New York Journal of Mathematics

New York J. Math. **14** (2008) 539–576.

On automorphisms of type *II* Arveson systems (probabilistic approach)

Boris Tsirelson

Abstract. We give a counterexample to the conjecture that the automorphisms of an arbitrary Arveson system act transitively on its normalized units.

Contents

Introduction

> *We do not know how to calculate the gauge group in this generality...*
>
> W. Arveson [1, Section 2.8]

Received April 25, 2008.
Mathematics Subject Classification. Primary 46L55, secondary 60G99.
Key words and phrases. product system, automorphism.
This research was supported by the israel science foundation (grant No. 683/05).

ISSN 1076-9803/08

At the moment, most important seems to us to answer the question whether the automorphisms of an arbitrary product system act transitively on the normalized units.

V. Liebscher [5, Section 11]

By an *Arveson system* I mean a product system as defined by Arveson [1, 3.1.1]. Roughly, it consists of Hilbert spaces H_t (for $0 < t < \infty$) satisfying $H_s \otimes H_t = H_{s+t}$. Classical examples are given by Fock spaces; these are *type I* systems, see [1, 3.3 and Part 2]. Their automorphisms are described explicitly, see [1, 3.8.4]. The group of automorphisms, called the *gauge group* of the Arveson system, for type I is basically the group of motions of the N-dimensional Hilbert space. The parameter $N \in \{0, 1, 2, \dots\} \cup \{\infty\}$ is the so-called (numerical) index; accordingly, the system is said to be of type $I_0, I_1, I_2, \dots$ or I_∞. All Hilbert spaces are complex (that is, over $\mathbb{C}$).

Some Arveson systems contain no type I subsystems; these are *type III* systems, see [1, Part 5]. An Arveson system is of *type II,* if it is not of type I, but contains a type I subsystem. (See [9, 6g and 10a] for examples.) In this case the greatest type I subsystem exists and will be called the *classical part* of the type II system. The latter is of type II_N where N is the index of its classical part.

Little is known about the gauge group of a type II system and its natural homomorphism into the gauge group of the classical part. In general, the homomorphism is not one-to-one, and its range is a proper subgroup. The corresponding subgroup of motions need not be transitive, which is the main result of this work (Theorem 1.10); it answers a question asked by Liebscher [5, Notes 3.6, 5.8 and Section 11 (question 1)] and (implicitly) Bhat [2, Definition 8.2]; see also [9], Question 9d3 and the paragraph after it. A partial answer is obtained by Markiewicz and Powers [6] using a different approach.

Elaborate constructions (especially, counterexamples) in a Hilbert space often use a coordinate system (orthonormal basis). In other words, the sequence space l_2 is used rather than an abstract Hilbert space. An Arveson system consists of Hilbert spaces, but we cannot choose their bases without sacrificing the given tensor product structure. Instead, we can choose maximal commutative operator algebras, which leads to the probabilistic approach. Especially, the white noise (or Brownian motion) will be used rather than an abstract type I_1 Arveson system.

1. Definitions, basic observations, and the result formulated

I do not reproduce here the definition of an Arveson system [1, 3.1.1], since we only need the special case

$$H_t = L_2(\Omega, \mathcal{F}_{0,t}, P) \tag{1.1}$$

corresponding to a noise.

Definition 1.2. A *noise* consists of a probability space $(\Omega, \mathcal{F}, P)$, sub-σ-fields $\mathcal{F}_{s,t} \subset \mathcal{F}$ given for all $s, t \in \mathbb{R}$, $s < t$, and a measurable action $(T_h)_h$ of $\mathbb{R}$ on Ω, having the following properties:

(a) $\mathcal{F}_{r,s} \otimes \mathcal{F}_{s,t} = \mathcal{F}_{r,t}$ whenever $r < s < t$,

(b) T_h sends $\mathcal{F}_{s,t}$ to $\mathcal{F}_{s+h,t+h}$ whenever $s < t$ and $h \in \mathbb{R}$,

(c) $\mathcal{F}$ is generated by the union of all $\mathcal{F}_{s,t}$.

See [9, 3d1] for details. As usual, all probability spaces are standard, and everything is treated mod 0. Item (a) means that $\mathcal{F}_{r,s}$ and $\mathcal{F}_{s,t}$ are (statistically) independent and generate $\mathcal{F}_{r,t}$. Invertible maps $T_h : \Omega \to \Omega$ preserve the measure P.

The white noise is a classical example; we denote it $(\Omega^{\text{white}}, \mathcal{F}^{\text{white}}, P^{\text{white}})$, $(\mathcal{F}^{\text{white}}_{s,t})_{s<t}$, $(T^{\text{white}}_h)_h$. It is generated by the increments of the one-dimensional Brownian motion $(B_t)_{-\infty<t<\infty}$, $B_t : \Omega \to \mathbb{R}$.

Given a noise, we construct Hilbert spaces H_t consisting of $\mathcal{F}_{0,t}$-measurable complex-valued random variables, see (1.1). The relation

$$H_s \otimes H_t = H_{s+t},$$

or rather a unitary operator $H_s \otimes H_t \to H_{s+t}$, emerges naturally:

$$\begin{aligned} H_{s+t} = L_2(\mathcal{F}_{0,s+t}) &= L_2(\mathcal{F}_{0,s} \otimes \mathcal{F}_{s,s+t}) = L_2(\mathcal{F}_{0,s}) \otimes L_2(\mathcal{F}_{s,s+t}) \\ &= L_2(\mathcal{F}_{0,s}) \otimes L_2(\mathcal{F}_{0,t}) = H_s \otimes H_t\,; \end{aligned}$$

the time shift T_s is used for turning $\mathcal{F}_{s,s+t}$ to $\mathcal{F}_{0,t}$. Thus, $(H_t)_{t>0}$ is an Arveson system. Especially, the white noise leads to an Arveson system $(H^{\text{white}}_t)_{t>0}$ (of type I_1, as will be explained).

For $X \in H_s$, $Y \in H_t$ the image of $X \otimes Y$ in H_{s+t} will be denoted simply XY (within this section).

We specialize the definition of a unit [1, 3.6.1] to systems of the form (1.1).

Definition 1.3. A *unit* (of the system (1.1)) is a family $(u_t)_{t>0}$ of nonzero vectors $u_t \in H_t = L_2(\mathcal{F}_{0,t}) \subset L_2(\mathcal{F})$ such that $t \mapsto u_t$ is a Borel measurable map $(0, \infty) \to L_2(\mathcal{F})$, and

$$u_s u_t = u_{s+t} \quad \text{for all } s, t > 0\,.$$

(In other words, the given unitary operator $H_s \otimes H_t \to H_{s+t}$ maps $u_s \otimes u_t$ to u_{s+t}.) The unit is *normalized,* if $\|u_t\| = 1$ for all t. (In general,

$$\|u_t\| = \exp(ct)$$

for some $c \in \mathbb{R}$.)

Here is the general form of a unit in $(H^{\text{white}}_t)_t$:

$$u_t = \exp(zB_t + z_1 t)\,; \quad z, z_1 \in \mathbb{C}\,;$$

it is normalized iff $(\operatorname{Re} z)^2 + \operatorname{Re} z_1 = 0$. The units generate $(H_t^{\text{white}})_t$ in the following sense: for every $t > 0$, H_t^{white} is the closed linear span of vectors of the form $(u_1)_{\frac{t}{n}}(u_2)_{\frac{t}{n}} \dots (u_n)_{\frac{t}{n}}$, where $u_1, \dots, u_n$ are units, $n = 1, 2, \dots$. Indeed, $L_2(\mathcal{F}_{0,t})$ is spanned by random variables of the form $\exp\bigl(\mathrm{i} \int_0^t f(s)\,\mathrm{d}B_s\bigr)$ where f runs over step functions $(0,t) \to \mathbb{R}$ constant on $\bigl(0, \frac{1}{n}t\bigr), \dots, \bigl(\frac{n-1}{n}t, t\bigr)$.

We specialize two notions, 'type I' and 'automorphism', to systems of the form (1.1).

Definition 1.4. A system of the form (1.1) is of *type I* if it is generated by its units.

We see that $(H_t^{\text{white}})_t$ is of type I.

Definition 1.5. An *automorphism* (of the system (1.1)) is a family $(\Theta_t)_{t>0}$ of unitary operators $\Theta_t : H_t \to H_t$ such that $\Theta_{s+t}(XY) = (\Theta_s X)(\Theta_t Y)$ for all $X \in H_s$, $Y \in H_t$, $s > 0$, $t > 0$, and the function $t \mapsto \langle \Theta_t X_t, Y_t \rangle$ is Borel measurable whenever $t \mapsto X_t$ and $t \mapsto Y_t$ are Borel measurable maps $(0, \infty) \to L_2(\mathcal{F})$ such that $X_t, Y_t \in L_2(\mathcal{F}_{0,t}) \subset L_2(\mathcal{F})$.

Basically, $\Theta_s \otimes \Theta_t = \Theta_{s+t}$. The group G of all automorphisms is called the *gauge group.* Clearly, G acts on the set of normalized units, $(u_t)_t \mapsto (\Theta_t u_t)_t$.

Automorphisms $\Theta_t = \Theta_t^{\text{trivial}(\lambda)} = \mathrm{e}^{\mathrm{i}\lambda t}$ (for $\lambda \in \mathbb{R}$), consisting of scalar operators, will be called *trivial;* these commute with all automorphisms, and are a one-parameter subgroup $G^{\text{trivial}} \subset G$. Normalized units $(u_t)_t$ and $(\mathrm{e}^{\mathrm{i}\lambda t} u_t)_t$ will be called *equilavent.* The factor group G/G^{trivial} acts on the set of all equivalence classes of normalized units.

We turn to the gauge group G^{white} of the classical system $(H_t^{\text{white}})_t$. Equivalence classes of normalized units of $(H_t^{\text{white}})_t$ are parametrized by numbers $z \in \mathbb{C}$, since each class contains exactly one unit of the form

$$u_t = \exp\bigl(zB_t - (\operatorname{Re} z)^2 t\bigr) .$$

The scalar product corresponds to the distance:

$$|\langle u_t^{(1)}, u_t^{(2)} \rangle| = \exp\bigl(-\tfrac{1}{2}|z_1 - z_2|^2 t\bigr)$$

for $u_t^{(k)} = \exp\bigl(z_k B_t - (\operatorname{Re} z_k)^2 t\bigr)$, $k = 1, 2$. The action of $G^{\text{white}}/G^{\text{trivial}}$ on equivalence classes boils down to its action on $\mathbb{C}$ by isometries. The orientation of $\mathbb{C}$ is preserved, since

$$\frac{\langle u_t^{(1)}, u_t^{(2)} \rangle \langle u_t^{(2)}, u_t^{(3)} \rangle \langle u_t^{(3)}, u_t^{(1)} \rangle}{|\langle u_t^{(1)}, u_t^{(2)} \rangle \langle u_t^{(2)}, u_t^{(3)} \rangle \langle u_t^{(3)}, u_t^{(1)} \rangle|} = \exp\bigl(\mathrm{i}tS(z_1, z_2, z_3)\bigr),$$

where $S(z_1, z_2, z_3) = \operatorname{Im}\bigl((z_2 - z_1)\overline{(z_3 - z_1)}\bigr)$ is twice the *signed* area of the triangle. So, $G^{\text{white}}/G^{\text{trivial}}$ acts on $\mathbb{C}$ by motions (see [1, 3.8.4]).

Shifts of $\mathbb{C}$ along the imaginary axis, $z \mapsto z + \mathrm{i}\lambda$ (for $\lambda \in \mathbb{R}$) emerge from automorphisms

$$\Theta_t = \Theta_t^{\mathrm{shift}(\mathrm{i}\lambda)} = \exp(\mathrm{i}\lambda B_t);$$

here the random variable $\exp(\mathrm{i}\lambda B_t) \in L_\infty(\mathcal{F}_{0,t}^{\mathrm{white}})$ is treated as the multiplication operator, $X \mapsto X \exp(\mathrm{i}\lambda B_t)$ for $X \in L_2(\mathcal{F}_{0,t}^{\mathrm{white}})$.

Shifts of $\mathbb{C}$ along the real axis, $z \mapsto z + \lambda$ (for $\lambda \in \mathbb{R}$) emerge from less evident automorphisms

$$\Theta_t^{\mathrm{shift}(\lambda)} X = D_t^{1/2} \cdot (X \circ \theta_t^\lambda); \tag{1.6}$$

here $\theta_t^\lambda : C[0,t] \to C[0,t]$ is the drift transformation $(\theta_t^\lambda b)(s) = b(s) - 2\lambda s$ (for $s \in [0,t]$), D_t is the Radon–Nikodym derivative of the Wiener measure shifted by θ_t^λ w.r.t. the Wiener measure itself,

$$D_t = \exp(2\lambda B_t - 2\lambda^2 t), \tag{1.7}$$

and $X \in L_2(\mathcal{F}_{0,t}^{\mathrm{white}})$ is treated as a function on $C[0,t]$ (measurable w.r.t. the Wiener measure). Thus,

$$(\Theta_t^{\mathrm{shift}(\lambda)} X)(b) = \exp\bigl(\lambda b(t) - \lambda^2 t\bigr) X(\theta_t^\lambda b).$$

By the way, these two one-parameter subgroups of G^{white} satisfy Weyl relations

$$\Theta_t^{\mathrm{shift}(\lambda)} \Theta_t^{\mathrm{shift}(\mathrm{i}\mu)} = \mathrm{e}^{-2\mathrm{i}\lambda\mu t} \Theta_t^{\mathrm{shift}(\mathrm{i}\mu)} \Theta_t^{\mathrm{shift}(\lambda)};$$

that is, $\Theta^{\mathrm{shift}(\lambda)} \Theta^{\mathrm{shift}(\mathrm{i}\mu)} = \Theta^{\mathrm{trivial}(-2\lambda\mu)} \Theta^{\mathrm{shift}(\mathrm{i}\mu)} \Theta^{\mathrm{shift}(\lambda)}$.

Rotations of $\mathbb{C}$ around the origin, $z \mapsto \mathrm{e}^{\mathrm{i}\lambda} z$ (for $\lambda \in \mathbb{R}$) emerge from automorphisms $\Theta^{\mathrm{rotat}(\lambda)}$. These will not be used, but are briefly described anyway. They preserve Wiener chaos spaces H_n,

$$\Theta_t^{\mathrm{rotat}(\lambda)} X = \mathrm{e}^{\mathrm{i}n\lambda} X \quad \text{for } X \in H_n \cap L_2(\mathcal{F}_{0,t}^{\mathrm{white}})\,;$$

the n-th chaos space $H_n \subset L_2(\mathcal{F}^{\mathrm{white}})$ consists of stochastic integrals

$$X = \int \cdots \int_{-\infty < s_1 < \cdots < s_n < \infty} f(s_1, \dots, s_n)\, \mathrm{d}B_{s_1} \dots \mathrm{d}B_{s_n}$$

where $f \in L_2(\mathbb{R}^n)$ (or rather, the relevant part of $\mathbb{R}^n$). One may say that $\Theta^{\mathrm{rotat}(\lambda)}$ just multiplies each $\mathrm{d}B_s$ by $\mathrm{e}^{\mathrm{i}\lambda}$.

Combining shifts and rotations we get all motions of $\mathbb{C}$. Accordingly, all automorphisms of $(H_t^{\mathrm{white}})_t$ are combinations of $\Theta^{\mathrm{shift}(\mathrm{i}\lambda)}$, $\Theta^{\mathrm{shift}(\lambda)}$, $\Theta^{\mathrm{rotat}(\lambda)}$ and $\Theta^{\mathrm{trivial}(\lambda)}$. More generally, the N-dimensional Brownian motion leads to the (unique up to isomorphism) Arveson system of type I_N and motions of $\mathbb{C}^N$. We need $N = 1$ only; $(H_t^{\mathrm{white}})_t$ is the Arveson system of type I_1.

Some noises are constructed as extensions of the white noise,

$$\mathcal{F}_{s,t} \supset \mathcal{F}_{s,t}^{\mathrm{white}} \tag{1.8}$$

(also T_h conforms to T_h^{white}). More exactly, it means that $B_t \in L_2(\mathcal{F})$ are given such that $B_t - B_s$ is $\mathcal{F}_{s,t}$-measurable for $-\infty < s < t < \infty$, and $B_t - B_s \sim N(0, t-s)$ (that is, the random variable $(t-s)^{-1/2}(B_t - B_s)$ has the standard normal distribution), and $B_0 = 0$, and $(B_t - B_s) \circ T_h = B_{t+h} - B_{s+h}$. Such $(B_t)_t$ may be called a Brownian motion adapted to the given noise. Then, of course, by $\mathcal{F}_{s,t}^{\text{white}}$ we mean the sub-σ-field generated by $B_u - B_s$ for all $u \in (s,t)$. The Arveson system $(H_t)_t$, $H_t = L_2(\mathcal{F}_{0,t})$, is an extension of the type I_1 system $(H_t^{\text{white}})_t$, $H_t^{\text{white}} = L_2(\mathcal{F}_{0,t}^{\text{white}})$,

$$H_t \supset H_t^{\text{white}}. \tag{1.9}$$

All units of $(H_t^{\text{white}})_t$ are also units of $(H_t)_t$. It may happen that $(H_t)_t$ admits no other units even though $\mathcal{F}_{s,t} \ne \mathcal{F}_{s,t}^{\text{white}}$, $H_t \ne H_t^{\text{white}}$. Then $(H_t)_t$ is of type II (units generate a nontrivial, proper subsystem), namely, of type II_1; $(H_t^{\text{white}})_t$ is the classical part of $(H_t)_t$, and the white noise is the classical part of the given noise. The automorphisms $\Theta^{\text{trivial}}(\lambda)$ and $\Theta^{\text{shift}(\mathrm{i}\lambda)}$ for $\lambda \in \mathbb{R}$ can be extended naturally from the classical part to the whole system (which does not exclude other possible extensions). For $\Theta^{\text{shift}(\lambda)}$ and $\Theta^{\text{rotat}(\lambda)}$ we have no evident extension. Moreover, these automorphisms need not have any extensions, as will be proved.

Two examples found by Warren [11], [12] are 'the noise of splitting' and 'the noise of stickiness'; see also [13] and [9, Section 2]. For the noise of splitting the gauge group restricted to the classical part covers all shifts of $\mathbb{C}$ (but only trivial rotations [10]), thus, it acts transitively on $\mathbb{C}$, therefore, on normalized units as well.

A new (third) example is introduced in Section 10 for proving the main result formulated as follows.

Theorem 1.10. *There exists an Arveson system of type II_1 such that the action of the group of automorphisms on the set of normalized units is not transitive.*

The proof is given in Section 11, after the formulation of Proposition 11.1.

The first version [8] of this paper raised some doubts [3, p. 6]. Hopefully they will be dispelled by the present version.

First of all, in Section 2 we reformulate the problem as a problem of isomorphism. Isomorphism of some models simpler than Arveson systems are investigated in Sections 3–9. In Section 11 we reduce the problem for Arveson systems to the problem for the simpler models. In combination with the new noise of Section 10 it proves Theorem 1.10.

2. Extensions of automorphisms and isomorphisms of extensions

Assume that a given noise $\big((\Omega,\mathcal{F},P),(\mathcal{F}_{s,t}),(T_h)\big)$ is an extension of the white noise (see (1.8) and the explanation after it) generated by a given Brownian motion $(B_t)_t$ adapted to the given noise. Assume that another noise $\big((\Omega',\mathcal{F}',P'),(\mathcal{F}'_{s,t}),(T'_h)\big)$ is also an extension of the white noise, according to a given adapted Brownian motion $(B'_t)_t$. On the level of Arveson systems we have two extensions of the type I_1 system:

$$H_t \supset H_t^{\text{white}}, \quad H'_t \supset H'^{\,\text{white}}_t;$$

here $H_t = L_2(\Omega, \mathcal{F}_{0,t}, P)$, $H_t^{\text{white}} = L_2(\Omega, \mathcal{F}_{0,t}^{\text{white}}, P)$, $\mathcal{F}_{0,t}^{\text{white}}$ being generated by the restriction of B to $[0,t]$. (H'_t and $H'^{\,\text{white}}_t$ are defined similarly.)

An isomorphism between the two Arveson systems $(H_t)_t$, $(H'_t)_t$ is defined similarly to 1.5 ($\Theta_t : H_t \to H'_t$, $\Theta_{s+t} = \Theta_s \otimes \Theta_t$, and the Borel measurability). If it exists, it is nonunique. In contrast, the subsystems $(H_t^{\text{white}})_t$ and $(H'^{\,\text{white}}_t)_t$ are *naturally* isomorphic:

$$\Theta_t^{\text{transfer}}\big(X(B|_{[0,t]})\big) = X(B'|_{[0,t]}) \quad \text{for all } X;$$

here $B|_{[0,t]}$ is treated as a $C[0,t]$-valued random variable on Ω, distributed $\mathcal{W}_t$ (the Wiener measure); similarly, $B'|_{[0,t]}$ is a $C[0,t]$-valued random variable on Ω', distributed $\mathcal{W}_t$; and X runs over $L_2(C[0,t], \mathcal{W}_t)$.

We define an *isomorphism between extensions* as an isomorphism $(\Theta_t)_t$ between Arveson systems that extends Θ^{transfer}, that is,

$$\Theta_t|_{H_t^{\text{white}}} = \Theta_t^{\text{transfer}} \quad \text{for all } t.$$

Adding a drift to the Brownian motion $(B_t)_t$ we get a random process $(B_t + \lambda t)_t$ locally equivalent, but globally singular to the Brownian motion. In terms of noises this idea may be formalized as follows.

Let $(\Omega, \mathcal{F}, \widetilde{P})$ be a probability space, $\mathcal{F}_{s,t} \subset \mathcal{F}$ sub-σ-fields, and $(T_h)_h$ a measurable action of $\mathbb{R}$ on Ω, satisfying Conditions (b) and (c) of Definition 1.2 (but not (a)). Let P, P' be (T_h)-invariant probability measures on $(\Omega, \mathcal{F})$ such that $P + P' = 2\widetilde{P}$, and 1.2(a) holds for each of the two measures P, P'. Then we have two noises $\big((\Omega, \mathcal{F}, P), (\mathcal{F}_{s,t}), (T_h)\big)$, $\big((\Omega, \mathcal{F}, P'), (\mathcal{F}_{s,t}), (T_h)\big)$. Assume also that the restrictions $P|_{\mathcal{F}_{s,t}}$ and $P'|_{\mathcal{F}_{s,t}}$ are equivalent (that is, mutually absolutely continuous) whenever $s < t$. This relation between two noises may be called a change of measure. The corresponding Arveson systems are naturally isomorphic (via multiplication by the Radon–Nikodym derivative):

$$\Theta_t^{\text{change}} : H_t \to H'_t, \quad \Theta_t^{\text{change}} \psi = D_t^{-1/2} \psi, \quad D_t = \frac{\mathrm{d}P'|_{\mathcal{F}_{0,t}}}{\mathrm{d}P|_{\mathcal{F}_{0,t}}}.$$

We are especially interested in a change of measure such that (recall (1.7))

$$D_t = \exp\big(2\lambda B_t - 2\lambda^2 t\big) \quad \text{for } t \in (0, \infty),$$

where $(B_t)_t$ is a Brownian motion adapted to the first noise, and $\lambda \in \mathbb{R}$ a given number. In this case $(B_t - 2\lambda t)_t$ is a Brownian motion adapted to the second noise. We take $B'_t = B_t - 2\lambda t$ and get two extensions of the white noise. In such a situation we say that the second extension results from the first one by the drift 2λ, denote Θ_t^{change} by $\Theta_t^{\text{change}(\lambda)}$ and $\Theta_t^{\text{transfer}}$ by $\Theta_t^{\text{transfer}(\lambda)}$.

Note that

$$\Theta_t^{\mathrm{transfer}(\lambda)}\big(X(B|_{[0,t]})\big) = (X \circ \theta_t^\lambda)(B|_{[0,t]})$$

for $X \in L_2(C[0,t], \mathcal{W}_t)$; as before, $\theta_t^\lambda : C[0,t] \to C[0,t]$ is the drift transformation, $(\theta_t^\lambda b)(s) = b(s) - 2\lambda s$ for $s \in [0,t]$, it sends the measure $D_t \cdot \mathcal{W}_t$ to $\mathcal{W}_t$.

The isomorphism $\Theta^{\mathrm{change}(\lambda)}$ between the two Arveson systems $(H_t)_t$, $(H'_t)_t$ is not an isomorphism of extensions (unless $\lambda = 0$), since its restiction to $(H_t^{\mathrm{white}})_t$ is not equal to $\Theta^{\mathrm{transfer}(\lambda)}$. Instead, by the lemma below, they are related via the automorphism $\Theta^{\mathrm{shift}(\lambda)}$ of $(H_t^{\mathrm{white}})_t$ introduced in Section 1.

Lemma 2.1.

$$\Theta^{\mathrm{change}(\lambda)}\Theta^{\mathrm{shift}(\lambda)} = \Theta^{\mathrm{transfer}(\lambda)},$$

that is,

$$\Theta_t^{\mathrm{change}(\lambda)}\Theta_t^{\mathrm{shift}(\lambda)}\psi = \Theta_t^{\mathrm{transfer}(\lambda)}\psi$$

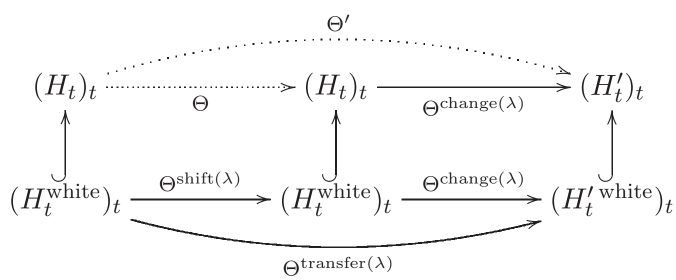
for all $\psi \in H_t^{\mathrm{white}}$ *and all* $t \in (0,\infty)$.

Proof. We take $X \in L_2(C[0,t], \mathcal{W}_t)$ such that $\psi = X\big(B|_{[0,t]}\big)$, then

$$\begin{aligned}\Theta_t^{\mathrm{change}(\lambda)}\Theta_t^{\mathrm{shift}(\lambda)}\psi &= D_t^{-1/2} \cdot D_t^{1/2} \cdot (X \circ \theta_t^\lambda)(B|_{[0,t]}) \\ &= (X \circ \theta_t^\lambda)(B|_{[0,t]}) = \Theta_t^{\mathrm{transfer}(\lambda)}\psi.\end{aligned}$$

□

The situation is shown on the diagram



and we see that the following conditions are equivalent:

- There exists an automorphism Θ of $(H_t)_t$ that extends $\Theta^{\mathrm{shift}(\lambda)}$.
- There exists an isomorphism Θ' between $(H_t)_t$ and $(H'_t)_t$ that extends $\Theta^{\mathrm{transfer}(\lambda)}$.

In other words, $\Theta^{\mathrm{shift}(\lambda)}$ can be extended to $(H_t)_t$ if and only if the two extensions of the type I_1 system are isomorphic.

Corollary 2.2. *In order to prove Theorem* 1.10 *it is sufficient to construct a noise, extending the white noise, such that for every* $\lambda \in \mathbb{R} \setminus \{0\}$ *the extension obtained by the drift* λ *is nonisomorphic to the original extension on the level of Arveson systems* (*that is, the corresponding extensions of the type* I_1 *Arveson system are nonisomorphic*).

Proof. In the group of all motions of the complex plane we consider the subgroup G of motions that correspond to automorphisms of $(H_t^{\text{white}})_t$ extendable to $(H_t)_t$. Real shifts $z \mapsto z + \lambda$ (for $\lambda \in \mathbb{R} \setminus \{0\}$) do not belong to G, as explained above. Imaginary shifts $z \mapsto z + \mathrm{i}\lambda$ (for $\lambda \in \mathbb{R}$) belong to G, since the operators $\Theta_t^{\text{shift}(\mathrm{i}\lambda)}$ of multiplication by $\exp(\mathrm{i}\lambda B_t)$ act naturally on H_t. It follows that G contains no rotations (except for the rotation by π) and therefore is not transitive. □

Thus, we need a *drift sensitive* extension. Such extension is constructed in Section 10 and its drift sensitivity is proved in Section 11.

3. Toy models: Hilbert spaces

Definitions and statements of Sections 3 and 4 will not be used formally, but probably help to understand the idea.

The phenomenon of a nonextendable isomorphism (as well as nonisomorphic extensions) is demonstrated in this section by a toy model, — a kind of product system of Hilbert spaces, simpler than Arveson system.

Definition 3.1. A *toy product system of Hilbert spaces* is a triple

$$(H_1, H_\infty, U),$$

where H_1, H_∞ are Hilbert spaces (over $\mathbb{C}$, separable), and

$$U : H_1 \otimes H_\infty \to H_\infty$$

is a unitary operator.

We treat it as a kind of product system, since

$$H_\infty \sim H_1 \otimes H_\infty \sim H_1 \otimes H_1 \otimes H_\infty \sim \cdots$$

where '$\sim$' means: may be identified naturally (using U).

An evident example: $H_\infty = (H_1, \psi_1)^{\otimes\infty}$ is the infinite tensor product of (an infinite sequence of) copies of H_1 relatively to (the copies of) a given vector $\psi_1 \in H_1$, $\|\psi_1\| = 1$. The equation $U(\psi \otimes \xi) = \xi$ has exactly one solution: $\psi = \psi_1$, $\xi = \psi_1^{\otimes\infty}$.

An uninteresting modification: $H_\infty = (H_1, \psi_1)^{\otimes\infty} \otimes H_0$ for some Hilbert space H_0.

A more interesting example: $H_\infty = (H_1, \psi_1)^{\otimes\infty} \oplus (H_1, \psi_2)^{\otimes\infty}$ is the direct sum of two such infinite tensor products, one relative to ψ_1, the other relative to another vector $\psi_2 \in H_1$, $\|\psi_2\| = 1$, $\psi_2 \neq \psi_1$. The equation $U(\psi \otimes \xi) = \xi$ has exactly two solutions: $\psi = \psi_1$, $\xi = \psi_1^{\otimes\infty}$ and $\psi = \psi_2$, $\xi = \psi_2^{\otimes\infty}$.

Definition 3.2. Let (H_1, H_∞, U) and (H_1', H_∞', U') be toy product systems of Hilbert spaces. An *isomorphism* between them is a pair $\Theta = (\Theta_1, \Theta_\infty)$ of unitary operators $\Theta_1 : H_1 \to H_1'$, $\Theta_\infty : H_\infty \to H_\infty'$ such that the diagram

$$\begin{array}{ccc} H_1 \otimes H_\infty & \xrightarrow{U} & H_\infty \\ \downarrow{\scriptstyle \Theta_1 \otimes \Theta_\infty} & & \downarrow{\scriptstyle \Theta_\infty} \\ H_1' \otimes H_\infty' & \xrightarrow{U'} & H_\infty' \end{array}$$

is commutative.

Thus,

$$\Theta_\infty \sim \Theta_1 \otimes \Theta_\infty \sim \Theta_1 \otimes \Theta_1 \otimes \Theta_\infty \sim \cdots$$

A unitary operator $\Theta_1 : H_1 \to H_1$ leads to an automorphism of

$$(H_1, \psi_1)^{\otimes\infty}$$

(that is, of the corresponding toy product system) if and only if $\Theta_1 \psi_1 = \psi_1$. Similarly, Θ_1 leads to an automorphism of $(H_1, \psi_1)^{\otimes\infty} \oplus (H_1, \psi_2)^{\otimes\infty}$ if and only if either $\Theta_1 \psi_1 = \psi_1$ and $\Theta_1 \psi_2 = \psi_2$, or $\Theta_1 \psi_1 = \psi_2$ and $\Theta_1 \psi_2 = \psi_1$.

Taking Θ_1 such that $\Theta_1 \psi_1 = \psi_1$ but $\Theta_1 \psi_2 \neq \psi_2$ we get an automorphism of $(H_1, \psi_1)^{\otimes\infty}$ that cannot be extended to an automorphism of

$$(H_1, \psi_1)^{\otimes\infty} \oplus (H_1, \psi_2)^{\otimes\infty}.$$

Similarly to Section 2 we may turn from extensions of automorphisms to isomorphisms of extensions. The system $(H_1, \psi_1)^{\otimes\infty} \oplus (H_1, \psi_2)^{\otimes\infty}$ is an extension of $(H_1, \psi_1)^{\otimes\infty}$ (in the evident sense). Another vector ψ_2' leads to another extension of $(H_1, \psi_1)^{\otimes\infty}$. We define an isomorphism between the two extensions as an isomorphism $(\Theta_1, \Theta_\infty)$ between the toy product systems $(H_1, \psi_1)^{\otimes\infty} \oplus (H_1, \psi_2)^{\otimes\infty}$ and $(H_1, \psi_1)^{\otimes\infty} \oplus (H_1, \psi_2')^{\otimes\infty}$ whose restriction to $(H_1, \psi_1)^{\otimes\infty}$ is trivial (the identity):

$$\begin{array}{ccc} (H_1, \psi_1)^{\otimes\infty} \oplus (H_1, \psi_2)^{\otimes\infty} & \xleftrightarrow{(\Theta_1, \Theta_\infty)} & (H_1, \psi_1)^{\otimes\infty} \oplus (H_1, \psi_2')^{\otimes\infty} \\ \nwarrow & & \nearrow \\ & (H_1, \psi_1)^{\otimes\infty}. & \end{array}$$

Clearly, Θ_1 must be trivial; therefore ψ_2' must be equal to ψ_2. Otherwise the two extensions are nonisomorphic.

4. Toy models: probability spaces

Definition 4.1. A *toy product system of probability spaces* is a triple

$$(\Omega_1, \Omega_\infty, \alpha),$$

where Ω_1, Ω_∞ are probability spaces (standard), and $\alpha : \Omega_1 \times \Omega_\infty \to \Omega_\infty$ is an isomorphism mod 0 (that is, an invertible measure preserving map).

Every toy product system of probability spaces $(\Omega_1, \Omega_\infty, \alpha)$ leads to a toy product system of Hilbert spaces (H_1, H_∞, U) as follows:

$$H_1 = L_2(\Omega_1); \quad H_\infty = L_2(\Omega_\infty);$$
$$(U\psi)(\cdot) = \psi(\alpha^{-1}(\cdot)).$$

Here we use the canonical identification

$$L_2(\Omega_1) \otimes L_2(\Omega_\infty) = L_2(\Omega_1 \times \Omega_\infty)$$

and treat a vector $\psi \in H_1 \otimes H_\infty$ as an element of $L_2(\Omega_1 \times \Omega_\infty)$.

An evident example: $\Omega_\infty = \Omega_1^\infty$ is the product of an infinite sequence of copies of Ω_1. It leads to $H_\infty = (H_1, \mathbf{1})^{\otimes\infty}$ where $H_1 = L_2(\Omega_1)$ and $\mathbf{1} \in L_2(\Omega_1)$ is the constant function, $\mathbf{1}(\cdot) = 1$.

An uninteresting modification: $\Omega_\infty = \Omega_1^\infty \times \Omega_0$ for some probability space Ω_0. It leads to $H_\infty = (H_1, \mathbf{1})^{\otimes\infty} \otimes H_0$, $H_0 = L_2(\Omega_0)$.

Here is a more interesting example. Let $X_1 : \Omega_1 \to \{-1, +1\}$ be a measurable function (not a constant). We define Ω_∞ as the set of all double sequences $\left(\begin{smallmatrix} \omega_1, & \omega_2, & \dots \\ s_1, & s_2, & \dots \end{smallmatrix}\right)$ such that $\omega_k \in \Omega_1$, $s_k \in \{-1, +1\}$ and $s_k = s_{k+1} X_1(\omega_k)$ for all k. Sequences $(\omega_1, \omega_2, \dots) \in \Omega_1^\infty$ are endowed with the product measure. The conditional distribution of the sequence $(s_1, s_2, \dots)$, given $(\omega_1, \omega_2, \dots)$, must be concentrated on the two sequences obeying the relation $s_k = s_{k+1} X_1(\omega_k)$. We give to these two sequences equal conditional probabilities, 0.5 to each. Thus, Ω_∞ is endowed with a probability measure. The map $\alpha : \Omega_1 \times \Omega_\infty \to \Omega_\infty$ is defined by

$$\alpha\left(\omega_1, \begin{pmatrix} \omega_2, & \omega_3, & \dots \\ s_2, & s_3, & \dots \end{pmatrix}\right) = \begin{pmatrix} \omega_1, & \omega_2, & \omega_3, & \dots \\ s_2 X_1(\omega_1), & s_2, & s_3, & \dots \end{pmatrix}.$$

Clearly, α is measure preserving.

This system $(\Omega_1, \Omega_\infty, \alpha)$ leads to a system (H_1, H_∞, U) of the form

$$(H_1, \psi_1)^{\otimes\infty} \oplus (H_1, \psi_2)^{\otimes\infty}$$

(up to isomorphism), as explained below. We have

$$H_1 = L_2(\Omega_1), \quad H_\infty = L_2(\Omega_\infty),$$
$$(U\psi)\begin{pmatrix} \omega_1, & \omega_2, & \omega_3, & \dots \\ s_1, & s_2, & s_3, & \dots \end{pmatrix} = \psi\left(\omega_1, \begin{pmatrix} \omega_2, & \omega_3, & \dots \\ s_2, & s_3, & \dots \end{pmatrix}\right).$$

The equation $U(\psi \otimes \xi) = \xi$ becomes

$$\psi(\omega_1)\xi\begin{pmatrix} \omega_2, & \omega_3, & \dots \\ s_2, & s_3, & \dots \end{pmatrix} = \xi\begin{pmatrix} \omega_1, & \omega_2, & \omega_3, & \dots \\ s_1, & s_2, & s_3, & \dots \end{pmatrix}.$$

One solution is evident: $\psi = \mathbf{1}_{\Omega_1}$, $\xi = \mathbf{1}_{\Omega_\infty}$. A less evident solution is, $\psi = X_1$, $\xi = S_1$, where S_1 is defined by $S_1\left(\begin{smallmatrix} \omega_1, & \omega_2, & \dots \\ s_1, & s_2, & \dots \end{smallmatrix}\right) = s_1$. (The equation is satisfied due to the relation $X_1(\omega_1)s_2 = s_1$.) We consider the system (H_1', H_∞', U') where $H_1' = H_1 = L_2(\Omega_1)$, $H_\infty' = (H_1', \mathbf{1}_{\Omega_1})^{\otimes\infty} \oplus (H_1', X_1)^{\otimes\infty}$

(U' being defined naturally) and construct an isomorphism $(\Theta_1, \Theta_\infty)$ between (H_1, H_∞, U) and (H_1', H_∞', U') such that

$$\Theta_\infty \mathbf{1}_{\Omega_\infty} = \mathbf{1}_{\Omega_1}^{\otimes\infty},$$
$$\Theta_\infty S_1 = X_1^{\otimes\infty}.$$

To this end we consider an arbitrary n and $\xi \in L_2(\Omega_1^n) = H_1^{\otimes n}$, define $\varphi, \psi \in L_2(\Omega_\infty)$ by

$$\varphi \begin{pmatrix} \omega_1, & \omega_2, & \dots \\ s_1, & s_2, & \dots \end{pmatrix} = \xi(\omega_1, \dots, \omega_n),$$
$$\psi \begin{pmatrix} \omega_1, & \omega_2, & \dots \\ s_1, & s_2, & \dots \end{pmatrix} = s_{n+1}\xi(\omega_1, \dots, \omega_n)$$

and, using the relation (or rather, the natural isomorphism)

$$H_\infty' = (H_1')^{\otimes n} \otimes H_\infty',$$

we let

$$\Theta_\infty \varphi = \xi \otimes \mathbf{1}_{\Omega_1}^{\otimes\infty}, \quad \Theta_\infty \psi = \xi \otimes X_1^{\otimes\infty},$$

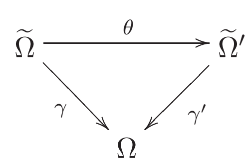
thus defining a unitary $\Theta_\infty : H_\infty \to H_\infty'$. (Further details are left to the reader.)

A more general construction is introduced in Section 5.

5. Binary extensions: probability spaces

Definition 5.1. (a) An *extension* of a probability space Ω consists of another probability space $\widetilde{\Omega}$ and a measure preserving map $\gamma : \widetilde{\Omega} \to \Omega$.

(b) Two extensions $(\widetilde{\Omega}, \gamma)$ and $(\widetilde{\Omega}', \gamma')$ of a probability space Ω are *isomorphic,* if there exists an invertible (mod 0) measure preserving map $\theta : \widetilde{\Omega} \to \widetilde{\Omega}'$ such that the diagram



is commutative. (Such θ will be called an *isomorphism of extensions.*)

(c) An extension of a probability space Ω is *binary,* if it is isomorphic to $(\Omega \times \Omega_\pm, \gamma)$, where $\Omega_\pm = \{-1, +1\}$ consists of two equiprobable atoms, and $\gamma : \Omega \times \Omega_\pm \to \Omega$ is the projection, $(\omega, s) \mapsto \omega$.

By a well-known theorem of V. Rokhlin, an extension is binary if and only if conditional measures consist of two atoms of probability 0.5. However, this fact will not be used.

Interchanging the two atoms we get an involution on $\widetilde{\Omega}$. Denoting it by $\widetilde{\omega} \mapsto -\widetilde{\omega}$ we have

$$-\widetilde{\omega} \neq \widetilde{\omega}, \quad -(-\widetilde{\omega}) = \widetilde{\omega}, \quad \gamma(-\widetilde{\omega}) = \gamma(\widetilde{\omega}) \quad \text{for } \widetilde{\omega} \in \widetilde{\Omega};$$

these properties characterize the involution. In the case $\widetilde{\Omega} = \Omega \times \Omega_\pm$ we have $-(\omega, s) = (\omega, -s)$ for $\omega \in \Omega$, $s = \pm 1$.

An isomorphism between two binary extensions boils down to an automorphism of $(\Omega \times \Omega_\pm, \gamma)$. The general form of such automorphism is $(\omega, s) \mapsto (\omega, sU(\omega))$ for $\omega \in \Omega$, $s = \pm 1$; here U runs over measurable functions $\Omega \to \{-1, +1\}$. The automorphism commutes with the involution, thus, every isomorphism of extensions intertwines the involutions,

$$\theta(-\widetilde{\omega}) = -\theta(\widetilde{\omega}) \quad \text{for } \widetilde{\omega} \in \widetilde{\Omega}.$$

Definition 5.2. (a) An *inductive system of probability spaces* consists of probability spaces Ω_n and measure preserving maps $\beta_n : \Omega_n \to \Omega_{n+1}$ for $n = 1, 2, \dots$

(b) Let $(\Omega_n, \beta_n)_n$ and $(\Omega'_n, \beta'_n)_n$ be two inductive systems of probability spaces. A *morphism* from $(\Omega_n, \beta_n)_n$ to $(\Omega'_n, \beta'_n)_n$ is a sequence of measure preserving maps $\gamma_n : \Omega_n \to \Omega'_n$ such that the infinite diagram

$$\begin{array}{ccccc} \Omega_1 & \xrightarrow{\beta_1} & \Omega_2 & \xrightarrow{\beta_2} & \cdots \\ \downarrow{\scriptstyle \gamma_1} & & \downarrow{\scriptstyle \gamma_2} & & \\ \Omega'_1 & \xrightarrow{\beta'_1} & \Omega'_2 & \xrightarrow{\beta'_2} & \cdots \end{array}$$

is commutative. If each γ_n is invertible, the morphism is an *isomorphism.*

(c) A morphism $(\gamma_n)_n$ is *binary,* if for every n the extension (Ω_n, γ_n) of Ω'_n is binary, and each β_n intertwines the corresponding involutions,

$$\beta_n(-\omega_n) = -\beta_n(\omega_n) \quad \text{for } \omega_n \in \Omega_n.$$

Given a binary morphism $(\gamma_n)_n$ from $(\Omega_n, \beta_n)_n$ to $(\Omega'_n, \beta'_n)_n$, we say that $(\Omega_n, \beta_n)_n$ is a *binary extension* of $(\Omega'_n, \beta'_n)_n$ (according to $(\gamma_n)_n$).

Definition 5.3. Let $(\Omega_n, \beta_n)_n$ be an inductive system of probability spaces, $(\widetilde{\Omega}_n, \widetilde{\beta}_n)_n$ its binary extension (according to $(\gamma_n)_n$), and $(\widetilde{\Omega}'_n, \widetilde{\beta}'_n)_n$ another binary extension of $(\Omega_n, \beta_n)_n$ (according to $(\gamma'_n)_n$). An *isomorphism* between the two binary extensions is an isomorphism $(\theta_n)_n$ between $(\widetilde{\Omega}_n, \widetilde{\beta}_n)_n$ and $(\widetilde{\Omega}'_n, \widetilde{\beta}'_n)_n$ treated as inductive systems of probability spaces, satisfying the following condition: for each n the diagram

$$\begin{array}{ccc} \widetilde{\Omega}_n & \xrightarrow{\theta_n} & \widetilde{\Omega}'_n \\ & {\scriptstyle \gamma_n} \searrow \quad \swarrow {\scriptstyle \gamma'_n} & \\ & \Omega_n & \end{array}$$

is commutative.

In other words, an isomorphism between the two binary extensions of the inductive system is a sequence $(\theta_n)_n$ where each θ_n is an isomorphism between the two binary extensions $(\widetilde{\Omega}_n, \widetilde{\gamma}_n)$ and $(\widetilde{\Omega}'_n, \widetilde{\gamma}'_n)$ of the probability space Ω_n, such that the diagram

$$\begin{array}{ccc} \widetilde{\Omega}_n & \xrightarrow{\widetilde{\beta}_n} & \widetilde{\Omega}_{n+1} \\ \downarrow{\scriptstyle \theta_n} & & \downarrow{\scriptstyle \theta_{n+1}} \\ \widetilde{\Omega}'_n & \xrightarrow{\widetilde{\beta}'_n} & \widetilde{\Omega}'_{n+1} \end{array}$$

is commutative for every n.

Lemma 5.4. *Let $(\Omega_n, \beta_n)_n$ be an inductive system of probability spaces.*

(a) *Let $X_n : \Omega_n \to \{-1, +1\}$ be measurable functions, and*

$$\widetilde{\Omega}_n = \Omega_n \times \Omega_\pm,$$

$$\left.\begin{aligned} \widetilde{\beta}_n(\omega_n, s_n) &= \big(\beta_n(\omega_n), s_n X_n(\omega_n)\big) \\ \gamma_n(\omega_n, s_n) &= \omega_n \end{aligned}\right\} \text{ for } \omega_n \in \Omega_n,\ s_n = \pm 1. \tag{5.5}$$

Then $(\widetilde{\Omega}_n, \widetilde{\beta}_n)$ is a binary extension of $(\Omega_n, \beta_n)_n$ (according to $(\gamma_n)_n$).

(b) *Every binary extension of $(\Omega_n, \beta_n)_n$ is isomorphic to the extension of the form* (5.5)*, for some $(X_n)_n$.*

Proof. (a) Clearly, $\widetilde{\beta}_n$ and γ_n are measure preserving, γ_n is binary, and $\gamma_{n+1}(\widetilde{\beta}_n(\omega_n, s_n)) = \beta_n(\omega_n) = \beta_n(\gamma_n(\omega_n, s_n))$.

(b) Let $(\widetilde{\Omega}_n, \widetilde{\beta}_n)_n$ be a binary extension of $(\Omega_n, \beta_n)_n$ according to $(\gamma_n)_n$. Without loss of generality we assume $\widetilde{\Omega}_n = \Omega_n \times \Omega_\pm$ and $\gamma_n(\omega_n, s_n) = \omega_n$. The relations

$$\gamma_{n+1}(\widetilde{\beta}_n(\omega_n, s_n)) = \beta_n(\gamma_n(\omega_n, s_n)) = \beta_n(\omega_n),$$
$$\widetilde{\beta}_n(-\omega_n) = -\widetilde{\beta}_n(\omega_n),$$

show that $\widetilde{\beta}_n$ is of the form $\widetilde{\beta}_n(\omega_n, s_n) = \big(\beta_n(\omega_n), s_n X_n(\omega_n)\big)$ for some measurable $X_n : \Omega_n \to \{-1, +1\}$. □

Given an inductive system $(\Omega_n, \beta_n)_n$ of probability spaces and two sequences $(X_n)_n$, $(Y_n)_n$ of measurable functions $X_n, Y_n : \Omega_n \to \{-1, +1\}$, the construction (5.5) gives us two binary extensions of $(\Omega_n, \beta_n)_n$. One extension, $(\widetilde{\Omega}_n, \widetilde{\beta}_n)_n$, $(\gamma_n)_n$, corresponds to $(X_n)_n$, the other extension, $(\widetilde{\Omega}'_n, \widetilde{\beta}'_n)_n$, $(\gamma'_n)_n$, corresponds to $(Y_n)_n$. We want to know, whether they are isomorphic or not.

For each n separately, the two binary extensions of the probability space Ω_n coincide: $\widetilde{\Omega}_n = \Omega_n \times \Omega_\pm = \widetilde{\Omega}'_n$, $\gamma_n(\omega_n, s_n) = \omega_n = \gamma'_n(\omega_n, s_n)$. Every isomorphism θ_n between them is of the form

$$\theta_n(\omega_n, s_n) = \big(\omega_n, s_n U_n(\omega_n)\big) \quad \text{for } \omega_n \in \Omega_n,\ s_n = \pm 1,$$

where $U_n : \Omega_n \to \{-1, +1\}$ is a measurable function. In order to form an isomorphism between the binary extensions of the inductive system, these

θ_n must satisfy the condition $\theta_{n+1}(\widetilde{\beta}_n(\widetilde{\omega}_n)) = \widetilde{\beta}'_n(\theta_n(\widetilde{\omega}_n))$, that is (recall (5.5)),

$$X_n(\omega_n)U_{n+1}(\beta_n(\omega_n)) = U_n(\omega_n)Y_n(\omega_n) \quad \text{for } \omega_n \in \Omega_n.$$

Given an inductive system $(\Omega_n, \beta_n)_n$ of probability spaces, we consider the commutative group $G((\Omega_n, \beta_n)_n)$ of all sequences $f = (f_n)_n$ of measurable functions $f_n : \Omega_n \to \{-1, +1\}$ treated mod 0; the group operation is the pointwise multiplication. We define the shift homomorphism $T : G((\Omega_n, \beta_n)_n) \to G((\Omega_n, \beta_n)_n)$ by

$$(Tf)_n(\omega_n) = f_{n+1}\big(\beta_n(\omega_n)\big) \quad \text{for } \omega_n \in \Omega_n.$$

According to (5.5), every $X \in G((\Omega_n, \beta_n)_n)$ leads to a binary extension of $(\Omega_n, \beta_n)_n$. We summarize the previous paragraph as follows.

Lemma 5.6. *The binary extensions corresponding to $X, Y \in G((\Omega_n, \beta_n)_n)$ are isomorphic if and only if $XT(U) = YU$ for some $U \in G((\Omega_n, \beta_n)_n)$.*

6. Binary extensions: Hilbert spaces

Given an extension of a probability space, $\gamma : \widetilde{\Omega} \to \Omega$, we have a natural embedding of Hilbert spaces, $L_2(\Omega) \subset L_2(\widetilde{\Omega})$, and a natural action of the commutative algebra $L_\infty(\Omega)$ on $L_2(\widetilde{\Omega})$. (L_2 and L_∞ over $\mathbb{C}$ are meant.) Assume that the extension is binary. Then the embedded subspace and its orthogonal complement are the 'even' and 'odd' subspaces w.r.t. the involution $\widetilde{\omega} \mapsto -\widetilde{\omega}$ on $\widetilde{\Omega}$; that is,

$$\psi \in L_2(\Omega) \quad \text{if and only if} \quad \psi(-\widetilde{\omega}) = \psi(\widetilde{\omega}) \text{ for almost all } \widetilde{\omega} \in \widetilde{\Omega};$$

$$\psi \in L_2(\widetilde{\Omega}) \ominus L_2(\Omega) \quad \text{if and only if} \quad \psi(-\widetilde{\omega}) = -\psi(\widetilde{\omega}) \text{ for almost all } \widetilde{\omega} \in \widetilde{\Omega}.$$

Lemma 6.1. *Let $\gamma : \widetilde{\Omega} \to \Omega$ and $\gamma' : \widetilde{\Omega}' \to \Omega$ be two binary extensions of a probability space Ω. Then the following two conditions on a unitary operator $\Theta : L_2(\widetilde{\Omega}') \to L_2(\widetilde{\Omega})$ are equivalent:*

(a) *Θ is trivial on $L_2(\Omega)$, and intertwines the two actions of $L_\infty(\Omega)$. In other words,*

$$\Theta\psi = \psi \quad \text{for all } \psi \in L_2(\Omega),$$

$$\Theta(h \cdot \psi) = h \cdot (\Theta\psi) \quad \text{for all } \psi \in L_2(\widetilde{\Omega}'),\ h \in L_\infty(\Omega).$$

(b) *There exists an isomorphism of extensions $\theta : \widetilde{\Omega} \to \widetilde{\Omega}'$ and $h \in L_\infty(\Omega)$, $|h(\cdot)| = 1$, such that*

$$\Theta\psi = \psi \circ \theta \quad \text{for all } \psi \in L_2(\Omega),$$

$$\Theta\psi = h \cdot (\psi \circ \theta) \quad \text{for all } \psi \in L_2(\widetilde{\Omega}') \ominus L_2(\Omega).$$

Proof. (b) $\Longrightarrow$ (a): evident. (a) $\Longrightarrow$ (b): Without loss of generality we assume that $\widetilde{\Omega} = \widetilde{\Omega}' = \Omega \times \Omega_\pm$ and $\gamma(\omega, s) = \gamma'(\omega, s) = \omega$. The Hilbert space $L_2(\widetilde{\Omega}) \ominus L_2(\Omega)$ consists of functions of the form $(\omega, s) \mapsto sf(\omega)$ where f runs over $L_2(\Omega)$. Thus, $L_2(\widetilde{\Omega}) \ominus L_2(\Omega)$ is naturally isomorphic to $L_2(\Omega)$, and the isomorphism intertwines the actions of $L_\infty(\Omega)$. The operator Θ maps $L_2(\widetilde{\Omega}') \ominus L_2(\Omega)$ onto $L_2(\widetilde{\Omega}) \ominus L_2(\Omega)$ and leads to an operator $L_2(\Omega) \to L_2(\Omega)$ that commutes with $L_\infty(\Omega)$ and is therefore the multiplication by a function $h \in L_\infty(\Omega)$. □

An inductive system of probability spaces $(\Omega_n, \beta_n)_n$ leads evidently to a decreasing sequence of Hilbert spaces,

$$L_2(\Omega_1) \hookleftarrow L_2(\Omega_2) \hookleftarrow \cdots$$

Similarly, a morphism from $(\Omega_n, \beta_n)_n$ to $(\Omega'_n, \beta'_n)_n$ leads to a commutative diagram of Hilbert space embeddings

$$\begin{array}{ccccc} L_2(\Omega_1) & \hookleftarrow & L_2(\Omega_2) & \hookleftarrow & \cdots \\ \uparrow & & \uparrow & & \\ L_2(\Omega'_1) & \hookleftarrow & L_2(\Omega'_2) & \hookleftarrow & \cdots \end{array}$$

The commutative algebra $L_\infty(\Omega'_n)$ acts on $L_2(\Omega'_n)$ and $L_2(\Omega_n)$, and the embedding $L_2(\Omega'_n) \to L_2(\Omega_n)$ intertwines these two actions.

Lemma 6.2. *Let $(\Omega_n, \beta_n)_n$ be an inductive system of probability spaces, $(\widetilde{\Omega}_n, \widetilde{\beta}_n)_n$ its binary extension (according to $(\gamma_n)_n$), and $(\widetilde{\Omega}'_n, \widetilde{\beta}'_n)_n$ another binary extension of $(\Omega_n, \beta_n)_n$ (according to $(\gamma'_n)_n$). Then the following two conditions are equivalent:*

(a) *The two binary extensions are isomorphic.*
(b) *There exist unitary operators*

$$\Theta_n : L_2(\widetilde{\Omega}'_n) \to L_2(\widetilde{\Omega}_n)$$

such that for every n, Θ_n intertwines the actions of $L_\infty(\Omega_n)$ on $L_2(\widetilde{\Omega}_n)$ and $L_2(\widetilde{\Omega}'_n)$, and the following two diagrams are commutative:

$$\begin{array}{ccc} L_2(\widetilde{\Omega}_n) & \xleftarrow{\Theta_n} & L_2(\widetilde{\Omega}'_n) \\ \nwarrow & & \nearrow \\ & L_2(\Omega_n) & \end{array} \qquad \begin{array}{ccc} L_2(\widetilde{\Omega}_n) & \hookleftarrow & L_2(\widetilde{\Omega}_{n+1}) \\ \uparrow \Theta_n & & \uparrow \Theta_{n+1} \\ L_2(\widetilde{\Omega}'_n) & \hookleftarrow & L_2(\widetilde{\Omega}'_{n+1}). \end{array}$$

Proof. (a) $\Longrightarrow$ (b): evident. (b) $\Longrightarrow$ (a): For each n separately we have two binary extensions $(\widetilde{\Omega}_n, \gamma_n)$, $(\widetilde{\Omega}'_n, \gamma'_n)$ of the probability space Ω_n, and a unitary operator $\Theta_n : L_2(\widetilde{\Omega}'_n) \to L_2(\widetilde{\Omega}_n)$ that satisfies Condition (a) of

Lemma 6.1. On the other hand, due to Lemma 5.4 we may assume that

$$\begin{aligned}\widetilde{\Omega}_n &= \Omega_n \times \Omega_\pm = \widetilde{\Omega}'_n,\\ \gamma_n(\omega_n, s_n) &= \omega_n = \gamma'_n(\omega_n, s_n),\\ \widetilde{\beta}_n(\omega_n, s_n) &= \big(\beta_n(\omega_n), s_n X_n(\omega_n)\big),\\ \widetilde{\beta}'_n(\omega_n, s_n) &= \big(\beta_n(\omega_n), s_n Y_n(\omega_n)\big).\end{aligned}$$

Now Lemma 6.1 gives us $h_n \in L_\infty(\Omega_n)$, $|h_n(\cdot)| = 1$, such that

$$\Theta_n \psi = h_n \cdot \psi$$

for all $\psi \in L_2(\Omega_n \times \Omega_\pm) \ominus L_2(\Omega_n)$. In other words, if $\psi(\omega_n, s_n) = s_n f(\omega_n)$ then $(\Theta_n\psi)(\omega_n, s_n) = s_n f(\omega_n) h_n(\omega_n)$; here f runs over $L_2(\Omega_n)$. By commutativity of the second diagram, $(\Theta_{n+1}\psi)\circ\widetilde{\beta}_n = \Theta_n(\psi\circ\widetilde{\beta}'_n)$ for $\psi \in L_2(\widetilde{\Omega}'_{n+1})$. For the case $\psi(\omega_{n+1}, s_{n+1}) = s_{n+1} f(\omega_{n+1})$ we have, first,

$$\begin{aligned}\big((\Theta_{n+1}\psi)\circ\widetilde{\beta}_n\big)(\omega_n, s_n) &= (\Theta_{n+1}\psi)\big(\beta_n(\omega_n), s_n X_n(\omega_n)\big)\\ &= s_n X_n(\omega_n) f(\beta_n(\omega_n)) h_{n+1}(\beta_n(\omega_n)),\end{aligned}$$

and second,

$$\begin{aligned}(\psi\circ\widetilde{\beta}'_n)(\omega_n, s_n) = \psi\big(\beta_n(\omega_n), s_n Y_n(\omega_n)\big) &= s_n Y_n(\omega_n) f(\beta_n(\omega_n)),\\ \Theta_n(\psi\circ\widetilde{\beta}'_n)(\omega_n, s_n) &= s_n Y_n(\omega_n) f(\beta_n(\omega_n)) h_n(\omega_n).\end{aligned}$$

They are equal, which means that $X_n(\omega_n) h_{n+1}(\beta_n(\omega_n)) = Y_n(\omega_n) h_n(\omega_n)$, that is,

$$(h_{n+1}\circ\beta_n)\cdot X_n = h_n \cdot Y_n.$$

By Lemma 5.6 it is sufficient to find measurable functions

$$U_n : \Omega_n \to \{-1, +1\}$$

such that

$$(U_{n+1}\circ\beta_n)\cdot X_n = U_n \cdot Y_n \quad \text{for all } n.$$

We choose a Borel function $\varphi : \mathbb{T} \to \{-1, +1\}$, where $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$, such that $\varphi(-z) = -\varphi(z)$ for all $z \in \mathbb{T}$. For example, $\varphi(\mathrm{e}^{\mathrm{i}\alpha}) = +1$ for $\alpha \in [0, \pi)$ but -1 for $\alpha \in [\pi, 2\pi)$. The functions $U_n(\cdot) = \varphi(h_n(\cdot))$ satisfy the needed equation, since $X_n(\cdot) = \pm 1$, $Y_n(\cdot) = \pm 1$. □

7. Products of binary extensions

Definitions and statements of this section are used only in Section 11 (in the proof of Lemma 11.10).

Special measures are taken in the next definition in order to keep the product binary (rather than quaternary).

Definition 7.1. Let $(\widetilde{\Omega}_k, \gamma_k)$ be a binary extension of a probability space Ω_k for $k = 1, 2$; $\Omega = \Omega_1 \times \Omega_2$; and $A \subset \Omega$ a measurable set. The *product* of these two binary extensions (according to A) is the extension $(\widetilde{\Omega}, \gamma)$ of Ω defined as follows:

$$\widetilde{\Omega} = \underbrace{\{(\widetilde{\omega}_1, \omega_2) : (\gamma_1(\widetilde{\omega}_1), \omega_2) \in A\}}_{\widetilde{A}} \uplus \underbrace{\{(\omega_1, \widetilde{\omega}_2) : (\omega_1, \gamma_2(\widetilde{\omega}_2)) \in \Omega \setminus A\}}_{\widetilde{\Omega} \setminus \widetilde{A}},$$

the measure on $\widetilde{A}$ is induced from (the product measure on) $\widetilde{\Omega}_1 \times \Omega_2$, on $\widetilde{\Omega} \setminus \widetilde{A}$ — from $\Omega_1 \times \widetilde{\Omega}_2$;

$$\gamma(\widetilde{\omega}_1, \omega_2) = (\gamma_1(\widetilde{\omega}_1), \omega_2), \quad \gamma(\omega_1, \widetilde{\omega}_2) = (\omega_1, \gamma_2(\widetilde{\omega}_2)).$$

Here and henceforth ω_k runs over Ω_k, and $\widetilde{\omega}_k$ runs over $\widetilde{\Omega}_k$.

Clearly, the extension $(\widetilde{\Omega}, \gamma)$ is binary.

Let a binary extension $(\widetilde{\Omega}, \gamma)$ of $\Omega = \Omega_1 \times \Omega_2$ be the product of two binary extensions $(\widetilde{\Omega}_k, \gamma_k)$, $k = 1, 2$ (according to a given $A \subset \Omega$). Then we have a natural embedding of Hilbert spaces,

$$L_2(\widetilde{\Omega}) \subset L_2(\widetilde{\Omega}_1) \otimes L_2(\widetilde{\Omega}_2); \tag{7.2}$$

it arises from the natural measure preserving map $\widetilde{\Omega}_1 \times \widetilde{\Omega}_2 \to \widetilde{\Omega}$,

$$(\widetilde{\omega}_1, \widetilde{\omega}_2) \mapsto \begin{cases} (\widetilde{\omega}_1, \gamma_2(\widetilde{\omega}_2)) & \text{if } (\gamma_1(\widetilde{\omega}_1), \gamma_2(\widetilde{\omega}_2)) \in A, \\ (\gamma_1(\widetilde{\omega}_1), \widetilde{\omega}_2) & \text{otherwise.} \end{cases}$$

The restriction of the embedding (7.2) to $L_2(\Omega)$ is just the tensor product of the two embeddings $L_2(\Omega_k) \subset L_2(\widetilde{\Omega}_k)$, $k = 1, 2$, since the corresponding composition map $\widetilde{\Omega}_1 \times \widetilde{\Omega}_2 \to \widetilde{\Omega} \to \Omega$ is just $\gamma_1 \times \gamma_2$.

The projection map $\widetilde{A} \to \widetilde{\Omega}_1$, $(\widetilde{\omega}_1, \omega_2) \mapsto \widetilde{\omega}_1$, need not be measure preserving, but anyway, generates a sub-σ-field $\mathcal{F}_1$ on $\widetilde{A}$.

Lemma 7.3. *Let $(\widetilde{\Omega}_k, \gamma_k)$ and $(\widetilde{\Omega}'_k, \gamma'_k)$ be two binary extensions of a probability space Ω_k (for $k = 1, 2$), $\Omega = \Omega_1 \times \Omega_2$, $A \subset \Omega$ a measurable set, $\Theta_k : L_2(\widetilde{\Omega}_k) \to L_2(\widetilde{\Omega}'_k)$ unitary operators, each satisfying Condition* (a) *of Lemma* 6.1*. Then $\Theta_1 \times \Theta_2$ maps $L_2(\widetilde{\Omega})$ onto $L_2(\widetilde{\Omega}')$, $L_2(\widetilde{A})$ onto $L_2(\widetilde{A}')$, and $L_2(\widetilde{A}, \mathcal{F}_1)$ onto $L_2(\widetilde{A}', \mathcal{F}'_1)$.*

It is meant that

$$L_2(\widetilde{A}, \mathcal{F}_1) \subset L_2(\widetilde{A}) \subset L_2(\widetilde{A}) \oplus L_2(\widetilde{\Omega} \setminus \widetilde{A}) = L_2(\widetilde{\Omega}) \subset L_2(\widetilde{\Omega}_1) \otimes L_2(\widetilde{\Omega}_2),$$

$$L_2(\widetilde{A}', \mathcal{F}'_1) \subset L_2(\widetilde{A}') \oplus L_2(\widetilde{\Omega}' \setminus \widetilde{A}') = L_2(\widetilde{\Omega}') \subset L_2(\widetilde{\Omega}'_1) \otimes L_2(\widetilde{\Omega}'_2).$$

The reader may prove Lemma 7.3 via Lemma 6.1, but the proof below does not use Lemma 6.1.

Proof. The operator $\Theta = \Theta_1 \otimes \Theta_2$ intertwines the actions of $L_\infty(\Omega_1)$ and $L_\infty(\Omega_2)$, therefore, also the actions of $L_\infty(\Omega_1 \times \Omega_2)$. In particular,

$$\Theta \mathbf{1}_{\widetilde{A}} \psi = \mathbf{1}_{\widetilde{A}'} \Theta \psi \quad \text{for } \psi \in L_2(\widetilde{\Omega}_1 \times \widetilde{\Omega}_2).$$

The space $L_2(\widetilde{A})$ is the closure of linear combinations of vectors of the form $\psi = \mathbf{1}_{\widetilde{A}}(\varphi_1 \otimes \varphi_2)$, where $\varphi_1 \in L_2(\widetilde{\Omega}_1)$ and $\varphi_2 \in L_2(\Omega_2)$. For such ψ we have

$$\Theta\psi = \Theta\mathbf{1}_{\widetilde{A}}(\varphi_1 \otimes \varphi_2) = \mathbf{1}_{\widetilde{A}'}\Theta(\varphi_1 \otimes \varphi_2) = \mathbf{1}_{\widetilde{A}'}(\Theta_1\varphi_1 \otimes \Theta_2\varphi_2) \in L_2(\widetilde{A}'),$$

since $\Theta_1\varphi_1 \in L_2(\widetilde{\Omega}'_1)$ and $\Theta_2\varphi_2 = \varphi_2 \in L_2(\Omega_2)$. Therefore

$$\Theta(L_2(\widetilde{A})) \subset L_2(\widetilde{A}').$$

The special case $\varphi_2 = \mathbf{1}$ gives $\Theta(L_2(\widetilde{A}, \mathcal{F}_1)) \subset L_2(\widetilde{A}', \mathcal{F}'_1)$. The same holds for Θ^{-1}, thus, the inclusions are in fact equalities. Similarly,

$$\Theta(L_2(\widetilde{\Omega} \setminus \widetilde{A})) = L_2(\widetilde{\Omega}' \setminus \widetilde{A}').$$

It follows that $\Theta(L_2(\widetilde{\Omega})) = L_2(\widetilde{\Omega}')$. □

8. Some necessary conditions of isomorphism

Let μ_1 be a probability measure on the space $\mathbb{R}^\infty$ (of all infinite sequences of reals), $\beta : \mathbb{R}^\infty \to \mathbb{R}^\infty$ the shift, $\beta(x_1, x_2, \dots) = (x_2, x_3, \dots)$, and μ_n the image of μ_1 under β^{n-1}. Probability spaces $\Omega_n = (\mathbb{R}^\infty, \mu_n)$ with maps $\beta_n = \beta$ are an inductive system of probability spaces.

Let Borel functions $f_n : \mathbb{R} \to \{-1, +1\}$ be given. We define

$$X_n : \Omega_n \to \{-1, +1\},$$
$$X_n(x_n, x_{n+1}, \dots) = f_n(x_n),$$

and consider the corresponding binary extension of $(\Omega_n, \beta_n)_n$. Another sequence of functions $g_n : \mathbb{R} \to \{-1, +1\}$ leads to another binary extension. According to Lemma 5.6 the two binary extensions are isomorphic if and only if there exist $U_n : \Omega_n \to \{-1, +1\}$ such that

$$U_{n+1}(x_{n+1}, x_{n+2}, \dots) = U_n(x_n, x_{n+1}, \dots) f_n(x_n) g_n(x_n). \tag{8.1}$$

Functions that do not depend on x_n, that is, functions of the form

$$(x_1, x_2, \dots) \mapsto \varphi(x_1, \dots, x_{n-1}, x_{n+1}, x_{n+2}, \dots)$$

are a subspace $H_n \subset L_2(\mu_1)$. We consider vectors $\psi_n \in L_2(\mu_1)$,

$$\psi_n(x_1, x_2, \dots) = f_n(x_n) g_n(x_n), \tag{8.2}$$

and the distance between ψ_n and H_n.

Lemma 8.3. *The condition*

$$\operatorname{dist}(\psi_n, H_n) \to 0 \quad \text{as } n \to \infty$$

is necessary for the two binary extensions to be isomorphic.

Proof. Let U_n satisfy (8.1), then

$$U_n(x_n, x_{n+1}, \dots) = U_1(x_1, x_2, \dots) h_1(x_1) \dots h_{n-1}(x_{n-1}),$$

where $h_n(x) = f_n(x) g_n(x)$. We have

$$\begin{aligned}\psi_n(x_1, x_2, \dots) &= h_n(x_n) = U_n(x_n, x_{n+1}, \dots) U_{n+1}(x_{n+1}, x_{n+2}, \dots) \\ &= U_1(x_1, x_2, \dots) h_1(x_1) \dots h_{n-1}(x_{n-1}) U_{n+1}(x_{n+1}, x_{n+2}, \dots).\end{aligned}$$

Taking into account that H_n is invariant under multiplication by any (bounded measurable) function of $x_1, \dots, x_{n-1}$ and $x_{n+1}, x_{n+2}, \dots$ we see that

$$\operatorname{dist}(\psi_n, H_n) = \operatorname{dist}(U_1, H_n).$$

The latter converges to 0, since H_n contains all functions of $x_1, \dots, x_{n-1}$. $\square$

The conditional distribution of x_n given $x_1, \dots, x_{n-1}$ and $x_{n+1}, x_{n+2}, \dots$ (assuming that $(x_1, x_2, \dots)$ is distributed μ_1) is a probability measure ν_n on $\mathbb{R}$; this ν_n is random in the sense that it depends on $x_1, \dots, x_{n-1}$ and $x_{n+1}, x_{n+2}, \dots$ (whose distribution is the marginal of μ_1).

Here is a useful condition on μ_1:

$$\exists \varepsilon > 0 \;\; \forall n \;\; \mathbb{P}\big(\nu_n \text{ is } \varepsilon\text{-good}\big) \ge \varepsilon, \tag{8.4}$$

where a probability measure ν on $\mathbb{R}$ is called ε-good if

$$\exists x \in \mathbb{R} \;\; \forall A \quad \nu(A) \ge \varepsilon \operatorname{mes}\big(A \cap (x, x+\varepsilon)\big) \tag{8.5}$$

(A runs over Borel subsets of $\mathbb{R}$; 'mes' stands for Lebesgue measure).

Usually ν has a density; then (8.5) requires the density to exceed ε on some interval of length ε.

Lemma 8.6. *Let μ_1 satisfy* (8.4), *and numbers $\varepsilon_n \in (0, \infty)$ satisfy $\varepsilon_n \to 0$. Then there exist Borel functions $f_n : \mathbb{R} \to \{-1, +1\}$ such that for every $c \in \mathbb{R} \setminus \{0\}$, defining $g_n : \mathbb{R} \to \{-1, +1\}$ by $g_n(x) = f_n(x + c\varepsilon_n)$ we get ψ_n* (*see* (8.2)) *violating the necessary condition of Lemma* 8.3 (*and therefore, $(f_n)_n$ and $(g_n)_n$ lead to two nonisomorphic binary extensions*).

Proof. We take $\lambda_1, \lambda_2, \dots$ such that

$$\lambda_n \varepsilon_n = \begin{cases} 1 & \text{for } n \text{ odd}, \\ \sqrt{2} & \text{for } n \text{ even}, \end{cases}$$

and define

$$f_n(x) = \sigma(\lambda_n x),$$

where

$$\sigma(x) = \begin{cases} -1 & \text{for } x \in \cup_{k \in \mathbb{Z}} [k - 0.5, k), \\ +1 & \text{for } x \in \cup_{k \in \mathbb{Z}} [k, k + 0.5). \end{cases}$$

Let $c \in \mathbb{R}$ be given, $c \ne 0$. It is sufficient to prove that at least one of two claims

$$\limsup_n \operatorname{dist}(\psi_{2n}, H_{2n}) > 0, \quad \limsup_n \operatorname{dist}(\psi_{2n-1}, H_{2n-1}) > 0$$

holds. Here

$$\psi_n(x_1, x_2, \dots) = h_n(x_n) = f_n(x_n)g_n(x_n) = f_n(x_n)f_n(x_n + c\varepsilon_n) \\ = \sigma(\lambda_n x_n)\sigma(\lambda_n x_n + c\lambda_n\varepsilon_n).$$

The function h_n is periodic, with period $1/\lambda_n$. The mean value M_n of h_n over the period is

$$M_n = \lambda_n \int_0^{1/\lambda_n} h_n(x)\,\mathrm{d}x = \int_0^1 \sigma(u)\sigma(u + c\lambda_n\varepsilon_n)\,\mathrm{d}u.$$

It reaches ± 1 when $2c\lambda_n\varepsilon_n \in \mathbb{Z}$; otherwise $-1 < M_n < 1$. The relations $2c \in \mathbb{Z}$ and $2c\sqrt{2} \in \mathbb{Z}$ are incompatible, therefore at least one of two claims

$$\sup_n |M_{2n}| < 1, \quad \sup_n |M_{2n-1}| < 1$$

holds. (Of course, M_{2n} and M_{2n-1} do not depend on n, but this fact does not matter.) It is sufficient to prove that

$$\sup_n |M_{2n}| < 1 \quad \text{implies} \quad \limsup_n \operatorname{dist}(\psi_{2n}, H_{2n}) > 0,$$
$$\sup_n |M_{2n-1}| < 1 \quad \text{implies} \quad \limsup_n \operatorname{dist}(\psi_{2n-1}, H_{2n-1}) > 0.$$

The former implication will be proved (the latter is similar). Assume the contrary: $\sup_n |M_{2n}| < 1$ and $\operatorname{dist}(\psi_{2n}, H_{2n}) \to 0$.

For any probability measure ν on $\mathbb{R}$, the squared distance in the space $L_2(\nu)$ between the function h_n and the one-dimensional space of constant functions is

$$\int \Big(h_n - \int h_n\,\mathrm{d}\nu \Big)^2 \mathrm{d}\nu = \int h_n^2\,\mathrm{d}\nu - \Big(\int h_n\,\mathrm{d}\nu \Big)^2 = 1 - \Big(\int h_n\,\mathrm{d}\nu \Big)^2.$$

We use the random measure ν_n, take the average and recall the definition of H_n:

$$\mathbb{E}\bigg(1 - \Big(\int h_n\,\mathrm{d}\nu_n \Big)^2 \bigg) = \operatorname{dist}^2(\psi_n, H_n).$$

Taking into account that $\operatorname{dist}(\psi_{2n}, H_{2n}) \to 0$ we see that $|\int h_{2n}\,\mathrm{d}\nu_{2n}| \to 1$ in probability. In order to get a contradiction to (8.4) it is sufficient to prove that $\limsup_n \sup_\nu |\int h_{2n}\,\mathrm{d}\nu| < 1$, where ν runs over all ε-good measures (recall (8.4) and (8.5)). Or, equivalently,

$$\liminf_n \inf_\nu \nu\big(h_{2n}^{-1}(-1)\big) > 0, \quad \liminf_n \inf_\nu \nu\big(h_{2n}^{-1}(+1)\big) > 0.$$

The former will be proved (the latter is similar). By (8.5),

$$\nu\big(h_{2n}^{-1}(-1)\big) \ge \varepsilon \operatorname{mes}\big(h_{2n}^{-1}(-1) \cap (x, x+\varepsilon)\big).$$

For large n the period $1/\lambda_{2n}$ of the function h_{2n} is $\ll \varepsilon$, therefore

$$\begin{aligned}\frac{1}{\varepsilon}\operatorname{mes}\bigl(h_{2n}^{-1}(-1)\cap(x,x+\varepsilon)\bigr) &\ge \frac{1}{2}\cdot\lambda_{2n}\int_0^{1/\lambda_{2n}}\frac{1-h_{2n}(x)}{2}\,\mathrm{d}x\\ &= \frac{1}{2}\cdot\frac{1-M_{2n}}{2}\\ &\ge \frac{1}{4}\bigl(1-\sup_n|M_{2n}|\bigr)>0. \qquad\square\end{aligned}$$

Remark 8.7. The functions f_n constructed in the proof of Lemma 8.6 depend only on the numbers ε_n, not on the measure μ_1.

Let a probability measure μ on $C[0,1]$ be given. Random variables $A(t)$ on the probability space $\Omega = (C[0,1],\mu)$ defined by $A(t)(a) = a(t)$ for $a \in C[0,1]$, $t \in [0,1]$, are a random process. For every n the restriction map $C[0,1] \to C[0,3^{-n}]$ sends μ to some μ_n. Probability spaces

$$\Omega_n = (C[0,3^{-n}],\mu_n)$$

with restriction maps are an inductive system.

Given Borel functions $f_n : \mathbb{R} \to \{-1,+1\}$, we define random variables $X_n : \Omega_n \to \{-1,+1\}$ by $X_n = f_n(A(2\cdot 3^{-n-1}))$. The corresponding binary extension may be visualized as follows. We consider pairs (a,s) of a function $a \in C[0,1]$ and another function $s : (0,1] \to \{-1,+1\}$ constant on each $[2\cdot 3^{-n-1}, 2\cdot 3^{-n})$ and such that $s(2\cdot 3^{-n}-)s(2\cdot 3^{-n}) = f_{n-1}(a(2\cdot 3^{-n}))$ for all n. We get a pair of random processes $A(\cdot)$, $S(\cdot)$ satisfying

$$\frac{S(2\cdot 3^{-n})}{S(2\cdot 3^{-n}-)} = f_{n-1}(A(2\cdot 3^{-n})).$$

Their restrictions to $[0,3^{-n}]$ give $\widetilde{\Omega}_n$. For each t (separately), the random variable $S(t)$ is independent of the process $A(\cdot)$ and takes on the two equiprobable values ± 1.

As before, given also $g_n : \mathbb{R} \to \{-1,+1\}$ (thus, another binary extension), we define $\psi_n \in L_2(\Omega)$ by

$$\psi_n = f_n(A(2\cdot 3^{-n-1}))g_n(A(2\cdot 3^{-n-1})).$$

We consider the subspaces $H_n \subset L_2(\Omega)$ consisting of functions of $A(t)$ for $t \in [0,3^{-n-1}] \cup [3^{-n},1]$ only (in other words, functions of the restrictions of sample paths to $[0,3^{-n-1}] \cup [3^{-n},1]$).

Lemma 8.8. *The condition* $\operatorname{dist}(\psi_n, H_n) \to 0$ *is necessary for the two binary extensions to be isomorphic.*

The proof, similar to the proof of Lemma 8.3, is left to the reader.

Similarly, $C[-1,1]$ may be used (instead of $C[0,1]$), with

$$\Omega_n = (C[-1,3^{-n}],\mu_n);$$

the process $S(\cdot)$ jumps at $2\cdot 3^{-n}$, as before. Now H_n consists of functions of the restriction $A|_{[-1,3^{-n-1}]\cup[3^{-n},1]}$ of A to $[-1,3^{-n-1}]\cup[3^{-n},1]$ (rather than $[0,3^{-n-1}]\cup[3^{-n},1]$). Lemma 8.8 remains true.

The conditional distribution of $A(2\cdot 3^{-n-1})$ given $A|_{[-1,3^{-n-1}]\cup[3^{-n},1]}$ is too concentrated (when n is large) for being ε-good (recall (8.5)). A useful condition on μ stipulates rescaling by $3^{n/2}$:

(8.9) there exists $\varepsilon>0$ such that for every n,

$$\text{the conditional distribution of } 3^{n/2}A(2\cdot 3^{-n-1}) \text{ given } A|_{[-1,3^{-n-1}]\cup[3^{-n},1]} \text{ is } \varepsilon\text{-good with probability} \ge \varepsilon.$$

Here is a counterpart of Lemma 8.6 for $\varepsilon_n = 3^{-n/2}$.

Lemma 8.10. *Let μ satisfy* (8.9)*. Then there exist Borel functions $f_n:\mathbb{R}\to\{-1,+1\}$ such that for every $c\in\mathbb{R}\setminus\{0\}$, defining $g_n:\mathbb{R}\to\{-1,+1\}$ by $g_n(x)=f_n(x+3^{-n}c)$ we get $\psi_n=f_n(A(2\cdot 3^{-n-1}))g_n(A(2\cdot 3^{-n-1}))$ violating the necessary condition of Lemma* 8.8 (*and therefore, two nonisomorphic binary extensions*).

The proof, similar to the proof of Lemma 8.6, is left to the reader.

9. A binary extension of Brownian motion

The space $C[0,1]$ of all continuous functions $b:[0,1]\to\mathbb{R}$, endowed with the Wiener measure $\mathcal{W}$, is a probability space. Random variables $B(t)$ on $(C[0,1],\mathcal{W})$, defined for $t\in[0,1]$ by $B(t)(b)=b(t)$, are the Brownian motion on $[0,1]$. Almost surely, a Brownian sample path on $[0,1]$ has a unique (global) minimum,

$$\min_{t\in[0,1]} B(t) = B(\tau),$$

τ being a measurable function on $(C[0,1],\mathcal{W})$, $0<\tau(\cdot)<1$ a.s.

We define another random process A, on the time interval $[-1,1]$, by

$$\begin{aligned} A(t) &= B\big(\min(1,\tau+t)\big) - B(\tau) \quad \text{for } t\in[0,1],\\ A(t) &= B\big(\max(0,\tau+t)\big) - B(\tau) \quad \text{for } t\in[-1,0]. \end{aligned}$$

A $\mathcal{W}$-measurable map $C[0,1]\to C[-1,1]$ is thus introduced. The map is one-to-one (mod 0), since $B(\cdot)$ is nonconstant on every time interval, almost surely.

Proposition 9.1. *The process A satisfies* (8.9)*.*

The proof is given after three lemmas.

The conditional distribution of the process B given the restriction $A|_{[-1,\varepsilon]}$ (for a given $\varepsilon\in(0,1)$) is the same as the conditional distribution of the process B given τ and $B|_{[0,\tau+\varepsilon]}$, since the two corresponding measurable partitions of $(C[0,1],\mathcal{W})$ are equal (mod 0). This conditional distribution is a probability measure on the set of Brownian sample paths b such that

$$\begin{aligned} b(t) &= x(t) \quad \text{for } t \in [0, s+\varepsilon], \\ b(t) &> x(s) \quad \text{for } t \in [s+\varepsilon, 1]; \end{aligned}$$

here $s \in (0, 1-\varepsilon)$ is a given value of τ, and $x \in C[0, s+\varepsilon]$ is a given sample path of $B|_{[0,\tau+\varepsilon]}$; of course, s is the unique minimizer of x. We assume that $s < 1-\varepsilon$, since the other case is trivial (the conditional distribution is a single atom).

The corresponding conditional distribution of $B|_{[s+\varepsilon,1]}$ is a probability measure on the set of functions $b \in C[s+\varepsilon, 1]$ such that

$$\begin{aligned} b(s+\varepsilon) &= x(s+\varepsilon), \\ b(t) &> x(s) \quad \text{for } t \in [s+\varepsilon, 1]. \end{aligned}$$

This set depends only on the three numbers $s+\varepsilon$, $x(s+\varepsilon)$, and $x(s)$. One may guess that the considered measure on this set also depends on these three numbers only (rather than the whole function x). The following well-known lemma confirms the guess and gives a simple description of the measure.

Lemma 9.2. *The conditional distribution of $B|_{[s+\varepsilon,1]}$ given that $\tau = s$ and $B_{[0,\tau+\varepsilon]} = x$ is equal to the conditional distribution of $B|_{[s+\varepsilon,1]}$ given that $B(s+\varepsilon) = x(s+\varepsilon)$ and $B(t) > x(s)$ for $t \in [s+\varepsilon, 1]$.*

Proof. We take n such that $\frac{1}{n} < \varepsilon$. Let $k \in \{1, \dots, n-1\}$. The conditional distribution of $B|_{[\frac{k}{n},1]}$ given $B|_{[0,\frac{k}{n}]}$ depends only on $B\bigl(\frac{k}{n}\bigr)$ (by the Markov property of B) and is just the distribution of the Brownian motion starting from $\bigl(\frac{k}{n}, B\bigl(\frac{k}{n}\bigr)\bigr)$. Therefore the conditional distribution of $B|_{[\frac{k}{n},1]}$ given both $B|_{[0,\frac{k}{n}]}$ and $\frac{k-1}{n} < \tau < \frac{k}{n}$ is the distribution of the Brownian motion starting from $\bigl(\frac{k}{n}, B\bigl(\frac{k}{n}\bigr)\bigr)$ and conditioned to stay above the minimum of the given path on $[0, \frac{k}{n}]$. (Indeed, a measurable partition of the whole probability space induces a measurable partition of a given subset of positive probability, and conditional measures for the former partition induce conditional measures for the latter partition.) Now it is easy to condition further on $B|_{[\frac{k}{n},\tau+\varepsilon]}$ and combine all k together. □

Lemma 9.2 gives the conditional distribution of $B|_{[\tau+\varepsilon,1]}$ given τ and $B|_{[0,\tau+\varepsilon]}$. Now we turn to the conditional distribution of $B|_{[\tau+\varepsilon,\tau+3\varepsilon]}$ given τ, $B|_{[0,\tau+\varepsilon]}$ and $B|_{[\tau+3\varepsilon,1]}$ (in the case $\tau + 3\varepsilon < 1$). We are especially interested in $B(\tau + 2\varepsilon)$.

Lemma 9.3. *The conditional distribution of $B(\tau+2\varepsilon)-B(\tau)$, given τ (such that $\tau + 3\varepsilon < 1$), $B|_{[0,\tau+\varepsilon]}$ and $B|_{[\tau+3\varepsilon,1]}$, has the density*

$$x \mapsto \frac{\bigl(1-\mathrm{e}^{-2ax/\varepsilon}\bigr)\bigl(1-\mathrm{e}^{-2bx/\varepsilon}\bigr)}{1-\mathrm{e}^{-ab/\varepsilon}} \cdot \frac{1}{\sqrt{\pi\varepsilon}} \exp\left(-\frac{1}{\varepsilon}\Bigl(x-\frac{a+b}{2}\Bigr)^2\right) \quad \text{for } x > 0,$$

where $a = B(\tau+\varepsilon) - B(\tau)$ *and* $b = B(\tau+3\varepsilon) - B(\tau)$.

Proof. Using Lemma 9.2 we turn to an equivalent question: a Brownian motion starting from $\big(s+\varepsilon, B(s+\varepsilon)\big)$ is conditioned to stay above $B(s)$ on $[s+\varepsilon, s+3\varepsilon]$, and is known on $[s+3\varepsilon, 1]$ (which means another conditioning, of course); we need the (conditional) distribution of $B(s+2\varepsilon)$. Omitting for a while the condition $B_{[s+\varepsilon,s+3\varepsilon]}(\cdot) > B(s)$ we get the so-called Brownian bridge, — the Brownian motion on $[s+\varepsilon, s+3\varepsilon]$ with given boundary values $B(s+\varepsilon)$, $B(s+3\varepsilon)$. (Later we'll condition the bridge to stay above $B(s)$.)

For the bridge, $B(s+2\varepsilon)$ has normal distribution $N\big(\frac{B(s+\varepsilon)+B(s+3\varepsilon)}{2}, \frac{\varepsilon}{2}\big)$. Given $B(s+2\varepsilon)$ we get two independent bridges, one on $[s+\varepsilon, s+2\varepsilon]$, the other on $[s+2\varepsilon, s+3\varepsilon]$. The bridge on $[s+\varepsilon, s+2\varepsilon]$ stays above $B(s)$ with the probability (calculated via the reflection principle)

$$\frac{p_\varepsilon(a-x) - p_\varepsilon(a+x)}{p_\varepsilon(a-x)} = 1 - \exp\Big(-\frac{2}{\varepsilon}ax\Big),$$

where $a = B(s+\varepsilon) - B(s)$, $x = B(s+2\varepsilon) - B(s)$, $b = B(s+3\varepsilon) - B(s)$, and

$$p_\varepsilon(u) = \frac{1}{\sqrt{2\pi\varepsilon}} \exp\Big(-\frac{u^2}{2\varepsilon}\Big).$$

It remains to write similar formulas on $[s+2\varepsilon, s+3\varepsilon]$ and all of $[s+\varepsilon, s+3\varepsilon]$, and apply the Bayes formula

$$p_{X|A}(x) = \frac{\mathbb{P}\big(A \,\big|\, X=x\big) p_X(x)}{\mathbb{P}\big(A\big)}$$

for the conditional density $p_{X|A}(\cdot)$ of a random variable X given an event A. Namely, $X = B(s+2\varepsilon) - B(s) \sim N\big(\frac{a+b}{2}, \frac{\varepsilon}{2}\big)$, $p_X(x) = p_{\varepsilon/2}\big(x - \frac{a+b}{2}\big)$, A is the event $B_{[s+\varepsilon,s+3\varepsilon]}(\cdot) > B(s)$, $\mathbb{P}\big(A \,\big|\, X=x\big) = \big(1-\mathrm{e}^{-2ax/\varepsilon}\big)\big(1-\mathrm{e}^{-2bx/\varepsilon}\big)$, $\mathbb{P}\big(A\big) = 1 - \mathrm{e}^{-ab/\varepsilon}$. □

Lemma 9.4. *There exists* $\varepsilon > 0$ *such that for all* $a, b \in (0,\infty)$ *the probability measure on* $(0,\infty)$ *that has the density*

$$x \mapsto \frac{\big(1-\mathrm{e}^{-2ax}\big)\big(1-\mathrm{e}^{-2bx}\big)}{1-\mathrm{e}^{-ab}} \cdot \frac{1}{\sqrt{\pi}} \exp\Big(-\Big(x - \frac{a+b}{2}\Big)^2\Big)$$

is ε*-good* (*as defined by* (8.5)).

Proof. It is sufficient to prove that

$$\inf_{a,b\in(0,\infty)} \inf_{x\in[\frac{a+b}{2}+1, \frac{a+b}{2}+2]} p_{a,b}(x) > 0,$$

where $p_{a,b}(\cdot)$ is the given density. Assume the contrary: there exist a_n, b_n, x_n such that $b_n \ge a_n > 0$, $\frac{a_n+b_n}{2} + 1 \le x_n \le \frac{a_n+b_n}{2} + 2$, and $p_{a_n,b_n}(x_n) \to 0$. Then

$$\frac{\big(1-\mathrm{e}^{-2a_nx_n}\big)\big(1-\mathrm{e}^{-2b_nx_n}\big)}{1-\mathrm{e}^{-a_nb_n}} \to 0.$$

It follows that $1 - \mathrm{e}^{-2a_n x_n} \to 0$, $a_n x_n \to 0$, $a_n\bigl(\frac{a_n+b_n}{2} + 1\bigr) \to 0$, $a_n \to 0$, $a_n b_n \to 0$, and
$$\frac{\bigl(1-\mathrm{e}^{-2a_n x_n}\bigr)\bigl(1-\mathrm{e}^{-2b_n x_n}\bigr)}{a_n b_n} \to 0\,.$$
Therefore
$$\frac{1-\mathrm{e}^{-2a_n}}{a_n} \cdot \frac{1-\mathrm{e}^{-2b_n}}{b_n} \to 0; \qquad \frac{1-\mathrm{e}^{-2b_n}}{b_n} \to 0;$$
$b_n \to \infty$, $b_n x_n \to \infty$; $1 - \mathrm{e}^{-2b_n x_n} \to 1$;
$$\frac{1-\mathrm{e}^{-2a_n x_n}}{a_n b_n} \to 0; \qquad \frac{1-\mathrm{e}^{-a_n b_n}}{a_n b_n} \to 0,$$
in contradiction to $a_n b_n \to 0$. □

Proof of Proposition 9.1. Lemma 9.3 (for $\varepsilon = 3^{-n-1}$) gives us the conditional distribution of $A(2 \cdot 3^{-n-1})$ given τ and $A|_{[-1,3^{-n-1}]\cup[3^{-n},1]}$, but only for the case $\tau + 3^{-n} < 1$. Lemma 9.4 states that the corresponding distribution of $3^{n/2}A(2 \cdot 3^{-n-1})$ is ε-good. It remains to note that $\mathbb{P}\bigl(\tau + 3^{-n} < 1\bigr) \ge \mathbb{P}\bigl(\tau < 2/3\bigr) \ge \varepsilon$. □

Combining Proposition 9.1 and Lemma 8.10 we get a binary extension of the inductive system (of probability spaces) formed by the restrictions $A|_{[-1,3^{-n}]}$ of the process A. In terms of the Brownian motion B the inductive system is formed by $B|_{[0,\tau+3^{-n}]}$, and the binary extension may be visualized by a random function $S : (\tau, 1) \to \{-1, +1\}$ constant on
$$[\tau + 2 \cdot 3^{-n-1}, \tau + 2 \cdot 3^{-n}) \cap (0, 1)$$
for each n and such that
$$\frac{S(\tau + 2 \cdot 3^{-n})}{S(\tau + 2 \cdot 3^{-n}-)} = f_{n-1}\bigl(B(\tau + 2 \cdot 3^{-n}) - B(\tau)\bigr) \tag{9.5}$$
for all n such that $\tau + 2 \cdot 3^{-n} < 1$. Here $f_n : \mathbb{R} \to \{-1, +1\}$ are the functions given by Lemma 8.10. They are constructed as to make the binary extension *sensitive to drift* in the following sense. For every $c \in \mathbb{R} \setminus \{0\}$ the binary extension constructed via
$$f_n\bigl(B(\tau + 2 \cdot 3^{-n}) - B(\tau) + c \cdot 2 \cdot 3^{-n}\bigr)$$
is not isomorphic to that for $c = 0$.

10. A new noise extending the white noise

> *This is a noise richer than white noise: in addition to the increments of a Brownian motion B it carries a countable collection of independent Bernoulli random variables which are attached to the local minima of B.*
>
> J. Warren [11, the end]

> *...magically, this independent random variable has appeared from somewhere! Indeed, it really has appeared from thin air, because...*

it is not present at time 0*!*

L.C.G. Rogers, D. Williams [7, p. 156]

The two ideas mentioned above will be combined; at every local minimum of the Brownian motion B, a new random variable will appear from thin air. That is, the binary extension, performed in Section 9 at the global minimum, will be performed at every local minimum, thus achieving locality and stationarity required from a noise, while retaining the drift sensitivity achieved in Section 9 (as will be shown in Section 11).

A new random sign attached to a local minimum at τ may be thought of as a random choice of one of the two functions $S : (\tau, \tau + \varepsilon_1) \to \{-1, +1\}$ constant on $[\tau + \varepsilon_{n+1}, \tau + \varepsilon_n)$ (for each n) and such that

$$S(\tau + \varepsilon_n) = S(\tau + \varepsilon_n -) f_n\big(B(\tau + \varepsilon_n) - B(\tau)\big) \tag{10.1}$$

(the numbers $\varepsilon_n \downarrow 0$ and the functions $f_n : \mathbb{R} \to \{-1, +1\}$ being chosen appropriately). Given a time interval $(0, t)$, for each local minimizer $\tau \in (0, t)$ we describe the new random sign by the value $S(t)$ (of the corresponding function S), denoted however by $\eta_t(\tau)$. Relation (10.1) turns into the relation (10.8) between $\eta_s(\tau)$ and $\eta_{s+t}(\tau)$.

Before attaching something to the local minima we enumerate them. For every time interval $(a, b) \subset \mathbb{R}$ there exists a *measurable enumeration* of local minima on (a, b), — a sequence of $\mathcal{F}^{\text{white}}_{a,b}$-measurable random variables $\tau_1, \tau_2, \dots : \Omega \to (a, b)$ such that for almost all ω the Brownian path $t \mapsto B_t(\omega)$ has a local minimum at each $\tau_k(\omega)$, no other local minima exist on (a, b), and the numbers $\tau_1(\omega), \tau_2(\omega), \dots$ are pairwise different a.s. Here is a simple construction for $(a, b) = (0, 1)$ taken from [9, 2e]. First, $\tau_1(\omega)$ is the minimizer on the whole $(0, 1)$ (unique a.s.). Second, if $\tau_1(\omega) \in (0, 1/2)$ then $\tau_2(\omega)$ is the minimizer on $(1/2, 1)$, otherwise — on $(0, 1/2)$. Third, $\tau_3(\omega)$ is the minimizer on the first of the four intervals $(0, 1/4)$, $(1/4, 1/2)$, $(1/2, 3/4)$ and $(3/4, 1)$ that contains neither $\tau_1(\omega)$ nor $\tau_2(\omega)$. And so on.

All measurable enumerations $(\tau'_k)_k$ result from one of them $(\tau_k)_k$ in the sense that

$$\tau'_k(\omega) = \tau_{\sigma_\omega(k)}(\omega) \quad \text{a.s.}$$

for some (unique, in fact) *random permutation* $\sigma : \Omega \to S_\infty$, that is, an $\mathcal{F}^{\text{white}}_{a,b}$-measurable random variable σ valued in the group S_∞ of all bijective maps $\{1, 2, \dots\} \to \{1, 2, \dots\}$ (equipped with its natural Borel σ-field). See also [9, 2e].

Each τ_k is a measurable selector of the set of all local minimizers; for short, let us say just a *selected minimum.* Here is the general form of a selected minimum τ in terms of a given enumeration $(\tau_k)_k$:

$$\begin{aligned} &\tau(\omega) = \tau_k(\omega) \quad \text{for } \omega \in A_k, \\ &\qquad \text{where } (A_1, A_2, \dots) \text{ is a countable measurable partition of } \Omega. \end{aligned} \tag{10.2}$$

Every selected minimum may serve as (say) the first element of some enumeration.

Given two adjacent time intervals (a,b) and (b,c), we may choose measurable enumerations $(\tau'_k)_k$ and $(\tau''_k)_k$ of local minima on (a,b) and (b,c) respectively, and combine them into a measurable enumeration $(\tau_k)_k$ on (a,c), say,

$$\tau_{2k-1} = \tau'_k, \quad \tau_{2k} = \tau''_k \quad \text{for } k = 1,2,\dots \tag{10.3}$$

taking into account that the point b is a.s. not a local minimizer.

Now we can attach independent random signs to the local minima. Let

$$\Omega_1^{\text{white}} \subset C[0,1]$$

be a set of full Wiener measure such that for every $\omega_1^{\text{white}} \in \Omega_1^{\text{white}}$ the set $\operatorname{LocMin}(\omega_1^{\text{white}})$ of all local minimizers of the path ω_1^{white} is a dense countable subset of $(0,1)$. We introduce the set $\{-1,+1\}^{\operatorname{LocMin}(\omega_1^{\text{white}})}$ of all functions $\eta_1 : \operatorname{LocMin}(\omega_1^{\text{white}}) \to \{-1,+1\}$ and consider the disjoint union Ω_1 of these sets over all ω_1^{white},

$$\Omega_1 = \Big\{ (\omega_1^{\text{white}}, \eta_1) : \omega_1^{\text{white}} \in \Omega_1^{\text{white}},\ \eta_1 \in \{-1,+1\}^{\operatorname{LocMin}(\omega_1^{\text{white}})} \Big\}.$$

Every measurable enumeration $(\tau_k)_k$ of the local minima on $(0,1)$ gives us a one-to-one correspondence

$$\Omega_1 \leftrightarrow \Omega_1^{\text{white}} \times \{-1,+1\}^\infty,$$
$$(\omega_1^{\text{white}}, \eta_1) \leftrightarrow \big(\omega_1^{\text{white}}, \big(\eta_1(\tau_k(\omega_1^{\text{white}}))\big)_k\big);$$

here $\{-1,+1\}^\infty = \{-1,+1\}^{\{1,2,\dots\}}$ is the set of all infinite sequences of numbers ±1. (As usual, a set of Wiener measure 0 in Ω_1^{white} may be neglected.) We take the uniform probability distribution m on $\{-1,+1\}$ (giving equal probabilities 0.5 to -1 and $+1$), equip $\{-1,+1\}^\infty$ with the product measure m^∞, and $\Omega_1^{\text{white}} \times \{-1,+1\}^\infty$ — with the Wiener measure multiplied by m^∞. Then, using the one-to-one correspondence, we transfer the probability measure (and the underlying σ-field) to Ω_1. The choice of an enumeration $(\tau_k)_k$ does not matter, since m^∞ is invariant under permutations.

Now Ω_1 is a probability space. Similarly, Ω_t becomes a probability space for every $t \in (0,\infty)$. Given $s,t \in (0,\infty)$, we get a natural isomorphism

$$\Omega_s \times \Omega_t \longleftrightarrow \Omega_{s+t}, \tag{10.4}$$
$$\big((\omega_s^{\text{white}}, \eta_s), (\omega_t^{\text{white}}, \eta_t)\big) \longleftrightarrow (\omega_{s+t}^{\text{white}}, \eta_{s+t})$$

where $(\omega_s^{\text{white}}, \omega_t^{\text{white}}) \longleftrightarrow \omega_{s+t}^{\text{white}}$ is the usual composition of Brownian paths, and

$$\eta_{s+t}(\tau) = \begin{cases} \eta_s(\tau) & \text{if } \tau < s, \\ \eta_t(\tau - s) & \text{if } \tau > s. \end{cases} \tag{10.5}$$

(The notation is not good for the case $s = t$, since ω_s and ω_t are still treated as different variables; hopefully it is not too confusing.) The composition $(\eta_s, \eta_t) \longleftrightarrow \eta_{s+t}$ is described conveniently in terms of an enumeration of the form (10.3) for $a = 0$, $b = s$, $c = s + t$:

$$\eta_{s+t}(\tau_{2k-1}) = \eta_s(\tau'_k), \quad \eta_{s+t}(\tau_{2k}) = \eta_t(\tau''_k - s) \quad \text{for } k = 1, 2, \dots \tag{10.6}$$

(of course, all these η and τ depend implicitly on the underlying ω^{white}).

We have a noise (an extension of the white noise). It is described above via probability spaces Ω_t satisfying $\Omega_s \times \Omega_t = \Omega_{s+t}$ rather than sub-σ-fields $\mathcal{F}_{s,t}$ (on a single Ω) satisfying $\mathcal{F}_{r,s} \otimes \mathcal{F}_{s,t} = \mathcal{F}_{r,t}$, but these are two equivalent languages (see [9, 3c1 and 3c6]), and the corresponding Arveson system is just $H_t = L_2(\Omega_t)$.

However, it is not yet the new, drift sensitive noise that we need. Rather, it is Warren's noise of splitting. The binary extension performed at each τ should follow the construction of Section 9. To this end we retain the probability spaces Ω_t constructed before, but replace the straightforward isomorphisms (10.4)–(10.5) with less evident, 'twisted' isomorphisms. Namely, (10.5) is replaced with

$$\eta_{s+t}(\tau) = \eta_t(\tau - s) \quad \text{if } \tau > s, \tag{10.7}$$

$$\eta_{s+t}(\tau) = \eta_s(\tau) \prod_{n:\tau+\varepsilon_n \in (s,s+t]} f_n\big(B(\tau + \varepsilon_n) - B(\tau)\big) \quad \text{if } \tau < s. \tag{10.8}$$

As before, all these η and τ depend implicitly on the underlying ω^{white}, and $B(s)(\omega_t^{\text{white}}) = \omega_t^{\text{white}}(s)$ for $s \in [0, t]$.

The new noise is thus constructed. Its parameters $(\varepsilon_n)_n$ and $(f_n)_n$ will be chosen later. (In fact, $\varepsilon_n = 2 \cdot 3^{-n-1}$, and f_n are given by Lemma 8.10.)

The classical part of the new noise is exhausted by the white noise, which can be proved via the predictable representation property, see [9, 4d].

In order to examine the impact of drift on the new noise we need the relation

$$\operatorname{LocMin}(\omega_t^{\text{white}}) = \operatorname{LocMin}\big(\theta_t^\lambda(\omega_t^{\text{white}})\big) \tag{10.9}$$

(for all t, λ and almost all $\omega_t^{\text{white}} \in \Omega_t^{\text{white}}$); as before, $\theta_t^\lambda : C[0, t] \to C[0, t]$ is the drift transformation, $(\theta_t^\lambda b)(s) = b(s) - 2\lambda s$. The relation (10.9) follows from the well-known fact that all local minima of the Brownian motion are sharp (a.s.) in the sense that

$$\frac{B(t) - B(\tau)}{|t - \tau|} \to \infty \quad \text{as } t \to \tau,\, t \neq \tau$$

whenever τ is a local minimizer. See [4, Section 2.10, Items 7,8]. (In fact, $|t - \tau|$ may be replaced with $\sqrt{|t - \tau|} / \ln^2 |t - \tau|$.)

It is easy to guess that a drift corresponds to a shift of the functions f_n. The proof (rather boring) is given below.

Lemma 10.10. *Let numbers $\lambda \in \mathbb{R}$, $\varepsilon_n \downarrow 0$ and Borel functions*

$$f_n, g_n : \mathbb{R} \to \{-1, +1\}$$

satisfy

$$g_n(x) = f_n(x + 2\lambda\varepsilon_n) \quad \text{for all } x \in \mathbb{R} \text{ and } n.$$

Let two extensions of the white noise be constructed as before, one corresponding to $(f_n)_n$ and $(\varepsilon_n)_n$, the other corresponding to $(g_n)_n$ and $(\varepsilon_n)_n$. Then the second extension results from the first one by the drift 2λ (as defined in Section 2), up to isomorphism of extensions.

Proof. The probability spaces Ω_t and measure preserving maps $\Omega_t \to \Omega_t^{\text{white}}$ are the same for both extensions, however, the corresponding isomorphisms $\alpha_f, \alpha_g : \Omega_s \times \Omega_t \to \Omega_{s+t}$ differ; α_f, used in the first extension, involves f_n (recall (10.8)), while α_g, used in the second extension, involves g_n instead of f_n.

We introduce the third extension, resulting from the first one by the drift 2λ, and seek an isomorphism between the second and third extensions.

The third extension uses the same Ω_t but with probability measures P'_t different from the probability measures P_t used by the first and second extensions; namely,

$$\frac{\mathrm{d}P'_t}{\mathrm{d}P_t} = D_t = \exp(2\lambda B_t - 2\lambda^2 t).$$

The white noise extended by the third extension is generated by the Brownian motion $B'_t = B_t - 2\lambda t$. Note also that the third extension uses α_f.

The probability space Ω_t consists of pairs $(\omega_t^{\text{white}}, \eta_t)$ where

$$\omega_t^{\text{white}} \in \Omega_t^{\text{white}} \subset C[0,t]$$

and $\eta_t \in \{-1,+1\}^{\text{LocMin}(\omega_t^{\text{white}})}$. The drift transformation θ_t^λ may be treated as a measure preserving map

$$\theta_t^\lambda : (\Omega_t^{\text{white}}, D_t \cdot \mathcal{W}_t) \to (\Omega_t^{\text{white}}, \mathcal{W}_t).$$

Using (10.9) we define $\widetilde{\theta}_t^\lambda : \Omega_t \to \Omega_t$ by $\widetilde{\theta}_t^\lambda(\omega_t^{\text{white}}, \eta_t) = (\theta_t^\lambda \omega_t^{\text{white}}, \eta_t)$ and get a measure preserving map

$$\widetilde{\theta}_t^\lambda : (\Omega_t, P'_t) \to (\Omega_t, P_t).$$

Clearly, $B'_s = B_s \circ \widetilde{\theta}_t^\lambda$ for $s \in [0,t]$. It remains to check that $\widetilde{\theta}_s^\lambda \times \widetilde{\theta}_t^\lambda = \widetilde{\theta}_{s+t}^\lambda$ in the sense that the diagram

$$\begin{array}{ccc} \Omega_s \times \Omega_t & \xrightarrow{\widetilde{\theta}_s^\lambda \times \widetilde{\theta}_t^\lambda} & \Omega_s \times \Omega_t \\ \downarrow{\scriptstyle \alpha_f} & & \downarrow{\scriptstyle \alpha_g} \\ \Omega_{s+t} & \xrightarrow{\widetilde{\theta}_{s+t}^\lambda} & \Omega_{s+t} \end{array}$$

is commutative. Let $\omega_s = (\omega_s^{\text{white}}, \eta_s) \in \Omega_s$ and $\omega_t = (\omega_t^{\text{white}}, \eta_t) \in \Omega_t$. We have $\alpha_f(\omega_s, \omega_t) = (\omega_{s+t}^{\text{white}}, \eta_{s+t})$, where $\omega_{s+t}^{\text{white}}$ is the usual composition of

ω_s^{white} and ω_t^{white}, while η_{s+t} is obtained from η_s and η_t according to (10.7), (10.8). Thus,

$$\widetilde{\theta}_{s+t}^{\lambda}\big(\alpha_f(\omega_s,\omega_t)\big) = \widetilde{\theta}_{s+t}^{\lambda}(\omega_{s+t}^{\text{white}}, \eta_{s+t}) = \big(\theta_{s+t}^{\lambda}(\omega_{s+t}^{\text{white}}), \eta_{s+t}\big).$$

On the other hand,

$$(\widetilde{\theta}_s^{\lambda} \times \widetilde{\theta}_t^{\lambda})(\omega_s,\omega_t) = \big(\widetilde{\theta}_s^{\lambda}(\omega_s), \widetilde{\theta}_t^{\lambda}(\omega_t)\big) = \big((\theta_s^{\lambda}(\omega_s^{\text{white}}), \eta_s), (\theta_t^{\lambda}(\omega_t^{\text{white}}), \eta_t)\big).$$

Clearly, $\alpha_g\big((\theta_s^{\lambda}(\omega_s^{\text{white}}), \eta_s), (\theta_t^{\lambda}(\omega_t^{\text{white}}), \eta_t)\big) = \big(\theta_{s+t}^{\lambda}(\omega_{s+t}^{\text{white}}), \eta'_{s+t}\big)$ for some η'_{s+t} (since $\theta_s^{\lambda} \times \theta_t^{\lambda} = \theta_{s+t}^{\lambda}$). Finally, $\eta'_{s+t} = \eta_{s+t}$ by (10.7), (10.8) and the equality

$$\begin{aligned} &g_n\big(\theta_{s+t}^{\lambda}(\omega_{s+t}^{\text{white}})(\tau+\varepsilon_n) - \theta_{s+t}^{\lambda}(\omega_{s+t}^{\text{white}})(\tau)\big) \\ &= g_n\big(\omega_{s+t}^{\text{white}}(\tau+\varepsilon_n) - \omega_{s+t}^{\text{white}}(\tau) - 2\lambda\varepsilon_n\big) \\ &= f_n\big(\omega_{s+t}^{\text{white}}(\tau+\varepsilon_n) - \omega_{s+t}^{\text{white}}(\tau)\big). \end{aligned}$$

□

11. The binary extension inside the new noise

According to Section 9, the Brownian motion B leads to an inductive system of probability spaces formed by the restrictions of B to the time intervals $[0, \tau + 3^{-n}] \cap [0,1]$, where τ is the (global) minimizer of B on $[0,1]$. Further, every sequence $(f_n)_n$ of Borel functions $f_n : \mathbb{R} \to \{-1,+1\}$ leads to a binary extension of this inductive system. The extension is formed by the restrictions of B and S_f to $[0, \tau + 3^{-n}] \cap [0,1]$; here $S_f : (\tau, 1) \to \{-1,+1\}$ is a random function satisfying (9.5).

On the other hand, according to Section 10, $(f_n)_n$ (in combination with $\varepsilon_n = 2 \cdot 3^{-n-1}$) leads to a noise that extends the white noise. The noise is formed by the Brownian motion B and the random variables $\eta_t(\tau)$; here τ runs over all local minimizers of B on $(0,t)$. In turn, the noise leads to an Arveson system that extends the type I_1 Arveson system of the white noise.

These constructions of Sections 9 and 10 are related as follows.

Proposition 11.1. *If two sequences $(f_n)_n$, $(g_n)_n$ of Borel functions*

$$\mathbb{R} \to \{-1,+1\}$$

lead to isomorphic extensions of the type I_1 Arveson system (of the white noise), then they lead to isomorphic binary extensions of the inductive system of probability spaces.

The proof is given after the proof of Proposition 11.4.

Proof of Theorem 1.10. The binary extension, constructed in Section 9 using the functions f_n given by Lemma 8.10 (combined with Proposition 9.1), is not isomorphic to the extension that corresponds to the shifted functions $g_n(x) = f_n(x + 3^{-n}c)$, unless $c = 0$. By Proposition 11.1, $(f_n)_n$ and $(g_n)_n$ lead to nonisomorphic extensions (constructed in Section 10) of the type I_1 Arveson system (of the white noise). By Lemma 10.10, these nonisomorphic extensions result from one another by a drift. This drift sensitivity implies Theorem 1.10 by Corollary 2.2. □

Comparing (9.5) and (10.8) we see that the function $t \mapsto \eta_t(\tau)$ behaves like the function S_f. (Here τ is the global minimizer of B on $[0,1]$.) In other words, we may let for some (therefore, all) n such that $\tau + 3^{-n} < 1$,

$$S_f(\tau + 3^{-n}) = \eta_{\tau+3^{-n}}(\tau),$$

thus defining a measure preserving map from the probability space $\Omega^{\text{noise}(f)}$ of the noise on $[0,1]$ to the probability space $\Omega^{\text{bin}(f)}$ of the binary extension on $[0,1]$;

$$\Omega^{\text{noise}(f)} \to \Omega^{\text{bin}(f)};$$

here $f = (f_n)_n$ is the given sequence of functions. Accordingly, we have a natural embedding of Hilbert spaces,

$$L_2(\Omega^{\text{bin}(f)}) \to L_2(\Omega^{\text{noise}(f)}).$$

Striving to prove Proposition 11.1 we assume existence of an isomorphism $\Theta = (\Theta_t)_t$ between the two Arveson systems,

$$\Theta_t : L_2(\Omega_t^{\text{noise}(f)}) \to L_2(\Omega_t^{\text{noise}(g)}), \tag{11.2}$$

$$\Theta_t \text{ is trivial on } L_2(\Omega_t^{\text{white}}). \tag{11.3}$$

Note that $\Omega^{\text{noise}(f)} = \Omega_1^{\text{noise}(f)}$.

Proposition 11.4. Θ_1 *maps the subspace* $L_2(\Omega^{\text{bin}(f)})$ *of* $L_2(\Omega^{\text{noise}(f)})$ *onto the subspace* $L_2(\Omega^{\text{bin}(g)})$ *of* $L_2(\Omega^{\text{noise}(g)})$.

The proof is given after Lemma 11.5.

The structure of $L_2(\Omega^{\text{noise}(f)})$ is easy to describe:

$$L_2(\Omega^{\text{noise}(f)}) = H_0^f \oplus H_1^f \oplus H_2^f \oplus \cdots ,$$

where H_n^f (called the n-th superchaos space) consists of the random variables of the form

$$\sum_{k_1 < \dots < k_n} \eta_1(\tau_{k_1}) \dots \eta_1(\tau_{k_n}) \varphi_{k_1,\dots,k_n},$$

$$\varphi_{k_1,\dots,k_n} \in L_2(\Omega_1^{\text{white}}), \qquad \sum_{k_1 < \dots < k_n} \|\varphi_{k_1,\dots,k_n}\|^2 < \infty,$$

where $(\tau_k)_k$ is a measurable enumeration of the local minimizers of B on $(0,1)$ (the choice of the enumeration does not matter). See [10, (3.1)] for the case $f_n(\cdot) = 1$ (Warren's noise of splitting); the same argument works in general. Note that $H_0^f = L_2(\Omega_1^{\text{white}})$.

It is well-known that the superchaos spaces may be described in terms of the Arveson system, and therefore Θ_1 maps H_n^f onto H_n^g. We need the first superchaos space only; here is a simple argument for this case:

$$H_1 = \Big\{ \psi \in L_2(\Omega^{\text{noise}}) : \forall t \in (0,1) \;\; \psi = Q_{0,t}\psi + Q_{t,1}\psi \Big\};$$

here $Q_{0,t}$ is the orthogonal projection of $L_2(\Omega^{\text{noise}}) = L_2(\Omega_t^{\text{noise}}) \otimes L_2(\Omega_{1-t}^{\text{noise}})$ onto the subspace $L_2(\Omega_t^{\text{noise}}) \otimes L_2(\Omega_{1-t}^{\text{white}})$, and $Q_{t,1}$ onto

$$L_2(\Omega_t^{\text{white}}) \otimes L_2(\Omega_{1-t}^{\text{noise}}).$$

We have

$$\begin{aligned}\Theta_1\big(L_2(\Omega_t^{\text{noise}(f)}) \otimes L_2(\Omega_{1-t}^{\text{white}})\big) &= \Theta_t\big(L_2(\Omega_t^{\text{noise}(f)})\big) \otimes \Theta_{1-t}\big(L_2(\Omega_{1-t}^{\text{white}})\big) \\ &= L_2(\Omega_t^{\text{noise}(g)}) \otimes L_2(\Omega_{1-t}^{\text{white}})\end{aligned}$$

by (11.3); therefore $\Theta_1 Q_{0,t}^f = Q_{0,t}^g \Theta_1$. Similarly, $\Theta_1 Q_{t,1}^f = Q_{t,1}^g \Theta_1$. It follows that

$$\Theta_1 H_1^f = H_1^g.$$

Similarly, $L_2(\Omega_t^{\text{noise}}) = H_0(t) \oplus H_1(t) \oplus H_2(t) \oplus \cdots$ (the upper index, be it f or g, is omitted). Identifying $L_2(\Omega_1)$ with $L_2(\Omega_t) \otimes L_2(\Omega_{1-t})$ we have

$$H_1 = \underbrace{H_1(t) \otimes H_0(1-t)}_{Q_{0,t}H_1} \oplus \underbrace{H_0(t) \otimes H_1(1-t)}_{Q_{t,1}H_1}.$$

The commutative algebra $L_\infty(\Omega_1^{\text{white}})$ acts naturally on H_1:

$$h \cdot \sum_k \eta_1(\tau_k)\varphi_k = \sum_k \eta_1(\tau_k) h \cdot \varphi_k \quad \text{for } h \in L_\infty(\Omega_1^{\text{white}}).$$

Also the commutative algebra $L_\infty(0,1)$ acts naturally on H_1. In particular, $\mathbf{1}_{(0,t)}$ acts as $Q_{0,t}$, and $\mathbf{1}_{(t,1)}$ acts as $Q_{t,1}$. In general,

$$h \cdot \sum_k \eta_1(\tau_k)\varphi_k = \sum_k \eta_1(\tau_k) h(\tau_k)\varphi_k \quad \text{for } h \in L_\infty(0,1).$$

(The choice of enumeration $(\tau_k)_k$ does not matter.) The two actions commute, and may be combined into the action of $L_\infty(\mu)$ (on H_1) for some measure μ on $\Omega_1^{\text{white}} \times (0,1)$:

$$h \cdot \sum_k \eta_1(\tau_k)\varphi_k(\cdot) = \sum_k \eta_1(\tau_k) h(\cdot,\tau_k)\varphi_k(\cdot) \quad \text{for } h \in L_\infty(\mu).$$

The measure μ may be chosen as

$$\int h \, \mathrm{d}\mu = \mathbb{E} \sum_k \frac{1}{k^2} h(B, \tau_k)$$

(or anything equivalent).

Lemma 11.5. *The diagram*

$$\begin{array}{ccc} H_1^f & \xrightarrow{\Theta_1} & H_1^g \\ \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle h} \\ H_1^f & \xrightarrow{\Theta_1} & H_1^g \end{array}$$

is commutative for every $h \in L_\infty(\mu)$.

Proof. Given $[a,b] \subset [0,1]$, we define a subalgebra $\Gamma(a,b) \subset L_\infty(\mu)$ as consisting of the functions of the form

$$h(\omega_{0,1}^{\text{white}}, t) = \begin{cases} h'(\omega_{0,a}^{\text{white}}, \omega_{b,1}^{\text{white}}) & \text{for } t \in (a,b), \\ 0 & \text{for } t \in (0,a) \cup (b,1), \end{cases}$$

where $h' \in L_\infty(\Omega_a^{\text{white}} \times \Omega_{1-b}^{\text{white}})$, and $\omega_{0,1}^{\text{white}} \in \Omega_1^{\text{white}}$ is treated as the triple $(\omega_{0,a}^{\text{white}}, \omega_{a,b}^{\text{white}}, \omega_{b,1}^{\text{white}})$ according to the natural isomorphism between Ω_1^{white} and $\Omega_a^{\text{white}} \times \Omega_{b-a}^{\text{white}} \times \Omega_{1-b}^{\text{white}}$. For each $n = 1, 2, \dots$ we define a subalgebra $\Gamma_n \subset L_\infty(\mu)$ by

$$\Gamma_n = \sum_{k=1}^{2^n} \Gamma\Big(\frac{k-1}{2^n}, \frac{k}{2^n}\Big).$$

It is easy to see that Γ_n corresponds to a measurable partition; in other words, $\Gamma_n = L_\infty(\Omega_1^{\text{white}} \times (0,1), \mathcal{E}_n, \mu)$ for some sub-σ-field $\mathcal{E}_n$ of the σ-field $\mathcal{E}$ of all μ-measurable sets. We have $\mathcal{E}_n \uparrow \mathcal{E}$, that is, $\mathcal{E}_1 \subset \mathcal{E}_2 \subset \cdots$ and $\mathcal{E}$ is the least sub-σ-field containing all $\mathcal{E}_n$, which follows from the fact that $\Gamma_1 \cup \Gamma_2 \cup \dots$ contains a countable set that separates points of $\Omega_1^{\text{white}} \times (0,1)$.

If $\Theta_1 h_n = h_n \Theta_1$ (as operators $H_1^f \to H_1^g$) for all n, and $h_n \to h$ almost everywhere, and $\sup_n \|h_n\|_\infty < \infty$, then $\Theta_1 h = h \Theta_1$. Thus, it is sufficient to prove the equality $\Theta_1 h = h\Theta_1$ for all $h \in \Gamma_1 \cup \Gamma_2 \cup \dots$. Without loss of generality we may assume that $h \in \Gamma(a,b)$ for some a, b. Moreover, I assume that $b = 1$, leaving the general case to the reader. Thus,

$$h(\omega_{0,1}^{\text{white}}, t) = h'(\omega_{0,a}^{\text{white}}) \mathbf{1}_{(a,1)}(t).$$

We recall that $\Theta_1 = \Theta_a \otimes \Theta_{1-a}$, $H_1 = H_1(a) \otimes H_0(1-a) \oplus H_0(a) \otimes H_1(1-a)$, and note that

$$\Theta_1\big(H_1^f(a) \otimes H_0^f(1-a)\big) = H_1^g(a) \otimes H_0^g(1-a),$$
$$\Theta_1\big(H_0^f(a) \otimes H_1^f(1-a)\big) = H_0^g(a) \otimes H_1^g(1-a).$$

The subspaces $H_1^f(a) \otimes H_0^f(1-a)$ and $H_1^g(a) \otimes H_0^g(1-a)$ are annihilated by h, thus, $\Theta_1 h$ and $h\Theta_1$ both vanish on $H_1^f(a) \otimes H_0^f(1-a)$. On the other subspace, $H_0^f(a) \otimes H_1^f(1-a)$, h acts as $h' \otimes \mathbf{1}$, while Θ_1 acts as $\mathbf{1} \otimes \Theta_{1-a}$. Therefore $\Theta_1 h = h\Theta_1$. □

Proof of Proposition 11.4. We apply Lemma 11.5 to $h \in L_\infty(\mu)$ defined by

$$h(\omega_1^{\text{white}}, t) = \begin{cases} 1 & \text{if } \tau(\omega_1^{\text{white}}) = t, \\ 0 & \text{otherwise,} \end{cases}$$

where τ is the (global) minimizer on $(0,1)$. This function acts on H_1 as the projection onto the subspace

$$\{\eta_1(\tau)\varphi : \varphi \in L_2(\Omega_1^{\text{white}})\} = L_2(\Omega^{\text{bin}}) \ominus L_2(\Omega^{\text{white}}). \qquad \square$$

Proof of Proposition 11.1. We have two binary extensions, $(\Omega_n^{\text{bin}(f)}, \widetilde{\beta}_n^f)_n$ and $(\Omega_n^{\text{bin}(g)}, \widetilde{\beta}_n^g)_n$, of an inductive system $(\Omega_n^{\text{white}}, \beta_n)_n$ of probability spaces (according to $(\gamma_n^f)_n$ and $(\gamma_n^g)_n$ respectively). Their isomorphism is ensured by Lemma 6.2, provided that Condition 6.2(b) is satisfied by some unitary operators $\Theta_n^{\text{bin}} : L_2(\Omega_n^{\text{bin}(f)}) \to L_2(\Omega_n^{\text{bin}(g)})$. Using the natural embeddings $L_2(\Omega_n^{\text{bin}(f)}) \subset L_2(\Omega^{\text{bin}(f)})$ and $L_2(\Omega_n^{\text{bin}(g)}) \subset L_2(\Omega^{\text{bin}(g)})$ we define all Θ_n^{bin} as restrictions of a single operator $\Theta^{\text{bin}} : L_2(\Omega^{\text{bin}(f)}) \to L_2(\Omega^{\text{bin}(g)})$. Using Proposition 11.4 we define Θ^{bin} as the restriction of Θ_1 to $L_2(\Omega^{\text{bin}(f)})$. It remains to prove that

$$\Theta^{\text{bin}}\big(L_2(\Omega_n^{\text{bin}(f)})\big) = L_2(\Omega_n^{\text{bin}(g)}), \tag{11.6}$$

$$\Theta_n^{\text{bin}} \text{ intertwines the actions of } L_\infty(\Omega_n^{\text{white}}) \text{ on } L_2(\Omega_n^{\text{bin}(f)}) \text{ and } L_2(\Omega_n^{\text{bin}(g)}), \tag{11.7}$$

$$\Theta_n^{\text{bin}} \text{ is trivial on } L_2(\Omega_n^{\text{white}}) \tag{11.8}$$

for all n.

By (11.3), Θ^{bin} is trivial on $L_2(\Omega^{\text{white}})$; (11.8) follows.

By Lemma 11.5, Θ^{bin} intertwines the actions of $L_\infty(\Omega^{\text{white}})$ on $L_2(\Omega^{\text{bin}(f)})$ and $L_2(\Omega^{\text{bin}(g)})$; (11.7) follows.

The proof of (11.6) is the point of Proposition 11.9 below. $\square$

Proposition 11.9. *The operator* Θ^{bin} *maps the subspace*

$$L_2(\Omega_n^{\text{bin}(f)}) \subset L_2(\Omega^{\text{bin}(f)})$$

onto the subspace

$$L_2(\Omega_n^{\text{bin}(g)}) \subset L_2(\Omega^{\text{bin}(g)}).$$

The proof is given after Lemma 11.10.

Recall that the elements of $L_2(\Omega_n^{\text{bin}(f)})$ are functions of the restrictions of B and S_f to $[0, \tau + 3^{-n}] \cap [0,1]$.

For a given $t \in (0,1)$ we consider the sub-σ-field $\mathcal{F}_t^f$ on $\Omega^{\text{bin}(f)}$, generated by the restrictions of B and S_f to $[0,t]$. The elements of the subspace $L_2(\Omega^{\text{bin}(f)}, \mathcal{F}_t^f)$ are functions of $B|_{[0,t]}$ and $S_f|_{[0,t]}$ (that is, $S_f|_{(\tau,t]}$).

We know $L_\infty(\Omega^{\text{white}})$ acts on $L_2(\Omega^{\text{bin}(f)})$. In particular, for $0 < r < s < 1$, the function $\mathbf{1}_{(r,s)}(\tau)$ (that is, the indicator of $\{\omega^{\text{white}} : r < \tau(\omega^{\text{white}}) < s\}$)

acts as the projection onto a subspace $H^f_{r,s}$ of $L_2(\Omega^{\mathrm{bin}(f)})$. The same holds for g. We have $\Theta^{\mathrm{bin}}(H^f_{r,s}) \subset H^g_{r,s}$, since Θ^{bin} intertwines the two actions of $L_\infty(\Omega^{\mathrm{white}})$. We define

$$H^f_{r,s,t} = H^f_{r,s} \cap L_2(\Omega^{\mathrm{bin}(f)}, \mathcal{F}^f_t) \quad \text{for } 0 < r < s < t < 1.$$

Lemma 11.10. $\Theta^{\mathrm{bin}}(H^f_{r,s,t}) \subset H^g_{r,s,t}$.

Proof. The binary extension $\Omega^{\mathrm{bin}(f)}$ is constructed on the time interval $(0,1)$, but the same can be made on the time interval $(0,t)$, giving a binary extension $\Omega^{\mathrm{bin}(f,t)}$ of $\Omega^{\mathrm{white}}_t$, using the (global) minimizer τ_t on $(0,t)$; sometimes $\tau_t = \tau_1$, sometimes $\tau_t \ne \tau_1$.

The binary extension $\Omega^{\mathrm{bin}(f)}$ is the product (recall Definition 7.1) of two binary extensions, $\Omega^{\mathrm{bin}(f,t)}$ and $\Omega^{\mathrm{bin}(f,1-t)}$, according to the set

$$A \subset \Omega^{\mathrm{white}}_1 = \Omega^{\mathrm{white}}_t \times \Omega^{\mathrm{white}}_{1-t},$$
$$A = \Big\{ \omega^{\mathrm{white}}_1 : \tau_1(\omega^{\mathrm{white}}_1) = \tau_t(\omega^{\mathrm{white}}_1|_{[0,t]}) \Big\}.$$

We know that $\Theta_1 = \Theta_t \otimes \Theta_{1-t}$. Similarly to Proposition 11.4,

$$\Theta_t(L_2(\Omega^{\mathrm{bin}(f,t)})) = L_2(\Omega^{\mathrm{bin}(g,t)});$$

we define $\Theta^{\mathrm{bin},t} : L_2(\Omega^{\mathrm{bin}(f,t)}) \to L_2(\Omega^{\mathrm{bin}(g,t)})$ as the restriction of Θ_t and observe that Θ^{bin} is the restriction of $\Theta^{\mathrm{bin},t} \otimes \Theta^{\mathrm{bin},1-t}$ to

$$L_2(\Omega^{\mathrm{bin}(f)}) \subset L_2(\Omega^{\mathrm{bin}(f,t)}) \otimes L_2(\Omega^{\mathrm{bin}(f,1-t)})$$

(recall (7.2)).

By Lemma 7.3, $\Theta^{\mathrm{bin}}(L_2(\widetilde{A}, \mathcal{F}_1)) = L_2(\widetilde{A}', \mathcal{F}'_1)$, where the sets $\widetilde{A} \subset \Omega^{\mathrm{bin}(f)}$, $\widetilde{A}' \subset \Omega^{\mathrm{bin}(g)}$ correspond to the inequality $\tau < t$, the sub-σ-field $\mathcal{F}_1$ on $\widetilde{A}$ is induced by the sub-σ-field $\mathcal{F}^f_t$ on $\Omega^{\mathrm{bin}(f)}$, and $\mathcal{F}'_1$ on $\widetilde{A}'$ — by $\mathcal{F}^g_t$.

Taking into account that $(r,s) \subset (0,t)$ we get $H_{r,s,t} \subset L_2(\widetilde{A}, \mathcal{F}_1)$. Therefore $\Theta^{\mathrm{bin}}(H^f_{r,s,t}) \subset L_2(\widetilde{A}', \mathcal{F}'_1)$. On the other hand,

$$\Theta^{\mathrm{bin}}(H^f_{r,s,t}) \subset \Theta^{\mathrm{bin}}(H^f_{r,s}) \subset H^g_{r,s}.$$

It remains to note that $L_2(\widetilde{A}', \mathcal{F}'_1) \cap H^g_{r,s} \subset H^g_{r,s,t}$. □

Proof of Proposition 11.9. If r,s,t and n satisfy $t \le r + 3^{-n}$ then

$$H^g_{r,s,t} \subset L_2\big(\Omega^{\mathrm{bin}(g)}_n\big)$$

(since $t \le \tau(\cdot) + 3^{-n}$ for all relevant points), and therefore

$$\Theta^{\mathrm{bin}}\big(H^f_{r,s,t}\big) \subset L_2\big(\Omega^{\mathrm{bin}(g)}_n\big).$$

The elements of $L_2\big(\Omega^{\mathrm{bin}(f)}_n\big)$ are functions of the restrictions of B and S_f to $[0, \tau + 3^{-n}] \cap [0,1]$. For every N such that $\frac{1}{N} < 3^{-n}$ consider the functions of the restrictions of B and S_f to $[0, \tau + 3^{-n} - \frac{1}{N}] \cap [0,1]$; these are $L_2(\Omega_n^{\mathrm{bin}(f)}, \mathcal{E}_N)$ for some sub-σ-field $\mathcal{E}_N$, and

$$\bigcup_N L_2(\Omega_n^{\mathrm{bin}(f)}, \mathcal{E}_N) \quad \text{is dense in } L_2(\Omega_n^{\mathrm{bin}(f)}),$$

since $\mathcal{E}_N \uparrow \mathcal{E}$ (a similar argument is used in the proof of Lemma 11.5; note that S_f jumps at $\tau + 2\cdot 3^{-n}$, not $\tau + 3^{-n}$). In order to prove Proposition 11.9 it remains to prove that

$$\Theta^{\mathrm{bin}}\big(L_2(\Omega_n^{\mathrm{bin}(f)}, \mathcal{E}_N)\big) \subset L_2(\Omega^{\mathrm{bin}(g)})$$

for all N (satisfying $\frac{1}{N} < 3^{-n}$).

Clearly,

$$L_2(\Omega^{\mathrm{bin}(f)}) = H_{0,\frac{1}{N}} \oplus \cdots \oplus H_{\frac{N-1}{N},1}$$

(for every N). Every $\psi \in L_2(\Omega^{\mathrm{bin}(f)})$ is of the form

$$\psi = \psi_1 + \cdots + \psi_N, \quad \psi_k \in H_{\frac{k-1}{N},\frac{k}{N}}.$$

If $\psi \in L_2(\Omega_n^{\mathrm{bin}(f)}, \mathcal{E}_N)$ then $\psi_k \in L_2(\Omega^{\mathrm{bin}(f)}, \mathcal{F}_{\frac{k-1}{N}+3^{-n}})$ (since

$$\tau(\cdot) + 3^{-n} - \frac{1}{N} < \frac{k-1}{N} + 3^{-n}$$

for all relevant points), thus, $\psi_k \in H^f_{\frac{k-1}{N},\frac{k}{N},\frac{k-1}{N}+3^{-n}}$. Taking into account that $\Theta^{\mathrm{bin}}\Big(H^f_{\frac{k-1}{N},\frac{k}{N},\frac{k-1}{N}+3^{-n}}\Big) \subset L_2(\Omega_n^{\mathrm{bin}(g)})$ we see that

$$\Theta^{\mathrm{bin}}(\psi) \in L_2(\Omega_n^{\mathrm{bin}(g)}).$$ □

References

[1] Arveson, William. Noncommutative dynamics and E-semigroups. Springer Monographs in Mathematics. *Springer-Verlag, New York*, 2003. x+434 pp. ISBN: 0-387-00151-4. MR1978577 (2004g:46082), Zbl 1032.46001.

[2] Bhat, B. V. Rajarama. Cocycles of CCR flows. *Memoirs Amer. Math. Soc.* **149** (2001), no. 709, x+114 pp. MR1804156 (2002e:46083), Zbl 0976.46050.

[3] Bhat, B. V. Rajarama; Liebscher, Volkmar; Skeide, Michael. A problem of Powers and the product of spatial product systems. arXiv:0801.0042v1 (2008).

[4] Itô, Kiyoshi; McKean, Henry P., Jr. Diffusion processes and their sample paths. Die Grundlehren der Mathematischen Wissenschaften, Band 125. *Springer-Verlag, Berlin-Heidelberg-New York*, 1965. xvii+321 pp. MR0199891 (33#8031), Zbl 0127.09503.

[5] Liebscher, Volkmar. Random sets and invariants for (type II) continuous tensor product systems of Hilbert spaces. arXiv:math.PR/0306365v1 (2003).

[6] Markiewicz, Daniel; Powers, Robert T. Local unitary cocycles of E_0-semigroups. *J. Funct. Anal.* (2008), to appear, doi:10.1016/j.jfa.2008.07.009; also arXiv:0805.2821.

[7] ROGERS, L. C. G.; WILLIAMS DAVID. Diffusions, Markov processes, and martingales. Vol. 2: Itô calculus. Wiley Series in Probability and Mathematical Statistics. *Wiley, New York*, 1987. xiv+475 pp. ISBN: 0-471-91482-7. MR0921238 (89k:60117), Zbl 0627.60001.

[8] TSIRELSON, BORIS. On automorphisms of type *II* Arveson systems (probabilistic approach). arXiv:math.OA/0411062v1 (2004).

[9] TSIRELSON, BORIS. Nonclassical stochastic flows and continuous products. *Probability Surveys* **1** (2004), 173–298. MR2068474 (2005m:60080).

[10] TSIRELSON, BORIS. Automorphisms of the type II_1 Arveson system of Warren's noise. arXiv:math.OA/0612303v1 (2006).

[11] WARREN, JON. Splitting: Tanaka's SDE revisited. arXiv:math.PR/9911115v1 (1999).

[12] WARREN, JON. The noise made by a Poisson snake. *Electronic Journal of Probability* **7** (2002), no. 21, 21 pp. MR1943894 (2004c:60180), Zbl 1007.60052.

[13] WATANABE, S. Stochastic flow and noise associated with the Tanaka stochastic differential equation. *Ukrainian Math. J.* **52** (2000), 1346–1365. MR1816931 (2001m:60140), Zbl 0971.60062.

SCHOOL OF MATHEMATICS, TEL AVIV UNIVERSITY, TEL AVIV 69978, ISRAEL
tsirel@post.tau.ac.il http://www.tau.ac.il/~tsirel/

This paper is available via http://nyjm.albany.edu/j/2008/14-25.html.