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Equivariant KK-theory for semimultiplicative sets

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Abstract. A semimultiplicative set G is a set which has a partially defined associative multiplication. We associate a reduced C^*-algebra $C_r^*(G)$ to G and define reduced crossed products $A \rtimes G$. Moreover, we introduce a G-equivariant KK-theory and show the existence of a Kasparov product.

Contents

1. Introduction

In this work we introduce and analyze some rudiments of semimultiplicative sets in connection with C^*-algebras. Semimultiplicative sets appear in [1], and the somewhat stronger notion of a semigroupoid is due to Exel [3]. A semimultiplicative set G is a set which is endowed with a partially defined associative multiplication (Definition 1). That means we allow, as in groupoids, that a product xy may or may not be defined. When G is a group then there exists a left regular representation $\lambda : G \to B(\ell^2(G))$. In

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a similar way we define a left regular representations for semimultiplicative sets G (Definition 2), thereby canceling all invalid multiplications in G. This concept can also be used to define a reduced product $A \rtimes_\alpha G$ for any left G-C^*-algebra A, that is, a C^*-algebra A which is endowed with a morphism $\alpha : G \to \mathrm{End}(A)$. (Technically, one uses the right regular representation in this case.)

In the second and final part of this work we focus on the K-theory of G-C^*-algebras. We introduce a G-equivariant KK-theory $KK^G(A,B)$ (Definitions 16, 17 and 22) for Hilbert C^*-algebras A, B and discrete countable semimultiplicative sets G. A Hilbert C^*-algebra A is a C^*-algebra which is endowed with a left action $G \to \mathrm{End}(A)$ and a right action $G \to \mathrm{End}(A)$ under which A becomes a G-Hilbert A-module in the natural way. When G happens to be a group, then any G-C^*-algebra is a Hilbert C^*-algebra and our equivariant KK-theory has a similarity to Kasparov's equivariant KK-theory for discrete groups G, the difference being that the underlying G-actions on Hilbert modules need not be full but degenerate (a "unit problem" so to say). See Lemma 7 and its preceding paragraph for the details. Our main work is to prove the existence of a Kasparov product for KK^G (Theorems 2 and 3), and to show its functoriality and associativity (Section 7).

An ongoing study of continuous semimultiplicative sets and their crossed products seems to be necessary to find the right continuity assumptions in KK^G, and actually we aim to continue our investigation in this direction. Kasparov's equivariant KK-theory [7] was generalized by Le Gall for groupoids G in [11] and [12], see also Tu [15] for an overview. Since discrete semimultiplicative sets generalize discrete groupoids it is tempting to compare Le Gall's theory with ours when G is a groupoid (though there seems to be an obvious difference already in the group case), but we will not go into that in this paper.

We give a brief overview of this paper. The Sections 2 and 3 are dedicated to semimultiplicative sets, some of their basic examples, and their crossed products. In Sections 4–6 we introduce $\widetilde{KK^G}(A,B)$ (cycles divided out by operator equivalence) for left G-C^*-algebras A, B and prove the existence of a Kasparov product for $\widetilde{KK^G}$. By definition we require a left and a right G-action for Hilbert modules, though, we are only provided with a left action for the C^*-algebras. This anomaly turns out to be a weak point in the theory, with bad functorial properties, whence in the last Section 7 we consider $KK^G(A,B)$ exclusively for Hilbert C^*-algebras A and B. Comparing the category of G-C^*-algebras and Hilbert C^*-algebras, the latter one seems be the "smooth" one when working in KK^G (at least in the approaches presented here). In the KK-theory part of this paper we closely follow Kasparov's exposition in [8] and Skandalis' paper [14]. This sometimes goes without saying. Influencing in general was also Valette's book [16], which is also recommendable as an introduction to KK-theory.

2. Semimultiplicative sets

Definition 1. A *semimultiplicative set* G is a set which is endowed with a partially defined associative multiplication, that is, there exists a subset $G^{(2)} \subseteq G \times G$ and a multiplication $G^{(2)} \to G$, $(a,b) \mapsto ab$ such that whenever $(ab)c$ or $a(bc)$ is defined then both $(ab)c$ and $a(bc)$ are defined and are equal.

If one also requires in the last definition that $(ab)c$ is defined whenever both ab and bc are defined then one would speak of a *semigroupoid*, see Exel [3]. Let L_g denote the left multiplication operator on G, that is, $L_g(h) = gh$ for $g, h \in G$. Its domain is $\{h \mid gh \text{ is defined}\}$. Write R_g for the right multiplication operator. We say that G has *injective left* (resp. *right*) *multiplication* if L_g (resp. R_g) is injective for all $g \in G$. We write $(e_g)_{g\in G}$ for the canonical base in $\ell^2(G)$.

Definition 2. Assume that G has injective left multiplication. The *left regular representation* of G is the map $\lambda : G \to B(\ell^2(G))$ given by

$$\lambda_g\left(\sum_{h\in G} \alpha_h e_h\right) = \sum_{h\in G,\, gh \text{ is defined}} \alpha_h e_{gh},$$

where α_h are scalars in $\mathbb{C}$. The C^*-subalgebra of $B(\ell^2(G))$ generated by $\lambda(G)$ is called the *reduced C^*-algebra* of G and denoted by $C^*_r(G)$.

Analogously, for G with injective right multiplication we can define a *right regular representation* $\rho : G \to B(\ell^2(G))$ in the obvious way.

We are going to give some simple examples of semimultiplicative sets. Clearly, groups, groupoids, semigroups, semigroupoids and multiplicative sets are semimultiplicative sets. If R is a ring then $R\backslash\{0\}$ is a semimultiplicative set under multiplication (however not a semigroupoid in general). The set of natural numbers under addition is a semimultiplicative set (and semigroup), and its reduced C^*-algebra is the Toeplitz algebra.

If Λ is a higher rank graph [9], that is one has a degree map d mapping Λ in $\mathbb{N}^k$, then the truncated graph $\Lambda^{(\le N)} = \{a \in S \mid d(a) \le N\}$ is a semimultiplicative set (that is, a product ab is defined if and only if $d(ab) \le N$) which has injective left multiplication. This is not a semigroupoid.

For $N \ge 0$, the real interval $[0, N]$ is a semimultiplicative set (but not a semigroupoid) under addition. That means, we let the composition $a \circ b$ be defined if and only if $a + b \in [0, N]$, and in this case we put $a \circ b = a + b$. More generally, the interval $[0, N \cdot 1]$ in a C^*-algebra is a semimultiplicative set.

Take the nonnegative reals $\mathbb{R}_+$ and the compact interval $[0, N]$ of reals whose elements are formally written as $t\mu$ for $t \in [0, N]$. Then $\mathbb{R}_+ \sqcup [0, N]$ is a semimultiplicative set (but not a semigroupoid) under the composition

$a \circ b = a + b$ if both a and b are in $\mathbb{R}_+$, and $a \circ b\mu = (a+b)\mu$ if $a \in \mathbb{R}_+, b \in [0, N]$ and $a + b \leq N$. Other compositions are not allowed.

If G is a semimultiplicative set with injective multiplication, and λ is the left regular representation, then the set of nonzero words in the letters $\lambda(G) \cup \lambda(G)^*$ is a semimultiplicative set $G^{(*)}$. Though one is now provided with a left inverse $\lambda(s)^*$ for $\lambda(s)$, in general $G^{(*)}$ need not to be a groupoid, see the next examples.

Consider the two graphs

$$
\begin{tikzcd}
\bullet a \arrow[d, "g"'] & \bullet b \arrow[d, shift right, "g"'] \\
\bullet c & \bullet d \arrow[u, shift right, "h"']
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \bullet a \arrow[dr, "h"] & \\
\bullet b \arrow[r] & \bullet c \arrow[r, "h"'] & \bullet d
\end{tikzcd}
$$

Assume the left graph is realized in a C^*-algebra where a, b, c, d are mutually orthogonal projections, and g, h are partial isometries with source and range projections as indicated in the diagram ($s(g) = a + b, r(g) = c + d$). Let G be the semimultiplicative set which consists of all nonzero products in the letters $\{a, b, s, r, g, h\}$. Then G is not a semigroupoid since $ga \neq 0$ and $hg \neq 0$ but $hga = 0$. The semimultiplicative set G associated to the right graph has injective left multiplication and is a semigroupoid but not a groupoid, as naturally choosing the source and range maps causes a problem: the composition ha exists but $s(h) = a + c \neq a = r(a)$.

Take a semimultiplicative set G and a family $(G_i)_{i \in I}$ of copies of G. Set $H = \bigsqcup_{i \in I} G_i$ and write $\pi_i : G \to G_i$ for the canonical bijection. We define a multiplication on H by $\pi_i(x)\pi_j(y) = \pi_j(xy)$ for all $x, y \in G, i, j \in I$ and whenever xy is defined. Then H is a semimultiplicative set which has injective left multiplication if G does so. The left regular representation λ, however, is not injective, as $L_{\pi_i(x)} = L_{\pi_j(x)}$ for all $x \in G, i, j \in I$.

Though the emphasis of this paper lies on discrete semimultiplicative sets, we will give one continuous example. Take the real interval $[0, N]$ as our semimultiplicative set G as described above. The formal convolution

$$a * b = \text{“} \int_{[0,N]} a(s)\delta_s ds \int_{[0,N]} b(t)\delta_t dt \text{”}$$

leads us to the convolution product

$$(a * b)(t) = \int_0^t a(s)b(t - s)ds.$$

It is straightforward to check that this convolution product is associative. So the continuous left regular representation $\lambda : C(G) \to B(L^2[0, N])$ given by $\lambda(f) = f * g$ for $g \in L^2[0, N]$ yields a reduced C^*-algebra $C^*_r(G) = C^*(\lambda(C(G)))$ associated to the continuous semimultiplicative set G.

3. Crossed products

For the rest of this paper G denotes a semimultiplicative set, which is occasionally regarded as a discrete topological set.

Definition 3. A *morphism* (resp. *antimorphism*) $\sigma : G \to H$ between two semimultiplicative sets G and H is a map satisfying $\sigma(gh) = \sigma(g)\sigma(h)$ (resp. $\sigma(gh) = \sigma(h)\sigma(g)$) for all $(g, h) \in G^{(2)}$.

Definition 4. If X is a linear space then a *left* (resp. *right*) *linear action* α on X is a morphism (resp. antimorphism) $\alpha : G \to L(X)$, where $L(X)$ denotes the set of linear maps on X. If X is a C^*-algebra then an *action* α on X is a linear action on X such that $\alpha(g)$ is a $*$-homomorphism for all $g \in G$. In this case we call X a *left* (resp. *right*) *G-C^*-algebra*.

Definition 5. A *left action* of a semimultiplicative set G on a set X is given by a subset $(G \times X)^{(2)} \subseteq G \times X$ and a multiplication $(G \times X)^{(2)} \to X$, $(g, x) \mapsto gx$ such that $(gh)x$ is defined if and only if $g(hx)$ and gh is defined, and then $(gh)x = g(hx)$, for all $g, h \in G, x \in X$.

Example 1. (a) Suppose G has injective left multiplication, and X is a discrete set endowed with a left action by G. We obtain a right G-action on the C^*-algebra $C_0(X)$ by letting

$$(fg)(x) = 1_{\{gx \text{ is defined}\}} f(gx)$$

for $f \in C_0(X)$ and $g, x \in G$. Similarly, a left action on $C_0(X)$ is given by

$$(gf)(gx) = f(x), \quad (gf)(y) = 0 \text{ if } y \neq gx.$$

(b) Analogously, if G has injective right multiplication and X has a right G-action then $C_0(X)$ is a left and right G-C^*-algebra by

$$(gf)(x) = 1_{\{xg \text{ is defined}\}} f(xg),$$
$$(fg)(xg) = f(x), \quad (fg)(y) = 0 \text{ if } y \neq xg.$$

Definition 6. Assume that G has injective left multiplication. Suppose that A is a C^*-algebra which is endowed with a right G-action and which is essentially represented on a Hilbert space H. Let $U : G \to B(\ell^2(G, H))$ and the C^*-representation $\pi : A \to B(\ell^2(G, H))$ be given by

$$\pi(a)(\xi e_h) = ((ah)\xi)e_h, \quad U_g(\xi e_h) = 1_{\{gh \text{ is defined}\}} \xi e_{gh},$$

where ξe_g stands for the function $h \mapsto 1_{\{h=g\}}\xi$ $(g, h \in G, \xi \in H, a \in A)$. Then the *reduced crossed product* $G \ltimes_r A$ is defined as the C^*-subalgebra of $B(\ell^2(G, H))$ generated by $U_G\pi(A) = \{U_g\pi(a) \mid a \in A, g \in G\}$.

Definition 7. Assume that G has injective right multiplication. Suppose that A is a left G-C^*-algebra essentially represented on H. Let $V, U : G \to B(\ell^2(G, H))$ and $\pi : A \to B(\ell^2(G, H))$ be given by

$$\pi(a)(\xi e_h) = ((ha)\xi)e_h, \quad V_g(\xi e_h) = 1_{\{hg \text{ is defined}\}} \xi e_{hg}, \quad U_g = V_g^*,$$

$(g, h \in G, \xi \in H, a \in A)$. Then the *reduced crossed product* $A \rtimes_r G$ is defined as the C^*-subalgebra of $B(\ell^2(G, H))$ generated by

$$\pi(A)U_G = \{\pi(a)U_g \mid a \in A, g \in G\}.$$

If $A = \mathbb{C}$ and the action of G on $\mathbb{C}$ is trivial, that is, $ag = a$ for all $a \in A, g \in G$, then the reduced C^*-algebra $C_r^*(G)$ coincides with the reduced crossed product $G \ltimes_r \mathbb{C}$. If $\pi' : A \to B(H')$ is another essential representation then a canonical unitary $W : \ell^2(G, H) \to \ell^2(G, H')$ shows that the definition of the reduced product does not depend on the representation π up to $*$-isomorphism.

Definition 8. A *left action* of G on a Hilbert space H is a morphism $U : G \to B(H)$ such that each U_g is a partial isometry $(g \in G)$. We call H with such an action U a (left) G-Hilbert space. The action is called *strong* if $U_g U_h = 0$ for all undefined compositions gh.

Definition 9. If A is a right G-C^*-algebra and H a left G-Hilbert space, then a $*$-homomorphism $\pi : A \to B(H)$ is called *equivariant* if

$$U_g^* U_g \pi(ag) = U_g^* \pi(a) U_g, \; \pi(a) U_g U_g^* = U_g U_g^* \pi(a), \; \pi(a) U_g^* U_g = U_g^* U_g \pi(a)$$

for all $a \in A, g \in G$. If the action on A is from the left then the first identity has to be replaced by $U_g U_g^* \pi(ga) = U_g \pi(a) U_g^*$.

It is easily verified that π of Definition 6 is an equivariant representation, and the action U on $\ell^2(G, H)$ is strong. If the action on A satisfies $(ag)h = 0$ for all $a \in A$ whenever gh is not defined $(g, h \in G)$ then one would have $U_g^* \pi(a) U_g = \pi(ag)$ for all $a \in A, g \in G$ for the representation of Definition 6. However, this requirement is too restrictive for us as we also want to consider the trivial action on $\mathbb{C}$.

The next lemma links convolution algebras and equivariant representations.

Lemma 1. *Endowing $C_c(G, A)$ with the convolution product given by*

$$(g \cdot a)(h \cdot b) = gh \cdot ((ah)b) 1_{\{gh \text{ is defined}\}},$$

where $a, b \in A$, $g, h \in G$, and $g \cdot a$ denotes the map $h \mapsto 1_{\{h=g\}} a$, the map $\sigma(g \cdot a) = U_g \pi(a)$ extends to an algebra homomorphism from $C_c(G, A)$ to $\operatorname{span}(U_G \pi(A))$ *for any equivariant representation (π, U) with strong action U.*

Proof. Straightforward. □

Definition 10. A right (resp. left) G-action $\alpha : G \to \operatorname{End}(A)$ on a C^*-algebra A is called *left-invertible* (resp. *right-invertible*) if for all $g \in G$ there is a $T_g \in \operatorname{End}(A)$ such that $\alpha(h)\alpha(g)T_g = \alpha(h)$ for all $h \in G$ for which gh (resp. hg) exists.

If G has injective right multiplication then we may introduce a virtual inverse g^{-1} for each $g \in G$, and write $xg^{-1} = h$ if $x = hg$, and let xg^{-1} otherwise undefined. (In general one does not obtain a semimultiplicative set in this way; take for example $[0, N]$ as a counterexample: $[-N, N]$ is not a semimultiplicative set.) If we suggestively write $T_g = \alpha(g^{-1})$ in the last definition then it becomes clear that invertibility of a left G-action α is the counterpart to injective right multiplication in G.

Lemma 2. *Assume that G has injective left (resp. right) multiplication and A is a right (resp. left) G-C^*-algebra whose G-action is left-invertible (resp. right-invertible). (We will write $g^{-1} := T_g$ for any choice T_g as in Definition 10.) Then the representation of Definition 6 (resp. 7) satisfies*

$$U_g \pi(a) U_g^* = \pi(g^{-1}(a)) U_g U_g^*$$

(resp. $U_g^* \pi(a) U_g = \pi(g^{-1}(a)) U_g^* U_g$) for all $a \in A, g \in G$.

Proof. Straightforward. □

4. Equivariant *KK*-theory

In the rest of this paper all C^*-algebras are supposed to be graded. A C^*-algebra B is graded if there is a grading automorphism $\varepsilon : B \to B$, $\varepsilon^2 = 1$. The grading is called trivial if $\varepsilon = 1$. An element $a \in B$ has degree $i = 0, 1$ if $\varepsilon(b) = (-1)^i b$. (Notation: $\partial b = i$.) All homomorphisms in the category of graded C^*-algebras are graded, i.e., commute with ε. All commutators are graded, that is, $[a, b] = ab - (-1)^{\partial a \cdot \partial b} ab$ for homogenous elements a, b, and the commutator is extended by linearity to all a, b. A Hilbert module $\mathcal{E}$ over a C^*-algebra B is always supposed to be graded, that is, there is a grading linear map $\varepsilon : \mathcal{E} \to \mathcal{E}$, $\varepsilon^2 = 1$, which is compatible with the grading of B, i.e., $\varepsilon(xb) = \varepsilon(x)\varepsilon(b)$ and $\varepsilon(\langle x, y \rangle) = \langle \varepsilon(x), \varepsilon(y) \rangle$, for all $x, y \in \mathcal{E}, b \in B$. The space of linear maps $L(\mathcal{E})$ on $\mathcal{E}$ is graded by the grading operator $\varepsilon(T) = \varepsilon T \varepsilon$, $T \in L(\mathcal{E})$. We write $\mathcal{L}(\mathcal{E})$ for the C^*-algebra of adjointable operators $T : \mathcal{E} \to \mathcal{E}$, and $\mathcal{K}(\mathcal{E}) \subseteq \mathcal{L}(\mathcal{E})$ for the C^*-algebra of compact operators, that is, $\mathcal{K}(\mathcal{E})$ is generated by the elements $\theta_{\xi,\eta} \in \mathcal{L}(\mathcal{E})$, $\theta_{\xi,\eta}(x) = \xi\langle \eta, x \rangle$, for all $\xi, \eta \in \mathcal{E}$, see Kasparov [6] or the books [10], [5]. We write $\mathcal{M}(A)$ for the multiplier algebra of a C^*-algebra A, see also [6], Theorem 1, for an isomorphism $\mathcal{M}(\mathcal{K}(\mathcal{E})) \cong \mathcal{L}(\mathcal{E})$.

For the rest of this paper we (may) drop the associativity requirement on G, that is, G is only a set together with a subset $G^{(2)} \subseteq G \times G$ and a function $G^{(2)} \to G$. However, we still call G a semimultiplicative set. All semimultiplicative sets G are supposed to be discrete and countable (thus locally compact, σ-compact Hausdorff spaces). All algebras and C^*-algebras are left G-C^*-algebras (if nothing else is said). All homomorphisms $\sigma : A \to B$ between C^*-algebras A, B are supposed to be $*$-homomorphisms which are graded (i.e., commute with ε) and equivariant (i.e., $\sigma(ga) = g\sigma(a)$ for all $g \in G, a \in A$).

Definition 11. Let B be a G-C^*-algebra. An *action* of G on a Hilbert B-module $\mathcal{E}$ consists of a left linear G-action $U : G \to L(\mathcal{E})$ on $\mathcal{E}$, and a right linear G-action $V : G \to L(\mathcal{E})$, which we denote by $U^* = V$, satisfying

$$U_g U_g^* U_g = U_g, \quad U_g^* U_g U_g^* = U_g^*,$$
$$\langle U_g x, y\rangle = g\langle x, U_g^* y\rangle, \quad U_g(xb) = (U_g x)(gb)$$

for all $x, y \in \mathcal{E}, b \in B, g \in G$. U_g and U_g^* must respect the grading (i.e., commute with ε) for each $g \in G$. Further we require g to be isometric on

$$B_g = \overline{\operatorname{span}}\{\langle U_g^* U_g x, y\rangle \in B \mid x, y \in \mathcal{E}\}$$

for all $g \in G$. Given such maps U and V we call $\mathcal{E}$ a *G-Hilbert B-module*.

One may observe that B_g is a two-sided closed ideal (without G-action) in B, see Lemma 3 below. Notice that Definition 11 consistently redefines G-Hilbert spaces when $B = \mathbb{C}$ with the trivial action and grading.

Lemma 3. *Let $\mathcal{E}$ be a G-Hilbert module with action U. Then each U_g is a partial isometry on $\mathcal{E}$ with self-adjoint source and range projections $U_g^* U_g$ and $U_g U_g^*$ respectively in $\mathcal{L}(\mathcal{E})$, and inverse partial isometry U_g^*. Moreover, $\langle x, U_g y\rangle = g\langle U_g^* x, y\rangle$ and $U_g^*(xg(b)) = U_g^*(x)b$ for all $x, y \in \mathcal{E}, g \in G, b \in B$.*

Proof. Let us begin with proving the following claim:

$$\langle x, U_g y\rangle = \langle U_g y, x\rangle^* = (g\langle y, U_g^* x\rangle)^* = g\langle U_g^* x, y\rangle.$$

Then one has $g\langle U_g^* U_g U_g^* U_g x, y\rangle = g\langle U_g^* U_g x, U_g^* U_g y\rangle$, and by injectivity of g on B_g this shows that $\langle U_g^* U_g x, y\rangle = \langle U_g^* U_g x, U_g^* U_g y\rangle$. This shows that $U_g^* U_g$ is selfadjoint and hence in $\mathcal{L}(\mathcal{E})$. Each U_g is a partial isometry, that means, $\|U_g(U_g^* U_g x)\| = \|U_g^* U_g x\|$ and $U_g(1 - U_g^* U_g) = 0$. The last claim follows from $U_g^* U_g U_g^*(xg(b)) = U_g^*(U_g U_g^*(x)g(b)) = U_g^*(U_g(U_g^*(x)b))$. □

Definition 12. A *Hilbert C^*-algebra A* is a G-C^*-algebra which is also a G-Hilbert module over A with inner product $\langle x, y\rangle = x^* y$ and action $U_g(x) = g(x)$ for all $x \in A, g \in G$. We also require that U_g^* is a $*$-homomorphism for all $g \in G$.

The algebra $C_0(X)$ of Example 1 is a Hilbert C^*-algebra. Any C^*-algebra A with trivial action $g(a) = a$, $a \in A, g \in G$, is a Hilbert C^*-algebra.

Definition 13. Given a G-Hilbert module $\mathcal{E}$, we endow $\mathcal{L}(\mathcal{E})$ with the left linear action $g(T) = U_g T U_g^*$ and the right linear action $g^{-1}(T) = U_g^* T U_g$ for $g \in G, T \in \mathcal{L}(\mathcal{E})$.

$\mathcal{L}(\mathcal{E})$ and subalgebras of it are usually not regarded as G-algebras, as the action is not a C^*-action. Note that $g^{-1}(T)$ is indeed adjointable: from $g\langle U_g^* T U_g x, y\rangle = g\langle x, U_g^* T^* U_g y\rangle$ for all $x, y \in \mathcal{E}, g \in G$, the injectivity of g on B_g and self-adjointness of $U_g^* U_g$ it follows $\langle U_g^* T U_g x, y\rangle = \langle x, U_g^* T^* U_g y\rangle$. With Lemma 3 one checks that $g(T), g^{-1}(T) \in \mathcal{K}(\mathcal{E})$ for all $g \in G$ and compact operators $T \in \mathcal{K}(\mathcal{E})$.

Definition 14. A subalgebra A of $\mathcal{L}(\mathcal{E})$ is called *G-invariant* if for all $g \in G$ the sets $g(A), g^{-1}(A), U_g U_g^* A, A U_g U_g^*, U_g^* U_g A, A U_g^* U_g$ are subsets of A.

Definition 15. If A is a left G-C^*-algebra and $\mathcal{E}$ is a G-Hilbert B-module, then a $*$-homomorphism $\pi : A \to \mathcal{L}(\mathcal{E})$ is called *equivariant* if

$$U_g U_g^* \pi(ga) = U_g \pi(a) U_g^*,$$
$$U_g U_g^* \pi(a) = \pi(a) U_g U_g^*, \quad U_g^* U_g \pi(a) = \pi(a) U_g^* U_g$$

for all $a \in A, g \in G$. Moreover, we require that $U_g^* \pi(A) U_g \subseteq U_g^* U_g \pi(A)$ for all $g \in G$ ('G^{-1}-invariance').

In the rest of this article all Hilbert modules are supposed to be G-Hilbert modules, and all homomorphisms from C^*-algebras into $\mathcal{L}(\mathcal{E})$ are supposed to be equivariant. We call a Hilbert B-module $\mathcal{E}$ together with an equivariant $*$-homomorphism $\varphi : A \to \mathcal{L}(\mathcal{E})$ a *Hilbert (A, B)-bimodule.* (Notice that $a \cdot x := \varphi(a)(x)$ makes $\mathcal{E}$ a left A-module.) With some abuse of notation we shall often identify elements of A with operators on $\mathcal{E}$.

Example 2. If $\mathbb{C}$ is endowed with the trivial action and $\mathcal{E}$ is a G-Hilbert B-module then $\mathcal{E}$ is a G-Hilbert $(\mathbb{C}, B)$-bimodule.

Any C^*-algebra A with the trivial action is a Hilbert (A, A)-bimodule.

Consider C^*-algebras A, B (without G-action) and a homomorphism $\sigma : A \to B$. Let X and G be as in Example 1. Then $A_1 = C_0(X, A) \cong C_0(X) \otimes A$ and $B_1 = C_0(X, B)$ are Hilbert C^*-algebras, and $C_0(X, B)$ is a Hilbert (A_1, B_1)-bimodule with A_1-action $(ab)(x) = \sigma(a(x))b(x)$ ($a \in A_1, b \in B_1, x \in X$).

Somewhat more generally, one may consider a family $\mathcal{B} = (B_x)_{x \in X}$ of C^*-algebras with a family of isomorphisms $\phi_{gx,x} : B_x \to B_{gx}$ whenever gx is defined ($g \in G$) such that $\phi_{hgx,gx} \circ \phi_{gx,x} = \phi_{hgx,x}$ whenever $(hg)x$ is defined. Then the (continuous) sections $\Gamma_0(\mathcal{B})$ of $\mathcal{B}$ vanishing at infinity are a Hilbert C^*-algebra under the G-action $\beta_g(b_x \delta_x) = 1_{\{gx \text{ is defined}\}} \phi_{gx,x}(b_x) \delta_{gx}$. One may also consider another C^*-family $\mathcal{A} = (A_x; \psi_x)_{x \in X}$ and a family of homomorphisms $\sigma_x : A_x \to B_x$ ($x \in X$) satisfying $\phi_{gx,x} \sigma_x = \sigma_{gx} \psi_{gx,x}$ to obtain a G-Hilbert $(\Gamma_0(\mathcal{A}), \Gamma_0(\mathcal{B}))$-bimodule $\Gamma_0(\mathcal{B})$.

If G has injective right multiplication and A is a C^*-algebra with invertible left G-action then the representation (π, U) of Definition 7 is equivariant in the sense of Definition 15 by Lemma 2.

Let S be an inverse semigroup and α an S-action on a C^*-algebra A in the sense of Sieben [13], i.e., a morphism of S into the partial actions on A. Assume there exist commuting Hilbert-module-self-adjoint projections $Q_{ss^*}, Q_{s^*s} \in \mathrm{End}(A)$ projecting onto the range and source, respectively, of α_s ($s \in S$). Then $\beta_s = \alpha_s Q_{s^*s} \in \mathrm{End}(A)$ is a S-Hilbert C^*-action on A. Indeed, note that $(\alpha_t \circ \alpha_s) Q_{s^*t^*ts} = \alpha_t Q_{t^*t} \alpha_s Q_{s^*t^*ts}$ ($s, t \in S$), and so $Q_{s^*t^*ts} = \alpha_{s^*} Q_{t^*t} \alpha_s Q_{s^*t^*ts}$. Hence $\alpha_t Q_{t^*t} \alpha_s Q_{s^*s} = \alpha_t Q_{t^*t} Q_{ss^*} \alpha_s Q_{s^*s} = \alpha_t \alpha_s \alpha_{s^*} Q_{t^*t} \alpha_s Q_{s^*s} = \alpha_{ts} Q_{s^*t^*ts}$.

For the definitions of the internal and skew tensor products of Hilbert modules see Kasparov [7], Section 2. The grading operator for tensor products of Hilbert modules or C^*-algebras is the diagonal grading operator $\varepsilon \otimes \varepsilon$. We denote the skew commutative (minimal) tensor product between C^*-algebras A, B (see Kasparov [7], Section 2) by $A \otimes B$ (that is $(a_1 \otimes b_1)(a_2 \otimes b_2) = (-1)^{\partial b_1 \cdot \partial a_2}(a_1 a_2 \otimes b_1 b_2)$, $(a \otimes b)^* = (-1)^{\partial a \cdot \partial b}(a^* \otimes b^*)$ for $a, a_i \in A, b, b_i \in B$). We endow $A \otimes B$ with the diagonal action $g(a \otimes b) = g(a) \otimes g(b)$ for all $g \in G, a \in A, b \in B$.

Lemma 4. *If $\mathcal{E}_i$ are G-Hilbert B_i-modules $(i = 1, 2)$ and $\varphi : B_1 \to \mathcal{L}(\mathcal{E}_2)$ is an equivariant $*$-homomorphism (not necessarily satisfying the G^{-1}-invariance) then the internal tensor product $\mathcal{E}_1 \otimes_{B_1} \mathcal{E}_2$ is a G-Hilbert B_2-module under the diagonal-action $U^{(1)} \otimes U^{(2)}$. If $\mathcal{E}_1$ is a G-Hilbert (A, B_1)-bimodule, then $\mathcal{E}_1 \otimes_{B_1} \mathcal{E}_2$ is a G-Hilbert (A, B_2)-bimodule (under the A-action $\pi : A \to \mathcal{L}(\mathcal{E}_1 \otimes_{B_1} \mathcal{E}_2)$, $\pi(a) = a \otimes 1$).*

Proof. Consider the algebraic tensor product $\mathcal{E}_1 \odot \mathcal{E}_2$ with its natural structure of a B_2-module and with the B_2-scalar product given by the formula

$$\langle x_1 \odot x_2, y_1 \odot y_2 \rangle = \langle x_2, \varphi(\langle x_1, y_1 \rangle) y_2 \rangle$$

for all $x_1, y_1 \in \mathcal{E}_1, x_2, y_2 \in \mathcal{E}_2$. Factoring out the B_2-submodule

$$\mathcal{N} = \{ z \in \mathcal{E}_1 \odot \mathcal{E}_2 \mid \langle z, z \rangle = 0 \}$$

and then completing the factor module in the norm $\|z\| = \|\langle z, z \rangle\|^{1/2}$ we obtain a Hilbert B_2-module which is denoted by $\mathcal{E}_1 \otimes_{B_1} \mathcal{E}_2$. This tensor product will be endowed with a G-action that comes from the diagonal action $(U_g^{(1)} \odot U_g^{(2)})(x_1 \odot x_2) = U_g^{(1)}(x_1) \odot U_g^{(2)}(x_2)$ on $\mathcal{E}_1 \odot \mathcal{E}_2$, where the "adjoint" operator to $W_g = U_g^{(1)} \odot U_g^{(2)}$ is given by $W_g^* = (U_g^{(1)})^* \odot (U_g^{(2)})^*$. Indeed, it is straightforward to compute that

$$\langle (U_g^{(1)} \odot U_g^{(2)})(x_1 \odot x_2), y_1 \odot y_2 \rangle = g \langle x_1 \odot x_2, (U_g^{(1)*} \odot U_g^{(2)*})(y_1 \odot y_2) \rangle$$

for all $x_1, y_1 \in \mathcal{E}_1, x_2, y_2 \in \mathcal{E}_2, g \in G$. It is also straightforward to check that $(U_g^{(1)} \odot U_g^{(2)})^* (U_g^{(1)} \odot U_g^{(2)})$ is self-adjoint, idempotent and seminorm-contractive on $\mathcal{E}_1 \odot \mathcal{E}_2$ by a similar argument usually used to show that $\mathcal{L}(\mathcal{E}_1) \otimes 1 \subseteq \mathcal{L}(\mathcal{E}_1 \otimes \mathcal{E}_2)$ (see for instance Lance [10], Section 4). Hence for $x, y \in \mathcal{E}_1 \odot \mathcal{E}_2$ one has

$$\langle W_g x, W_g y \rangle = g \langle x, W_g^* W_g y \rangle = g \langle W_g^* W_g x, W_g^* W_g y \rangle,$$

and since g is isometric on $\langle W_g^* W_g x, y \rangle \in \mathrm{span}_{a,b} \langle U_g^{(2)*} U_g^{(2)} a, b \rangle$, one gets $\|W_g x\| = \|W_g^* W_g x\| \le \|x\|$, and consequently also $\|W_g^* x\| = \|W_g W_g^* x\| \le \|x\|$ by $W_g^* W_g W_g^* = W_g^*$. Thus W_g and W_g^* leave $\mathcal{N}$ invariant and their linear quotient maps extend by continuity to linear maps on $\mathcal{E}_1 \otimes_{B_1} \mathcal{E}_2$ denoted by $U_g^{(1)} \otimes U_g^{(2)}$ and $U_g^{(1)*} \otimes U_g^{(2)*}$ which make $\mathcal{E}_1 \otimes_{B_1} \mathcal{E}_2$ a G-Hilbert module. $\square$

Lemma 5. *If $\mathcal{E}_i$ are G-Hilbert B_i-modules ($i = 1, 2$) then the skew tensor product $\mathcal{E}_1 \otimes \mathcal{E}_2$ is a G-Hilbert $(B_1 \otimes B_2)$-module under the diagonal action of G. If $\mathcal{E}_i$ are G-Hilbert (A_i, B_i)-bimodules ($i = 1, 2$), then $\mathcal{E}_1 \otimes \mathcal{E}_2$ is a G-Hilbert $(A_1 \otimes A_2, B_1 \otimes B_2)$-bimodule.*

Proof. This may be proved similarly as Lemma 4. We only discuss the G^{-1}-invariance of Definition 15: Let U denote the diagonal action on $\mathcal{E}_1 \otimes \mathcal{E}_2$ and $\pi : A_1 \otimes A_2 \to \mathcal{L}(\mathcal{E}_1 \otimes \mathcal{E}_2)$ the canonical homomorphism. It is clear that $U_g^* \pi(A_1 \odot A_2) U_g \subseteq U_g^* U_g \pi(A_1 \otimes A_2) =: X$ for all $g \in G$. But then also $U_g^* \pi(A_1 \otimes A_2) U_g \subseteq X$, as X is the closed image of the $*$-homomorphism σ, where $\sigma(x) = U_g^* U_g \pi(x)$. □

For a Hilbert (A, B)-bimodule $\mathcal{E}$ and a subset $C \subseteq \mathcal{L}(\mathcal{E})$, we denote

$$Q_C(\mathcal{E}) = \{T \in \mathcal{L}(\mathcal{E}) \mid [T, c] \in \mathcal{K}(\mathcal{E}),\ \forall c \in C\},$$
$$I_C(\mathcal{E}) = \{T \in \mathcal{L}(\mathcal{E}) \mid Tc \text{ and } cT \text{ in } \mathcal{K}(\mathcal{E}),\ \forall c \in C\}.$$

Definition 16. Let A and B be G-C^*-algebras. A *cycle over* (A, B) is a pair $(\mathcal{E}, T)$, where $\mathcal{E}$ is a countably generated G-Hilbert (A, B)-bimodule, and T is an operator in $Q_A(\mathcal{E})$ of degree 1 such that

$$T - T^*, \quad T^2 - 1, \quad \varphi_1(g) = U_g T U_g^* - U_g U_g^* T U_g U_g^* = g(T) - gg^{-1}(T),$$
$$\varphi_2(g) = U_g U_g^* T - T U_g U_g^*, \quad \varphi_3(g) = U_g^* U_g T - T U_g^* U_g$$

belong to $I_A(\mathcal{E})$ for all $g \in G$. We shall not distinguish between cycles $(\mathcal{E}_1, T_1)$ and $(\mathcal{E}_2, T_2)$ if there is an isometric, grading preserving isomorphism $u : \mathcal{E}_1 \to \mathcal{E}_2$ of G-Hilbert (A, B)-bimodules with $T_2 = u T_1 u^{-1}$. The set of all cycles will be denoted by $\mathbb{E}^G(A, B)$. A cycle $(\mathcal{E}, T)$ will be called *degenerate* if the elements

$$[a, T], \quad a(T - T^*), \quad a(T^2 - 1), \quad a\varphi_1(g), \quad a\varphi_2(g), \quad a\varphi_3(g)$$

are 0 for all $a \in A$, $g \in G$. The set of degenerate cycles is denoted by $\mathcal{D}^G(A, B)$.

Lemma 6. *If $(\mathcal{E}, T) \in \mathbb{E}^G(A, B)$ then $U_g^* T U_g - U_g^* U_g T U_g^* U_g \in I_A(\mathcal{E})$ for all $a \in A$.*

Proof. By the rules of Definition 15 it is straightforward to check that $ag^{-1}(T) - ag^{-1}g(T) = g^{-1}(g(a)gg^{-1}(T) - g(a)g(T)) \in \mathcal{K}(\mathcal{E})$. □

We define an addition of cycles $(\mathcal{E}_1, T_1), (\mathcal{E}_2, T_2) \in \mathbb{E}^G(A, B)$ by taking the direct sum: $(\mathcal{E}_1, T_1) \oplus (\mathcal{E}_2, T_2) = (\mathcal{E}_1 \oplus \mathcal{E}_2, T_1 \oplus T_2)$.

Definition 17. Two cycles $(\mathcal{E}_0, T_0)$ and $(\mathcal{E}_1, T_1)$ over (A, B) are *operatorially homotopic* if $\mathcal{E}_0 = \mathcal{E}_1$ and there exists a norm continuous path $t \mapsto T_t \in \mathcal{L}(\mathcal{E}_0)$ ($t \in [0, 1]$) such that for each $t \in [0, 1]$ the pair $(\mathcal{E}_0, T_t)$ is a cycle over (A, B). Two cycles $(\mathcal{E}_0, T_0)$ and $(\mathcal{E}_1, T_1)$ in $\mathbb{E}^G(A, B)$ are *operatorially equivalent* if there are degenerate cycles $(\mathcal{F}_0, S_0), (\mathcal{F}_1, S_1) \in \mathcal{D}^G(A, B)$ such that $(\mathcal{E}_0, T_0) \oplus (\mathcal{F}_0, S_0)$ is operatorially homotopic to $(\mathcal{E}_1, T_1) \oplus (\mathcal{F}_1, S_1)$. The set $\widetilde{KK^G}(A,B)$ is defined as the quotient of $\mathbb{E}^G(A,B)$ by the equivalence relation given by operatorial equivalence.

Proposition 1. $\widetilde{KK^G}(A,B)$ *is an abelian group with addition given by direct sum.*

Proof. One proves this along the lines of [7], Section 4, Theorem 1, or [14], Proposition 4. $\square$

We remark that the G-action of a Hilbert module can be completely degenerate to zero, and cycles of such Hilbert modules in the sense of Definition 16 coincide with cycles in the sense of Kasparov [8] for the trivial group. One may circumvent this difference by restricting to unital semi-multiplicative sets G (possibly by adjoining a unit) and requiring that the unit of G always acts as the identity on C^*-algebras and Hilbert modules. Otherwise we have the following elementary observation.

Lemma 7. *If G is a group and $\mathcal{E}$ is a G-Hilbert module then $U_gU_g^* = U_eU_e^*$ and $U_g^*U_g = U_e^*U_e$ for all $g \in G$, and U_e^* is the adjoint of $U_e \in \mathcal{L}(\mathcal{E})$. If* $\ker(U_e) = 0$ *then $U_g^*U_g = U_gU_g^* = 1$ and $U_g^* = U_{g^{-1}}$ for all $g \in \mathcal{E}$ (and thus $\mathcal{E}$ is a G-Hilbert module in the sense of* [7]*).*

Proof. The first claim follows from $U_hU_h^* = U_{gg^{-1}h}U_h^* = U_gU_g^*U_gU_{g^{-1}h}U_h^* = U_gU_g^*U_hU_h^*$ and similarly $U_gU_g^* = U_gU_g^*U_hU_h^*$ for all $g,h \in G$. Further, $U_e = (U_eU_e^*)(U_e^*U_e) \in \mathcal{L}(\mathcal{E})$ and its adjoint is $(U_e^*U_e)(U_eU_e^*) = U_e^*$. For the last claim, $P = U_e^*U_e = U_g^*U_g$ is a full selfadjoint projection and hence $P = 1$. Moreover, by Lemma 3 all U_g are bijective and consequently $U_g^* = U_g^{-1} = U_{g^{-1}}$. $\square$

5. Kasparov's technical theorem

If nothing else is said, approximate units are supposed to be positive, increasing and all their elements having degree 0. If A is a subalgebra and Δ a subset of an algebra B then Δ *derives* A if $[a,d] \in A$ for all $a \in A, d \in \Delta$. (All commutators are graded.) In this section we prove a modification of the so-called Kasparov technical theorem, see Kasparov [7], Section 3. We follow closely Kasparov [8], Section 1.4, a simplification of Kasparov's original proof due to Higson [4]. If X is a locally compact Hausdorff space and A a C^*-algebra then we also write $A(X)$ for the C^*-algebra $C_0(X,A)$.

Lemma 8. *Let $\mathcal{E}$ be a G-Hilbert module with G-action U, A a G-invariant σ-unital subalgebra of $\mathcal{L}(\mathcal{E})$, Y a σ-compact locally compact Hausdorff space, and $\varphi : Y \to \mathcal{L}(\mathcal{E})$ a function such that $[\varphi(y), a] \in A$ for all $a \in A, y \in Y$, and $y \mapsto [\varphi(y), a]$ is a continuous function on Y (norm topology in $\mathcal{L}(\mathcal{E})$) for all $a \in A$. Then there is a countable approximate unit $(u_i) \subseteq A$ for A*

such that the limits

$$\lim_{i\to\infty} \|[u_i, \varphi(y)]\|,$$
$$\lim_{i\to\infty} \|U_g u_i U_g^* - u_i U_g U_g^*\|, \quad \lim_{i\to\infty} \|u_i U_g U_g^* - U_g U_g^* u_i\|,$$
$$\lim_{i\to\infty} \|U_g^* u_i U_g - u_i U_g^* U_g\|, \quad \lim_{i\to\infty} \|u_i U_g^* U_g - U_g^* U_g u_i\|,$$

are 0 *for all* $y \in Y, g \in G$. *These limits are uniform on compact subsets of* Y *and* G *respectively.*

Proof. Let $X_1 \subseteq X_2 \subseteq \cdots$ be an increasing sequence of open sets in G, with compact closures and $\bigcup_n X_n = G$. Let also $Y_1 \subseteq Y_2 \subseteq \cdots$ be a similar sequence in Y and $(v_i) \subseteq A$ a (positive increasing) countable approximate unit for A. Using induction, suppose that we have already constructed $u_1 \le u_2 \le \cdots \le u_n$ out of finite convex linear combinations of elements of v_i, and the following conditions are fulfilled:

$$\|u_k v_j - v_j\| \le 1/k, \quad \|[u_k, \varphi(y)]\| \le 1/k, \quad \|u_k U_g U_g^* - U_g U_g^* u_k\| \le 1/k,$$
$$\|u_k U_g^* U_g - U_g^* U_g u_k\| \le 1/k, \quad \|g(u_k) - gg^{-1}(u_k)\| \le 1/k$$

for all $j \le k, y \in \overline{Y}_k, g \in \overline{X}_k, k \le n$. To construct u_{n+1}, note that $v_i \ge u_n$ for all $i \ge m$ for some $m \ge 1$. Let Λ be the convex hull of $\{v_m, v_{m+1}, \dots\}$. Denote by Z the disjoint union of $\{v_1, \dots, v_{n+1}\}, \overline{Y}_{n+1}$ and three copies $\overline{X}_{n+1}^{(1)}, \overline{X}_{n+1}^{(2)}, \overline{X}_{n+1}^{(3)}$ of $\overline{X}_{n+1}$. For any $v \in \Lambda$ let $a_v \in A(Z)$ be the function defined by

$$a_v(v_j) = vv_j - v_j, \quad a_v(y) = [v, \varphi(y)], \quad a_v(g) = U_g U_g^* v - v U_g U_g^*,$$
$$a_v(h) = U_h^* U_h v - v U_h^* U_h, \quad a_v(l) = l(v) - ll^{-1}(v)$$

for all $1 \le j \le n+1, y \in \overline{Y}_{n+1}, g \in \overline{X}_{n+1}^{(1)}, h \in \overline{X}_{n+1}^{(2)}, l \in \overline{X}_{n+1}^{(3)}$. Suppose that there is no element $u_{n+1} \in \Lambda$ with the required properties. Since the set of functions $\{a_v \mid v \in \Lambda\}$ is convex, the separation theorem gives a bounded linear functional f on $A(Z)$ with $|f(a_v)| \ge 1$ for all $v \in \Lambda$. This leads to a contradiction in the following way.

Write $B = \mathcal{L}(\mathcal{E})$, and denote by $B(Z)''$ and $A(Z)''$ the universal enveloping von Neumann algebras of $B(Z)$ and $A(Z)$, respectively, and identify $A(Z)''$ as a subset of $B(Z)''$. Regarding v_i as an element in $B(Z)$ (constant function with value v_i), we have $v_i \uparrow p$ in the weak operator topology for some element $p \in B(Z)''$. Since Z is compact, by a simple compactness argument we see that v_i is an approximate unit for $A(Z)$, and so p is a unit for $A(Z)''$. Write $\varphi' \in B(Z)$ for the function $\varphi'|_{\overline{Y}_{n+1}} = \varphi|_{\overline{Y}_{n+1}}$ and $\varphi'|_{Z\setminus\overline{Y}_{n+1}} = 0$. Since $[p, \varphi'] \in A(Z)''$,

$$[p, \varphi'] = [p^2, \varphi'] = p[p, \varphi'] + [p, \varphi']p = 2[p, \varphi'],$$

which implies that $[p, \varphi'] = 0$. Define $\psi(z) = U_z U_z^*$ for $z \in \overline{X}_{n+1}^{(1)} \subseteq G$ and $\psi(z) = 0$ for other z. By the G-invariance of A, $[p, \psi] \in A(Z)''$, and by the same argument as before we thus obtain $[p, \psi] = 0$. For $a \in A(Z)$, write $\sigma(a), \sigma^{-1}(a) \in A(Z)$ for the functions

$$\sigma(a)(z) = z(a(z)) = U_z a(z) U_z^*,$$
$$\sigma^{-1}(a)(z) = z^{-1}(a(z)) = U_z^* a(z) U_z,$$

for $z \in \overline{X}_{n+1}^{(3)} \subseteq G$, and $\sigma(a)(z) = \sigma^{-1}(a)(z) = 0$ for other z. As

$$\|\sigma(v_i)\sigma\sigma^{-1}(a) - \sigma\sigma^{-1}(a)\| \leq \|v_i\sigma^{-1}(a) - \sigma^{-1}(a)\|,$$
$$\|\sigma\sigma^{-1}(v_i)\sigma\sigma^{-1}(a) - \sigma\sigma^{-1}(a)\| \leq \|v_i\sigma^{-1}(1)a - \sigma^{-1}(1)a\|$$

for all $a \in A(Z)$, and since $\sigma^{-1}(a), \sigma^{-1}(1)a \in A(Z)$, the sequences $\sigma(v_i) = \sigma\sigma^{-1}\sigma(v_i)$ and $\sigma\sigma^{-1}(v_i)$ are (not necessarily increasing and positive) approximate units for the C^*-subalgebra $A_\sigma = \overline{\sigma\sigma^{-1}(A(Z))} \subseteq A(Z)$. Hence the weak operator topology limits α, β of $\sigma(v_i)$ and $\sigma\sigma^{-1}(v_i)$ in $B(Z)''$ (if the sequence $\sigma(v_i)$ does not converge, we go over to a weak operator topology convergent subsequence $\sigma(v_{k_i})$) are units of A_σ'', and so $\alpha = \beta$.

The above calculations show that the weak operator topology limit of a_{v_i} vanishes in $A(Z)''$. Hence $\lim_i f(a_{v_i}) = 0$ (by a well-known linear topological identification of $A(Z)''$ with the bidual space $A(Z)^{**}$), which is a contradiction. Obviously, the constructed sequence u_k satisfies the claim. $\square$

In the next theorem we regard $\mathcal{M}(J)$ as a subalgebra of $\mathcal{L}(\mathcal{E})$, see [10], Proposition 2.1.

Theorem 1. *Let $\mathcal{E}$ be a Hilbert module, J a nondegenerate σ-unital G-invariant subalgebra of $\mathcal{L}(\mathcal{E})$, A_1 a σ-unital G-invariant subalgebra of $\mathcal{M}(J)$, and A_2 a σ-unital subalgebra (without G-action) of $\mathcal{M}(J)$. Let Δ be a norm-separable subset of $\mathcal{M}(J)$ which derives A_1. Let Ω be a σ-compact locally compact Hausdorff space, and $\varphi, \psi : \Omega \to \mathcal{M}(J)$ be bounded functions. Assume that*

$$A_1A_2,\ A_1\varphi(\Omega),\ \psi(\Omega)A_1 \subseteq J,$$

and the functions

$$\omega \mapsto a\varphi(\omega), \quad \omega \mapsto \varphi(\omega)a, \quad \omega \mapsto a\psi(\omega), \quad \omega \mapsto \psi(\omega)a$$

are continuous on Ω, with respect to the norm topology in $\mathcal{M}(J)$, for all $a \in A_1 + J$. Then there are positive elements $M_1, M_2 \in \mathcal{M}(J)$ of degree 0 such that $M_1 + M_2 = 1$,

$$M_i a_i,\ [M_i, d],\ M_2\varphi(\omega),\ \psi(\omega)M_2 \subseteq J,$$

$$g(M_i) - gg^{-1}(M_i),\ [g(1), M_i], [g^{-1}(1), M_i] \subseteq J$$

for all $a_i \in A_i, d \in \Delta, g \in G, \omega \in \Omega$ $(i = 1, 2)$, and the functions

$$\omega \mapsto M_2\varphi(\omega), \quad \omega \mapsto \psi(\omega)M_2$$

are norm continuous on Ω.

Proof. The proof is similar to the proof of the Theorem of Subsection 1.4 on page 151 of Kasparov's paper [8], with some adaption we discuss now. In Kasparov's paper G is a group, and somewhere in the proof of the theorem one chooses approximate units $(u_i) \subseteq A_1$ for A_1 and $(v_i) \subseteq J$ for J according to the lemma on page 152 in Kasparov's paper satisfying (among other things)

$$\|g(u_n) - u_n\| \leq 2^{-n}, \quad \forall n, \forall g \in \overline{X}_n, \quad \text{and} \tag{3}$$

$$\|g(b_n) - b_n\| \leq 2^{-n}, \quad \forall n, \forall g \in \overline{X}_n, \tag{6}$$

where b_n is defined by $b_n = (v_n - v_{n-1})^{1/2}$. The sought element $M_2 \in \mathcal{L}(\mathcal{E})$ is defined as the series $\sum_{n \geq 1} b_n u_n b_n$ which converges in the strict topology. By the estimates (3) and (6) one gets the estimate

$$\|g(b_n u_n b_n) - b_n u_n b_n\| \leq 3 \cdot 2^{-n}$$

for all $n \geq 1$ (see the bottom of page 153 in Kasparov's paper).

We modify Kasparov's proof as follows. At first, Kasparov's stated theorem deals only with one function φ. But it is quite obvious how to modify the proof that one can handle both functions φ and ψ. Next, Kasparov's function φ has domain G. At the beginning of the proof he writes G as $G = \bigcup_{n \in \mathbb{N}} X_n$ with open subsets $X_n \subseteq G$ with compact closures. We modify the proof in that we also choose a union $\Omega = \bigcup_{n \in \mathbb{N}} \Omega_n$ of open subsets $\Omega_n \subseteq \Omega$ with compact closures, and substitute X_n by Ω_n everywhere there where X_n acts as a domain of φ or ψ. Instead of the subset $W_n \subseteq J$ defined under point (4) in Kasparov's proof, we take

$$W_n = \{k, u_n h_2, u_{n+1} h_2\} \cup u_n \varphi(\overline{\Omega}_n) \cup u_{n+1} \varphi(\overline{\Omega}_{n+1}) \cup \psi(\overline{\Omega}_n) u_n \cup \psi(\overline{\Omega}_{n+1}) u_{n+1}.$$

Next, we choose the mentioned approximate units (u_i) and (v_i) by Lemma 8 in such a way that we have the estimates

$$\begin{aligned} &\|g(u_n) - gg^{-1}(u_n)\| + \|U_g U_g^* u_n - u_n U_g U_g^*\| + \|U_g^* U_g u_n - u_n U_g^* U_g\| \\ &\leq 2^{-n}, \quad \forall n, \forall g \in \overline{X}_n, \quad \text{and} \end{aligned} \tag{3}$$

$$\begin{aligned} &\|g(c_n) - gg^{-1}(c_n)\| + \|U_g U_g^* c_n - c_n U_g U_g^*\| + \|U_g^* U_g c_n - c_n U_g^* U_g\| \\ &\leq (1/100) 2^{-n}/N_n^2, \quad \forall n, \forall g \in \overline{X}_n, \end{aligned} \tag{6'}$$

rather than the estimates (3) and (6) in Kasparov's paper. Thereby denote $c_n = b_n^2$, let $\sum_{k=0}^{\infty} \alpha_k (x-1)^k = x^{1/2}$ be the power series of $x^{1/2}$ at 1, and choose N_n such that $\sum_{k=N_n+1}^{\infty} |\alpha_k| \leq (1/100) 2^{-n}$ for all $n \in \mathbb{N}$. Note that $\|b_n - \sum_{k=0}^{N_n} \alpha_k (c_n - 1)^k\| \leq (1/100) 2^{-n}$ for all $n \in \mathbb{N}$. From (6′) we thus deduce

$$\begin{aligned} &\|g(b_n) - gg^{-1}(b_n)\| + \|U_g U_g^* b_n - b_n U_g U_g^*\| + \|U_g^* U_g b_n - b_n U_g^* U_g\| \\ &\leq 2^{-n}, \quad \forall n, \forall g \in \overline{X}_n \end{aligned} \tag{6}$$

(mainly by similar estimates we show next). This leads one to the following estimate.

$$\begin{aligned}
&\|U_g b_n u_n b_n U_g^* - U_g b_n U_g^* U_g u_n U_g^* U_g b_n U_g^*\| \\
&\le \|U_g(U_g^* U_g b_n - b_n U_g^* U_g) u_n b_n U_g^*\| \\
&\quad + \|U_g b_n U_g^* U_g (U_g^* U_g u_n - u_n U_g^* U_g) b_n U_g^*\| \\
&\quad + \|U_g b_n U_g^* U_g u_n U_g^* U_g (U_g^* U_g b_n - b_n U_g^* U_g) U_g^*\| \le 3 \cdot 2^{-n}.
\end{aligned}$$

for all $g \in G, n \in \mathbb{N}$. A similar estimate yields

$$\|gg^{-1}(b_n u_n b_n) - gg^{-1}(b_n) gg^{-1}(u_n) gg^{-1}(b_n)\| \le 3 \cdot 2^{-n}.$$

Hence

$$\begin{aligned}
&\|g(b_n u_n b_n) - gg^{-1}(b_n u_n b_n)\| \\
&\le 6 \cdot 2^{-n} + \|g(b_n) g(u_n) g(b_n) - gg^{-1}(b_n) gg^{-1}(u_n) gg^{-1}(b_n)\| \\
&\le 6 \cdot 2^{-n} + \|(g(b_n) - gg^{-1}(b_n)) g(u_n) g(b_n)\| \\
&\quad + \|gg^{-1}(b_n)(g(u_n) - gg^{-1}(u_n)) g(b_n)\| \\
&\quad + \|gg^{-1}(b_n) gg^{-1}(u_n)(g(b_n) - gg^{-1}(b_n))\| \le 9 \cdot 2^{-n}.
\end{aligned}$$

Also,

$$\|U_g U_g^* b_n u_n b_n - b_n u_n b_n U_g U_g^*\| \le 3 \cdot 2^{-n},$$
$$\|U_g^* U_g b_n u_n b_n - b_n u_n b_n U_g^* U_g\| \le 3 \cdot 2^{-n}.$$

Other things of Kasparov's proof need not to be changed. □

Corollary 1. *Let M_1 and M_2 be the operators of Theorem* 1*, and assume that $J = \mathcal{K}(\mathcal{E})$. Then all claims of Theorem* 1 *(excepting $M_1 + M_2 = 1$) hold also for $M_i^{1/2}$ rather than M_i $(i = 1, 2)$.*

Proof. As $g(M_i^n) \equiv g(M_i)^n$ and $g(1)M_i^n \equiv (g(1)M_i)^n$ modulo J, writing $M_i^{1/2}$ as a power series shows that $g(M_i^{1/2}) - g(M_i)^{1/2} \in J$, $g(1)M_i^{1/2} - (g(1)M_i)^{1/2} \in J$ and $[d, M_i^{1/2}] \in J$ for all $d \in \Delta$. Hence, modulo J we get

$$\begin{aligned}
g(M_i^{1/2}) - gg^{-1}(M_i^{1/2}) &\equiv g(M_i)^{1/2} - gg^{-1}(M_i)^{1/2} \\
&= (g(M_i) - gg^{-1}(M_i))(g(M_i)^{1/2} + gg^{-1}(M_i)^{1/2})^{-1} \\
&\equiv 0.
\end{aligned}$$

Next, $M_i^{1/2} a_i = M_i^{-1/2} M_i a_i \in J$, and similarly we treat the remaining cases. □

6. The Kasparov cap product

Given a G-Hilbert module $\mathcal{E}$, we define $\mathcal{A}(\mathcal{E})$ as the smallest G-invariant C^*-subalgebra of $\mathcal{L}(\mathcal{E})$ generated by 1. Note that $\partial U_g = \partial U_g^* = 0$ for all $g \in G$, whence all elements of $\mathcal{A}(\mathcal{E})$ are zero graded.

Lemma 9. *If $\pi : A \to \mathcal{L}(\mathcal{E})$ is an equivariant representation then $\pi(A)$ commutes with $\mathcal{A}(\mathcal{E})$.*

If $(\mathcal{E}, T) \in \mathbb{E}^G(A, B)$ then $a[x, T] \in \mathcal{K}(\mathcal{E})$ for all $a \in A, x \in \mathcal{A}(\mathcal{E})$.

Proof. Let W be the smallest set in $\mathcal{L}(\mathcal{E})$ which is closed under taking products, is invariant under G and G^{-1}, and contains 1. Since W is self-adjoint, $\mathcal{A}(\mathcal{E}) = \overline{\operatorname{span}} W$. That $\pi(A)$ commutes with W can be proved by induction for expressions in W. Suppose that $x \in W$ and $\pi(a)x = x\pi(a)$ for all $a \in A$. Fix $a \in A, g \in G$. Then $U_g^* \pi(a) U_g = U_g^* U_g \pi(b)$ for some $b \in A$ by Definition 15, and thus $\pi(a)U_g = U_g \pi(b)$ and $\pi(b)U_g^* = U_g^* \pi(a)$. Hence, $\pi(a)U_g x U_g^* = U_g x U_g^* \pi(a)$. This proves the first claim of the lemma.

Let $(\mathcal{E}, T) \in \mathbb{E}^G(A, B)$. Take $x \in W$ and assume that $a(xT - Tx) \in \mathcal{K}(\mathcal{E})$ for all $a \in A$. Fix $a \in A, g \in G$, and write $b = U_g^* a U_g = U_g^* U_g b'$ for some $b' \in A$ by Definition 15. Multiplying

$$a(U_g T U_g^* - T U_g U_g^*) = U_g U_g^* U_g b' T U_g^* - a T U_g U_g^* \in \mathcal{K}(\mathcal{E})$$

from the right with U_g (thereby noticing that $b' T U_g^* U_g \equiv b' U_g^* U_g T$ modulo $\mathcal{K}(\mathcal{E})$), one gets $a(U_g T - T U_g) \in (\mathcal{K}(\mathcal{E}) U_g + U_g \mathcal{K}(\mathcal{E}))$. In a similar way, by multiplying $a(U_g^* T U_g - T U_g^* U_g) \in \mathcal{K}(\mathcal{E})$ (see Lemma 6) from the right with U_g^*, one obtains $a(U_g^* T - T U_g^*) \in (\mathcal{K}(\mathcal{E}) U_g^* + U_g^* \mathcal{K}(\mathcal{E}))$. With these formulas, the formulas $aU_g = U_g b, U_g^* a = b U_g^*, ax = xa, bx = xb, b = U_g^* U_g b'$, and the invariance of $\mathcal{K}(\mathcal{E})$ under G and G^{-1}, it is straightforward to compute that $a(g(x)T - Tg(x)) \equiv g(b'(xT - Tx)) \equiv 0$ modulo $\mathcal{K}(\mathcal{E})$. A similar computation shows that $a(g^{-1}(x)T - Tg^{-1}(x)) \in \mathcal{K}(\mathcal{E})$. □

Definition 18. Let $\mathcal{E}_1$ be a Hilbert B_1-module, $\mathcal{E}_2$ a Hilbert (B_1, B_2)-bimodule, and $\mathcal{E}_{12} = \mathcal{E}_1 \otimes_{B_1} \mathcal{E}_2$. For any $\xi \in \mathcal{E}_1$, define $\theta_\xi \in \mathcal{L}(\mathcal{E}_2, \mathcal{E}_{12})$ by $\theta_\xi(\eta) = \xi \otimes \eta$ (with an adjoint given by $\theta_\xi^*(\xi_1 \otimes \eta) = \langle \xi, \xi_1 \rangle \cdot \eta$). Let $T_2 \in \mathcal{L}(\mathcal{E}_2)$. An element $T_{12} \in \mathcal{L}(\mathcal{E}_{12})$ will be called a *T_2-connection on $\mathcal{E}_{12}$* if for any $\xi \in \mathcal{E}_1$ both

$$\theta_\xi T_2 - (-1)^{\partial\xi \cdot \partial T_2} T_{12} \theta_\xi,$$
$$\theta_\xi T_2^* - (-1)^{\partial\xi \cdot \partial T_2} T_{12}^* \theta_\xi$$

are in $\mathcal{K}(\mathcal{E}_2, \mathcal{E}_{12})$. (Adjoining the last element gives $T_2 \theta_\xi^* - (-1)^{\partial\xi \cdot \partial T_2} \theta_\xi^* T_{12}$.)

Note that the definition of T_2-connections does not involve a G-structure. Connections were introduced by Connes and Skandalis in [2]. By Lemma 9, $1 \otimes T$ is a well-defined operator in $\mathcal{L}(\mathcal{E}_{12})$ for $T \in \mathcal{A}(\mathcal{E}_2)$.

Lemma 10. *With the notation from the previous definition, let T_{12} be a T_2-connection.*

If $T_2 \in I_{B_1}(\mathcal{E}_2)$ then $T_{12} \in I_{\mathcal{K}(\mathcal{E}_1) \otimes 1}(\mathcal{E}_{12})$.

If $(\mathcal{E}_2, T_2) \in \mathbb{E}^G(B_1, B_2)$ *then*

$$T_{12} \in Q_{\mathcal{K}(\mathcal{E}_1)\otimes\mathcal{A}(\mathcal{E}_2)}(\mathcal{E}_{12}), \tag{1}$$

$$g(T_{12}) - gg^{-1}(T_{12}) \in I_{\mathcal{K}(\mathcal{E}_1)\otimes 1}(\mathcal{E}_{12}), \tag{2}$$

$$(\mathcal{K}(\mathcal{E}_1)\otimes 1)(g(1)T_{12}^{\mu} - gg^{-1}(T_{12}^{\mu})) \in \mathcal{K}(\mathcal{E}_{12}), \tag{3}$$

$$(\mathcal{K}(\mathcal{E}_1)\otimes 1)(g^{-1}(1)T_{12}^{\mu} - g^{-1}g(T_{12}^{\mu})) \in \mathcal{K}(\mathcal{E}_{12}) \tag{4}$$

for all $\mu \in \{1, *\}$.

If $T_1 \in \mathcal{L}(\mathcal{E}_1)$ *then* $(\mathcal{K}(\mathcal{E}_1)\otimes 1)[T_{12}, T_1 \otimes 1] \in \mathcal{K}(\mathcal{E}_{12})$.

Proof. First notice that $\theta_{\xi,\eta}\otimes 1 = \theta_\xi\theta_\eta^*$ for all $\xi,\eta \in \mathcal{E}_1$. Take $\xi \in \mathcal{E}_1$ and write $b_i = \langle\xi,\xi\rangle(\langle\xi,\xi\rangle + 1/i)^{-1}$. Then $\theta_\xi = \lim_{i\to\infty}\theta_{\xi b_i}$, and so $\theta_\xi T_2 = \lim_i \theta_\xi b_i T_2$. Thus, if $T_2 \in I_{B_1}(\mathcal{E}_2)$, then $\theta_\xi T_2, \theta_\xi T_2^* \in \mathcal{K}(\mathcal{E}_{12}, \mathcal{E}_2)$, and consequently $T_{12}\theta_\xi\theta_\eta^*$ and $T_{12}^*\theta_\xi\theta_\eta^*$ are in $\mathcal{K}(\mathcal{E}_{12}, \mathcal{E}_2)$, which proves that $T_{12} \in I_{\mathcal{K}(\mathcal{E}_1)\otimes 1}(\mathcal{E}_{12})$.

If $T_2 \in \mathbb{E}^G(B_1, B_2)$, then by Lemma 9 one has $\theta_\xi T_2 x = \lim_i \theta_\xi b_i T_2 x \equiv \lim_i \theta_\xi b_i x T_2 = \theta_\xi x T_2$ modulo $\mathcal{K}(\mathcal{E}_2, \mathcal{E}_{12})$ for all $x \in \mathcal{A}(\mathcal{E}_2)$. Hence

$$\theta_\xi x T_2 - (-1)^{\partial\xi\cdot\partial T_2} T_{12}\theta_\xi x \in \mathcal{K}(\mathcal{E}_2, \mathcal{E}_{12})$$

for all $x \in \mathcal{A}(\mathcal{E}_2)$. Modulo $\mathcal{K}(\mathcal{E}_{12})$, this gives

$$\begin{aligned}\theta_\xi x\theta_\eta^* T_{12} \equiv \theta_\xi x T_2\theta_\eta^*(-1)^{\partial\eta\cdot\partial T_2} &\equiv T_{12}\theta_\xi x\theta_\eta^*(-1)^{\partial\xi\cdot\partial T_2}(-1)^{\partial\eta\cdot\partial T_2}\\ &\equiv T_{12}\theta_\xi x\theta_\eta^*(-1)^{\partial(\theta_\xi x\theta_\eta)\cdot\partial T_2}\end{aligned}$$

for all $x \in \mathcal{A}(\mathcal{E}_2)$, for the last identity noticing that $\partial(\theta_\xi) = \partial\xi$ and $\partial(x) = 0$. If $\partial(T_{12}) = \partial(T_2)$, this proves that $[\theta_\xi x\theta_\eta^*, T_{12}] \in \mathcal{K}(\mathcal{E}_{12})$. If $\partial(T_{12}) \neq \partial(T_2)$ then $\partial(\theta_\xi T_2) \neq \partial(T_{12}\theta_\xi)$, which shows that T_{12} is a 0-connection, and in this case $[\theta_\xi x\theta_\eta^*, T_{12}] \in \mathcal{K}(\mathcal{E}_{12})$ is obvious. Noticing $\theta_\xi x\theta_\eta^* = (1\otimes x)\theta_\xi\theta_\eta^*$, this shows (1).

If $T_2 \in \mathbb{E}^G(B_1, B_2)$, and $S_g = g(T_2) - gg^{-1}(T_2)$, then $B_1 S_g \subseteq \mathcal{K}(\mathcal{E}_2)$. By $\theta_\xi = \lim_i \theta_{\xi b_i}$ we get $\theta_\xi S_g, \theta_\xi S_g^* \in \mathcal{K}(\mathcal{E}_2, \mathcal{E}_{12})$. Denote the G-action on $\mathcal{E}_1$ and $\mathcal{E}_2$ by U and V, respectively. Fix $g \in G, \xi \in \mathcal{E}_1$ and assume that $\xi = U_g U_g^*\xi$. Then

$$\begin{aligned}&\theta_\xi g(T_2) - (-1)^{\partial(\xi)\cdot\partial(T_2)} g(T_{12})\theta_\xi\\ &\quad = (U_g\otimes V_g)(\theta_{U_g^*\xi}T_2 - (-1)^{\partial(\xi)\cdot\partial(T_2)}T_{12}\theta_{U_g^*\xi})V_g^* \in \mathcal{K}(\mathcal{E}_2, \mathcal{E}_{12}).\end{aligned}$$

Similarly, $\theta_\xi gg^{-1}(T_2) - (-1)^{\partial(\xi)\cdot\partial(T_2)} gg^{-1}(T_{12})\theta_\xi \in \mathcal{K}(\mathcal{E}_2, \mathcal{E}_{12})$. Hence,

$$(g(T_{12}) - gg^{-1}(T_{12}))\theta_\xi\theta_\eta^* \in \mathcal{K}(\mathcal{E}_{12}).$$

This identity holds even for any $\xi \in \mathcal{E}_1$, since

$$(g(T_{12}) - gg^{-1}(T_{12}))\theta_\xi = (g(T_{12}) - gg^{-1}(T_{12}))\theta_{U_g U_g^*\xi}$$

for all $\xi \in \mathcal{E}_1$. The identity also holds for T_{12} replaced by T_{12}^*, and hence we checked (2). The claims (3) and (4) are proved similarly. (Recall Lemma 6 when checking (4).) For the last claim, it is straightforward to compute that $[T_{12}^*, T_1^* \otimes 1]\theta_\xi \theta_\eta^* \in \mathcal{K}(\mathcal{E}_{12})$. □

The following lemma is Lemma 2.7 of Kasparov's paper [8].

Lemma 11. *If $\mathcal{E}_1$ is countably generated and $T_2 \in Q_{B_1}(\mathcal{E}_2)$ then there exists a T_2-connection T_{12} on $\mathcal{E}_{12}$.*

Definition 19. Let A, B_1, B_2 be G-C^*-algebras. An element $(\mathcal{E}_{12}, T_{12}) \in \mathbb{E}^G(A, B_2)$ is called a *Kasparov* (or *cap*) *product* of $(\mathcal{E}_1, T_1) \in \mathbb{E}^G(A, B_1)$ and $(\mathcal{E}_2, T_2) \in \mathbb{E}^G(B_1, B_2)$, if $\mathcal{E}_{12} = \mathcal{E}_1 \otimes_{B_1} \mathcal{E}_2$, T_{12} is a T_2-connection on $\mathcal{E}_{12}$, and $a[T_1 \otimes 1, T_{12}]a^* \geq 0$ in the quotient $\mathcal{L}(\mathcal{E}_{12})/\mathcal{K}(\mathcal{E}_{12})$ for all $a \in A$.

Lemma 12. *Let $\mathcal{E}$ be a Hilbert (A, B)-bimodule, $(\mathcal{E}, F), (\mathcal{E}, F') \in \mathbb{E}^G(A, B)$, and assume that $a[F, F']a^* \geq 0$ in $\mathcal{L}(\mathcal{E})/\mathcal{K}(\mathcal{E})$ for all $a \in A$. Then $(\mathcal{E}, F)$ and $(\mathcal{E}, F')$ are operatorially homotopic.*

Proof. The proof is the same as in [14], Lemma 11, and we shall only check the aspects involving G. As in Skandalis' paper the operatorial homotopy is given by the path $F_t = (1 + (\cos t)(\sin t)P)^{-1/2}((\cos t)F + (\sin t)F') \in \mathcal{L}(\mathcal{E})$ ($t \in [0, \pi/2]$) for some operator $P \geq 0$ satisfying $[F, F'] - P \in I_A(\mathcal{E})$. By Skandalis' proof, $(\mathcal{E}, F_t) \in \mathbb{E}(A, B)$. Given $F_1, \ldots, F_n \in \{F, F'\}$, $a \in A$, and $b \in A$ such that $U_g^* a U_g = U_g^* U_g b$, one has

$$\begin{aligned} ag(F_1 \ldots F_n) = g(bU_g^* U_g F_1 \ldots F_n) &\equiv g(bU_g^* U_g F_1 U_g^* U_g \ldots U_g^* U_g F_n) \\ &= ag(F_1) \ldots g(F_n) \end{aligned}$$

modulo $\mathcal{K}(\mathcal{E})$, as $bF_i \equiv (-1)^{\partial b} F_i b$. By induction on n one gets

$$\begin{aligned} &ag(F_1 \ldots F_n) - agg^{-1}(F_1 \ldots F_n) \\ &\equiv ag(F_1) \ldots g(F_n) - agg^{-1}(F_1) \ldots gg^{-1}(F_n) \equiv 0 \end{aligned}$$

modulo $\mathcal{K}(\mathcal{E})$. Using power series, there are fixed scalars $\alpha_n, \beta_n, \gamma_n, \delta_n \in \mathbb{C}$ ($n \geq 0$) such that for any $c \in A$ there exist $K_n \in \mathcal{K}(\mathcal{E})$ ($n \geq 0$) such that

$$cF_t = \sum_{n \geq 0} \alpha_n c(FF')^n F + \beta_n c(F'F)^n F + \gamma_n c(FF')^n F' + \delta_n c(FF')^n F' + K_n.$$

Note that the series is (still) absolutely convergent. In order to show that $ag(F_t) - agg^{-1}(F_t) \in \mathcal{K}(\mathcal{E})$, it is enough to show that $ag((FF')^n F) - agg^{-1}((FF')^n F) \in \mathcal{K}(\mathcal{E})$ for all $n \geq 0$ (and similarly for the other terms). But we have checked this. □

Theorem 2. *If A is separable then the Kasparov product of $(\mathcal{E}_1, T_1) \in \mathbb{E}^G(A, B_1)$ and $(\mathcal{E}_2, T_2) \in \mathbb{E}^G(B_1, B_2)$ exists and is unique up to operatorial homotopy. The Kasparov product induces a bilinear map*

$$\otimes_{B_1} : \widetilde{KK^G}(A, B_1) \otimes \widetilde{KK^G}(B_1, B_2) \to \widetilde{KK^G}(A, B_2)$$

denoted by $x \otimes y \mapsto x \otimes_{B_1} y$.

Proof. *Existence.* By Lemma 11 there is a T_2-connection $\widetilde{T}_2$ of degree 1 on $\mathcal{E}_{12}$. Put $J = \mathcal{K}(\mathcal{E}_{12})$, and $A_1 = J + \mathcal{K}(\mathcal{E}_1) \otimes \mathcal{A}(\mathcal{E}_2)$. A_1 is closed and σ-unital by [7], Section 3, Lemma 2. Note that A_1 is G-invariant. Denote by A_2 the C^*-subalgebra (without G-action) of $\mathcal{L}(\mathcal{E}_{12})$ generated by the elements

$$\widetilde{T}_2 - \widetilde{T}_2^*, \widetilde{T}_2^2 - 1, [\widetilde{T}_2, T_1 \otimes 1], [\widetilde{T}_2, a]$$

for all $a \in A$. Let

$$\Delta = \{T_1 \otimes 1, \widetilde{T}_2\} \cup A.$$

It is clear that Δ derives A_1, see Lemma 10. Of course, $\widetilde{T}_2 - \widetilde{T}_2^*$ is a $(T_2 - T_2^*)$-connection, and $\widetilde{T}_2^2 - 1$ is a $(T_2^2 - 1)$-connection. Noting $T_2 - T_2^* \in I_{B_1}(\mathcal{E}_2)$, and writing

$$\mathcal{K}(\mathcal{E}_1) \otimes \mathcal{A}(\mathcal{E}_2) = (1 \otimes \mathcal{A}(\mathcal{E}_2))(\mathcal{K}(\mathcal{E}_1) \otimes 1),$$

we get $A_1(\widetilde{T}_2 - \widetilde{T}_2^*) \subseteq J$ by Lemma 10. Similarly, $A_1(\widetilde{T}_2^2 - 1) \subseteq J$. By Lemma 10, one has $A_1[\widetilde{T}_2, T_1 \otimes 1] \subseteq J$ and $A_1[\widetilde{T}_2, a] \subseteq J$ for all $a \in A$. It thus follows that $A_1 A_2 \subseteq J$. Define

$$\begin{aligned}
&\varphi_1(g) = g(\widetilde{T}_2) - gg^{-1}(\widetilde{T}_2),\ \varphi_2(g) = g(1)\widetilde{T}_2 - gg^{-1}(\widetilde{T}_2),\\
&\varphi_3(g) = g^{-1}(1)\widetilde{T}_2 - g^{-1}g(\widetilde{T}_2),\ \psi_1(g) = \widetilde{T}_2 g(1) - gg^{-1}(\widetilde{T}_2),\\
&\psi_2(g) = \widetilde{T}_2 g^{-1}(1) - g^{-1}g(\widetilde{T}_2)
\end{aligned}$$

for all $g \in G$. We may combine $\varphi_1, \varphi_2, \varphi_3$ and $\psi_1, \psi_2, 0$ to one function φ and ψ, respectively, with domain Ω being a threefold disjoint copy of G. We apply Theorem 1 to obtain $M_1, M_2 \in \mathcal{L}(\mathcal{E}_{12})$ and set

$$T_{12} = M_1^{1/2}(T_1 \otimes 1) + M_2^{1/2}\widetilde{T}_2.$$

It is well established (and straightforward to check) that $(\mathcal{E}_{12}, T_{12})$ is in $\mathbb{E}(A, B_2)$ (without the set G), which is why will focus on those additional relations showing even $(\mathcal{E}_{12}, T_{12}) \in \mathbb{E}^G(A, B_2)$. The other properties which show that T_{12} is a Kasparov product are deduced as in Skandalis [14], Theorem 12. Denote the G-action on $\mathcal{E}_1$ and $\mathcal{E}_2$ by U and V, respectively, and the diagonal action $U \otimes V$ on $\mathcal{E}_{12}$ by W. Write $\hat{M}_i = M_i^{1/2}$ $(i = 1, 2)$. Since $[\hat{M}_i, A] \subseteq J$ by Corollary 1, and for any $a \in A$, $W_g W_g^* a = a W_g W_g^*$ and there is a $b \in A$ such that $W_g^* a W_g = W_g^* W_g b$ by Definition 15,

$$a W_g \hat{M}_i W_g^* = W_g W_g^* a W_g \hat{M}_i W_g^* = W_g b \hat{M}_i W_g^* \equiv W_g \hat{M}_i W_g^* a$$

modulo J for all $i = 1, 2$. Similarly, $agg^{-1}(\hat{M}_i) \equiv gg^{-1}(\hat{M}_i)a$ modulo J for all $a \in A, i = 1, 2$. Since $[\hat{M}_i, W_g^* W_g] \in J$, one has $g(\hat{M}_i T) \equiv g(\hat{M}_i) g(T)$

modulo J for any operator $T \in \mathcal{L}(\mathcal{E}_{12})$. Thus

$$\begin{aligned}
&ag(\hat{M}_1(T_1 \otimes 1)) - agg^{-1}(\hat{M}_1(T_1 \otimes 1)) \\
&\equiv ag(\hat{M}_1)g(T_1 \otimes 1) - agg^{-1}(\hat{M}_1)gg^{-1}(T_1 \otimes 1) \\
&\equiv g(\hat{M}_1)ag(T_1 \otimes 1) - gg^{-1}(\hat{M}_1)agg^{-1}(T_1 \otimes 1) \\
&\equiv (g(\hat{M}_1) - gg^{-1}(\hat{M}_1))ag(T_1 \otimes 1) + \\
&\quad + gg^{-1}(\hat{M}_1)a(g(T_1 \otimes 1) - gg^{-1}(T_1 \otimes 1)) \\
&\equiv gg^{-1}(\hat{M}_1)gg^{-1}(a)gg^{-1}(g(T_1 \otimes 1) - gg^{-1}(T_1 \otimes 1)) \\
&\equiv gg^{-1}(\hat{M}_1 a(g(T_1 \otimes 1) - gg^{-1}(T_1 \otimes 1))) \equiv 0
\end{aligned}$$

modulo J, since $\hat{M}_1 A_1 \subseteq J$, for all $a \in A, g \in G$. A similar computation yields

$$ag(\hat{M}_2\widetilde{T}_2) - agg^{-1}(\hat{M}_2\widetilde{T}_2) \equiv agg^{-1}(\hat{M}_2(g(\widetilde{T}_2) - gg^{-1}(\widetilde{T}_2))) \equiv 0$$

modulo J, since $\hat{M}_2\varphi_1(g) \in J$, for all $a \in A, g \in G$. Thus we have proved that $a(g(T_{12}) - gg^{-1}(T_{12})) \in J$. Similar calculations show that also

$$(g(T_{12}) - gg^{-1}(T_{12}))a \in J.$$

Next,

$$\begin{aligned}
ag(1)\hat{M}_1(T_1 \otimes 1) - a\hat{M}_1(T_1 \otimes 1)g(1) &\equiv \hat{M}_1(ag(1)(T_1 \otimes 1) - a(T_1 \otimes 1)g(1)) \\
&\equiv 0
\end{aligned}$$

modulo J, since $\hat{M}_1 A_1 \subseteq J$, for all $a \in A, g \in G$. Note that $\hat{M}_2 gg^{-1}(\widetilde{T}_2) = \hat{M}_2 W_g W_g^* \widetilde{T}_2 W_g W_g^* \equiv W_g W_g^* \widetilde{T}_2 W_g W_g^* \hat{M}_2 = gg^{-1}(\widetilde{T}_2)\hat{M}_2$ modulo J for all $g \in G$. Since $\hat{M}_2\varphi_2(g) \subseteq J$ and $\psi_1(g)\hat{M}_2 \subseteq J$, one gets

$$\begin{aligned}
ag(1)\hat{M}_2\widetilde{T}_2 - a\hat{M}_2\widetilde{T}_2 g(1) &\equiv a\hat{M}_2(g(1)\widetilde{T}_2 - gg^{-1}(\widetilde{T}_2)) \\
&\quad + a\hat{M}_2(gg^{-1}(\widetilde{T}_2) - \widetilde{T}_2 g(1)) \\
&\equiv a(gg^{-1}(\widetilde{T}_2) - \widetilde{T}_2 g(1))\hat{M}_2 \equiv 0
\end{aligned}$$

modulo J for all $a \in A, g \in G$. It is thus evident that $a(g(1)T_{12} - T_{12}g(1)) \in J$, and by a quite similar computation, that $a(g^{-1}(1)T_{12} - T_{12}g^{-1}(1)) \in J$ for all $a \in A, g \in G$. We have checked that $(\mathcal{E}_{12}, T_{12}) \in \mathbb{E}^G(A, B_2)$.

Uniqueness. Consider two Kasparov products $(\mathcal{E}_{12}, F), (\mathcal{E}_{12}, F')$. In the above existence proof we defined sets A_1, A_2, Δ and $\Phi = \{\varphi_1, \varphi_2, \varphi_3, \psi_1, \psi_2\}$ with respect to a given T_2-connection $\widetilde{T}_2$. To express dependence on T_1 and $\widetilde{T}_2$, let us rename this sets as $A_1^{(T_1,\widetilde{T}_2)}, A_2^{(T_1,\widetilde{T}_2)}, \Delta^{(T_1,\widetilde{T}_2)}$ and $\Phi^{(T_1,\widetilde{T}_2)}$. Now define $J = \mathcal{K}(\mathcal{E}_{12})$, $A_1 = A_1^{(T_1,F)}$, A_2 to be the C^*-algebra (without G-action) generated by $A_2^{(T_1,F)} \cup A_2^{(T_1,F')} \cup \{F - F'\}$; $\Delta = \Delta^{(T_1,F)} \cup \Delta^{(T_1,F')}$, and $\Phi = \Phi^{(T_1,F)} \sqcup \Phi^{(T_1,F')}$. Applying Theorem 1 with these parameters we obtain operators M_1, M_2, and we set $F'' = M_1^{1/2}(T_1 \otimes 1) + M_2^{1/2}F$. One has $F'' \in \mathbb{E}^G(A,B)$, and $a[F,F'']a^* \geq 0$ and $a[F',F'']a^* \geq 0$ in $\mathcal{L}(\mathcal{E})/\mathcal{K}(\mathcal{E})$. (Confer the proof in Skandalis [14], Theorem 12.) The conclusion follows by Lemma 12.

Passage to $\widetilde{KK^G}$. If $(\mathcal{E}_1,T_1)$ or $(\mathcal{E}_2,T_2)$ is degenerate then so is their Kasparov product. (See the proof in Skandalis [14], Theorem 12.) We have to show that the Kasparov product respects operator homotopies. Let $(\mathcal{E}_1,T_1^t) \in \mathbb{E}^G(A,B_1)$ and $(\mathcal{E}_2,T_2^t) \in \mathbb{E}^G(B_1,B_2)$ ($t \in [0,1]$) be two operatorial homotopies. Choose a norm continuous path $\widetilde{T}_2^t \in \mathcal{L}(\mathcal{E}_{12})$ ($t \in [0,1]$) such that each $\widetilde{T}_2^t$ is a T_2^t-connection. Define $J = \mathcal{K}(\mathcal{E}_{12})$, $A_1 = A_1^{(T_1,\widetilde{T}_2^0)}$, A_2 to be the C^*-algebra (without G-action) generated by $\bigcup_{t\in[0,1]} A_2^{(T_1^t,\widetilde{T}_2^t)}$; $\Delta = \bigcup_{t\in[0,1]} \Delta^{(T_1^t,\widetilde{T}_2^t)}$, $\Omega = G \times [0,1]$, $\varphi_1(g,t) = g(\widetilde{T}_2^t) - gg^{-1}(\widetilde{T}_2^t)$ for all $(g,t) \in \Omega$, and similarly $\varphi_2,\varphi_3,\psi_1,\psi_2$ (see above). Entering these parameters in Theorem 1 yields operators M_1, M_2 and a desired operatorial homotopy $(\mathcal{E}_{12}, M_1^{1/2}(T_1^t \otimes 1) + M_2^{1/2}\widetilde{T}_2^t)$. □

7. The Kasparov cup-cap product

In this section all C^*-algebras are assumed to be Hilbert C^*-algebras. We define a slightly modified KK-theory for Hilbert C^*-algebras in that we redefine Hilbert modules and equivariant maps, taking into account the structure of Hilbert C^*-algebras. We denote the actions on a Hilbert C^*-algebra B by $g \mapsto g(b)$ and $g \mapsto g^{-1}(b)$ for all $b \in B, g \in G$. All $*$-homomorphisms between Hilbert C^*-algebras are assumed to be equivariant with respect to both actions g and g^{-1} (that is, we require that $\pi g = g\pi$ and $\pi g^{-1} = g^{-1}\pi$ for all $g \in G$).

Definition 20. A *G-Hilbert B-module* $\mathcal{E}$ over a Hilbert C^*-algebra B is a G-Hilbert B-module in the sense of Definition 11 satisfying

$$\langle U_g^* x, y\rangle = g^{-1}\langle x, U_g y\rangle, \quad U_g^*(xb) = U_g^*(x)g^{-1}(b)$$

for all $x,y \in \mathcal{E}, b \in B, g \in G$.

With respect to the last definition: The injectivity of g on B_g implies that $U_g^*U_g$ is self-adjoint by Lemma 3. This implication can be reversed, as $\langle U_g^*U_g x, y\rangle = g^{-1}g\langle x, U_g^*U_g y\rangle = g^{-1}g\langle U_g^*U_g x, y\rangle$.

If A is a Hilbert C^*-algebra then $P = g^{-1}g$ is idempotent and self-adjoint, and thus the range of P is an ideal in A. This shows, for instance, that $A = C[0,1]$ with $P(f) = g(f) = g^{-1}(f) = f(1)1$ for all $f \in A, g \in G$ is not a Hilbert C^*-algebra.[1] Actually,[2] the elements gg^{-1} and $g^{-1}g$ for any Hilbert C^*-algebra are elements of the center of the multiplier algebra by identifying $\mathcal{M}(A)$ with $\mathcal{L}(A)$; $[gg^{-1}, A] = 0$ is proven in Lemma 13.

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[1] I thank Christian Voigt for this "P is idempotent implies range(P) is an ideal?"—counterexample.

[2] Remarked by the referee.

Definition 21. If A is a Hilbert C^*-algebra and $\mathcal{E}$ is a G-Hilbert B-module, then a $*$-homomorphism $\pi : A \to \mathcal{L}(\mathcal{E})$ is called *equivariant* if it is equivariant in the sense of Definition 15 and

$$U_g^* U_g \pi(g^{-1}(a)) = U_g^* \pi(a) U_g$$

holds for all $a \in A, g \in G$.

Though we have now redefined G-Hilbert modules and equivariant representations for Hilbert C^*-algebras, we can more or less continue with Sections 4–6 without change. Indeed, we only have to ensure that all constructions related to Hilbert modules and equivariant representations enjoy the above redefinitions, and these are only the tensor product constructions and direct sums of Hilbert modules.

Lemma 13. *If A, B are Hilbert C^*-algebras and $\pi : A \to B$ is a homomorphism then $\widetilde{\pi} : A \to \mathcal{L}(B)$, $\widetilde{\pi}(a)(b) = \pi(a)b$ $(a \in A, b \in B)$ is an equivariant homomorphism. In particular, B is a Hilbert (B, B)-bimodule.*

Proof. For instance, by Lemma 3 $U_g U_g^*$ is selfadjoint and thus

$$U_g U_g^* \widetilde{\pi}(a)(b) = U_g U_g^*(\pi(a)b) = (U_g U_g^*(\pi(a^*)))^* U_g U_g^*(b) = \pi(a) U_g U_g^*(b). \qquad \square$$

By Lemmas 13 and 4 we may form the tensor product $\mathcal{E} \otimes_{B_1} B_2$ if $\mathcal{E}$ is a Hilbert B_1-module and $\varphi : B_1 \to B_2$ a homomorphism between Hilbert C^*-algebras B_1 and B_2.

Lemma 14. *If $\mathcal{E}_1$ is a Hilbert (A, B_1)-bimodule, $\mathcal{E}_2$ a Hilbert (B_2, B_3)-bimodule and $f : B_1 \to B_2$ is a homomorphism then*

$$\pi : \mathcal{E}_1 \otimes_{B_1} B_2 \otimes_{B_2} \mathcal{E}_2 \to \mathcal{E}_1 \otimes_{B_1} \mathcal{E}_2, \quad \pi(x_1 \otimes b_2 \otimes x_2) = x_1 \otimes f(b_2) x_2$$

is an isomorphism of Hilbert (A, B_3)-bimodules.

If A is unital then $\sigma : A \otimes_A \mathcal{E} \to \mathcal{E}$, $\sigma(a \otimes x) = ax$ is an isomorphism of Hilbert (A, B_1)-bimodules.

Proof. Without the G-structure this is well established. It is straightforward to compute that π and σ intertwine the G-actions. $\square$

Definition 22. An element $(\mathcal{E}, T) \in \mathbb{E}^G(A, B[0,1])$ generates a path $t \mapsto (\mathcal{E}_t, T_t) \in \mathbb{E}^G(A, B)$ $(t \in [0,1])$ obtained by evaluation at each $t \in [0,1]$, that is, $\mathcal{E}_t = \mathcal{E} \otimes_{B \otimes C[0,1]} B, T_t = T \otimes 1$, where $B \otimes C[0,1] \to B$ is evaluation at time t. This path and the pair $(\mathcal{E}, T)$ itself will be called a *homotopy* between $(\mathcal{E}_0, T_0)$ and $(\mathcal{E}_1, T_1)$. The set $KK^G(A, B)$ is defined as the quotient of $\mathbb{E}^G(A, B)$ by the equivalence relation given by homotopy.

Proposition 2. *$KK^G(A, B)$ is a quotient of $\widetilde{KK^G}(A, B)$. $KK^G(A, B)$ and $\widetilde{KK^G}(A, B)$ are abelian groups with addition given by direct sum.*

Proof. One proves this along the lines of [7], Section 4, Theorem 1, or [14], Proposition 4. $\square$

Definition 23. Let A_1, A_2, B be Hilbert C^*-algebras, and $f : A_1 \to A_2$ a homomorphism. Then f induces a map $f^* : \mathbb{E}^G(A_2, B) \to \mathbb{E}^G(A_1, B)$ by $f^*((\mathcal{E}, T)) = (f^*(\mathcal{E}), T)$, where $f^*(\mathcal{E})$ is the Hilbert (A_1, B)-bimodule $\mathcal{E}$ with A_1-action $A_1 \xrightarrow{f} A_2 \to \mathcal{L}(\mathcal{E})$. The map f^* passes to the quotients KK^G and $\widetilde{KK^G}$, and we keep the notation f^* for these maps.

Definition 24. Let A, B_1, B_2 be Hilbert C^*-algebras and $g : B_1 \to B_2$ a homomorphism. Then g induces a map $g_* : \mathbb{E}^G(A, B_1) \to \mathbb{E}^G(A, B_2)$ given by $g_*((\mathcal{E}, T)) = (\mathcal{E} \otimes_{B_1} B_2, T \otimes 1)$. The map g_* passes to the quotients KK^G and $\widetilde{KK^G}$, and we keep the notation g_* for these maps.

For Definition 24 one needs:

Lemma 15. *Let $\mathcal{E}$ be a Hilbert B_1-module, B_2 a Hilbert C^*-algebra, and $T \in \mathcal{K}(\mathcal{E})$. Then $T \otimes 1$, $T \otimes U_g U_g^*$, $T \otimes U_g^* U_g \in \mathcal{K}(\mathcal{E} \otimes_{B_1} B_2)$ for all $g \in G$.*

Proof. The proof is the same as in [7], page 523, or [5], Lemma 1.2.8, taking $U_g U_g^*$, $U_g^* U_g$ rather than 1. □

Definition 25. Let D be a σ-unital Hilbert C^*-algebra. Define

$$\tau_D : \mathbb{E}^G(A, B) \to \mathbb{E}^G(A \otimes D, B \otimes D)$$

by $\tau_D(\mathcal{E}, T) = (\mathcal{E} \otimes D, T \otimes 1)$ (where $\mathcal{E} \otimes D$ denotes the skew tensor product). The map τ_D passes to the quotients KK^G and $\widetilde{KK^G}$, and these homomorphisms are also denoted by τ_D.

Theorem 3. *There is a Kasparov product as stated in Theorem 2, and this product also induces a bilinear map*

$$\otimes_{B_1} : KK^G(A, B_1) \otimes KK^G(B_1, B_2) \to KK^G(A, B_2).$$

Proof. That the Kasparov product respects homotopy may be proved in the same way as in Skandalis [14], Theorem 12. □

Proposition 3. *Let A_1, A_2 be separable Hilbert C^*-algebras, and $f : A_1 \to A_2$ and $g : B_1 \to B_2$ homomorphisms.*

If $x \in KK^G(A_2, B)$ and $y \in KK^G(B, B_1)$ (or $\widetilde{KK^G}$) then

$$f^*(x) \otimes_B y = f^*(x \otimes_B y).$$

If $x \in KK^G(A_1, B_1)$ and $y \in KK^G(B_2, B_3)$ (or $\widetilde{KK^G}$) then

$$g_*(x) \otimes_{B_2} y = x \otimes_{B_1} g^*(y).$$

If $x \in KK^G(A_1, B)$ and $y \in KK^G(B, B_1)$ (or $\widetilde{KK^G}$) then

$$g_*(x \otimes_B y) = x \otimes_B g_*(y).$$

Proof. The proof is the same as [14], Proposition 13. For the second identity one uses Lemma 14. □

Definition 26. Let A_2, B_1 be σ-unital Hilbert C^*-algebras and A_1, A_2 be separable. The *cup-cap product*

$$\otimes_D : KK^G(A_1, B_1 \otimes D) \otimes KK^G(D \otimes A_2, B_2) \to KK^G(A_1 \otimes A_2, B_1 \otimes B_2)$$

is defined by the formula $x_1 \otimes_D x_2 = \tau_{A_2}(x_1) \otimes_{B_1 \otimes D \otimes A_2} \tau_{B_1}(x_2)$. The cup-cap product for $\widetilde{KK^G}$ is defined in the same way.

Lemma 16. *If $x \in KK^G(A, B)$ (or $\widetilde{KK^G}(A, B)$) and $f : A' \to A$ and $g : B \to B'$ are homomorphisms, then $\tau_D(f^*(x)) = (f \otimes 1)^*(\tau_D(x))$ and $\tau_D(g_*(x)) = (g \otimes 1)_*(\tau_D(x))$.*

Proof. Let $x = (\mathcal{E}, T)$. For the second claim we use

$$(\mathcal{E} \otimes D) \otimes_{B \otimes D} (B' \otimes D) \cong (\mathcal{E} \otimes_B B') \otimes (D \otimes_D D)$$

(see Kasparov [7], Section 2, page 523). □

Lemma 17. *If $x \in KK^G(A, B)$ and $f : D_1 \to D_2$ is a homomorphism then $(1 \otimes f)^*(\tau_{D_2}(x)) = (1 \otimes f)_*(\tau_{D_1}(x))$.*

Proof. One checks that the proof of Skandalis [14], Lemma 7, works also in our setting. □

Proposition 4. *Let B_1, B_1', B_2', D' be σ-unital Hilbert C^*-algebras and A_1, A_1', A_2, A_2' be separable. Let $f_1 : A_1' \to A_1$, $f_2 : A_2' \to A_2$, $g_1 : B_1 \to B_1'$, $g_2 : B_2 \to B_2'$, $h : D \to D'$ be homomorphisms. Then the cup-cap product of Definition 26 satisfies*

$$f_1^*(g_1 \otimes 1)_*(x_1) \otimes_D (1 \otimes f_2)^* g_{2*}(x_2) = (f_1 \otimes f_2)^*(g_1 \otimes g_2)_*(x_1 \otimes_D x_2),$$
$$(h \otimes 1)_*(x_1) \otimes_{D'} x_2 = x_1 \otimes_D (h \otimes 1)^*(x_2),$$

with the restriction that if f_2 or g_1 are not trivial (i.e., are not the identity map) then this only holds in KK^G.

Proof. This is some computation by applying the formulas of Proposition 3 and Lemmas 16 and 17. □

Let $1 \in \widetilde{KK^G}(\mathbb{C}, \mathbb{C})$ (or $KK^G(\mathbb{C}, \mathbb{C})$) be given by the Hilbert $(\mathbb{C}, \mathbb{C})$-bimodule $\mathbb{C}$ with trivial grading and action, and the zero operator.

Proposition 5. *Let A be separable and $x \in KK^G(A, B)$ (or $\widetilde{KK^G}(A, B)$). Then $x \otimes_{\mathbb{C}} 1 = x$. If A is unital and $g(1_A) = 1_A$ for all $g \in G$ then $1 \otimes_{\mathbb{C}} x = x$.*

Proof. One proves this along the lines of [14], Proposition 17. □

Theorem 4. *Suppose that A is a separable and B a σ-unital Hilbert C^*-algebra. Then the map $\widetilde{KK^G}(A, B) \to KK^G(A, B)$ is an isomorphism.*

Proof. The proof is the same as Theorem 19 of Skandalis [14]. □

Theorem 5. *Assume that B_1, B_2 are σ-unital Hilbert C^*-algebras, A_1, A_2, A_3, D_1 are separable Hilbert C^*-algebras, and*

$$x_1 \in KK^G(A_1, B_1 \otimes D_1),$$
$$x_2 \in KK^G(D_1 \otimes A_2, B_2 \otimes D_2),$$
$$x_3 \in KK^G(D_2 \otimes A_3, B_3).$$

Then

$$(x_1 \otimes_{D_1} x_2) \otimes_{D_2} x_3 = x_1 \otimes_{D_1} (x_2 \otimes_{D_2} x_3).$$

Proof. One proves this along the lines of [14], Theorem 21. □

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