} \equiv \frac{3^{4L-2}-1}{8} \equiv \tfrac{1}{9}\bigl(\tfrac{2^7}{527} - 3\bigr) \equiv 192725,$$

where we use `Maple` at the last step. This easily implies $L \equiv 3 \bmod 8$, and so we let $L = 8b + 3$. Again using `Maple` and working mod 2^{18} we compute

$$\sum_{i=1}^{6} \tbinom{16b+5}{i} 8^{i-1} - 192725 \equiv 2^{10}u_0 + 2^4u_1b + 2^{10}u_2b^2 + 2^{17}b^3,$$

with u_i odd. Thus we must have $b \equiv 64 \bmod 128$, and so $L \equiv 515 \bmod 2^{10}$. Several more steps of this type lead to the desired value of L.

Thus, in the terminology of 4.3, we have proved the following result.

Proposition 3.8. *If $L = 90627$, then there is an isomorphism of Adams modules*

$$K^*_{CR}(\Sigma^{8L+3}T_{K/2} \wedge M(2^{21})) \approx K^*_{CR}(\Phi_1\mathrm{DI}(4)).$$

Theorem 1.2 follows immediately from this using the remarkable [8, 5.3], which says, among other things, that $K/2$-local spectra X having some $K^i(X) = 0$ are determined up to equivalence by the Adams module $K^*_{CR}(X)$. Theorem 1.1 follows immediately from Theorem 1.2 and the following result.

Proposition 3.9. *For all integers i,*

$$\pi_{8i+d}(T_{K/2} \wedge M(2^{21})) \approx \begin{cases} \mathbb{Z}/2 \oplus \mathbb{Z}/2^{\min(21,\nu(i)+4)} & d=-2 \\ \mathbb{Z}/2^{\min(21,\nu(i)+4)} & d=-1 \\ 0 & d=0,1 \\ \mathbb{Z}/8 & d=2 \\ \mathbb{Z}/2 \oplus \mathbb{Z}/8 & d=3 \\ \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2 & d=4,5. \end{cases}$$

Proof. For the most part, these groups are immediate from the groups $\pi_*(T_{K/2})$ given in [8, 8.8] and the exact sequence

$$\xrightarrow{2^{21}} \pi_{j+1}(T_{K/2}) \to \pi_j(T_{K/2} \wedge M(2^{21})) \to \pi_j(T_{K/2}) \xrightarrow{2^{21}}. \tag{3.10}$$

All that needs to be done is to show that the following short exact sequences, obtained from (3.10), are split.

$$\begin{aligned}(3.11)\qquad & 0 \to \mathbb{Z}/2 \to \pi_{8i+3}(T_{K/2}\wedge M(2^{21})) \to \mathbb{Z}/8 \to 0\\ & 0 \to \mathbb{Z}/2\oplus\mathbb{Z}/2 \to \pi_{8i+4}(T_{K/2}\wedge M(2^{21})) \to \mathbb{Z}/2 \to 0\\ & 0 \to \mathbb{Z}/2 \to \pi_{8i+5}(T_{K/2}\wedge M(2^{21})) \to \mathbb{Z}/2\oplus\mathbb{Z}/2 \to 0\\ & 0 \to \mathbb{Z}/2^{\min(21,\nu(i)+4)} \to \pi_{8i-2}(T_{K/2}\wedge M(2^{21})) \to \mathbb{Z}/2 \to 0.\end{aligned}$$

Let $Y = T_{K/2}\wedge M(2^{21})$. We consider the exact sequence for $\pi_*(Y\wedge M(2))$,

$$(3.12)\qquad \xrightarrow{2} \pi_{i+1}(Y) \to \pi_i(Y\wedge M(2)) \to \pi_i(Y) \xrightarrow{2} .$$

If the four sequences (3.11) are all split, then by (3.12) the groups $\pi_{8i+d}(Y\wedge M(2))$ for $d = 2,3,4,5,-2$ have orders 2^3, 2^5, 2^6, 2^5, and 2^3, respectively, but if any of the sequences (3.11) fails to split, then some of the orders $|\pi_{8i+d}(Y\wedge M(2))|$ will have values smaller than those listed here.

By (3.2),

$$\pi_i(Y\wedge M(2)) \approx \pi_{i+1}(T_{K/2}\wedge M(2)) \oplus \pi_i(T_{K/2}\wedge M(2)).$$

By (3.5), since localization preserves cofibrations and $(A_1)_{K/2} = *$, there is an equivalence

$$(3.13)\qquad \Sigma^4 M_{K/2} \simeq T_{K/2}\wedge M(2),$$

and hence

$$(3.14)\qquad \pi_i(Y\wedge M(2)) \approx \pi_{i-3}(M_{K/2}) \oplus \pi_{i-4}(M_{K/2}).$$

By [10, 4.2],

$$\pi_{8i+d}(M_{K/2}) = \begin{cases} 0 & d = 4,5\\ \mathbb{Z}/2 & d = -2,3\\ \mathbb{Z}/2\oplus\mathbb{Z}/2 & d = -1,2\\ \mathbb{Z}/4\oplus\mathbb{Z}/2 & d = 0,1.\end{cases}$$

This is the sum of two "lightning flashes," one beginning in $8d-2$ and the other in $8d-1$. Substituting this information into (3.14) yields exactly the orders which were shown in the previous paragraph to be true if and only if all the exact sequences (3.11) split. □

4. Determination of $KO^*(\Phi_1\mathrm{DI}(4))$

In this section, we prove Theorem 2.6, which shows how ψ^k in $K^*(B\mathrm{DI}(4))$ leads to the determination of $KO^*(\Phi_1\mathrm{DI}(4))$. Our presentation here follows suggestions in several e-mails from Pete Bousfield.

The first result explains how $KO^*(B\mathrm{DI}(4))$ follows from $K^*(B\mathrm{DI}(4))$.

Theorem 4.1. *There are classes g_8, g_{12}, and g_{24} in $KO^0(B\mathrm{DI}(4))$ such that $c(g_i)=\xi_i$, with ξ_i as in* 2.1, *and*

$$KO^*(B\mathrm{DI}(4)) \approx KO^*[\![g_8, g_{12}, g_{24}]\!].$$

The Adams operations ψ^2 and ψ^3 mod decomposables on the basis of g_i's are as in 2.1.

Proof. In [2, 2.1], it is proved that if there is a torsion-free subgroup $F^* \subset KO^*(X)$ such that $F^* \otimes K^*(\mathrm{pt}) \to K^*(X)$ is an isomorphism, then so is $F^* \otimes KO^*(\mathrm{pt}) \to KO^*(X)$. The proof is a Five Lemma argument using exact sequences in [17, p. 257]. Although the result is stated for ordinary (not 2-completed) $KO^*(-)$, the same argument applies in the 2-completed context. If F^* is a multiplicative subgroup, then the result holds as rings. Our result then follows from 2.1, since the generators there are real. A similar proof can be derived from [5, 2.3]. □

Next we need a similar sort of result about $KO^*(\mathrm{DI}(4))$. We could derive much of what we need by an argument similar to that just used, using the result of [16] about $K^*(\mathrm{DI}(4))$ as input. However, as we will need this in a specific form in order to use it to draw conclusions about $KO^*(\Phi_1\mathrm{DI}(4))$, we begin by introducing much terminology from [5].

The study of united K-theory begins with two categories, which will then be endowed with additional structure. We begin with a partial definition of each, and their relationship. For complete details, the reader will need to refer to [5] or an earlier paper of Bousfield.

Definition 4.2 ([5, 2.1])**.** *A CR-module $M=\{M_C, M_R\}$ consists of $\mathbb{Z}$-graded* 2*-profinite abelian groups M_C and M_R with continuous additive operations*

$$M_C^* \xrightarrow[\approx]{B} M_C^{*-2}, \qquad M_R^* \xrightarrow[\approx]{B_R} M_R^{*-8}, \qquad M_C^* \xrightarrow[\approx]{t} M_C^*,$$

$$M_R^* \xrightarrow{\eta} M_R^{*-1}, \qquad M_R^* \xrightarrow{c} M_C^*, \qquad M_C^* \xrightarrow{r} M_R^*,$$

satisfying 15 *relations, which we will mention as needed.*

We omit the descriptor "2-adic," which Bousfield properly uses, just as we omit writing the 2-adic coefficients $\hat{\mathbb{Z}}_2$ which are present in all our K- and KO-groups.

Example 4.3. *For a spectrum or space X, the united* 2*-adic K-cohomology*

$$K_{CR}^*(X) := \{K^*(X), KO^*(X)\}$$

is a CR-module, with complex and real Bott periodicity, conjugation, the Hopf map, complexification, and realification giving the respective operations.

Definition 4.4. *A Δ-module $N = \{N_C, N_R, N_H\}$ is a triple of 2-profinite abelian groups N_C, N_R, and N_H with continuous additive operations*

$$N_C \xrightarrow[\approx]{t} N_C, \qquad N_R \xrightarrow{c} N_C, \qquad N_C \xrightarrow{r} N_R,$$

$$N_H \xrightarrow{c'} N_C, \qquad N_C \xrightarrow{q} N_H$$

satisfying nine relations.

Example 4.5. *For a CR-module M and an integer n, there is a Δ-module $\Delta^n M = \{M_C^n, M_R^n, M_R^{n-4}\}$ with $c' = B^{-2}c$ and $q = rB^2$. In particular, for a space X and integer n, there is a Δ-module $K_\Delta^n(X) := \Delta^n K_{CR}^*(X)$.*

Now we add additional structure to these definitions.

Definition 4.6 ([5, 4.3,6.1])**.** *A $\theta\Delta$-module is a Δ-module N together with homomorphisms $N_C \xrightarrow{\theta} N_C$, $N_R \xrightarrow{\theta} N_R$, and $N_H \xrightarrow{\theta} N_R$ satisfying certain relations listed in* [5, 4.3]. *An Adams Δ-module is a $\theta\Delta$-module N together with Adams operations $N \xrightarrow[\approx]{\psi^k} N$ for odd k satisfying the familiar properties.*

Example 4.7. *In the notation of Example* 4.5, *$K_\Delta^{-1}(X)$ is an Adams Δ-module with $\theta = -\lambda^2$.*

Definition 4.8 ([5, 2.6,3.1,3.2])**.** *A special ϕCR-algebra $\{A_C, A_R\}$ is a CR-module with bilinear $A_C^m \times A_C^n \to A_C^{m+n}$ and $A_R^m \times A_R^n \to A_R^{m+n}$ and also $A_C^0 \xrightarrow{\phi} A_R^0$ and $A_C^{-1} \xrightarrow{\phi} A_R^0$ satisfying numerous properties.*

Remark 4.9. The operations ϕ, which were initially defined in [7], are less familiar than the others. Two properties are $c\phi a = t(a)a$ and $\phi(a+b) = \phi a + \phi b + r(t(a)b)$ for $a, b \in A_C^0$. For a connected space X, $K_{CR}^*(X)$ is a special ϕCR-algebra.

The following important lemma is taken from [5].

Lemma 4.10 ([5, 4.5,4.6])**.** *For any $\theta\Delta$-module M, there is a universal special ϕCR-algebra $\hat{L}M$. This means that there is a morphism $M \xrightarrow{\alpha} \hat{L}M$ such that any morphism from M into a ϕCR-algebra factors as α followed by a unique ϕCR-algebra morphism. There is an algebra isomorphism $\hat{\Lambda}M_C \to (\hat{L}M)_C$, where $\hat{\Lambda}(-)$ is the* 2*-adic exterior algebra functor.*

In [5, 2.7], Bousfield defines, for a CR-algebra A, the indecomposable quotient $\hat{Q}A$. We apply this to $A = K_{CR}^*(B\mathrm{DI}(4))$, and consider the Δ-module $\hat{Q}K_\Delta^0(B\mathrm{DI}(4))$, analogous to [5, 4.10]. We need the following result, which is more delicate than the K_Δ^{-1}-case considered in [5, 4.10].

Lemma 4.11. *With $\theta = -\lambda^2$, the Δ-module $\hat{Q}K_\Delta^0(B\mathrm{DI}(4))$ becomes a $\theta\Delta$-module.*

Proof. First we need that θ is an additive operation. In [7, 3.6], it is shown that $\theta(x+y)=\theta(x)+\theta(y)-xy$ if $x,y\in KO^n(X)$ with $n\equiv 0 \bmod 4$. The additivity follows since we are modding out the product terms. (In the case $n\equiv -1 \bmod 4$ considered in [5, 4.10], the additivity of θ is already present before modding out indecomposables.)

There are five additional properties which must be satisfied by θ. That $\theta cx = c\theta x$ and $\theta tz = t\theta z$ are easily obtained from [7, 3.4]. That $\theta c'y = c\theta y$ follows from [8, 6.2(iii),6.4]. That $\theta qz=\theta rz$ follows from preceding [7, 3.10] by c, which is surjective for us. Here we use that $rc=2$ and $q=rB^2$. Finally, $\theta rz = r\theta z$ for us, since c is surjective; here we have used the result $\overline{\phi}cx=0$ given in [5, 4.3]. □

Now we obtain the following important description of the CR-algebra $K^*_{CR}(\mathrm{DI}(4))$.

Theorem 4.12. *There is a morphism of $\theta\Delta$-modules*

$$\hat{Q}K^0_\Delta(B\mathrm{DI}(4))\to \widetilde{K}^{-1}_\Delta(\mathrm{DI}(4))$$

which induces an isomorphism of special ϕCR-algebras

$$\hat{L}(\hat{Q}K^0_\Delta(B\mathrm{DI}(4)))\to K^*_{CR}(\mathrm{DI}(4)).$$

Proof. The map $\Sigma\mathrm{DI}(4)=\Sigma\Omega B\mathrm{DI}(4)\to B\mathrm{DI}(4)$ induces a morphism

$$K^0_\Delta(B\mathrm{DI}(4))\to K^{-1}_\Delta(\mathrm{DI}(4))$$

which factors through the indecomposable quotient $\hat{Q}K^0_\Delta(B\mathrm{DI}(4))$. In [16, 1.2], a general result is proved which implies that $K^*(\mathrm{DI}(4))$ is an exterior algebra on elements of $K^1(\mathrm{DI}(4))$ which correspond to the generators of the power series algebra $K^*(B\mathrm{DI}(4))$ under the above morphism followed by the Bott map. Thus our result will follow from [5, 4.9], once we have shown that the $\theta\Delta$-module $M:=\hat{Q}K^0_\Delta(B\mathrm{DI}(4))$ is robust([5, 4.7]). This requires that M is profinite, which follows as in the remark following [5, 4.7], together with two properties regarding $\overline{\phi}$, where $\overline{\phi}z:=\theta rz - r\theta z$ for $z\in M_C$. In our case, c is surjective, and so $\overline{\phi}=0$ as used in the previous proof.

One property is that M is torsion-free and exact. This follows from the Bott exactness of the CR-module $K^*_{CR}(B\mathrm{DI}(4))$ noted in [5, 2.2], and [5, 5.4], which states that, for any n, the Δ-module $\Delta^n N$ associated to a Bott exact CR-module N with N^n_C torsion-free and $N^{n-1}_C=0$ is torsion-free. The other property is $\ker(\overline{\phi})=cM_R+c'M_H+2M_C$. For us, both sides equal M_C since c is surjective and $\overline{\phi}=0$. □

Our Theorem 2.6 now follows from [5, 9.5] once we have shown that the Adams Δ-module $M:=\hat{Q}K^0_\Delta(B\mathrm{DI}(4))$ is "strong." ([5, 7.11]) This result ([5, 9.5]) requires that the space (here $\mathrm{DI}(4)$) be an H-space (actually $K/2_*$-durable, which is satisfied by H-spaces) and that it satisfies the conclusion of our 4.12. It then deduces that $KO^*(\Phi_1\mathrm{DI}(4))$ fits into an exact sequence which reduces to ours provided $M_R=M_C$ and $M_H=2M_C$. These equalities are implied by $\hat{Q}K^0_\Delta(B\mathrm{DI}(4))$ being exact, as was noted to be true in the previous proof, plus $t = 1$ and c surjective, as were noted to be true in 2.1. Indeed, the exactness property, ([5, 4.2]), includes that $cM_R + c'M_H = \ker(1 - t)$ and $cM_R \cap c'M_H = \mathrm{im}(1 + t)$. Another perceptible difference is that Bousfield's exact sequence is in terms of $\overline{M} := M/\bar{\phi}$, while ours involves M, but these are equal since, as already observed, $\bar{\phi} = 0$ since c is surjective.

Note also that the Adams operations in $\overline{M}$ in the exact sequence of [5, 9.5], which reduces to that in our 2.6, are those in the Adams Δ-module $\hat{Q}K^0_\Delta(B\mathrm{DI}(4))$, which are given in our 2.1. The morphism θ in [5, 9.5] or our 2.6 is $\frac{1}{2}\psi^2$, since this equals $-\lambda^2$ mod decomposables.

Finally, we show that our M is strong. One of the three criteria for being strong is to be robust, and we have already discussed and verified this. The second requirement for an Adams Δ-module to be strong is that it be "regular." This rather technical condition is defined in [5, 7.8]. In [5, 7.9], a result is proved which immediately implies that $\widetilde{K}^{-1}_\Delta(\mathrm{DI}(4))$ is regular. By 4.12, our M injects into $\widetilde{K}^{-1}_\Delta(\mathrm{DI}(4))$, and so by [5, 7.10], which states that a submodule of a regular module is regular, our M is regular.

The third requirement for M to be strong is that it be ψ^3-splittable ([5, 7.2]), which means that the quotient map $M \to M/\bar{\phi}$ has a right inverse. As we have noted several times, we have $\bar{\phi} = 0$, and so the identity map serves as a right inverse to the identity map. This completes the proof that our M is strong, and hence that [5, 9.5] applies to DI(4) to yield our Theorem 2.6.

References

[1] Andersen, Kasper; Grodal, Jesper. The classification of 2-compact groups. *J. Amer. Math. Soc.*, to appear.

[2] Anderson, Donald W. Universal coefficient theorems for K-theory. Preprint, 1971.

[3] Bendersky, Martin; Davis, Donald M. v_1-periodic homotopy groups of $SO(n)$. *Mem. Amer. Math. Soc.* **172** (2004), no. 815. MR2096862 (2005j:55017).

[4] Bendersky, Martin; Davis, Donald M.; Mahowald, Mark. Stable geometric dimension of vector bundles over even-dimensional real projective spaces. *Trans. Amer. Math. Soc.* **358** (2006) 1585–1603. MR2186987 (2006f:55020).

[5] Bousfield, A. K. On the 2-adic K-localizations of H-spaces. *Homology, Homotopy, and Applications* **9** (2007) 331–366. MR2299803 (2007m:55005).

[6] Bousfield, A. K. The K-theory localization and v_1-periodic homotopy groups of H-spaces. *Topology* **38** (1999) 1239–1264. MR1690156 (2000d:55022), Zbl 0933.57034.

[7] Bousfield, A. K. Kunneth theorems and unstable operations in 2-adic KO-cohomology. *K-theory*, to appear.

[8] Bousfield, A. K. On the 2-primary v_1-periodic homotopy groups of spaces. *Topology* **44** (2005) 381–413. MR2114954 (2005j:55018).

[9] Bousfield, A. K. A classification of K-local spectra. *J. Pure Appl. Algebra* **66** (1990) 121–163. MR1075335 (92d:55003), Zbl 0713.55007.

[10] Bousfield, A. K. The localization of spectra with respect to homology. Topology **18** (1979) 257–281. MR0551009 (80m:55006), Zbl 0417.55007.

[11] DAVIS, DONALD M. Representation types and 2-primary homotopy groups of certain compact Lie groups. *Homology, Homotopy, and Applications* **5** (2003) 297–324. MR2006403 (2005b:55030), Zbl 1031.55008.

[12] DAVIS, DONALD M.; MAHOWALD, MARK. Some remarks on v_1-periodic homotopy groups. *Adams Memorial Symposium on Algebraic Topology*, 2 (Manchester, 1990), 55–72. London Math. Society Lecture Notes, 176, *Cambridge Univ. Press, Cambridge*, 1992. MR1232199 (94h:55015), Zbl 0753.55004.

[13] DWYER, WILLIAM G.; WILKERSON, CLARENCE W. A new finite loop space at the prime two. *J. Amer. Math. Soc.* **6** (1993) 37–64. MR1161306 (93d:55011), Zbl 0769.55007.

[14] DWYER, WILLIAM G.; WILKERSON, CLARENCE W. Homotopy fixed-point methods for Lie groups and finite loop spaces *Ann. of Math.* (2) **139** (1994) 395–442. MR1274096 (95e:55019), Zbl 0801.55007.

[15] HOPKINS, MICHAEL J.; MAHOWALD, MARK; SADOFSKY, HAL. Constructions of elements in Picard groups. *Topology and representation theory* (Evanston, IL, 1992), 89–126. Contemp. Math., 158. *Amer. Math. Soc., Providence, RI*, 1994. MR1263713 (95a:55020), Zbl 0799.55005.

[16] JEANNERET, A.; OSSE, A. The K-theory of p-compact groups. *Comm. Math. Helv.* **72** (1997) 556–581. MR1600146 (99d:55007), Zbl 0895.55001.

[17] KAROUBI, MAX. Algebrès de Clifford et K-théorie. *Ann. Sci. École Norm. Sup.* (4) **1** (1968) 161–270. MR0238927 (39 #287), Zbl 0194.24101.

[18] OSSE, AKIMOU; SUTER, ULRICH. Invariant theory and the K-theory of the Dwyer–Wilkerson space. *Une dégustation topologique* [Topological morsels]: *homotopy theory in the Swiss Alps* (Arolla, 1999), 175–185. Contemp. Math., 265. *Amer. Math. Soc., Providence, RI*, 2000. MR1803957 (2001j:55005), Zbl 0970.55005.

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